Topological footprints of the Kitaev chain with long-range superconducting pairings at a finite temperature

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We study the one-dimensional Kitaev chain with long-range superconductive pairing terms at a finite temperature where the system is prepared in a mixed state in equilibrium with a heat reservoir maintained at a constant temperature T. In order to probe the footprint of the ground-state topological behavior of the model at finite temperature, we look at two global quantities extracted out of two geometrical constructions: the Uhlmann and the interferometric phase. Interestingly, when the long-range effect dominates, the Uhlmann phase approach fails to reproduce the topological aspects of the model in the pure-state limit; on the other hand, the interferometric phase which has a proper pure state reduction, shows a behavior independent of the ambient temperature.

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I. INTRODUCTION

A tremendous effort is now being focused at the experimental front to realize topological superconductors, which constitute an essential component of quantum computers and simulators. What makes the realization of a topological superconductor an absolute experimental necessity is that it hosts exotic Majorana modes (MMs) as zero-energy localized modes at its edges or boundaries. Such modes are absent in conventional nontopological superconductors. These exotic MMs are topologically protected against local perturbations and cannot be removed unless a global change in the ground-state properties in the form of a topological phase transition occurs. This robustness serves as a key property which enables them to be used as qubits to store and manipulate quantum information in a topological quantum computer without the chance of quick loss of information through decoherence. MMs have been proposed to exist in many systems like heterostructures of topological insulators and s-wave superconductors [1], cold fermion systems with Rashba spin-orbit coupling, Zeeman field, and an attractive s-wave interaction [2,3], and also heterostructures of spin-orbit-coupled semiconductor thin films [4,5] or nanowires [5–7] proximity coupled with s-wave superconductors and a Zeeman field. Although there have been also been claims of observation of MMs in a few experiments [8–17], they have as yet remained experimentally elusive.

On the other hand, recent experimental realization of long-range interacting quantum models with tunable long-range interactions (or a long-range pairing term) [18] has renewed interest in studying the equilibrium behavior as well as the nonequilibrium dynamics of quantum models with infinite-range interactions with interaction strength between two sites separated by a distance r falling off in a power-law fashion as $1/r^{\alpha}$ [19–33]. Let us recall that a power-law interacting ferromagnetic Ising chain has been studied for longer than the past four decades [34–46]; quantum phase transitions [47–49] in the corresponding quantum Ising chain with interaction decaying in a power-law fashion were also explored long ago [50].

Recently motivated by the short-range one-dimensional Kitaev chain [51], a long-range version of an *integrable p*-wave superconducting chain of fermions, with a long-range superconducting pairing term was proposed [19–21]; interestingly, in this model the 2×2 structure corresponding to each momentum value survives in spite of the power-law interacting superconducting term. It has been observed [21] that when the pairing terms decay faster, the model captures short-range topological superconducting physics; on the contrary, for slow decay of the long-range interactions given by $\alpha < 1$, the model supports a new unconventional topological phase of matter. In this new phase, the zero-energy MMs coalesce to form massive nonlocal edge states called massive Dirac modes which are otherwise absent in the standard Kitaev model. These new edge states lie within the bulk energy gap and are topologically protected against local perturbations that do not break fermionic parity and particle-hole symmetry and may eventually find novel applications in the field of topological quantum computations.

Furthermore, the open question of whether higher-order topological phase transitions can appear in symmetry-protected topological systems has been rigorously investigated by Cats *et al.* [52] A staircase to topological phase transitions of increasing order has been found in the long-range superconducting chain which is beyond the conventional second-order phase transition observed in a one-dimensional topological superconductor. Considering a grand potential within an adaptive Ehrenfest classification, the order of the phase transition is determined according to the derivative for which the grand potential has a divergence or a discontinuity. The jumps in the order of the transitions for the case of a topological superconductor with long-range pairings depend on α, which at unity results in the order of the topological transition becoming infinite.

