Phase transition in $SU(N) \times U(1)$ gauge theory with many fundamental bosons

Ankur Das*
Physics & Astronomy, University of Kentucky, 505 Rose St, Lexington, KY 40508, USA

(Received 19 March 2018; revised manuscript received 29 May 2018; published 25 June 2018)

Here we study the renormalization group flow of $SU(N) \times U(1)$ gauge theory with M-fundamental bosons in $4-\epsilon$ dimension by calculating the beta functions. We found a new stable fixed point in the zero mass plane for $M > M_{\text{crit}}$ by expanding up to $O(\epsilon)$. This indicates a second-order phase transition. We also calculated the critical exponents in both ϵ expansion and also in the large-M expansion.

DOI: 10.1103/PhysRevB.97.214429

I. INTRODUCTION

Phase transitions in gauge theories are very interesting because gauge theories appear as effective theories in many physical problems. Historically, in particle physics, gauge theories have been studied in detail because of their potential application to phenomenology. More recently, there are several examples of emergent gauge degrees of freedom in condensed matter physics [1–7]. Phase transitions in those theories hold very rich physics. We will be concerned solely with continuous gauge symmetries.

The simplest example of a phase transition in a continuous gauge theory is in $U(1)$ gauge theory with a single boson. This is the Ginzburg-Landau theory of superconductor-insulator transition [8]. Fluctuations around mean field were first studied by Coleman and Weinberg [9], who found that in $d = 4$, the theory undergoes a first-order phase transition. This conclusion was verified independently by Halperin, Lubensky, and Ma (HLM) [10], who also carried out a ϵ expansion in $d = 4 - \epsilon$ dimensions to first order in ϵ. They also showed $d = 3$ by integrating out the gauge degrees of freedom that the transition becomes weakly first order. Generalizing to M complex boson fields, they found for $M > M_{\text{crit}} = 182.95$ two more fixed points appear, as shown in Fig. 1. It is seen that for $M > M_{\text{crit}}$, there is a stable fixed point in the zero mass plane indicating a second-order phase transition. HLM also calculated the critical exponents for the transition in the ϵ expansion and in fixed dimension $d = 3$ in the large-M approximation.

The case of a $SU(2)$ gauge field coupled to M fundamental bosons has been studied more recently by Arnold and Yaffe [11]. They found a picture very similar to Fig. 1 in the ϵ expansion. To $O(\epsilon)$ they found that for $M > M_{\text{crit}} = 359$ there are two charged fixed points. One of them is attractive in the $b - g^2$ plane, again indicating a second-order phase transition. The $SU(2) \times U(1)$ case is known as the electroweak phase transition.

It is known from several numerical studies [12–14] in lattice gauge theory that in the case of $M = 1$ there exists a critical ratio of the couplings such that for $b/g^2 > C$ there is no phase transition at all, and for $b/g^2 < C$ the transition is first order. The second-order phase transition exists only if $b/g^2 = C$. The reason is that, for $b/g^2 > C$, no symmetry is broken in the $SU(2)$ transition.

But this picture changes in a very significant way when more than one species/flavor of boson are introduced (these transform as higher representations under the gauge group). In that case, as Fradkin and Shenker [15] show in lattice gauge theory, a phase transition does occur for all the values of the ratio of couplings. In a gauge theory with a nontrivial center, the center survives for higher representations in unitary gauge if the boson is in the adjoint representation. Introducing M species of bosons leads to a global $U(M)$ symmetry [16]. In the unitary gauge, the $SU(N)$ gauge symmetry breaks down but this $U(M)$ symmetry survives. The phase transition corresponds to spontaneous breaking of this $U(M)$ symmetry.

In this paper, we study $SU(N) \times U(1)$ theory with M flavors of bosons. Such a theory arises in a completely different context, the study of $SU(M)$ antiferromagnets on a square lattice [1].

The Hamiltonian of this model is

$$\mathcal{H} = \frac{J}{M} \sum_{\langle i,j \rangle} \hat{S}_{\alpha}^{\beta}(i) \hat{S}_{\beta}^{\alpha}(j), \tag{1}$$

where $\hat{S}_{\alpha}^{\beta}(i)$ are the generators of $SU(M)$ and $\langle i,j \rangle$ represents the nearest-neighbor sum on this bipartite square lattice. The representation of the spins sitting in two sublattices (A and B) can be described using the two integers describing the Young tableau, n_c and M. The representation of the spins is described in Fig. 2. For the A sublattice, the number of boxes in the column of the young tableau is N, where for the B sublattice, the boxes in the column are $M - N$. The number of boxes in every row is fixed to be n_c.

