Rapid Communications   Editors' Suggestion

Equipartition of the entanglement entropy

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The entanglement in a quantum system that possesses an internal symmetry, characterized by the S^z magnetization or $U(1)$ charge, is distributed among different sectors. The aim of this Rapid Communication is to gain a deeper understanding of the contribution to the entanglement entropy in each of those sectors for the ground state of conformal invariant critical one-dimensional systems. Surprisingly, we find that the entanglement entropy is equally distributed among the different magnetization sectors. Its value is given by the standard area law violating logarithmic term that depends on the central charge c, minus a double logarithmic correction related to the zero-temperature susceptibility. This result provides a method to estimate simultaneously the central charge c and the critical exponents of $U(1)$-symmetric quantum chains. The method is numerically simple and gives precise results for the spin-$\frac{1}{2}$ quantum XXZ chain. We also compute the probability distribution of the magnetization in contiguous sublattices.

Introduction. In recent years the study of entanglement in quantum many-body systems, and in quantum field theory, has been carried out intensively. As a result of it, many links have been established among previously disconnected areas of physics, computer science, and mathematics. These studies have led to a quantum information perspective of phase transitions and topological order, topics that belonged traditionally to condensed matter physics and statistical mechanics [1]. For most of the quantum critical systems in one spatial dimension, a precise characterization of entanglement has been achieved owing to the powerful methods of conformal field theory (CFT). In these systems, the area law of the von Neumann entanglement entropy (EE) of the ground state (GS), in a single interval [2], develops a logarithmic violation parametrized by the central charge c of the underlying CFT [3–7].

Employing ultracold atoms loaded in optical lattices, it is nowadays possible to simulate many one-dimensional quantum systems [8]. Quite recently, a measurement of entanglement was done using a one-dimensional optical lattice composed of a few ^{87}Rb atoms [9]. Since the number of atoms involved in these experiments is small, finite-size effects play an important role in measuring the EE. Fortunately, CFT predicts the leading finite-size correction of the Rényi entropy of the GS of a chain of L sites, which is given by [3–7]

$$S_{A,\text{CFT}}^{(n)} = c_n^{(b)} + \frac{c}{3b}\left(1+\frac{1}{n}\right)\ln\left[\frac{bL}{\pi}\sin\frac{\pi x}{L}\right], \quad (1)$$

where x is the size of the subsystem A, $b = 1,2$ for periodic/free boundary conditions (PBCs/FBCs), and $c_n^{(b)}$ is a nonuniversal constant. The EE corresponds to the choice $n = 1$.

Besides the central charge c, the entanglement properties of a quantum chain can also depend on the critical exponents or the operator content of the underlying CFT. This dependence

was previously observed in the entanglement for multiple intervals [10], scaling corrections [11], parity effects [12], and in the primary states and descendants of the CFT [13].

The aim of this Rapid Communication is to split the total entanglement into contributions coming from disjoint symmetry sectors. We carry out this analysis for the critical quantum chains that have a $U(1)$ symmetry that, in the scaling limit, develops a $U(1)$ Kac-Moody algebra (KM). Those are the models with a critical line with continuously varying exponents [14]. As a by-product of our calculation, we present herewith a simple method to evaluate simultaneously the central charge and the critical exponents for this class of quantum chains. We hope that these results could be tested in ultracold atom experiments as the ones performed in Ref. [9].

Let us start with a general quantum chain with L sites, whose Hamiltonian $\hat{H} = \sum_{i=1}^{L} h_{i,i+1}$ commutes with the magnetization operator $S^z = \sum_{i=1}^{L} S_i^z$, where S_i^z are spin s matrices. Let $|\psi\rangle$ be a common eigenstate of $\hat{H}$ and S^z, with eigenvalues E and M, respectively. We split the chain into disjoints blocks A ($i = 1,\ldots,x$) and B ($i = x+1,\ldots,L$), and compute the reduced density matrix $\rho_A = \text{tr}_B \rho$ ($\rho = |\psi\rangle\langle\psi|$). The magnetization operator also splits into the sum $S^z = S_A^z + S_B^z$. Then, tracing over the Hilbert subspace of the block B in the equation $[S^z, \rho] = 0$ yields $[S_A^z, \rho_A] = 0$. This implies

$$\rho_A = \oplus_m \tilde{\rho}_{A,m} = \oplus_m p_{A,m} \rho_{A,m}, \quad (2)$$

where $-sx \leqslant m \leqslant sx$, $\rho_{A,m}$ is a density matrix with eigenvalue m of S_A^z, and $p_{A,m} = \text{tr}\,\tilde{\rho}_{A,m} \geqslant 0$ is the probability of finding m in a measurement of S_A^z.

