Editors' Suggestion

Onset of Floquet thermalization

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In the presence of interactions, a closed, homogeneous (disorder-free) many-body system is believed to generically heat up to an "infinite temperature" ensemble when subjected to a periodic drive: in the spirit of the ergodicity hypothesis underpinning statistical mechanics, this happens as no energy or other conservation law prevents this. Here we present an interacting Ising chain driven by a field of time-dependent strength, where such heating begins only below a threshold value of the drive amplitude, above which the system exhibits nonergodic behavior. The onset appears at *strong, but not fast* driving. This in particular puts it beyond the scope of high-frequency expansions. The onset location shifts, but it is robustly present, across wide variations of the model Hamiltonian such as driving frequency and protocol, as well as the initial state. The portion of nonergodic states in the Floquet spectrum, while thermodynamically subdominant, has a finite entropy. We find that the magnetization as an *emergent* conserved quantity underpinning the freezing; indeed, the freezing effect is readily observed, as initially magnetized states remain partially frozen *up to infinite time*. This result, which resembles the Kolmogorov-Arnold-Moser theorem for classical dynamical systems, could be a valuable ingredient for extending Floquet engineering to the interacting realm.

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I. INTRODUCTION

Interacting many-body systems, by the ergodic hypothesis, generically thermalize, placing them in the purview of statistical mechanics and equilibrium thermodynamics [1]. Our understanding of the corresponding situation for nonequilibrium systems is still in flux. For perhaps the simplest class of nonequilibrium systems, namely periodically driven (Floquet) systems, thermalization physics at first glance looks maximally simple: removing time translation invariance destroys energy conservation, hence the concept of temperature—which means thermalization to a featureless "infinite-temperature" state [2,3].

Such Floquet systems have been predicted to be capable of sustaining new forms of spatiotemporal ordering when many-body localized as a result of strong quenched disorder [4]. The experimental search for such so-called discrete time crystals has been qualitatively more successful [5,6] than may have been anticipated: the collection of systems appearing to exhibit such order now even includes a dense periodic array of nuclear spins initialized in a thermal state [7].

All of this focuses the inquiry on settings that permit long-lived correlations and order to persist despite the presence of periodic driving even in the absence of quenched disorder. In periodically driven *noninteracting* systems, quantum heating can be suppressed [8–12] and an extensive number of periodically conserved quantities identified [13]. In turn, a prethermalization regime has been identified [14] that resembles a frozen nonthermal state [8], which can be described by a periodic (generalized) Gibbs' ensemble [13]. Tuning the drive parameters, and weakening the interactions, can substantially enhance the prethermalization period, still expected to remain finite [15–17]. In fact, for disorder-free systems, a transient but exponentially long-lived regime exhibiting discrete time-crystalline phenomenology has already been identified [18]. These constitute lower bounds on the thermalization timescales. For finite-size systems, an emergent integrability structure for strong drives has also been proposed as a way to avoid thermalization [19]. There is further evidence indicating the absence of heating at high drive frequencies in a variety of other settings [20–29] and in specially designed models [30,31].

Here, we address the question of whether there is an identifiable threshold for the ratio of driving and interaction strength, below which the system approaches a nontrivial steady state that depends on the drive and the initial state. We consider a spin chain subject to *strong, but not fast* driving, and we use remanent infinite-time magnetization of an initial magnetized state as a measure of failure to Floquet-thermalize. As the driving is increased from low strength, where standard Floquet thermalization is observed, we find a remarkably well-defined second regime, in which remanent magnetization is present even in the infinite-time limit. Its value is given by the Floquet diagonal ensemble average implied by the initial state. The location of this threshold moves, but its existence is stable to variations in state initialization, driving strength, driving protocol, and driving frequency.

In all cases, however, we are able to identify an emergent *approximately* conserved quantity—in the case we discuss at length, the magnetization itself—which becomes exactly conserved if the static part of the Hamiltonian is ignored. Thus, rather than an extensive set of integrals of motion, as is present in the case of the periodic Gibbs ensemble [13] and the Floquet many-body localized cases [32–34], all that appears to be needed to stop the system from heating up indefinitely is a single, approximately conserved quantity.

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While our numerical investigation on systems up to $L = 14$ spins naturally limits our capacity to extrapolate these results to the "thermodynamic" limit, there are indications that this is not only a finite-size effect. First, in plots of remanent magnetization versus driving strength, we identify a crossing point for curves for different L separating the ergodic and the nonthermal regimes. Second, the set of Floquet eigenstates exhibiting memory, while accounting only for a vanishing fraction of the total Hilbert space, extrapolates to have a finite entropy in the thermodynamic limit. This means that such states can still be straightforwardly selected by an initial condition, not unlike initializing a static system in a low-temperature configuration.