Although, throughout the last century, phases of matter have been very successfully characterized by taking recourse to a local order parameter in accordance with Landau's theory, the order parameter required to classify such one-dimensional (1D) topological superconductors studied here is, however, global in nature. Indeed at zero temperature, most of the phase diagram for the 1D Kitaev chain with long-range pairings can be understood using the conventional winding number used to classify a standard 1D p-wave topological superconductor. However, an intriguing question remains as to what extent the topological properties of such a long-range paired system would survive when coupled to a heat reservoir at some constant temperature T. We here note that Viyuela *et al.* [53] (see also Ref. [54]) introduced the Uhlmann geometric phase [55,56] as a tool to characterize symmetry-protected topological phases in 1D fermion systems described by a Gibbs ensemble. They not only illustrated that the Uhlmann phase acts as a global order parameter which can classify the two different topological phases in the standard 1D Kitaev chain but they also demonstrated that there exists a critical temperature at which the Uhlmann phase goes discontinuously and abruptly to zero. Furthermore, at small temperatures, they showed that the Uhlmann phase can also capture the expected behavior of topological phases in such fermionic systems. Subsequently, the behavior of a different geometric phase, introduced in the context of interferometry by Sjoqvist *et al.* [57], has also been studied in the context of the short-range 1D Kitaev chain, which shows contrasting behavior to that of the Uhlmann approach [58]. (For a review on these two approaches, see Ref. [59].) We note in passing that recently the interferometric phase approach was also found to be relevant in the context of mixed-state dynamical quantum phase transitions [60,61] and also in the context of mixed-state topology [62].

In this work, we therefore consider a 1D Kitaev chain with a long-range superconducting pairing term after it has thermalized by being in contact with a heat reservoir at temperature T and is effectively described by a Gibbs ensemble. In order to probe the topological aspects of the model considered, two disjoint approaches are pursued, namely, the Uhlmann geometric approach and the interferometric geometric approach. The two main questions addressed here through the two approaches are the following: (a) Can both approaches properly reproduce the topological phase diagram in the pure-state (or the zero-temperature) limit? (b) What do the two approaches reveal about the extent of the survival of the topological properties in this long-range superconducting scenario?

The paper is organized in the following fashion: In Sec. II, we review the topological phase diagram of the long-range Kitaev (LRK) chain. In Secs. III and IV, the LRK chain is studied at a finite temperature using the Uhlmann phase and interferometric phase approaches, respectively. Concluding comments are presented in Sec. V.

II. THE LONG-RANGE INTERACTING KITAEV CHAIN

Let us consider a simple model of spinless fermions on a 1D lattice with long-range p-wave superconducting pairings, known as the LRK chain. The Hamiltonian is of the form [19]

$$H = \sum_{n=1}^{N}\Bigg\{-t(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}) - \mu c_n^\dagger c_n + \sum_{l=1}^{N-1}\frac{\Delta}{d_l^\alpha}(c_{n+l}^\dagger c_n^\dagger + c_n c_{n+l})\Bigg\}, \tag{1}$$

where $t > 0$ is the hopping amplitude, μ is the chemical potential, $\Delta = |\Delta| e^{i\Theta}$ is known as the complex superconducting gap, and c_n's ($c_n^\dagger$'s) are the spin-polarized fermionic annihilation (creation) operators defined at every site n of the chain with total sites N. The superconducting pairing term being a function of the distance $d_l = \mathrm{Max}[l, L-l]$ between any two sites in the lattice is long-range interacting with the strength of interaction decaying with a decay exponent $\alpha > 0$. Although the total fermionic number is not conserved, the parity operator (total fermionic number modulo 2) commutes with the Hamiltonian and is conserved.

Focusing on the rather well-known short-range limit ($\alpha \to \infty$) [51], when the system is in the topological phase, there are two Majorana modes at each end of the open chain. The two MMs having the same degrees of freedom as an ordinary fermion can either be together occupied or unoccupied. Since the energy of the MMs is zero, these two possible states (occupied or unoccupied) are both ground states, thereby rendering the ground state of a short-range 1D Kitaev chain twofold degenerate with different parities: (i) a bulk with even fermion parity and unoccupied MMs, whereas (ii) populating the two Majorana modes at the edges (in addition to the bulk) amounts to a single ordinary fermion and odd parity. As we illustrate below, the LRK chain is *topologically* short ranged when the decay parameter $\alpha > 3/2$.

