Rapid Communications

Theory of the supercyclotron resonance and Hall response in anomalous two-dimensional metals

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Weakly disordered superconducting films can be driven into an anomalous low-temperature resistive state upon applying a magnetic field. Recent experiments on weakly disordered amorphous InO_x have established that both the Hall resistivity and the frequency of a cyclotronlike resonance in the anomalous metal are highly suppressed relative to the values expected for a conventional metal. We show that both of these observations can be understood from the flux flow dynamics of vortices in a superconductor with significant vortex pinning. Results for flux flow transport are obtained using a systematic hydrodynamic expansion, controlled by the diluteness of mobile vortices at low temperatures. Hydrodynamic transport coefficients are related to microscopics through Kubo formulas for the longitudinal and Hall vortex conductivities, as well as a "vortoelectric" conductivity.

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Introduction. In conventional metals, the Hall resistivity and the cyclotron frequency are key observables that can often be used as proxies for the density and mass of charge carriers, respectively. Recent measurements have probed these quantities in the anomalous metallic state of weakly disordered amorphous InO_x films. The low-temperature superconducting state in this material becomes metallic upon applying a magnetic field greater than $H_c \approx 2$ T. In the metallic phase at a field of 5 T the measured Hall resistivity is three orders of magnitude smaller than in the more conventional high-temperature state [1], and falls below experimental sensitivity at a lower field $H_{M2} > H_c$. Furthermore, the maximum of the frequency-dependent conductivity is at zero frequency to within experimental resolution [2]; for a conventional Drude peak this fact would require a cyclotron frequency at least four orders of magnitude smaller than expected based on normal state properties [3].

Anomalous metallic phases with resistive behavior similar to that of amorphous InO_x have been found in many two-dimensional systems, as thoroughly reviewed in Ref. [4]. All of these metals emerge continuously from a superconducting phase. A rapid drop in the resistivity occurs as the temperature is lowered, before saturating to a constant at low temperatures. This suggests an interpretation of these regimes as "failed superconductors." We will show in this Rapid Communication that the experimental facts outlined above can indeed be explained by the flow of phase-disordering vortices in the would-be superconductor.

Flux flow in magnetic fields. The extensive theoretical literature on the Hall effect due to flux flow in magnetic fields has considered a myriad of different physical effects (see, e.g., Refs. [5–15]). This reflects a diverse set of experimental results in different flux flow regimes and in different materials. Much of the existing discussion involves microscopic modeling of the forces acting on vortices. We will instead argue that the diluteness of the mobile vortices allows an alternative, unified, and completely systematic treatment based on hydrodynamic argumentation combined with Kubo formulas. Our result for the Hall resistivity will be

$$\rho_{yx} = \left(\sigma_{\rm n}^H + \sigma_o^H\right)\rho_{xx}^2 + \frac{n_v^{\rm eff}}{n_s}\frac{\hbar}{e^{\star 2}}. \tag{1}$$

In the remainder we set $\hbar = e^\star = 2e = 1$. The three terms in (1) respectively describe a Hall signal arising from currents in the vortex core, currents carried by Bogoliubov quasiparticles in the superfluid, and the comotion of supercurrent parallel to the vortex current. The Hall conductivity σ_o^H of the Bogoliubov quasiparticles is typically negligible due to their approximate particle-hole symmetry. Dominance of the first term, proportional to the Hall conductivity $\sigma_{\rm n}^H$ of the vortex cores, leads to the relation $\rho_{yx} \sim \rho_{xx}^2$ obtained by Vinokur *et al.* [10], and observed in some thermally activated flux flow [16–21]. This scaling arises because—as shown by Bardeen-Stephen [5] and rederived below—$\rho_{xx} \sim x$, the fraction of the area occupied by mobile vortex cores, is strongly temperature and field dependent, while $\sigma_{\rm n}^H$ is not. Dominance of the final "comotion" term, on the other hand, is crucial to understand experimental results on free flux flow. There, the density of vortices that comove with the superfluid $n_v^{\rm eff} \sim H$, the applied field, while n_s is the superfluid density. Thus $\rho_{yx} \sim H \sim x \sim \rho_{xx}$, as is observed [22,23], and predicted by Nozières and Vinen [6].