In the following, first we set the notation and provide a brief introduction to the Floquet concepts we have used. We then define our model and drive protocol. We characterize the ergodic and the nonthermal phases and the threshold between them using various measures, and we demonstrate robustness to variations of drive patterns and system parameters. We close with an outlook and suggestions for further investigations. In particular, the origin and nature of the sharp features in the memory as a function of driving strength merit further study.

II. FLOQUET BASICS

Let us decompose the time-dependent Hamiltonian $H(t)$ into a static interacting Hamiltonian H_0 and a time-periodic drive $H_D(t)$ with $[H_0,H_D] \neq 0$:

$$H(t) = H_0 + H_D(t). \tag{1}$$

The time evolution operator evolving a state through a period from $t = \epsilon$ to $t = \epsilon + T$ $(0 \leqslant \epsilon < T)$ is $U(\epsilon)$. Since $U(\epsilon)$ is unitary, it can always be expressed in terms of a Hermitian operator, the "Floquet Hamiltonian" H_{eff}, as

$$U(\epsilon) = e^{-iH_{\text{eff}}(\epsilon)}. \tag{2}$$

Formally,

$$\exp\left[-iH_{\text{eff}}(\epsilon)\right] = \mathcal{T}\exp\left(-i\int_{\epsilon}^{\epsilon+T} dt\, H(t)\right), \tag{3}$$

where $\mathcal{T}$ denotes time-ordering. Let $|\mu_i\rangle$ denote the ith "Floquet eigenstate" of H_{eff} corresponding to the "Floquet eigenvalue" (also known as quasienergy) μ_i.

A sequence of stroboscopic observations at instants $t = \epsilon, \epsilon + T, \ldots, \epsilon + nT$ (integer n) is identical to that produced by the dynamics under the time-independent Hamiltonian H_{eff}. This applies for every ϵ, hence we get a continuous family of stroboscopic series.

In the following, we are interested in long-time asymptotic behavior, so that temporal variations within a driving period are of secondary importance. Hence, we arbitrarily pick $\epsilon = 0$.

A. Infinite-time limit: Diagonal ensemble average

The nature of the asymptotic state under the drive can be understood as follows. Consider an initial state,

$$|\psi(0)\rangle = \sum_i c_i|\mu_i\rangle,$$

and the stroboscopic time series for an observable,

$$\hat{O} = \sum_{i,j} O_{ij}|\mu_i\rangle\langle\mu_j|,$$

$$\langle\psi(nT+\epsilon)|\hat{O}|\psi(nT+\epsilon)\rangle = \sum_{i,j} c_i c_j^* O_{ij} e^{-i(\mu_i-\mu_j)(nT+\epsilon)}. \tag{4}$$

As in the case of static Hamiltonians, under quite general and experimentally relevant conditions (see, e.g., Ref. [35]), at long times ($n \to \infty$) the off-diagonal ($i \neq j$) terms "average to zero" and the state of the system can hence be described by an effective "diagonal ensemble" (in the absence of synchronization, e.g., for discrete time crystals, this is replaced by a block diagonal ensemble [36]). This is captured by the mixed density matrix [37] $\hat{\rho}_{\text{DE}} = \sum_i |c_i|^2|\mu_i\rangle\langle\mu_i|$. Thus, the asymptotic properties of a periodically driven system are effectively given by a classical average (known as the diagonal ensemble average, or DEA) over the expectation values of the eigenstates of H_{eff},

$$\langle\hat{O}\rangle(\text{DEA}) = \sum_i |c_i|^2\langle\mu_i|\hat{O}|\mu_i\rangle. \tag{5}$$

Hence it is sufficient to study the nature of the eigenstates and eigenvalues of H_{eff}, or equivalently of $U(\epsilon)$, in order to obtain the long-time behavior.

III. THE DRIVE PROTOCOL AND THE MODEL

In this section, we introduce the notation, model Hamiltonian, drive protocol, and observables to be studied. We consider L spins on a chain. We chose a binary drive protocol, which switches periodically between a pair of rectangular pulses. The time-dependent Hamiltonian is

$$H(t) = H_0 + \text{sgn}(\sin\omega t)\, H_D, \tag{6}$$

with the two components

$$H_0 = -J\sum_i \sigma_i^x\sigma_{i+1}^x + \kappa\sum_i \sigma_i^x\sigma_{i+2}^x - h_0^x\sum_i^L \sigma_i^x - h^z\sum_i^L \sigma_i^z, \tag{7}$$

$$H_D = -h_D^x\sum_i^L \sigma_i^x. \tag{8}$$

The σ^α are Pauli matrices. For the results in the main text, we have chosen $J = 1$, $\kappa = 0.7$, $h^z = 1.2$, and $h_0^x = 0.02$. We use a periodic boundary condition, but tamper the boundary slightly by setting $J_{L,1} = 1.2J$ and $\kappa_{L-1,1} = 1.2\kappa$ to break translational invariance (and hence remove any remaining block-diagonal structure of the Hamiltonian). Here since we keep the interaction strengths constant during the drive, we use the drive amplitude itself as the tuning parameter. We have chosen our drive frequency $\omega = 0.1$ unless stated otherwise explicitly.