Throughout the rest of the paper, without the loss of generality, we set $t = \Delta = 1/2$ and, assuming periodic boundary conditions, one can implement a Fourier transformation to rewrite the Hamiltonian in the Nambu spinor basis, $\psi_k = (c_k, c_{-k}^\dagger)^T$. The thermodynamic limit of $N \to \infty$ yields [19–21]

$$H = \int_0^{2\pi} \frac{dk}{2\pi} \Psi_k^\dagger H_k \Psi_k, \tag{2}$$

where

$$H_k = -f_\alpha(k)\sigma_y - (\mu + \cos k)\sigma_z \tag{3}$$

and

$$f_\alpha(k) = \sum_{l=1}^{N-1} \frac{\sin(kl)}{l^\alpha}. \tag{4}$$

The eigenvalues of this Hamiltonian are

$$E_k^\pm = \pm\sqrt{(\mu + \cos k)^2 + (f_\alpha(k))^2}. \tag{5}$$

Moreover, to simplify matters, we consider the Hamiltonian in a rotated basis, with the Bloch vectors of the Hamiltonian lying on the equatorial plane so that H_k assumes the form

$$H_k = -\frac{\Delta_k}{2}\vec{n}_k \cdot \vec{\sigma}, \tag{6}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and

$$\vec{n}_k = \frac{2}{\Delta_k}(\mu + \cos k, f_\alpha(k), 0), \tag{7}$$

$$\Delta_k = 2|E_k^\pm|. \tag{8}$$

It is noteworthy that the LRK chain is classified under the BDI symmetry class of topological insulators and superconductors [63,64] and is particle-hole, time-reversal, and chiral symmetric. These symmetries restrict the movement of the Bloch vector $\vec{n}_k$ from the sphere S_2 to the circle S_1 on the x-y plane, resulting in a mapping from the Hamiltonians H_k on the Brillouin zone (BZ) $k \in S_1$ onto the winding vectors $\vec{n}_k \in S_1$. This mapping yields a topological Z_2 invariant called the winding number ω, which is the angle (modulo 2π) subtended by $\vec{n}_k$ when quasimomentum k is varied across the BZ from $-\pi$ to π, where

$$\omega = \frac{1}{2\pi} \oint \frac{\partial_k n_k^y}{n_k^x} dk. \tag{9}$$

Alternatively, one can consider an adiabatic transport of the system from a certain crystalline momentum via a reciprocal lattice vector. The eigenstate of the lower band of the system $|g_k\rangle$ then picks up a Berry-Zak phase φ_Z [65–67] which is generally quantized (0 or π) and has a one-to-one correspondence with the winding number [Eq. (9)] defined above:

$$\varphi_Z = i \oint \langle g_k|\partial_k|g_k\rangle dk. \tag{10}$$

In the thermodynamic limit, the polylogarithmic function $f_\alpha(k)$ in Eq. (4) that encodes all the information about the long-range pairing is divergent at $k = 0$ for $\alpha < 1$ and this results in the likewise divergence of the dispersion relation [see Eq. (5)] and the group velocity $(\partial E_\pm(k)/\partial k)$. Moreover, the impossibility in gauging away the divergence from $k = 0$ generates a topological singularity. Therefore, according to the behavior of $f_\alpha(k)$ at $k = 0$, the existence of three different topological sectors depending on the exponent α have been rigorously established by Viyuela *et al.* [21]:

(a) The $\alpha > 3/2$ sector, also known as the Majorana sector, is equivalent to the topological phase of the short-range Kitaev chain [51]. The $|\mu| > 1$ phase is topologically trivial and is marked by the absence of the Majorana zero modes (MZMs). On the other hand, in the region $\mu \in (-1,1)$, the MZMs are ever present (see Fig. 1). The presence of a $U(1)$ phase discontinuity at $k = 0$ in the eigenvector $|g_k\rangle$ and the function $f_\alpha(k)$ being not divergent yields the Z_2 invariant $\omega = \frac{\varphi_Z}{\pi} = 1$, which characterizes this phase.