The decomposition (2) is implemented normally in numerical methods, such as the density matrix renormalization group (DMRG) [15] and matrix product state (MPS), multiscale entanglement renormalization ansatz (MERA), etc. [16], to

reduce the memory resources needed for high-precision results. The latter equation implies

$$S_A = \sum_m p_{A,m} S_{A,m} + H_A, \tag{3}$$

where $S_A = -\text{tr}\,\rho_A \ln \rho_A$, $S_{A,m} = -\text{tr}\,\rho_{A,m} \ln \rho_{A,m}$, and $H_A = -\sum_m p_{A,m} \ln p_{A,m}$. Equation (3) means that the quantum entropy in the subsystem A is greater, in general, than the weighted sum of the entropies of the different magnetization sectors. This fact expresses the holistic nature of quantum entanglement. Actually, Eq. (3) can be seen as a special case of the general Holevo theorem in quantum information theory [17,18] that states that the maximum information we can extract from a general mixed state, $\rho = \sum_m p_m \rho_m$, is given by the difference $S(\rho) - \sum_m p_m S(\rho_m)$. In the case of Eq. (3) the maximum information is given by the Shannon entropy.

The critical chains studied in this Rapid Communication reveal surprisingly that the contributions $S_{A,m}$ to the entropy S_A are equal for low values of the magnetization $|m|$. We call this situation *equipartition of entanglement entropy*.

Analytic predictions. The previous discussion applies to any quantum lattice system with a $U(1)$ symmetry. In the following, we shall derive analytic predictions of $p_{A,m}$ and $S_{A,m}$, for critical spin-s quantum chains, such as the spin-s XXZ model. In a block A with x sites, Eq. (2) can be inverted to obtain

$$p_{A,m} \rho_{A,m} = \frac{1}{2sx+1} \sum_{n=-sx}^{sx} e^{\frac{2\pi i n}{2sx+1}(S_A^z - m)} \rho_A, \tag{4}$$

where the sum projects the density matrix ρ_A into the sector with $S_A^z = m$. In the limit $x \gg 1$, Eq. (4) becomes

$$p_{A,m} \rho_{A,m} = \int_{-1/2}^{1/2} d\phi\, e^{2\pi i \phi(S_A^z - m)} \rho_A. \tag{5}$$

Taking the trace over the states in A, and using $\text{tr}_A\,\rho_A = \text{tr}_A\,\rho_{A,m} = 1$, gives the probability distribution

$$p_{A,m} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\phi\, e^{-2\pi i m\phi}\, \text{tr}_A(e^{2\pi i \phi S_A^z} \rho_A). \tag{6}$$

Similarly, the nth power of Eq. (5) yields

$$p_{A,m}^n \text{tr}_A \rho_{A,m}^n = \prod_{j=1}^n \int_{-1/2}^{1/2} d\phi_j\, e^{-2\pi i m \sum_{j=1}^n \phi_j}$$
$$\times \text{tr}_A(e^{2\pi i S_A^z \sum_{j=1}^n \phi_j} \rho_A^n), \tag{7}$$

that, together with (6), provides the Rényi entropies $S_{A,m}^{(n)} = \frac{1}{1-n} \log \text{tr}_A \rho_{A,m}^n$. To find $\text{tr}_A(e^{2\pi i \phi S_A^z} \rho_A^n)$, we extend the general formalism to construct the entanglement Hamiltonian in CFTs [19] that we summarize below.