Now, we introduce boson (Schwinger boson [17,18]) operators $b^{\alpha a}(i)$ for sublattice A and $\bar{b}_{\alpha a}(j)$ on each sublattice B with the constraint,

$$b_{\alpha a}^{\dagger}(i) b^{\alpha b}(i) = \delta_a^b n_c, \text{ no sum on } i, \tag{2a}$$

$$\bar{b}^{\alpha a \dagger}(j) \bar{b}_{\alpha b}(j) = \delta_a^b n_c, \text{ no sum on } j. \tag{2b}$$

Now we represent the spin operator using the Schwinger bosons. To calculate the partition function, one can use the coherent states of these boson operators and represent the

*ada258@g.uky.edu

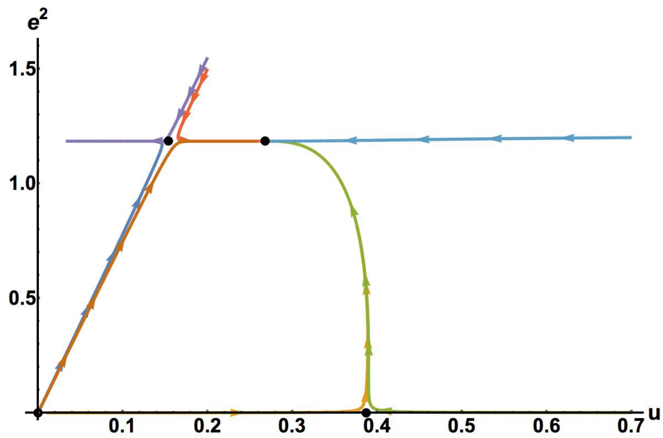

FIG. 1. Flow diagram in the $u - e^2 (= \alpha)$ plane for $M > M_{\rm cric}$. As one can see, there are four fixed points (the fixed points are also plotted in black dots). One can see the Gaussian fixed point and the well-known and famous Wilson-Fisher (WF) fixed point. But there are two new charged fixed points there which are present only for $M > M_{\rm cric}$. One of them is a stable fixed point. There exists also a charged fixed point which is not stable in this plane. This is what was found by Halperin-Lubansky-Ma [10].

partition function as a path integral over the coherent states. Then by introducing a Hubbard-Stratonovich field, one can break the four-boson term and can introduce Lagrange multiplier to put constraint Eqs. (2). Now expanding these new fields around the mean-field approximation, one can see that these fluctuations have a gauge-field component. Using this, one can write down the action around the mean field. After that, one needs to integrate out the gapped fields to reach the final action. One can then try to integrate the boson fields out to find the coupling constant dependence on the boson number. Then a gradient expansion of this action will lead to $SU(N) \times U(1)$ action with M fundamental boson. This tells us that the gauge group of the spins on the lattice gives the boson flavor when we expand around the mean field. Also, the representation of the spins as defined by N gives us the gauge group of the action around the mean field [1].

The phases $SU(M)$ antiferromagnet are known for $N = 1$. We want to check how the order of the phase transition depends on the number of flavors for the $SU(N) \times U(1)$. We want to check this in two ways. First, we can try to integrate out the gauge field, which we will do for $M = 1$ and $N = 2$ to show

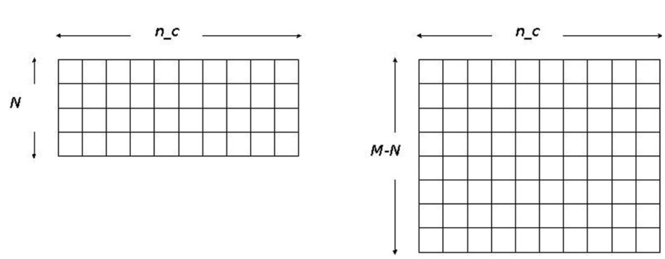

FIG. 2. The representation in terms of Young tableau of $SU(M)$ lie group of the spins on sublattice A and B. The number of boxes in every row is n_c, where the number of boxes in the column for the A sublattice is N and $M - N$ for B sublattice.

that for a single flavor in fundamental representation, there is no second-order transition, at least for $N = 2$. Next we want to study the theory that arises from the $SU(M)$ antiferromagnets. We will study the renormalisation group (RG) flow of this theory for arbitrary M and N and the fixed-point structure of the theory,

$$S[\psi, \vec{A}, \vec{W}^a] = \int d^3x \bigg[\left| \left(\partial_\mu - iyA_\mu - igT^a W^a_\mu \right) \psi \right|^2 + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} + a|\psi|^2 + \frac{b}{2} |\psi|^4 \bigg]. \tag{3}$$