(b) The $\alpha < 1$ sector is truly an emergent feature of the long-range nature of the hopping and is absent in the conventional 1D Kitaev model. In this sector, for $\mu > 1$ the system under open boundary condition is in a trivial phase, with no edge states, while for $\mu < 1$ this system hosts a topological massive Dirac fermion at the edges, as shown in the wave-function plot in Fig. 2(b) of [21]. This massive Dirac mode (MDM) appears solely due to the coupling induced between the two MZMs at the two distant edges due to the presence of long-range superconducting pairing and, thus, the MDM formed is highly nonlocal. Moreover, the MDM, although massive, is still topological and is thereby protected by the bulk gap. Furthermore, as the ground state of the system in this phase still retains its even parity, populating the MDM, which is the first excited state of the system, would now require a change in the fermionic parity from even to odd. Therefore, this highly nonlocal topological quasiparticle is also protected by the fermionic parity. Since no discrete symmetry has been broken, due to the inclusion of the long-range pairing, the system still belongs to the BDI symmetry class. The winding number ω, however, is modified by the topological singularity

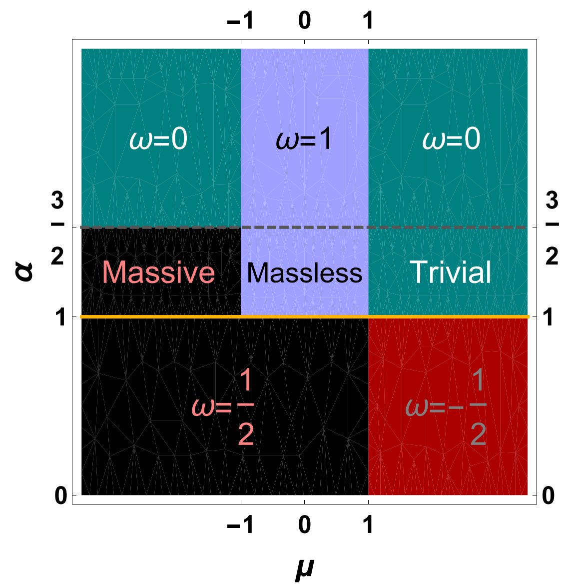

FIG. 1. Phase diagram of the LRK chain in the μ-α plane: for $\alpha > 3/2$ the phase diagram of this model is topologically equivalent to the short-range Kitaev chain, whereas for $\alpha < 1$ the model hosts massive Dirac edge modes for $\mu < 1$ and is characterized by a half-integer winding number. There is a crossover phase in between (for $1 < \alpha < 3/2$) with no well-defined winding number.

at $k = 0$. This happens because at $k = 0$ the adiabatic condition breaks down since both the energy dispersion relation $E_k^\pm$ in Eq. (5) and the quasiparticle group velocity $\partial_k E_k^\pm$ diverge as the Berry-Zak phase $\omega = \varphi_Z/\pi$ evolves under parallel transport. For the trivial phase $\mu > 1$, the winding number is $\omega = -1/2$, whereas for the massive Dirac fermion hosting a topological phase when $\mu < 1$, it turns out to be $\omega = +1/2$. Although the topological invariant is half integer, the difference of one unit exists between the two topologically different phases, indicating that a topological phase transition separates the two half-integer quantized topological phases.

(c) The third sector for $\alpha \in (1,3/2)$ not only hosts MZMs for $-1 < \mu < 1$ but also includes MDM for $\mu > 1$. The dispersion relation $E_k^\pm$ is no longer divergent; however, the group velocity $\partial_k E_k^\pm$ is still singular at $k = 0$ and, hence, a winding number cannot be defined for such a crossover sector.

In the rest of the article we are only going to focus on sectors (a) and (b) with well-defined winding numbers, to see how the topological invariant behaves when the chain is in constant contact with a thermal bath at temperature T, and is described by the Gibbs state

$$\rho(k) = \frac{e^{-H_k/T}}{\mathrm{Tr}e^{-H_k/T}} = \frac{1}{2}\left(\mathbb{1} + \tanh\frac{\Delta_k}{2T}\vec{n}_k \cdot \vec{\sigma}\right). \tag{11}$$

An important question that has been asked now is whether it is possible that a geometric phase factor can also be defined for mixed states, analogous to the Berry-Zak phase for pure states, which can serve as a topological invariant describing the phase structure of 1D chains at finite temperature. The work by Viyuela *et al.* [53] suggests that the Uhlmann geometric phase [55,56] can play this role. Alternatively, the behavior of interferometric geometric phase for mixed states defined in Ref. [57] was also studied by Andersson *et al.* [58] for the short-range 1D Kitaev chain and was shown to be a candidate approach. In this work, we illustrate that for the 1D Kitaev chain for a long-range pairing, although the interferometric approach still manages to capture the topological properties of the pure state, the Uhlmann approach fails miserably even to reproduce the correct pure-state limit.