In the presence of the transverse field, the Hamiltonian H_0 is known to be ergodic due to the four-fermionic interaction terms arising from the next-nearest-neighbor interactions under the

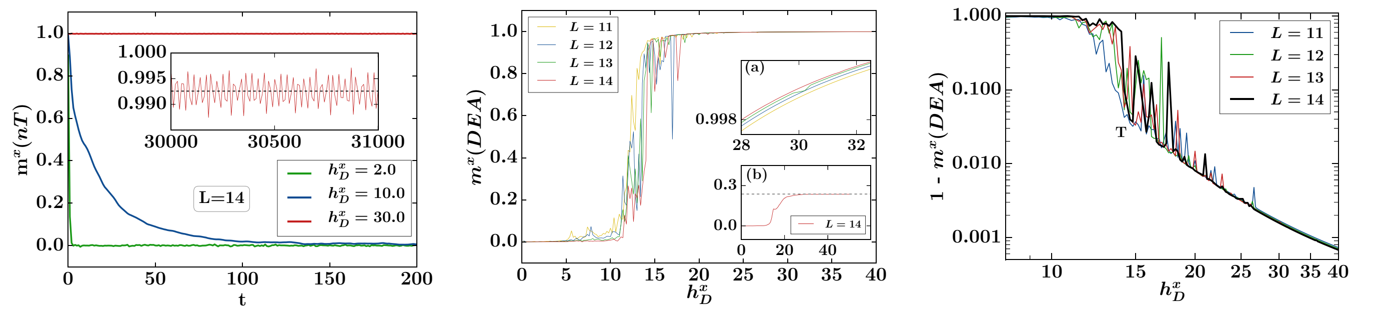

FIG. 1. Freezing and its onset threshold. Left frame: Stroboscopic time series of magnetization $m^x(t)$ for different driving strengths showing initial-state memory for strong driving. The inset zooms in on the long-time behavior; the black horizontal line denotes the DEA of the magnetization. Middle frame: Remnant magnetization as a function of driving strength for different system sizes. The high-field regime (top inset) shows an *increase* of the remnant magnetization with L. The bottom inset shows the DEA of m^x vs drive amplitude for a "generic" state (see the text for details) whose net initial magnetization is marked with the horizontal line, which remains almost unchanged for very strong drives. Right frame: Same data as the middle frame on a doubly logarithmic plot for $1 - m^x(\text{DEA})$. The deviation away from almost complete thermalization gets steeper and moves toward the right with increasing system size. The curves appear to accumulate from the left at a "threshold point" (**T**), which itself appears to move little as the system size is increased from $L = 11$ to 14.

spin to fermion mapping, and also due to the longitudinal field. We have explicitly verified that H_0 is ergodic for our case [38].

We initialize the simulation in the time domain with different initial states. Unless otherwise stated, we use the default choice of the ground state of $H(t = 0)$.

IV. NUMERICAL RESULTS

The central quantity is the longitudinal magnetization

$$m^x(t) = \frac{1}{L}\sum_i^L \langle\psi(t)|\sigma_i^x|\psi(t)\rangle. \tag{9}$$

We monitor its real-time dynamics in a stroboscopic time series. We diagnose nonthermalization/freezing via its long-time asymptotic behavior, the remnant magnetization, which we study as a function of various model parameters.

A. Onset of Floquet thermalization

In the following, we provide numerical evidence that for a strong (but not fast) drive, the system fails to Floquet-thermalize, instead retaining memory of its initially magnetized state. We then show that the onset of Floquet thermalization occurs at a fairly well-defined threshold driving strength.

The stroboscopic time series for the magnetization m^x is shown in Fig. 1, left frame. Already at short times, three representative trajectories for different driving strengths show strikingly different behavior. While for weak driving fields, the magnetization disappears almost immediately, for stronger ones the decay slows down. Finally, for h_D^x beyond a threshold value, the decay is arrested: even at the longest times, a remnant magnetization persists.