The reduced density matrix ρ_A in CFT is given by

$$\rho_A = \frac{1}{Z_1} e^{-2\pi K_A}, \quad Z_1 = \text{tr}_A\, e^{-2\pi K_A}, \tag{8}$$

where $K_A = \int_A dx\, T_{00}(x)/f'(x)$ is the entanglement Hamiltonian and T_{00} is a component of the stress tensor [19]. $f(z)$ is the conformal map from the Euclidean space-time, with a cut along the interval A and two boundaries, into an annulus of width W and height 2π. Taking the trace of the nth power in

(8) yields

$$\text{tr}_A\, \rho_A^n = \frac{Z_n}{Z_1^n}, \quad Z_n = \text{tr}_A\, e^{-2\pi n K_A}, \tag{9}$$

where Z_n is the Euclidean partition function of an n-sheeted cover of the original space-time with conical singularities around the end points of A. We propose the following extension of Eq. (9) to CFTs with a $U(1)$ KM symmetry,

$$\text{tr}_A(e^{2\pi i \phi J_0} \rho_A^n) = \frac{Z_n(\phi)}{Z_1^n} = \frac{\text{tr}_A(e^{2\pi i \phi J_0} e^{-2\pi n K_A})}{Z_1^n},$$

where J_0 is the zero mode of the $U(1)$ current $J(z)$. $Z_n(\phi)$ is the partition function, given in Eq. (9), but with fugacity $2\pi i \phi$. Since the eigenvalues of K_A are given by $\pi(\Delta_{p,m} - c/24)/W$, where $\Delta_{p,m}$ are the dimensions of the boundary operators [19] and m is the eigenvalue of J_0, we obtain

$$Z_n(\phi) = q^{-nc/24} \sum_{p,m} d_{p,m}\, q^{n\Delta_{p,m}} e^{2\pi i m \phi}, \tag{10}$$

where $q = e^{-2\pi^2/W}$ and $d_{p,m}$ is the degeneracy of the boundary operator (p,m). In the case of the ground state of the CFT, with periodic/free boundary conditions, the width W should be fixed to [recall Eq. (1)]

$$W = \frac{2}{b} \ln[b L_c(x)], \quad L_c(x) = \frac{L}{\pi} \sin \frac{\pi x}{L}. \tag{11}$$

For the thermal state at temperature $1/\beta$, $W = 2\ln\left(\frac{\beta}{\pi} \sinh \frac{\pi x}{\beta}\right)$ [19].

The first application of the analytic formula (10) is the Luttinger liquid which is a CFT with $c = 1$ and a $U(1)$ symmetry generated by the current operator $J(z) = i\sqrt{K} \partial\varphi(z)$, where $\varphi(z)$ is a chiral boson and K a constant. A state with charge $m \in Z + a$ (with $a = 0, \frac{1}{2}$) is associated with the vertex operator $e^{im\varphi(z)/\sqrt{K}}$, and has conformal weight $h_m = m^2/(2K)$. The partition function (10) reads in this case

$$Z_n(\phi) = \frac{1}{\eta(q^n)} \sum_{m \in Z+a} q^{\frac{nm^2}{2K}} e^{2\pi i m \phi} = \frac{\theta_{a,0}\left(\phi, \frac{n\tau}{K}\right)}{\eta(q^n)}, \tag{12}$$

where $\tau = i\pi/W$, $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n)$ is the Dedekind eta function, and $\theta_{a,c}(z,\tau) = \sum_{n \in Z} e^{\pi i \tau (n+a)^2 + 2\pi i n(z+c)}$ is a Jacobi theta function with characteristics. In the limit $L \gg 1$, one has $W \gg 1$ and therefore $q \sim 1$, so that a large number of terms contribute to Eq. (12). However, using the modular transformation $\tau \to -1/\tau$ [20],

$$\theta_{a,0}(z,\tau) = \sqrt{\frac{i}{2\tau}} e^{-i\frac{z^2}{2\tau}} \sum_{a'=0,\frac{1}{2}} e^{4\pi i a a'} \theta_{a',0}\left(-\frac{z}{\tau}, -\frac{2}{\tau}\right) \tag{13}$$

and $\eta(-1/\tau) = \sqrt{\tau/i}\, \eta(\tau)$, we obtain

$$Z_n(\phi) \sim e^{\frac{W}{n}\left(\frac{1}{12} - K\phi^2\right)}. \tag{14}$$

For special values of K, the CFT is rational and $Z_n(\phi)$ becomes a finite sum (e.g., if K^2 is an even number [21]),

$$Z_n(\phi) = \sum_j n^j \chi_j(q^n, \phi), \tag{15}$$

where n^j are non-negative integers that depend on the boundary conditions on the annulus. The coefficients $\chi_j(q^n, \phi)$ are denoted nonspecialized characters that are labeled by the representation j of an extended KM algebra. Their modular transformations [21,22] have been used to study the correlators in the multichannel Kondo model [23] and bulk susceptibilities [24,25].