III. THE UHLMANN APPROACH

Uhlmann's approach [55,56] is based upon considering pure states in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ which forms the total space of a fiber bundle over the mixed states on $\mathcal{H}_A$. Now a purification is performed such that

$$\rho = \mathrm{Tr}_B\,[|\psi\rangle\langle\psi|] = ww^\dagger, \tag{12}$$

where the trace is over the auxiliary space $\mathcal{H}_B$. This description contains a $U(N)$ gauge freedom since under $w \to wU(N)$, $\rho \to wU(N)U^\dagger(N)w^\dagger$ remains unchanged. A geometric phase can be associated to any curve in the base manifold once a parallelism condition for curves in the total space is defined. A lift $w(k)$ of $\rho(k)$ is said to be parallel if for every infinitesimal δk the probability for the transition from $\psi(k)$ to $\psi(k+\delta k)$ is identical to the fidelity of $\rho(k)$ and $\rho(k+\delta k)$:

$$|\mathrm{Tr}(w(k)^\dagger w(k+\delta k))|^2 = \mathrm{Tr}\sqrt{\rho(k)^{1/2}\rho(k+\delta k)\rho(k)^{1/2}}. \tag{13}$$

The parallelism condition is eventually described in terms of a connection $\mathcal{A}$ [68], which along the velocity fields of square-root lifts [69], i.e., $w(k) = \sqrt{\rho(k)}$, becomes

$$\mathcal{A}(\partial_k w) = \sum_{i,j} |u_i\rangle \frac{\langle u_i|[\partial_k w, w]|u_j\rangle}{p_i + p_j}\langle u_j|, \tag{14}$$

where the p_i and the $|u_i\rangle$ are the eigenvalues and eigenstates of ρ in Eq. (11) and

$$p_+ = \frac{1}{2}\left(1 + \tanh\frac{\Delta_k}{2T}\right), \quad p_- = \frac{1}{2}\left(1 - \tanh\frac{\Delta_k}{2T}\right). \tag{15}$$

The above formula simplifies for a two-level system into

$$\begin{aligned}\mathcal{A}(\partial_k w) = (\sqrt{p_+} - \sqrt{p_-})^2\{&|u_+\rangle\langle u_+|\partial_k u_-\rangle\langle u_-| \\ &+ |u_-\rangle\langle u_-|\partial_k u_+\rangle\langle u_+|\},\end{aligned} \tag{16}$$

where

$$|u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\varphi}\end{pmatrix}, \quad |u_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -e^{i\varphi}\end{pmatrix}, \tag{17}$$

and $\varphi = \arctan(n_y/n_x)$. As the connection in our case becomes Abelian,

$$\mathcal{A}(\partial_k w) = \frac{i}{2}(\partial_k\varphi)(\sqrt{p_+} - \sqrt{p_-})^2\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{18}$$

path ordering is automatically taken care of in computing

$$U = \exp\left(-\oint dk\,\mathcal{A}(\partial_k w)\right) = \begin{pmatrix}e^{iB} & 0\\ 0 & e^{-iB}\end{pmatrix}, \tag{19}$$

where

$$B = \frac{1}{2}\oint dk\,(\partial_k\varphi)(\sqrt{p_-} - \sqrt{p_+})^2. \tag{20}$$

Let us now remark that even though $\varphi(k)$ is periodic, the function B need not be periodic. Finally, we obtain the Uhlmann's geometric phase as the argument of the phase factor of the function

$$\begin{aligned}&\mathrm{Tr}(w(0)^\dagger w(0)U)\\ &\quad = \tfrac{1}{2}(\sqrt{p_+(0)} + \sqrt{p_-(0)})^2\cos B\\ &\qquad + \tfrac{1}{2}(\sqrt{p_1(0)} - \sqrt{p_2(0)})^2\cos(\varphi(0) + B).\end{aligned} \tag{21}$$