This remnant magnetization agrees with the DEA of the magnetization evaluated for the same system (see the inset). Note that the nonvanishing DEA is already in itself a signature of the lack of Floquet thermalization—in general, Floquet-thermalized eigenstates individually show no nontrivial correlations.

To locate the onset, the DEA of m^x as a function of the drive amplitude h_D^x is plotted in Fig. 1, middle frame. A threshold for nonzero remnant magnetization is observed, separating the ergodic ($m_{\text{DE}}^x \approx 0$) from the nonergodic regime.

The lower inset shows freezing for an initial state with a reduced polarization in the x direction. The black dotted line shows the initial value of m^x for the state, and the curve shows that for high enough h_D^x, the DEA of m^x almost coincides with it. In detail, this initial state is given as $|\psi_0\rangle = \sum_{i=1}^{2^L} c_i|i_x\rangle$, where $|i_x\rangle$ is the ith eigenstate of the longitudinal field part (computational basis states in the x direction, or x-basis states), by choosing $\text{Re}[c_i]$ and $\text{Im}[c_i]$ from a uniform distribution between -1 and $+1$, multiplying them by $e^{\beta m_i^x}$, where $\beta > 0$ and m_x^i is the longitudinal magnetization of $|i_x\rangle$, and finally normalizing the state. This gives a "generic" state with a bias toward positive longitudinal magnetization. For the plot in Fig. 1 (middle frame), we have chosen a random instance corresponding to $\beta = 1.75$. The right frame of Fig. 1 shows the DEA of $1 - m^x$ on a doubly logarithmic log-log plot zoomed in around the threshold for better visibility.

B. Floquet eigenstates and an emergent conservation law

1. Localization and magnetization

We now turn to the properties of the Floquet eigenstates obtained by numerically diagonalizing the time evolution operator $U(0)$, Eq. (2). We consider first their "localization" in Hilbert space, followed by their magnetization content.

To investigate the localization properties of the Floquet states in the x basis $\{|i_x\rangle\}$, we calculate the inverse participation ratio (IPR) in said basis defined as $\text{IPR}(|\mu_j\rangle) = \sum_{i=1}^{2^L} |\langle i_x|\mu_j\rangle|^4$. The left frame of Fig. 2 shows the IPR thus obtained, arranged in decreasing order. Indefinite heating corresponds to the states being delocalized in the eigenbasis of any local operator, which implies a uniformly small IPR given by the inverse dimension of Hilbert space, $1/D_H$. This is indeed what is observed for small drive fields. By contrast, for large drive fields, states appear that have an IPR close to 1, which indicates the presence of well-localized states, and hence the absence of Floquet thermalization for the corresponding part of the spectrum.