The general relationship (1) between the Hall and longitudinal resistivities both unifies previous results and establishes their domain of validity. It also allows for regimes in which no single term dominates. The first two terms in (1) lead to $\rho_{yx} \sim \rho_{xx}^2$ while the final term leads to $\rho_{yx} \sim \rho_{xx}$, if the density of comoving vortices $n_v^{\rm eff} \sim x$. The full expression, therefore, may well explain the range of scaling relations, $\rho_{yx} \sim \rho_{xx}^\beta$ with $1 \lesssim \beta \lesssim 2$, reported in the experimental literature [24–27]. Competition between effects captured by the first and last terms in (1) has previously been invoked to explain the observed change in sign of the Hall response in some flux flow regimes [8,9,12,28,29]. Charging of the vortex cores can lead to a sign reversal of $\sigma_{\rm n}^H$ [30–32].

We furthermore obtain expressions for the width Ω and frequency Ω_H of a "supercyclotron resonance" [33]. This resonance is due to superfluid and vortex flow in a magnetic field. It can coexist with a conventional cyclotron resonance (due to flow of the normal fluid component). We will find

$$\Omega = \frac{2x}{\sigma_{\rm n}} f_s \quad \text{and} \quad \Omega_H = -\frac{\partial j_v^x}{\partial u_\phi^x} \equiv \frac{n_v^{\rm eff}}{m_\star}. \qquad (2)$$

The result for Ω in (2) is precisely the Bardeen-Stephen expression for vortex diffusivity [5], with $\sigma_{\rm n}$ the conductivity of the vortex cores and f_s the superfluid stiffness. In (2), Ω_H is given by a static susceptibility; the second step in the expression defines $n_v^{\rm eff}$. The mass scale $m_\star$ is such that $f_s \equiv n_s/m_\star$. Therefore the phase gradient $u_\phi \equiv \nabla\phi = m_\star v_s$, with v_s the superfluid velocity. We have set $\hbar = 1$. Finally, j_v is the vortex current. With Galilean invariance, $n_v^{\rm eff} = n_v$ is the full density of mobile vortices. Ω_H is then precisely the frequency appearing due to the comotion of vortices and supercurrent in Nozières-Vinen [6]. More generally, pinning can strongly break Galilean invariance, so that the effective number of vortices that comove with the supercurrent $n_v^{\rm eff} < n_v$.

Experiments on anomalous metals. We can return now to the measurements on InO_x. The first observation is that $\Omega_H \lesssim 10^{-5}\ \Omega$ [3]. The consequences of this fact follow from (2), whereby $\Omega_H/\Omega = \frac{1}{2}\sigma_{\rm n} n_v^{\rm eff}/x n_s$. The area occupied by mobile vortex cores is $x \sim n_v \xi^2 \sim n_v/n_s$, where ξ is the superconducting correlation length. Here, $x \propto n_v$ because in a magnetic field we expect the flux through all the different vortices to be aligned. The above, together with $\sigma_{\rm n} \sim 10 e^2/h \sim 0.4 e^{\star 2}/\hbar$ [1,3], then implies

$$\frac{n_v^{\rm eff}}{n_v} \lesssim 10^{-5}. \qquad (3)$$

It follows that there is essentially vanishing parallel comotion of vortices and supercurrent, as quantified by the dissipationless susceptibility $\partial j_v^x/\partial u_\phi^x$ in (2). Indeed, strong pinning in InO_x causes $\rho_{xx} \sim x$ to vary by more than an order of magnitude as a function of applied field in the anomalous metal [1]. InO_x is therefore far from the "Nozières-Vinen" free flow regime.