Using $\varphi(0) = 0$, Eq. (21) reduces to

$$\begin{aligned}\mathrm{Tr}(w(0)^\dagger w(0)U) &= \cos(B)\\ &= \cos\left[\frac{1}{2}\oint dk\,(\partial_k\varphi)\left\{1 - \mathrm{sech}\left(\frac{\Delta_k}{2T}\right)\right\}\right].\end{aligned} \tag{22}$$

Let us first study the $T \to 0$ limit. In this limit the Berry-Zak phase for the pure-state case (see phase diagram in Fig. 1) should be

$$\varphi_Z = \begin{cases} 0, & |\mu| > 1 \text{ and } \alpha > \frac{3}{2}\\ \pi, & -1 < \mu < 1 \text{ and } \alpha > \frac{3}{2}\\ \text{not applicable}, & \forall\mu \text{ and } 1 < \alpha < \frac{3}{2}\\ -\frac{\pi}{2}, & \mu > 1 \text{ and } 1 < \alpha \leqslant 0\\ \frac{\pi}{2}, & \mu < 1 \text{ and } 1 < \alpha \leqslant 0. \end{cases}$$

In the $T \to 0$ limit, Eq. (22) becomes

$$\begin{aligned}&\mathrm{Tr}(w(0)^\dagger w(0)U)\\ &\quad = \lim_{T\to 0}\cos\left[\frac{1}{2}\oint dk\,(\partial_k\varphi)\left\{1 - \mathrm{sech}\left(\frac{\Delta_k}{2T}\right)\right\}\right]\\ &\quad = \cos\left[\frac{1}{2}\oint dk\,(\partial_k\varphi)\right];\end{aligned} \tag{23}$$

on the other hand,

$$\varphi_U = \mathrm{Arg}[\mathrm{Tr}(w(0)^\dagger w(0)U)] \tag{24}$$

reduces to

$$\varphi_U = \mathrm{Arg}\left[\cos\left(\frac{1}{2}\oint dk\,(\partial_k\varphi)\right)\right]. \tag{25}$$

This yields

$$\varphi_U = \begin{cases} 0, & |\mu| > 1 \text{ and } \alpha > \frac{3}{2}\\ \pi, & -1 < \mu < 1 \text{ and } \alpha > \frac{3}{2}\\ \text{not applicable}, & \forall\mu \text{ and } 1 < \alpha < \frac{3}{2}\\ \text{undefined}, & \mu > 1 \text{ and } 1 < \alpha \leqslant 0\\ \text{undefined}, & \mu < 1 \text{ and } 1 < \alpha \leqslant 0. \end{cases}$$

We observe that, although the Uhlmann phase φ_U equals the Berry-Zak phase φ_Z in the pure-state limit for all range of μ when $\alpha > 1$, $\mathrm{Tr}\,(w(0)^\dagger w(0)U) = 0$ for all μ in the strong long-range limit when $\alpha < 1$ results in φ_U being undefined. Therefore, one of the key results of our work is that for the 1D Kitaev chain with long-range hopping, the Uhlmann phase fails to detect the topological phase transition at $\mu = 1$ for $\alpha < 1$ in the pure-state limit. It is therefore necessary to resort to a different geometric approach that with a well-defined pure-state limit can predict the fate of the topological phases when the system is described by mixed quantum states.

IV. THE INTERFEROMETRIC PHASE

In the geometric interferometric phase approach by Sjoqvist *et al.*, a normalized state under purification is represented by $|w\rangle \in \mathcal{H}_w$, where $\mathcal{H}_w = \mathcal{H}_S \bigotimes \mathcal{H}_A$, $\mathcal{H}_S$ is the Hilbert space of the system, $\mathcal{H}_A$ is the Hilbert space spanned by ancillary states, and

$$|w\rangle = \sum_i \sqrt{p_i}|\psi_i\rangle \bigotimes |\psi_i'\rangle \tag{26}$$