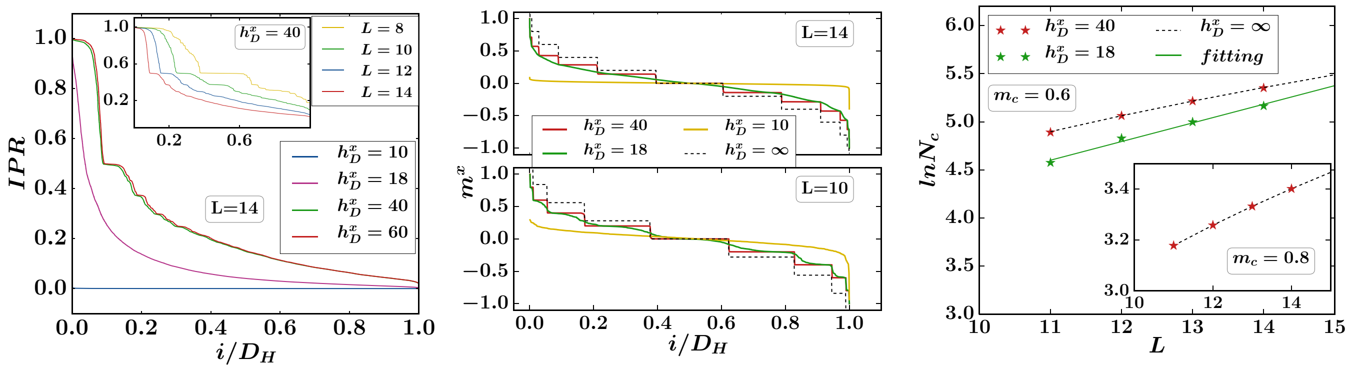

FIG. 2. Emergent conservation law for strong drives, as reflected in the Floquet eigenstates $|\mu_i\rangle$. Left frame: Values of the IPR in the x basis, arranged in decreasing order. Unbounded heating requires these states to be delocalized in the eigenbasis of any local operator. This is the case for the drive with amplitude below the threshold ($h_D^x = 10$,) *but not above* ($h_D^x = 18,40,60$). The inset shows a decreasing IPR for different system-sizes for $h_D^x = 40$ due to the emergent conservation law evidenced in the middle frame: m^x for the Floquet eigenstates arranged in decreasing order, for different values of h_D^x. Black dotted lines ($h_D^x = \infty$) show the values of m^x of the x-basis states (multiplied by a factor of 1.4 for visibility). For $h_D^x = 40$, clear steplike structures appear, indistinguishable from the steps of m^x for x-basis states for both system sizes $L = 10,14$ (see [38] for finer details of the L dependence of this matching). For a lower drive value $h_D^x = 18$, close to the threshold, the curve smoothes out, indicating weakening of the quasiconservation, yet highly polarized Floquet states are still substantial in number. For still lower values (e.g., $h_D^x = 10$), the curve finally flattens. The pronounced asymmetry in the Floquet magnetizations for lower values of h_D^x is due to the small asymmetry in the drive. Right frame: The log of the number N_c of Floquet eigenstates with polarization above a given value m_c is shown to grow approximately exponentially with system size, corresponding to (a vanishing fraction of states but with a) finite entropy. For large h_D^x ($h_D^x = 40$), the numerical data points fall almost exactly on the analytically calculated results (black dotted lines) corresponding to $h_D^x = \infty$ (see the matching of the steplike structures in the middle frame). For a lower value $h_D^x = 18$, a linear fit is done for the numerical data points.

Complementary information can be gleaned by considering the correlations encoded in the nonergodic states. The middle frame of Fig. 2 shows the magnetization of different Floquet eigenstates, m_i^x, ordered according to their size. In the ergodic regime, these curves are featureless and m_i^x is uniformly tiny, showing a tendency to increase with increasing drive strength. Deep into the nonergodic regime, large values of m_i^x appear, which together form plateaus. For the largest drives h_D^x, the plateaus correspond to essentially an integer number of spin flips, which indicates that the new basis is close to the computational basis in the x direction mentioned above. As the drive is decreased, the plateaus give way to a smooth curve, which, however, still makes a large excursion toward $m^x = \pm 1$ before assuming the featureless shape of the ergodic regime.

While the fraction of Floquet states with a magnetization above a certain value is thermodynamically vanishing, their entropy is nonetheless finite (see Fig. 2, middle and right frames). This is analogous to the case of a finite-temperature ensemble of a magnet in a field, where a nonzero magnetization arises as a thermodynamically vanishing fraction of magnetized states is preferentially populated, with their energy gain compensating for the entropy loss involved in concentrating the probability density on them. Here, the selection of the magnetized Floquet states arises via the state initialization. It is interesting to note that in this 1D system there would be no magnetization at any finite temperature: the observation of a finite magnetization at finite energy density is purely a nonequilibrium effect.

2. Emergence of m^x as a local quasiconserved quantity

We next address what we believe is the central feature underpinning the nonthermalization, namely the existence of a conserved quantity in the drive Hamiltonian in isolation. In our example, this is the magnetization in the x direction, m^x, which persists as a quasiconserved quantity even when the ratio of drive to static components of the Hamiltonian is finite.

The middle frame of Fig. 2 shows the value of m^x for the different Floquet eigenstates arranged by their size. For the strongest drives, the steps in this quantity are identical to the ones of the computational basis states in the x basis, i.e., the steps simply reflect the number of spins flipped.

The static part of the Hamiltonian then mixes the states with the same value of m^x, which is reflected in the nontrivial distribution of the IPR of the Floquet states (left frame of Fig. 2). The growth of the size of each m^x sector (except for the fully polarized one) is in turn reflected in a decrease of the IPR.

For lower driving strengths, $h_D^x = 18$, the steps get washed out, but the range of m^x continues to span practically the full range in the interval between -1 and 1. This feature disappears below the threshold, $h_D^x = 10$, where the curve flattens substantially.

While the fraction of Floquet states with a nonzero magnetization density vanishes with system size, these states nonetheless have nonzero entropy (Fig. 2, right panel), as is the case for magnetized states of a paramagnet generally.

The emergent quasiconserved nature of m^x, along with the straightforward possibility of initializing the system in a magnetized state, account for the main features of the results discussed in this work.

C. Robustness against variation of model and protocol parameters

We first address the existence of the onset for variants of the above model. We note that so far, no fine-tuning was necessary.