Second, to a good approximation, $\rho_{yx} \sim \rho_{xx}^2$, where the Hall signal is detectable [1]. This requires the final term in (1) to be negligible, so that $n_v^{\rm eff}/n_s \lesssim \rho_{yx}$. From here we can obtain $n_v^{\rm eff}/n_v \sim n_v^{\rm eff}/(n_s x) \lesssim \rho_{yx}/x \sim \rho_{yx}/(\rho_{xx}\sigma_{\rm n}) = \tan\theta^H/\sigma_{\rm n}$. We used the Bardeen-Stephen result $\rho_{xx} \sim x/\sigma_{\rm n}$, recovered below. The measured $\tan\theta^H$ becomes as small as 10^{-4} [1], leading to $n_v^{\rm eff}/n_v \lesssim 10^{-4}$, consistent with (3). The conclusion (3) is therefore reached from two independent experiments. Furthermore, the data show that for the anomalous metal, $\rho_{yx}/\rho_{xx}^2 = \sigma_{\rm n}^H \sim 2\times 10^{-6}\ \Omega^{-1}$ in (1)—recall that $\sigma_o^H \sim 0$ due to particle-hole symmetry of the Bogoliubov excitations. This is of the same magnitude as the Hall conductivity of the high-temperature normal state [1], and is consistent with the interpretation of $\sigma_{\rm n}^H$ as the Hall conductivity of the vortex cores. It follows that $\rho_{yx}/\rho_{yx}^{{\rm high}\,T} \sim (\rho_{xx}/\rho_{xx}^{{\rm high}\,T})^2$, and hence the observed $\rho_{yx} \sim 0.01\ \Omega$ at a field of 5 T, suppressed by almost three orders of magnitude relative to the high-temperature value, follows from the suppression of ρ_{xx} in the anomalous metal.

A condition analogous to (3) must also hold for the systems mentioned above where a $\rho_{yx} \sim \rho_{xx}^2$ scaling was previously observed [16–21]. The supercyclotron resonance will be easiest to observe in materials that instead exhibit free flux flow, with negligible pinning, so that $\rho_{yx} \sim \rho_{xx}$.

The Hall resistivity measurements further reveal a weak field dependence of $\sigma_{xy} = \rho_{yx}/\rho_{xx}^2$, with σ_{xy} possibly vanishing below a field $H_{\rm M2} > H_{\rm c}$ [1]. A strictly vanishing zero temperature σ_{xy} over some field range requires that the vortex core contribution $\sigma_{\rm n}^H = 0$ in (1), in addition to the vanishing of vortex/superfluid comotion implied by (3). Such particle-hole symmetry [1] is seen away from a flux flow regime in more disordered samples [34,35].

Hydrodynamic approach. Our analysis is anchored in the observation [2] of a narrow peak at zero frequency in the optical response $\sigma(\omega)$. This peak defines a lifetime that is around 10^5 times longer than that of the electronic quasiparticles in the material. Such a hierarchy of timescales allows a systematic hydrodynamic expansion of the collective response; all noncollective modes have decayed before the timescales of interest. Furthermore, the conductance peak narrows as the magnetic field is reduced towards the onset of superconductivity at $H_{\rm c}$. This strongly suggests that the appropriate low-energy description of the anomalous metal is superfluid hydrodynamics with a slow phase-relaxation timescale [2,33]. Phase relaxation requires the inclusion of vortices in the hydrodynamic description. The hydrodynamic variables are therefore the electrical and vortex currents j and j_v and the phase gradient $u_\phi = \nabla\phi$. The conductance peak in fact survives into the superconducting phase [3]—at the very end we explain how this can arise from the contribution of pinned vortices to the optical conductivity.