with $|\psi_i'\rangle \in \mathcal{H}_A$, and the index i runs over the dimensions of the Hilbert space $\mathcal{H}_S$ (or $\mathcal{H}_A$). Therefore, the original density matrix is obtained by tracing over the ancillary states:

$$\rho = \mathrm{Tr}_A\,(|w\rangle\langle w|). \tag{27}$$

Let the states $|w(k)\rangle$ be parametrized by a continuous parameter k, with $|w(k)\rangle$ tracing out a curve in the Hilbert space H_w. A metric is defined in H_w as the measure of distance between two states as $d = |||w(k_1)\rangle - |w(k_2)\rangle||$. Let us note that the two states $|w(k_1)\rangle$ and $|w(k_2)\rangle$ are said to be parallel if the distance between them is a minimum. But, the purification states $|w(k)\rangle$ also have a phase ambiguity or a $U(1)$ gauge freedom as under a gauge transformation $|w(k)\rangle \to e^{i\delta(k)}|w(k)\rangle$ produces the same density matrix and preserves inner products in the space $\mathcal{H}_w$, which needs to be fixed to generate a unique trajectory in $\mathcal{H}_w$. This gauge fixing is implemented by demanding that two infinitesimally separated states in $\mathcal{H}_w$ are parallel to each other. We should also note that under such a parallel transport the state of the system, $|\psi(k)\rangle$, is only affected while the ancillary states $|\psi'(k)\rangle$ are not. For the purifications, using the orthonormality of $|w(k)\rangle$, the corresponding parallel transport condition can be recast to the form

$$\langle w(k)|\partial_k w(k)\rangle = \mathrm{Tr}(\rho(0)V^\dagger(k)\partial_k V(k)) = 0, \tag{28}$$

where $\rho(0) = \sum_i p_i|\psi_i(0)\rangle\langle\psi_i(0)|$ and

$$V(k) = e^{-\int_0^k dk'\,\langle\psi(k')|\partial_{k'}\psi(k')\rangle}. \tag{29}$$

In summary, if we consider a family of density operators parametrized by k,

$$\rho(k) = \sum_i p_i(k)|\psi_i(k)\rangle\langle\psi_i(k)|, \tag{30}$$

such that for each k, the eigenvalues $p_i(k)$ are nondegenerate, the parallel gauge-fixing condition,

$$\langle\psi_i(k)|\partial_k\psi_i(k)\rangle = 0, \tag{31}$$

after a parallel transport across the whole 1D Brillouin zone, yields the interferometric phase [58] of $\rho(k)$,

$$\theta_g = \mathrm{Arg}\left[\sum_i \sqrt{p_i(0)p_i(2\pi)}\langle\psi_i(0)|V_i(2\pi)|\psi_i(2\pi)\rangle\right], \tag{32}$$

where $V_i(2\pi) = e^{-\oint dk'\,\langle\psi_i(k')|\partial_{k'}\psi_i(k')\rangle}$.

It is now straightforward to calculate this interferometric phase θ_g in the case of the 1D Kitaev model with long-range hoppings which is essentially a two-level quantum system for each independent k -mode. Using Eqs. (15), (8), and (5) and identifying $\psi_i(k)$ as $u_i(k)$ in Eq. (17), we finally obtain

$$\theta_g = \mathrm{Arg}\left[\exp\left(-\frac{i}{2}\oint dk'\,\frac{\partial\varphi}{\partial k'}\right)\sum_{i=\pm} p_i(0)\right] \tag{33}$$

$$= \frac{1}{2}\oint dk'\,\frac{\partial\varphi}{\partial k'} = \varphi_Z. \tag{34}$$

It can now easily be seen that not only does the interferometric geometric phase θ_g reduce to the Berry-Zak phase in the pure-state limit for all values of α and thus reproduce the phase diagram properly, but also θ_g is completely independent of the temperature of the bath. This happens as the phase accumulated by both the eigenstates under parallel transport across the k-space remains the same and is identical to the Berry-Zak phase φ_Z.