The second application we report deals with the spin s-isotropic exactly solvable model [26]. This is a critical system described by the Wess-Zumino-Witten (WZW) model $SU(2)_k$ at level $k = 2s$, and central charge $c = \frac{3k}{k+2}$. The model contains a similar $U(1)$ current operator, which is now $J^z(z) = i\sqrt{\frac{k}{2}}\partial\varphi(z)$. The primary fields are labeled by the total spin $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. The partition function (15) is a linear combination of the nonspecialized characters $\chi_j(q, \phi)$ of $SU(2)_k$ and using their modular transformations [21,22], we obtain [27]

$$Z_n(\phi) \sim e^{\frac{W}{n}\left(\frac{c}{12} - \frac{k}{2}\phi^2\right)}. \tag{16}$$

For the spin-$\frac{1}{2}$ chain, $c = 1$ and $k = 1$, and then (16) coincides with (14), for $K = \frac{1}{2}$, where the Luttinger liquid has an enhanced $SU(2)_1$ symmetry.

We can summarize both previous applications as

$$\text{tr}_A\left(e^{2\pi i\phi J_0}\rho_A^n\right) \propto e^{W\left[\frac{c}{12}\left(\frac{1}{n}-n\right)-\frac{\mathcal{K}}{n}\phi^2\right]}, \tag{17}$$

where $\mathcal{K} = K$ or $\mathcal{K} = \frac{k}{2}$ for the Luttinger liquid and the spin-s chain, respectively. Quite remarkably, this parameter satisfies the *universal relation* $\mathcal{K} = \pi v_s \chi$, where v_s is the spin-wave velocity, and χ is the zero-field susceptibility at zero temperature of these distinct spin chains [24,25]. The case $n = 1$ in Eq. (17) coincides with the full counting statistics (FCS) for the subsystem magnetization S_A^z [28,29]. Taking $n > 1$ provides a generalized FCS where the entanglement properties are taken into account.

Let us derive some consequences of Eq. (17) for finite chains with PBC [i.e., $b = 1$ in (11)]. For $n = 1$, we obtain the probability distribution

$$p_{A,m} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\phi \, e^{-2\pi i\phi m - \kappa\phi^2}, \quad \kappa = 2\mathcal{K}\ln[gL_c(x)]. \tag{18}$$

The constant g comes from the lattice cutoff in the chains that has not been included in (17). The highest probability corresponds to $m = 0$,

$$p_{A,0} = \sqrt{\frac{\pi}{\kappa}}\text{erf}\left(\sqrt{\kappa}/2\right), \tag{19}$$

where $\text{erf}(x)$ is the error function. $p_{A,m}$ can be approximated by replacing the integration limits in (18) by $\pm\infty$,

$$p_{A,m} \simeq \sqrt{\frac{\pi}{\kappa}}e^{-(\pi m)^2/\kappa}, \tag{20}$$

which is a distribution whose Shannon entropy,

$$H_A \sim \frac{1}{2}\ln\left(2\mathcal{K}\ln[gL_c(x)]\right), \tag{21}$$

quantifies our knowledge after measurement of sublattice magnetization. For a half block it will go as $\ln\ln L$, a remarkably slow increase with L. It is interesting to observe

that the relation $\mathcal{K} = \pi v_s \chi$ obtained in Refs. [24,25] can be derived from Eq. (20). The zero-field susceptibility is given by $\chi = \langle m^2 \rangle/(xT)$, where m is the magnetization of a region of length x and T is the temperature. Using Eq. (20), one finds $\langle m^2 \rangle = \frac{\kappa}{2\pi^2}$, where $\kappa = \mathcal{K}W = 2\ln\left(\frac{\beta}{\pi}\sinh\frac{\pi x}{\beta}\right)$ [notice that the expression of κ, defined in Eq. (18), is for $T = 0$, where $W = 2\ln L_c(x)$]. In the limit $x \gg \beta$ one finds $\langle m^2 \rangle \simeq \mathcal{K}xT/\pi$, which gives the relation $\mathcal{K} = \pi v_s \chi$. The same sort of computation provides, for example, the Gibbs entropy for the subsystem of size x that is given by $S_A = \frac{\pi c}{3}T$. Note also that since Eq. (17) is related with the zero-field susceptibility (which is related with the spin fluctuations), we would expect a connection between the entanglement and spin fluctuations. This is very interesting, since measurements of fluctuations are easier to do than the entanglement ones. Indeed, recently some authors have made this connection [29] (see also Ref. [30]).