We will study this model in the $\epsilon = 4 - d$ expansion in $O(\epsilon)$. It is known in the $U(1)$ case, a higher order analysis in ϵ can change the RG flow of the theory qualitatively [11,19,20]. The RG beta functions are not convergent for $\epsilon = 1$. These beta functions can be asymptotic in nature for $\epsilon = 1$; this has not yet been proved [20]. If the beta functions are asymptotic, then one can perform an analytic continuation of the beta functions (e.g., by the method of Padé approximation and Borel resummation). Using these methods, it has been found that a stable fixed point does exist for $\epsilon = 1$ and the critical value of M is modified. For $M = 1$, it has been seen that some flow can still escape to the negative ψ^4 coupling [19]. So there is a possibility that for all values of M there can be second-order phase transition.

The first-order transition for $\epsilon = 1$ in ϵ expansion has also been studied before using renormalized thermodynamic quantities [21,22].

The $U(1)$ theory has also been studied by expanding at $\varepsilon = d - 2$ up to order $O(\varepsilon)$ and it has been found that the transition is second order and a stable fixed point exists for all M [23]. Thus first-order $\epsilon = 4 - d$ expansion may not be valid for all values of M.

The beta functions can be calculated in several ways. One of them is functional renormalization group (FRG). The effective action Γ is defined as the Legendre transform of $W[J] = \ln Z[J] = \ln \int \mathcal{D}\varphi e^{-S[\varphi] + \int J\varphi}$ (this is a schematic variable representation, φ means all fields involved in the theory):

$$\Gamma[\phi] = \sup_J \left(\int J\varphi - W[J] \right). \tag{4}$$

This gives

$$0 = \frac{\delta}{\delta J} \left(\int J\phi - W[J] \right) \Rightarrow \phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J.$$

Thus ϕ corresponds to the expectation value of the φ and also $\frac{\delta \Gamma[\phi]}{\delta \phi} = J$, meaning that $\Gamma[\phi]$ governs the dynamics of the field expectation values taking into account all the fluctuations.

Parallel to this definition, we can define an infrared (IR) regulated functional

$$e^{W_k[J]} = \int^{\Lambda} \mathcal{D}\varphi e^{-S[\varphi] - \Delta S_k[\varphi] + \int J\varphi}, \tag{5}$$

where

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \varphi(-q) R_k(q) \varphi(q). \tag{6}$$

R_k is known as the regulating function. The interpolating effective action is defined as

$$\Gamma_k[\phi] = \sup_J\left(\int J\varphi - W_k[J]\right) - \Delta S_k[\phi]. \tag{7}$$

It should satisfy $R_{k\to 0} = 0$, which in turn implies $\Gamma_{k\to 0} = \Gamma$. Lets use a notation $\Gamma_k^{(n)}[\phi] = \frac{\delta^n \Gamma_k[\phi]}{\delta\phi\ldots\delta\phi}$. Now by defining $t = \ln\frac{k}{\Lambda}$, one can show by taking derivative,

$$\partial_t \Gamma_k[\phi] = \tfrac{1}{2}Tr\big[\partial_t R_k\big(\Gamma_k^{(2)}[\phi] + R_k\big)^{-1}\big]. \tag{8}$$

This is the RG flow equation. Now we want to study a gauge model. Here we have three different fields: the scalar field, the $U(1)$ gauge field, and the $SU(N)$ field. The generating functional for this case will be

$$\begin{aligned} e^{W_k[J,L^a_\mu,K_\mu;\bar{A_\mu},\bar{B^a_\mu}]} &= \int \mathcal{D}\varphi\mathcal{D}a_\mu\mathcal{D}b^a_\mu \exp -\Big\{S\big[\varphi,a_\mu,b^a_\mu;\bar{A_\mu},\bar{B^a_\mu}\big] \\ &+ \Delta S_k\big[\varphi,a_\mu,b^a_\mu;\bar{A_\mu},\bar{B^a_\mu}\big] + (GF)^{U(1)} + (GF)^{SU(N)} \\ &- \int\big(J^*\varphi + J\varphi^* + K_\mu a^\mu + L^{a\mu}b^a_\mu\big)\Big\}, \end{aligned} \tag{9}$$

where (GF) means a gauge fixing term. For the non-Abelian case, this will contain a contribution from the ghost fields [24]. And by ΔS_k, we mean sum of all the contributions from scalar field ΔS_k^s, $U(1)$ gauge part $\Delta S_k^{U(1)}$, and $SU(N)$ gauge part $\Delta S_k^{SU(N)}$. Now one can define the Γ_k similarly as before. From all this, similar to the previous case, one can write down the flow equation as

$$\partial_t\Gamma_k[\psi,A_\mu,B^a_\mu;\bar{A_\mu},\bar{B^a_\mu}] = \tfrac{1}{2}\mathrm{STr}[(\Gamma^{(2)} + R_k)^{-1}(\partial_t R_k)]. \tag{10}$$