Working within linear response and assuming homogeneous currents [36], the equations for the hydrodynamic variables in the presence of a uniform electric field E are completely fixed. The Josephson relation, allowing for transverse vortex flow, is (with $\hbar = e^\star = 1$)

$$\dot{u}_\phi^i = E^i + \epsilon^{ij} j_v^j. \qquad (4)$$

Here, ϵ^{ij} is antisymmetric with $\epsilon^{xy} = 1$. We must now express the electric and vortex currents in terms of the electric field and superfluid velocity. The most general relation that obeys the Onsager constraint is shown in the Supplemental Material [37] to be [38]

$$\begin{pmatrix} j_o^i \\ j_v^i \end{pmatrix} = \begin{pmatrix} \hat\sigma_o^{ij} & \hat\alpha_v^{ij} \\ \hat\alpha_v^{ij} & \hat\Omega^{ij}/f_s \end{pmatrix} \begin{pmatrix} E^j \\ f_s \epsilon^{jk} u_\phi^k \end{pmatrix}. \qquad (5)$$

Here, the normal component electric current $j_o \equiv j - f_s u_\phi$. This "generalized Ohm's law" introduces six transport coefficients: $\hat\sigma_o^{ij} = \sigma_o\delta^{ij} + \sigma_o^H\epsilon^{ij}$, $\hat\Omega^{ij} = \Omega\delta^{ij} + \Omega_H\epsilon^{ij}$, and $\hat\alpha_v^{ij} = \alpha_v\delta^{ij} + \alpha_v^H\epsilon^{ij}$. Their physical meaning is as follows: $\hat\Omega^{ij}$ is the vortex conductivity, $\hat\sigma_o^{ij}$ is the electrical conductivity of the normal (nonsuperfluid) component, and $\hat\alpha_v$ is a "vorto-electric" conductivity. We will drop the Hall component α_v^H in the remainder of our discussion—its only effect on charge transport is to produce a small shift in the superfluid stiffness $f_s = n_s/m_\star$. All of the earlier theoretical works referenced above contain equations analogous to the steady state relations (5). We have emphasized that these equations are justified when slow phase relaxation defines a separation of timescales that allows the collective response to be treated hydrodynamically. Furthermore, we will obtain precise Kubo formulas for the transport coefficients that we then evaluate systematically.

Solving for j_v and u_ϕ using (4) and (5) gives Ohm's law $j^i = \sigma^{ij} E^j$ with the low-frequency conductivities [33]

$$\sigma_{xx}(\omega) = f_s \frac{\left(1-\alpha_v^2\right)(-i\omega+\Omega) + 2\alpha_v \Omega_H}{(-i\omega+\Omega)^2+\Omega_H^2} + \sigma_o, \quad (6)$$

$$\sigma_{xy}(\omega) = f_s \frac{2\alpha_v(-i\omega+\Omega) - \left(1-\alpha_v^2\right)\Omega_H}{(-i\omega+\Omega)^2+\Omega_H^2} + \sigma_o^H. \quad (7)$$

These formulas predict a "supercyclotron resonance" due to the poles at $\omega_\star = \pm\Omega_H - i\Omega$, tying dc transport to an optical response. We see that Ω determines the superfluid relaxation rate. Positivity of entropy production requires $\alpha_v^2 \leqslant \sigma_o \Omega / f_s$.

Kubo formulas. The transport coefficients in (6) and (7) are given by Kubo formulas, derived in the Supplemental Material [37] using the hydrodynamic Green's functions that follow from Eqs. (4) and (5). The vortex conductivity is given by the retarded Green's function of the vortex current operator J_v. This can in turn be expressed in terms of the Green's function for the time derivative $\dot{J}_\phi = i[H, J_\phi]$ of the supercurrent,

$$\hat{\Omega}^{ij} = f_s \lim_{\omega\to 0}\lim_{x \ll 1} \frac{1}{\omega} \operatorname{Im} G^R_{J_v^i J_v^j}(\omega) \quad (8)$$

$$= f_s\left(\lim_{\omega\to 0}\lim_{x \ll 1} \frac{1}{\omega} \operatorname{Im} G^R_{\dot{J}_\phi^i \dot{J}_\phi^j}(\omega) - \chi_{\dot{J}_\phi^i J_\phi^j}\right). \quad (9)$$