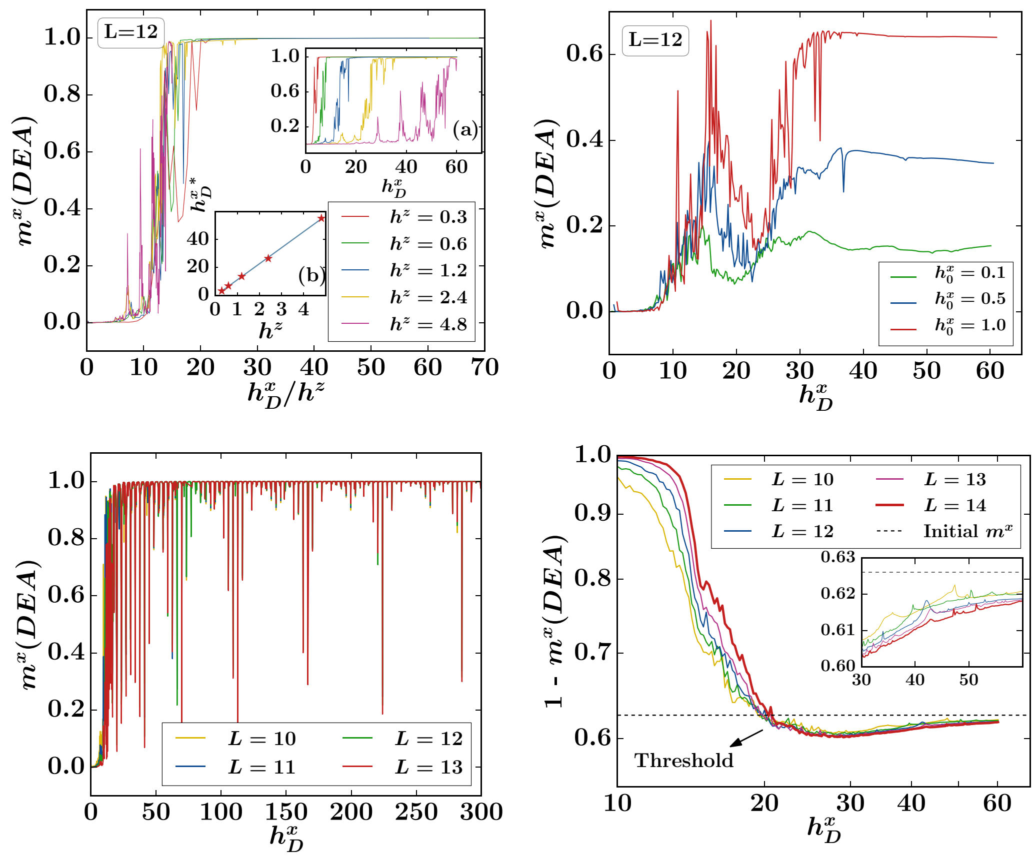

FIG. 3. Remnant magnetization in various settings. Top left: Dependence of a DC transverse field h^z that does not commute with the other, mutually commuting, terms of the model. h^z enhances thermalization (upper inset). The response approximately scales with h_D^x/h^z (main panel); in particular, the estimated threshold h_D^{x*} is approximately proportional to h^z (lower inset). Top right: Robustness of freezing with respect to the addition of a DC field h_0^x. Bottom left: Freezing for uneven division of the total drive period. For $0 \leqslant t < rT$, $h_D^x = +40$ while for $rT \leqslant t < T$, $h_D^x = -40$, where $r = 1/(\text{golden ratio})$. Deep freezing minima persist to high driving strengths but show little size dependence. Bottom right: Behavior for the initial state chosen as the ground state of the nonintegrable undriven part H_0, with h^z and h_0^x chosen to created an initial state with a small positive polarization $m^x(0) \approx 0.373\,90$. For large h_D^x, freezing increases somewhat with L.

The central demand was for the drive amplitude h_D^x to be the largest scale, while the other parameters of the Hamiltonian were chosen all to be in the same ballpark.

1. Role of the noncommuting term

First, the location of the thermalization threshold can be moved by varying the strength of the term in the static Hamiltonian H_0, which does not commute with the driving Hamiltonian H_D. Indeed, the top left frame of Fig. 3 shows that the threshold driving field is approximately proportional to the static transverse field strength h^z.

2. Drive shape and initial state

Also, we ask whether the "symmetry" of having a vanishing mean drive of zero for symmetric pulse shapes about zero is an important ingredient. Figure 3, top right frame, shows that the freezing is quite robust to the addition of a dc field of strength h_0^x. Indeed, the freezing actually grows with h_0^x.

Next, we consider a deviation of the drive protocol away from a time-symmetric switch in the sign of the driving term to one where more time is spent for one sign than the other (Fig. 3, bottom left frame). While the latter case has considerably more structure at high drives, in particular an apparently regular suppression of the remnant magnetization even above the onset threshold, the former curve basically acts as a high-magnetization envelope of the latter.