In the limit $\kappa \gg 1$, the Rényi entropies can be found using (7) and (17), and behave asymptotically as

$$\frac{\text{tr}\,\rho_{A,m}^n}{\text{tr}\,\rho_A^n} \propto \kappa^{\frac{1}{2}(n-1)}, \tag{22}$$

which implies

$$S_{A,m}^{(n)} \simeq S_{A,\text{CFT}}^{(n)} - \frac{1}{2}\ln\kappa. \tag{23}$$

Hence, the EE of the density matrix $\rho_{A,m}$ is dominated by the EE of the full density matrix ρ_A, with a reduction $-\frac{1}{2}\ln(2\mathcal{K}\ln[gL_c(x)])$ that is independent of the quantum number m. This is the *equipartition* of the EE mentioned above.

Numerical tests. We have considered the spin-$\frac{1}{2}$ XXZ Hamiltonian with PBC,

$$H = \sum_{n=1}^{L}\left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z\right), \tag{24}$$

in the critical regime $-1 < \Delta \leqslant 1$, whose low energy is described by a Luttinger liquid with parameter $K = \frac{1}{2}[1 - \frac{1}{\pi}\arccos(\Delta)]^{-1}$ [14,31]. Using the DMRG method we obtained the GS and the reduced density matrices ρ_A and $\rho_{A,m}$. We consider system sizes up to $L = 600$ under PBC and keeping up to $\tilde{m} = 3000$ states per block in the final sweep. We have done $\sim$6–10 sweeps, and the discarded weight was typically 10^{-10}–10^{-12} at that final sweep. To verify Eq. (17), we write it as

$$\ln\text{tr}\left(e^{2\pi i\phi S_A^z}\rho_A^n\right) = -\gamma_n(\phi)\ln[g_n L_c(x)] + d_n, \tag{25}$$

where g_n, d_n are nonuniversal constants ($g_1 \equiv g$), and

$$\gamma_n(\phi) = \alpha_n + \beta_n(\phi), \quad \alpha_n = \frac{c}{6}\left(n - \frac{1}{n}\right), \quad \beta_n(\phi) = \frac{2\mathcal{K}}{n}\phi^2. \tag{26}$$