The final contact term in (9) is a static susceptibility. The vortoelectric conductivity depends also on the normal component current operator $J_o \equiv J - f_s J_\phi$,

$$\alpha_v = \lim_{\omega\to 0}\lim_{x \ll 1} \frac{1}{\omega} \operatorname{Im} G^R_{J_v^y J_o^y}(\omega) \quad (10)$$

$$= \lim_{\omega\to 0}\lim_{x \ll 1} \frac{1}{\omega} \operatorname{Im} G^R_{\dot{J}_\phi^x J_o^y}(\omega). \quad (11)$$

Finally, the normal component conductivity $\hat{\sigma}_o^{ij}$ follows similarly from the Green's function for J_o. Writing the vortex conductivities in terms of $\dot{J}_\phi$ as in (9) and (11) will enable them to be directly related to a microscopic mechanism for phase relaxation.

In evaluating the Kubo formulas for the vortex conductivities $\hat{\Omega}$ and α_v it is necessary to take the limit $x \ll 1$—wherein mobile vortices occupy a small fraction of the sample area, ensuring slow phase relaxation—before the zero frequency limit. This can be seen explicitly from the hydrodynamic Green's functions given in the Supplemental Material [37]. In the remainder we evaluate (9) and (11) for phase relaxation due to vortex flux flow. In Ref. [33] the Bardeen-Stephen phase relaxation rate Ω was recovered in this way. We can now extend that result to obtain Ω_H and α_v.

Supercurrent relaxation due to flux flow. The supercurrent operator is given by the gradient of the phase integrated *outside* of vortex cores, where the phase is well defined, $J_\phi \equiv \int_{\mathbb{R}^2\setminus\text{cores}} \nabla\phi \, d^2x$. This definition holds in the limit of weak phase relaxation with dilute, independent vortices in an otherwise well-defined background phase—corresponding to the $x \ll 1$ limit in the Kubo formulas, taken prior to any low-frequency limit. The supercurrent operator is relaxed by charge fluctuations that are described by a "self-charging" term in the Hamiltonian, $H = \frac{1}{2\chi}\int n^2 \, d^2x$, where n is the charge density and χ the charge compressibility [39]. The commutator $[\phi(x), n(y)] = i\delta(x-y)$ and single-valuedness of the density operator n everywhere then leads to the expression

$$\dot{J}_\phi = \frac{2}{\chi}\int_{\text{cores}} \nabla n \, d^2x. \quad (12)$$

This operator relation can now be used to obtain the Green's functions (9) and (11). The factor of 2 in (12) was missed in our previous work [33], but is physically important. When computing $\dot{J}_\phi$ one must allow for the fact that the location of the core is time dependent; in this way only mobile vortices are seen to contribute. See the Supplemental Material [37] for details.

The operator relation (12) is at the heart of our approach. Taking the expectation value of (12) in a state with a single large vortex and using $\langle \nabla n \rangle = \chi \nabla\mu$ in the core leads to the standard classical relation between the vortex current and the microscopic electric field $-\nabla\mu$ in the core [40].

If (i) correlations between excitations in distinct vortex cores are neglected and (ii) the vortex cores are assumed to be large compared to the mean free path of the normal state in the core, then the Kubo formulas can be evaluated explicitly. Using the operator (12), the first contribution to (9) becomes

$$\frac{1}{\omega}\operatorname{Im} G^R_{\dot{J}_\phi^i \dot{J}_\phi^j}(\omega) = -x\frac{4}{\chi^2}\int_{\text{core}} \lim_{\omega\to 0} \partial_i\partial_j \frac{\operatorname{Im} G^R_{nn}(\omega, y)}{\omega} d^2y, \quad (13)$$