Further, we consider an initial state prepared as the ground state of a many-body problem (rather than a more simply prepared polarized state). This displays (Fig. 3, bottom right frame) all the salient features observed with the simply polarized ground state in Fig. 1, right frame.

3. Drive frequency

What is particularly worth emphasizing is that the nonergodic behavior is *not* a high-frequency phenomenon. While such freezing also exists in the limit of a driving frequency in excess of the many-body bandwidth of the finite-size system, it is not even the case that the nonergodicity necessarily grows with frequency. This is illustrated in Fig. 4, where the remnant magnetization is, if anything, more robust at small driving frequencies.

This is intriguing since at lower drive frequencies, Magnus-type high-frequency expansions are divergent. Hence, this is an example of the breakdown of a Magnus expansion which is not associated with unbounded heating.

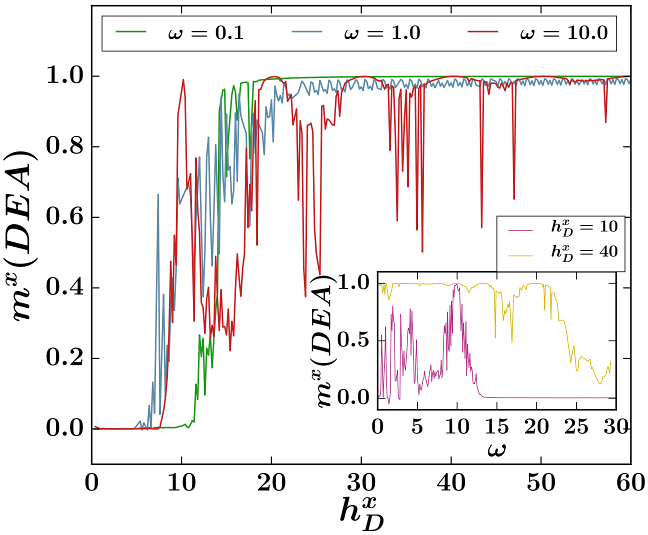

FIG. 4. Dependence of the remnant magnetization on the drive frequency ω for $L = 14$ initialized with the ground state of $H(0)$. The basic morphology (in particular, the two regimes and the threshold) remains the same over two decades in frequency. The freezing decreases as ω is increased, with the threshold only varying slowly with ω. The inset shows m^x vs ω for $h_D^x = 10$ (outside the frozen regime) and $h_D^x = 40$, where the weakening of freezing with increasing ω is evident.

D. Finite-size behavior

Our results indicate that an absence of thermalization in this driven interacting system might persist even in the infinite-size limit. While there are some dips of the freezing strength in the nonergodic regime complicating a sharp identification of a threshold, the onset nonetheless appears to sharpen with increasing system size. A closer view of the nonergodic regime (Fig. 1, middle frame, top inset) shows the smooth behavior of the remnant magnetization for the largest fields; this in fact *grows* with increasing system size. By contrast, for weak drives, the remnant magnetization tends to decrease with system size. This results in a crossing point as the curves for different system sizes of the deviation of the remnant magnetization from its initial value (Fig. 1, right frame) thus approximately cross at the threshold point. While it is hard entirely to rule out a slow drift to higher fields of the threshold with increasing system size, these observations suggest the possibility of a sharp transition at a finite threshold field in the thermodynamic limit.

Next, and most importantly, the step structures in the m^x of the Floquet states are almost indistinguishable from that of the x-basis states for *all system sizes* we investigated (Fig. 2, middle frame). This absence of a system-size dependence indicates that at large values of h_D^x, the drive does not mix the x-basis states of different m^x values. A decrease in the fraction of Floquet states with $m^x > m_c$ with system size is not because in larger systems the Floquet states are more delocalized between different magnetization sectors, but merely because the number of x-basis states in a given magnetization sector changes with the system size. Delocalization between different magnetization sectors is suppressed strongly *for all* system sizes at hand for h_D^x above the threshold. This is in keeping with the observations that on different types of initial states, Figs. 1 and 3, the freezing at the highest frequencies does not decrease with system size, and it gives a further indication that our results are not merely finite-size effects.

1. Ergodicity with and without driving

Though our results from various directions point toward the absence of thermalization in the thermodynamic limit, a limited many-body quantum finite-size numerical study cannot guarantee that. In this context, it is interesting to note that there has been a series of studies on ergodicity breaking in quantum systems in the context of quenches by Rigol and collaborators [39–41]. In these studies, rather than a time-dependent system, the properties of the eigenstates of a static local Hamiltonian are investigated as a function of integrability breaking parameters. These studies illustrate the importance of going to large system sizes in order to see ergodic behavior emerge. By contrast, in our case the static part of the Hamiltonian, H_0 [Eq. (8)], is already ergodic for the system sizes studied [38].