In our opinion, Eqs. (25) and (26) give us the most simple and numerically easier method to evaluate the central charge and the Luttinger parameter from reduced density matrices. The evaluation of $\rho_{A,m}$ with the DMRG does not require any additional numerical effort because it is already calculated in the evaluation of ρ_A. For $n = 1$, Eq. (25) yields a ϕ *extension* of the trace that provides the Luttinger parameter ($\alpha_1 = 0$, $\beta_1 = 2\mathcal{K}\phi^2$), and for $n > 1$ it gives the central charge [$\alpha_n = c(n - 1/n)/6$].
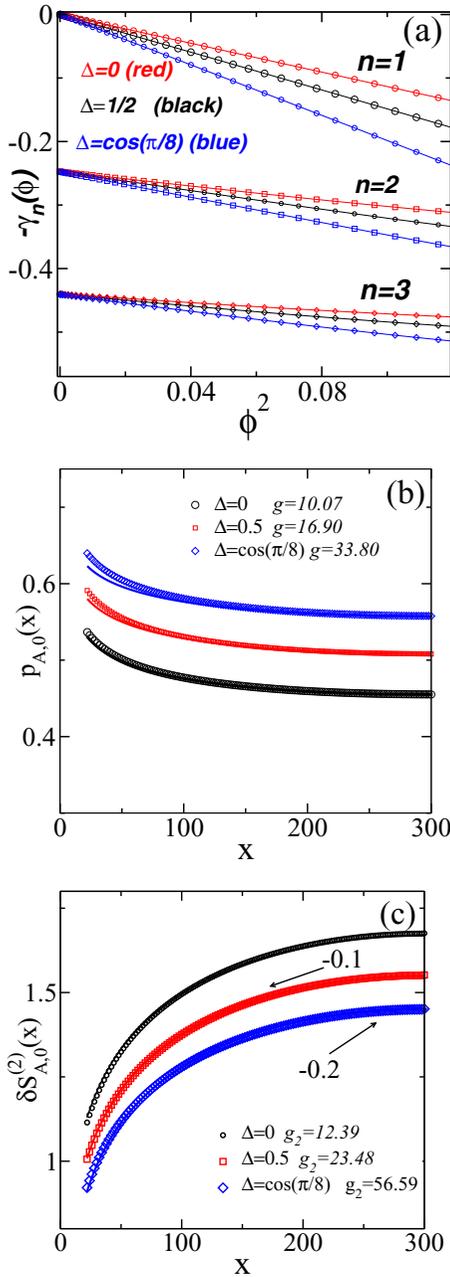
FIG. 1. DMRG results for XXZ quantum chain for several values of Δ and a chain of $L = 600$ sites. (a) $\gamma_n(\phi)$ vs ϕ^2 for $n = 1, 2,$ and 3. The data were obtained considering, for each ϕ, the sublattice sizes $x \in [80, 300]$. (b) $p_{A,0}(x)$ vs x. (c) $\delta S_{A,0}^{(2)}$ vs x. The numbers $-0.1, -0.2$ are vertical shifts to facilitate the picture. The values of g and g_2 in (b) and (c) are obtained from fitting the data of (a) to Eq. (25). The symbols are the numerical data and the lines in (b) and (c) are the theoretical predictions.

The DMRG data show clearly, for each ϕ, the linear dependence on $\ln L_c(x)$ in Eq. (25). We illustrate in Fig. 1(a) the function $\gamma_n(\phi)$ obtained from Eq. (25) for several values of Δ and n. Table I summarizes the results for the estimated values of α_n and $\beta_n(\phi)$. Notice the excellent agreement between numerical and theoretical results.

TABLE I. The values of α_n and β_n/ϕ^2 obtained by fitting the DMRG data of Fig. 1(a) to Eq. (26) for the spin-$\frac{1}{2}$ XXZ chain for $\Delta = 0, 0.5, \cos\left(\frac{\pi}{8}\right)$. The values in parentheses are the predicted ones in Eq. (26).

Δ		$n = 1$	$n = 2$	$n = 3$
0	α_n	0.00 (0)	0.25 (0.25)	0.44 (0.444...)
0	β_n/ϕ^2	1.99 (2)	0.99 (1)	0.66 (0.666...)
0.5	α_n	0.00 (0)	0.25 (0.25)	0.44 (0.444...)
0.5	β_n/ϕ^2	1.48 (1.5)	0.75 (0.75)	0.48 (0.5)
$\cos(\pi/8)$	α_n	0.00 (0)	0.25 (0.25)	0.44 (0.444...)
$\cos(\pi/8)$	β_n/ϕ^2	1.13 (1.1428)	0.57 (0.5714)	0.35 (0.3805)

We also test Eq. (19) for the XXZ spin-$\frac{1}{2}$ chain using g as a fitting parameter. In the case of the XX model, the exact value is given by $g = 2e^{1+\gamma} = 9.68\ldots$ [27]. Figure 1(b) illustrates the excellent agreement between the numerical and the analytical prediction (19) for $p_{A,0}$ as a function of x for three values of Δ.

Finally, we present the results for the Rényi-2 entropy $S_{A,0}^{(2)}$. We found that $S_{A,0}^{(2)} = S_A^{(2)} - f(\kappa, \kappa_2)$ [27], where

$$f(\kappa, \kappa_2) = \ln\left[\frac{\kappa}{\pi\kappa_2} \frac{\left[\sqrt{2\pi\kappa_2}\,\mathrm{erf}\left(\frac{\sqrt{\kappa_2}}{2}\right) - 2 + 2e^{-\kappa_2/2}\right]}{\left[\mathrm{erf}\left(\frac{\sqrt{\kappa}}{2}\right)\right]^2}\right]$$

and $\kappa_2 = 2K \ln[g_2 L_c(x)]$. The asymptotic behavior was already shown in (23). Figure 1(c) shows the DMRG data for $\delta S_{A,0}^{(2)} = S_{A,0}^{(2)} + f(\kappa, \kappa_2) = c/4 \ln[2L_c(x)] + c_2^2$ in the XXZ spin-$\frac{1}{2}$ chain with $L = 600$, where we use the values of g_1 and g_2 found in Fig. 1(a), shown in the insets of Figs. 1(b) and 1(c).