with x the fraction of the total area covered by mobile vortex cores. The integral is over a single core. The control parameter in this entire computation is $x \ll 1$, so that dilute vortices lead to slow phase relaxation. The large core assumption allowed the Green's function in the core to be translationally invariant so that $G^R(x,y) = G^R(x-y)$. In the large core limit the charge density diffuses so that $G^R_{nn}(\omega,k) = \sigma_\mathrm{n} k^2/(-i\omega + Dk^2)$. The conductivity of the normal state in the core $\sigma_\mathrm{n} = \chi D$, with D the diffusivity. The integral in (13) is then easily evaluated to give

$$\frac{1}{\omega}\operatorname{Im} G^R_{\dot{J}_\phi^i \dot{J}_\phi^j}(\omega) = \frac{2x}{\sigma_\mathrm{n}}\delta^{ij}. \quad (14)$$

The susceptibility term in (9) can be written

$$\chi_{\dot{J}_\phi^i J_\phi^j} = \frac{1}{f_s}\frac{\partial \dot{u}_\phi^i}{\partial u_\phi^j} = \frac{\epsilon^{ik}}{f_s}\frac{\partial j_v^k}{\partial u_\phi^j}. \quad (15)$$

The first equality uses $\chi_{AB} = \partial\langle A\rangle/\partial s_B$. Here, s_B is the source for B, and in the case at hand $s_{u_\phi} = f_s u_\phi$. The second equality uses the Josephson relation (4). The electric field term, which is in fact $E - \nabla\mu$ in general, drops out because E is held fixed and $\chi_{J_\phi \nabla\mu} = 0$ at any nonzero temperature (where the response at low wave vector k is nonsingular, so that $\chi_{J_\phi \nabla_i \mu} \sim k_i \to 0$). Putting (14) and (15) together gives the results for Ω and Ω_H stated in (2) above. Finally, the inclusion of correlations between distinct vortex cores and finite size corrections to Green's functions in the cores (i.e., lifting the two assumptions made above) do not lead to additional contributions to Ω_H, as we note in the Supplemental Material [37].

With the same assumptions, the vortoelectric conductivity similarly gets a contribution from inside the vortex cores given by

$$\alpha_v = -\frac{x}{\chi}\int_{\text{core}} \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\, G^R_{n\,\epsilon^{ij}\partial_i j_j}(\omega,y)d^2y. \tag{16}$$

The contribution from outside of the cores turns out to be suppressed by powers of x compared to the inside-core contribution, as we show in the Supplemental Material [37]. The Green's function in the core appearing in (16) again follows from the diffusive normal state dynamics. It is given by $G^R_{n\,\epsilon^{ij}\partial_i j_j}(\omega,k) = -i\omega\sigma_{\text{n}}^H k^2/(-i\omega + Dk^2)$ and is derived in the Supplemental Material [37]. Here, σ_{n}^H is the Hall conductivity of the normal state in the core. Using this Green's function we obtain

$$\alpha_v = x\frac{\sigma_{\text{n}}^H}{\sigma_{\text{n}}} = -x\tan\theta_{\text{n}}^H. \tag{17}$$

Here, θ_{n}^H is the Hall angle of the normal state.

Conductivity and resistivity. Inserting the flux flow results (2) and (17) into the hydrodynamic expressions (6) and (7) gives the dc conductivities at small x,

$$\sigma_{xx} = \frac{\sigma_{\text{n}}}{2x}, \tag{18}$$

$$\sigma_{xy} = \sigma_{\text{n}}^H + \sigma_o^H + \sigma_{xx}^2\frac{n_v^{\text{eff}}}{n_s}. \tag{19}$$

The final term in (19) is larger than the first two by a factor of $1/x$, because $\sigma_{xx}^2 \sim 1/x^2$ and $n_v^{\text{eff}} \sim x$. We saw in our earlier discussion, however, that the other terms can dominate when n_v^{eff} is suppressed. Assuming $\sigma_{xy} \ll \sigma_{xx}$ then gives the Hall resistivity (1).