V. DISCUSSION AND CONCLUDING COMMENTS

The 1D Kitaev chain with short-range (nearest-neighbor) superconducting pairings is a model of a p-wave topological superconductor [51] which possesses a topological phase characterized by a Z_2 topological invariant in the zero-temperature limit. After coupling this model to a bath maintained at a constant temperature T, its topological behavior has been thoroughly investigated in the works of Viyuela *et al.* [53] and Andersson *et al.* [58]. While the former works used the Uhlmann phase approach to provide an order parameter, the latter resorted to the geometric interferometric phase to ascertain its topological aspects as both these approaches correctly reproduce the pure-state topological nature of this model. The Uhlmann phase approach predicts the presence of a critical temperature T_c beyond which the system loses its topological behavior. But it also has a memory effect which prevents it from determining the fate of the edge modes at finite temperatures. The interferometric phase, on the other hand, does not detect any phase transition in temperature, but it correctly captures the zero-temperature phase portrait of this model.

In our work, we have considered a generalized version of the 1D Kitaev chain [19] with a superconducting pairing which is now long ranged. The phase diagram of this model is different as it hosts a new massive Dirac phase characterized by a half-integer winding number and is the sole result of the long-ranged nature of the superconducting term. Having prepared the state of this system in a (mixed) Gibbs state which is in thermal equilibrium with a bath at finite temperature T, the effect of the long-ranged nature of the interaction on the topological behavior is probed using both the aforementioned geometric approaches. We interestingly observe that the Uhlmann phase approach in the extreme long-range limit ($\alpha < 1$) fails to detect the zero-temperature behavior of this model. In the presence of the long-range hopping terms, there exists a singularity in the Hamiltonian and in turn in its energy spectrum at $k = 0$ for $\alpha < 1$. The behavior of these divergences indeed affects the definition of the winding numbers that classify the emergent long-range phases. In spite of the fact that they are measured on a closed 1D loop, winding numbers should assume only integer values; in our case, the existence of semi-integer winding numbers for $\alpha < 1$ has been observed. The root cause behind the generation of such semi-integer winding numbers can be traced to the fact that, although the eigenvectors themselves are orthonormalized and nonsingular, the $\partial_k|u_k\rangle$ term in Eqs. (10) and (16) are both singular for $\alpha < 1$. These observations eventually lead to the existence of two purely long-range phases at $\alpha < 1$ totally disjoint from the short-range ones [70].

This can elaborately be understood by first considering the system to be in the short-range trivial phase where $|u_k\rangle$ is continuous throughout the BZ. On the other hand, when the system lies in the short-range topological phase, there is a $U(1)$ phase discontinuity at the singular point $k = 0$, i.e., $|u_{k\to 0^+}\rangle = e^{i\pi}|u_{k\to 0^-}\rangle$. This $U(1)$ phase shift at $k = 0$ results in the accumulation of the Berry phase by the system after an adiabatic transport across the whole BZ. However, for $\alpha < 1$, the topological singularity at $k = 0$ makes the winding vector ill defined at that point, even though its contribution to the winding number can still be determined. The divergence of the function $f_\alpha(k)$ at $k = 0$ as $f_\alpha(k_-) \to -\infty$ and $f_\alpha(k_+) \to +\infty$ results in the crumbling of the adiabatic condition and the divergence of the quasiparticle group velocity at $k = 0$. Due to the breakdown of adiabaticity, no longer does the system only pick up just a $U(1)$ phase after a closed loop in BZ; the eigenvectors at $k = 0$ are now related via a phase-shift unitary jump (depending upon the value of $|\mu|$) which cannot simply be gauged away via a unitary phase transformation [21]. The above considerations can easily be extended to Eq. (16), where the Uhlmann connection has been evaluated. The terms $\langle u_\pm|\partial_k u_\mp\rangle$ in the Uhlmann connection itself become singular at $k = 0$, which eventually makes the argument of cosine in φ_U [in Eq. (25)] $\pm\pi/2$, but the cosine being an even function fails to discern between the two, rendering the Uhlmann phase insufficient in capturing the long-range phase transition at $\mu = 1$.

On the other hand, the interferometric phase approach, although it correctly reproduces the pure-state topological limit, invariably fails to capture any topological phase transition with temperature. Our study, therefore, establishes that both the Uhlmann and interferometric phase approaches are inadequate in describing the finite-temperature topology of a LRK chain.

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