Twist fields. Although we have derived the analytic results using the modular properties of nonspecialized characters, we think that the twist field method of Refs. [6,32] can be extended to this case. This is suggested by Eq. (17), whose right-hand side is proportional to $L_c(x)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \times L_c(x)^{-\frac{2K}{n}\phi^2}$. The first factor comes from the correlator $\langle \mathcal{T}_n \mathcal{T}_{-n} \rangle$ of the twist field $\mathcal{T}_{\pm n}$ with scaling dimensions $\Delta_{\mathcal{T}_n} = \bar{\Delta}_{\mathcal{T}_n} = \frac{1}{24}\left(n - \frac{1}{n}\right)$, and the second factor corresponds to the correlator $\langle O_{\phi,n} O_{-\phi,n} \rangle$ of a field $O_{\phi,n}$ with scaling dimensions $\Delta_{O_{\phi,n}} = \bar{\Delta}_{O_{\phi,n}} = \frac{K}{2n}\phi^2$. The field $O_{\phi,n}$ is a generalized string-order parameter with angle $2\pi\phi$ [33], which for $n = 1$ and $\phi = \frac{1}{2}$ has the two-point correlator described above [34]. We expect that the generalized string-order fields provide an extension of the twist fields that are reminiscent of the ones used in nonunitary CFTs where the ground state is not the CFT vacuum [35]. Double log corrections to the EE have been discussed in the context of nonunitary CFTs [35,36], and in the noncompact Liouville theory with $c = 1$ [36].

Conclusions. We have shown that for critical Hamiltonians, with a $U(1)$ KM symmetry, the bipartite entanglement of the projected states exhibits universal properties related to the underlying CFT such as the Luttinger parameter K, or the level k of the KM algebra $SU(2)_k$. The numerical determination of the parameter K using entanglement measures is quite difficult and imprecise. We have presented here a simple way to compute K together with the central charge c, through the projected density matrices. We have also derived the

probabilities of measuring a given magnetization in a part of the system, a problem that is related to the full counting statistics which we generalize to deal with entanglement effects.

We believe that the results obtained in this Rapid Communication can be measured in experiments with ultracold atoms. For that, it is necessary to measure $\mathrm{tr}_A(e^{2\pi i \phi S_A^z} \rho_A^n) = \sum_m e^{2\pi i \phi m} \mathrm{tr}_A(\rho_{A,m}^n)$. In principle, this quantity can be measured using two different schemes, proposed recently in Refs. [37,38]. In the scheme of Ref. [37], it is necessary to build n copies of the state ρ. Since $\mathrm{tr}_A(\rho_{A,m}^n) = \mathrm{tr}_A(V_n \rho_{A,m}^{\otimes n})$, where V_n is the shift operator [37], we only need to measure the expectation value $\langle V_n \rangle$ on n copies, for a fixed value of m. Note that expectation values can be measured in optical lattices [37]. On the other hand, the scheme proposed in Ref. [38] uses a random measurement protocol in a single copy and for the reconstruction it explores the decomposition of the density matrix into disjoint blocks with different quantum numbers. This scheme seems to be a natural route to measure $\mathrm{tr}_A(\rho_{A,m}^n)$ for a fix value of m.

Note that the generalization of our approach to systems with higher rank KM algebras such as $\mathrm{SU}(n)_k$ is straightforward and will be reported elsewhere [27]. Finally, we would also to point out that the results obtained in this Rapid Communication apply only to critical theories. They can be extended to the massive theories, obtained by adding relevant perturbation to the critical ones. The reduced density matrix for an interval whose size is smaller that the correlation length ξ coincides with the critical one, except that the cord length $L_c(x)$ is now replaced by the ratio ξ/a, where a is the lattice spacing. The equipartition of the entanglement entropy will also hold for this more general class of models.

Noted added. Recently, we became aware of Refs. [39,40] that also consider the problem studied here.

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