Final remark. The hydrodynamic theory can be extended into the superconducting phase, and explains how dynamical depinning of vortices leads to the zero frequency conductance peaks observed in Ref. [3]. Ignoring the (small) parity-odd terms, the optical conductivity (6) is a simple Lorentzian $\sigma(\omega) = f_s/(-i\omega + \Omega)$. We have noted that Ω is the vortex conductivity. A simple model of vortex pinning is to let $\Omega \to \Omega(\omega) = \omega\Omega/(\omega + i\omega_o)$. Here, ω_o is a pinning frequency. This form arises in the limit of strong momentum relaxation from the general hydrodynamics of pinned lattices [41]. The upshot is then the optical conductivity

$$\sigma(\omega) = \frac{f_s}{\Omega+\omega_o}\left(\frac{\omega_o}{-i\omega} + \frac{\Omega}{-i\omega+\Omega+\omega_o}\right). \tag{20}$$

A superconducting delta function arises once the pinning frequency ω_o becomes nonzero. It is accompanied by a zero frequency Lorentzian peak whose width is continuous across the superconducting-anomalous metal transition (which is driven by $\omega_o \to 0$, not $\Omega \to 0$). This is what the data show [3], further supporting the picture of the anomalous metal as being due to the flux flow of mobile vortices. Indeed, zero field amorphous InO_x shows a canonical Berezinskii-Kosterlitz-Thouless (BKT) transition as a function of temperature. The conductance peak in the high-temperature BKT phase [42] is due to mobile unpaired vortices, and is continuously connected in the phase diagram to the conductance peak seen in the anomalous metal [2,3].

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[1] N. P. Breznay and A. Kapitulnik, Sci. Adv. **3**, e1700612 (2017).
[2] W. Liu, L. Pan, J. Wen, M. Kim, G. Sambandamurthy, and N. P. Armitage, Phys. Rev. Lett. **111**, 067003 (2013).
[3] Y. Wang, I. Tamir, D. Shahar, and N. P. Armitage, Phys. Rev. Lett. **120**, 167002 (2018).
[4] A. Kapitulnik, S. A. Kivelson, and B. Spivak, arXiv:1712.07215.
[5] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).
[6] P. Nozières and W. F. Vinen, Philos. Mag. **14**, 667 (1966).
[7] Z. D. Wang and C. S. Ting, Phys. Rev. Lett. **67**, 3618 (1991).
[8] A. T. Dorsey, Phys. Rev. B **46**, 8376 (1992).
[9] N. B. Kopnin, B. I. Ivlev, and V. A. Kalatsky, J. Low Temp. Phys. **90**, 1 (1993).
[10] V. M. Vinokur, V. B. Geshkenbein, M. V. Feigel'man, and G. Blatter, Phys. Rev. Lett. **71**, 1242 (1993).
[11] Z. D. Wang, J. Dong, and C. S. Ting, Phys. Rev. Lett. **72**, 3875 (1994).
[12] A. van Otterlo, M. Feigel'man, V. Geshkenbein, and G. Blatter, Phys. Rev. Lett. **75**, 3736 (1995).
[13] N. B. Kopnin and V. M. Vinokur, Phys. Rev. Lett. **83**, 4864 (1999).
[14] T. Kita, Phys. Rev. B **64**, 054503 (2001).
[15] E. Arahata and Y. Kato, J. Low Temp. Phys. **175**, 346 (2014).
[16] A. V. Samoilov, Phys. Rev. Lett. **71**, 617 (1993).
[17] P. J. M. Wöltgens, C. Dekker, and H. W. de Wijn, Phys. Rev. Lett. **71**, 3858 (1993).
[18] A. V. Samoilov, A. Legris, F. Rullier-Albenque, P. Lejay, S. Bouffard, Z. G. Ivanov, and L.-G. Johansson, Phys. Rev. Lett. **74**, 2351 (1995).
[19] H.-C. Ri, R. Gross, F. Gollnik, A. Beck, R. P. Huebener, P. Wagner, and H. Adrian, Phys. Rev. B **50**, 3312 (1994).
[20] S. Okuma and N. Kokubo, Phys. Rev. B **56**, 410 (1997).
[21] W. N. Kang, H.-J. Kim, E.-M. Choi, H. J. Kim, K. H. P. Kim, and S.-I. Lee, Phys. Rev. B **65**, 184520 (2002).
[22] T. W. Jing and N. P. Ong, Phys. Rev. B **42**, 10781 (1990).
[23] S. Bhattacharya, M. J. Higgins, and T. V. Ramakrishnan, Phys. Rev. Lett. **73**, 1699 (1994).
[24] J. Luo, T. P. Orlando, J. M. Graybeal, X. D. Wu, and R. Muenchausen, Phys. Rev. Lett. **68**, 690 (1992).
[25] W. N. Kang, D. H. Kim, S. Y. Shim, J. H. Park, T. S. Hahn, S. S. Choi, W. C. Lee, J. D. Hettinger, K. E. Gray, and B. Glagola, Phys. Rev. Lett. **76**, 2993 (1996).
[26] M. Cagigal, J. Fontcuberta, M. Crusellas, J. Vicent, and S. Piñol, Physica C: Superconductivity **248**, 155 (1995).