A question that arises naturally, therefore, is whether it is possible to induce nonergodic behavior by driving an ergodic system. This is indeed possible, as exhibited by the famous problem of a periodically driven quantum kicked rotator (see, e.g., [42] and references therein), where energy absorption is bounded by quantum interference even where the static system is chaotic (ergodic). Analogous phenomenology cannot be ruled out in a many-body system in principle, and indeed it is that question that has partly motivated this work [43].

V. DISCUSSION

We have studied the onset of Floquet thermalization in a driven interacting spin chain. We have found a fairly sharp threshold for the drive strength, above which Floquet thermalization does not take place. The threshold value varies in different manners with parameters such as pulse shape, drive frequency, or the (noncommuting) transverse field strength, but the freezing persists robustly under all these variations. The question of the existence of such a threshold is of fundamental importance, with a related issue appearing for classical dynamical systems, where the Kolmogorov-Arnold-Moser theorem deals with the onset of chaotic behavior upon breaking of integrability.

An open question concerns the origin, and in particular the L dependence, of the dips in the frozen component even beyond the threshold in the m^x versus h_D^x plots: the dips touching the x axis correspond to points of thermalization. While their occurrence for certain discrete values of h_D^x has no significant consequence, if their number diverges with L, this may lead to a destruction of the frozen regime. For drives with pulse durations evenly placed about $T/2$, the dips disappear rapidly with increasing h_D^x. Such dips are, however, observed to persist even for very strong amplitudes for the case of drive with uneven division of the drive period (Fig. 3, bottom left). In this case, the total drive period is divided into two parts, T/GR and $T(1 - 1/\mathrm{GR})$, where GR is the Golden ratio. While the depth of the dips seems to increase with L, their number and locations remain surprisingly independent of L, which points against their proliferation. Regarding an extrapolation to the thermodynamic limit, we refer to our discussion at the end of the previous section.

Comparison of the magnetization and IPR of the Floquet states in the frozen regime allows one to conclude that the magnetization itself plays the role of a quasiconserved quantity, which becomes exactly conserved in the limit of infinitely strong driving. However, the emergence of only a single conserved quantity does not rule out nontrivial steady states, as can be gleaned from the structure of Floquet eigenstates in the frozen regime: these states have definite m^x values, yet they are not fully localized in the x basis. It is also interesting to note that a single local conserved quantity such as m^x does not preclude a nonlocal H_{eff}, yet it is sufficient to result in a nonthermal Floquet spectrum.

While our driving term in isolation is integrable, it appears that the existence of a conserved quantity is all that is required for the existence of the frozen regime. A study of a nonintegrable drive with an emergent conservation law is therefore an obvious item for future work.

This nonergodicity is *not* a high-frequency phenomenon. Instead, it is particularly well-developed at lower driving frequencies, which *a priori* renders attempts to construct a Magnus-type high-frequency expansion problematic. Instead, nonergodicity is primarily associated with strong driving. Note that for the driving term in isolation, the instantaneous eigenvectors of the Hamiltonian are time-independent, while the instantaneous eigenvalues change; this suggests the development of a perturbation theory controlled by the instantaneous gap, rather than a high frequency. It would also be interesting to investigate the connections of this problem to the case of weakly driven interacting systems with approximate conservation laws [44].

The role of emergent conservation laws may in particular be important for experimental studies of driven many-body systems. Indeed, a first sighting of the physics we have analyzed here has occurred in the context of an experiment of Floquet many-body localization [26], where the possibility of a finite threshold for delocalization was also noted for the low-disorder limit. The main ingredient we have identified, namely an emergent conservation law, turns out also to have been present in that situation. Analogously, for the searches of time crystals taking place at present, it will be interesting to investigate if emergent conservation laws do, or can, play a role there as well.

Finally, while periodic driving is expected to heat a system and hence delocalize it, drive-induced destructive quantum interference can produce just the opposite effect. A competition of these might result in unexpected freezing behavior, as has been observed in a quantum counterpart of classically chaotic systems, namely in the kicked rotators (see, e.g., [42]). Such a suppression of heating [43] might not be impossible in a quantum many-body system in which interactions lead to ergodicity. An absence of unbounded heating under periodic driving could be a step in that direction, and the availability of emergent approximate conservation laws may turn out to be a useful ingredient for many-body Floquet engineering.

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