[27] W. N. Kang, B. W. Kang, Q. Y. Chen, J. Z. Wu, S. H. Yun, A. Gapard, J. Z. Qu, W. K. Chu, D. K. Christen, R. Kerchner, and C. W. Chu, Phys. Rev. B **59**, R9031(R) (1999).

[28] S. J. Hagen, C. J. Lobb, R. L. Greene, M. G. Forrester, and J. H. Kang, Phys. Rev. B **41**, 11630 (1990).

[29] S. J. Hagen, C. J. Lobb, R. L. Greene, and M. Eddy, Phys. Rev. B **43**, 6246 (1991).

[30] S. J. Hagen, A. W. Smith, M. Rajeswari, J. L. Peng, Z. Y. Li, R. L. Greene, S. N. Mao, X. X. Xi, S. Bhattacharya, Q. Li, and C. J. Lobb, Phys. Rev. B **47**, 1064 (1993).

[31] K.-i. Kumagai, K. Nozaki, and Y. Matsuda, Phys. Rev. B **63**, 144502 (2001).

[32] H. Ueki, M. Ohuchi, and T. Kita, J. Phys. Soc. Jpn. **87**, 044704 (2018).

[33] R. A. Davison, L. V. Delacrétaz, B. Goutéraux, and S. A. Hartnoll, Phys. Rev. B **94**, 054502 (2016).

[34] N. P. Breznay, M. A. Steiner, S. A. Kivelson, and A. Kapitulnik, Proc. Natl. Acad. Sci. USA **113**, 280 (2016).

[35] M. A. Steiner, N. P. Breznay, and A. Kapitulnik, Phys. Rev. B **77**, 212501 (2008).

[36] Spatial inhomogeneity may be an important ingredient of anomalous metals [4]. Hydrodynamics is a powerful framework for inhomogeneous dynamics, but we focus on the simplest, homogeneous situation in this work.

[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.97.220506 for details of mathematical derivations.

[38] Here and throughout we have defined Ω_H with a different sign relative to our previous work [33]. We have also renamed ρ_v in our previous work as α_v, because ρ_v carried misleading connotations.

[39] This is the most relevant Hamiltonian for phase relaxation in the large core limit that we will be considering. Other terms can become relevant away from this limit.

[40] M. Tinkham, *Introduction to Superconductivity* (Courier Corporation, North Chelmsford, MA, 1996).

[41] L. V. Delacrétaz, B. Goutéraux, S. A. Hartnoll, and A. Karlsson, Phys. Rev. B **96**, 195128 (2017).

[42] W. Liu, M. Kim, G. Sambandamurthy, and N. P. Armitage, Phys. Rev. B **84**, 024511 (2011).