Here I have used two notations, one is that field dependencies on the right hand side (RHS) have been suppressed. Second, the meaning of STr (supertrace) is that the ghost fields contribute with a negative sign [24]. Now this equation will be gauge invariant if the background gauge field transforms the same way the as the dynamical gauge field. This forms the basis for the functional renormalization analysis for gauge theories [24–32].

FRG is a very special method as these flow equations are nonperturbative. However, one can reproduce the perturbative results to a given order of ϵ from the full FRG equations for Landau-Ginzburg theory of superconductors [26]. The critical exponents of the superconductor-insulator phase transition are also different in these two methods (FRG and ϵ-expansion) and also the qualitative behavior of the flows changes significantly. FRG shows that the stable fixed point exists for all values of M [25,30], which is not seen in ϵ-expansion [10]. This suggests that the type-II superconductor with sufficiently strong scalar coupling will have a second-order phase transition. Recently, it has been found that FRG approach can find corrections to scaling in the critical theory of deconfined criticality, which agrees well with some quantum Monte Carlo studies [27].

The phase transition in $SU(M)$ magnets (Heisenberg model) has been studied numerically before. Kawashima and Tanabe [33] found evidence of emergent $U(1)$ symmetry of the ground-state space of the $SU(M)$ Heisenberg model with the fundamental representation. Beach *et al.* [34] developed a quantum Monte Carlo algorithm to simulate this model for continuous M in total singlet basis and found a phase transition between Neél and VBC columnar phase occurring at $M_c = 4.57(5)$. They also identified the phase transition to be second order with critical exponents, $z = 1$ and $\beta/\nu = 0.81(3)$.

II. EFFECT OF GAUGE FLUCTUATIONS

First, we will try to integrate out the gauge field to see what happens for $M = 1$(in the unitary gauge) to the action defined as

$$\begin{aligned} S[\psi,\vec{A},\vec{W}^a] = \int d^3x\Big[&\big|\big(\partial_\mu - iyA_\mu - igT^aW^a_\mu\big)\psi\big|^2 \\ &+ \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} + a|\psi|^2 + \frac{b}{2}|\psi|^4\Big], \end{aligned} \tag{11}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{12}$$

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu, \tag{13}$$

and as usual,

$$a = \frac{a'(T - T_c)}{T_c}. \tag{14}$$

This, in the pure $U(1)$ case, leads to a weak first-order phase transition as the gauge field around mean-field approximation of the order parameter picks up a mass (in other words, this will give us Meissner effect with a penetration depth defined by the mass) [10].

The ψ field has N components. Now, the minimum of this action is when all the fluctuations of fields are zero and $|\psi| =$ const. This value of the constant is well known, i.e.,

$$|\psi| = \pm\sqrt{-\frac{a}{b}}. \tag{15}$$

Now for $N = 2$, one can choose a gauge to make, $\psi_1 = 0$ and $\psi_2 = \sqrt{-a/b}$.

Now the question comes of the Ginzburg criteria. Now we can expand the ψ field around the mean-field point, i.e., $\psi_1(x) = \psi_{1R}(x) + i\psi_{1I}(x)$ and $\psi_2 = \sqrt{-a/b} + \psi_{2R}(x) + i\psi_{2I}(x)$. Then, by putting this back into the action, we get that only the $\psi_{2R}(x)$ field becomes massive with mass $2|a|$. Then one can calculate the partition function. The calculation of the partition function can be done in the momentum space much more easily. But the partition function is not good enough as it's not measurable. So one needs to calculate some physical quantities for the analysis. The expectation is that the correction term to the physical quantity from the fluctuation of the fields must be smaller than the value calculated from the mean-field approximation. That leads to the condition known as the Ginzburg criteria. This calculation is exactly similar to the $U(1)$ case. In our case, we can calculate the average energy or the specific heat. This result gives exactly same relation as the $U(1)$ case,

$$\frac{T - T_c}{T_c} < \frac{1}{32\pi^2}\frac{b^2T_c^2}{a'}. \tag{16}$$

In the case of superconductor theory, we actually know the microscopic theory (BCS theory) and from there one can exactly find these coefficients a,b in terms of microscopic parameters [8]. This ensures that the Ginzburg criteria is met and we can actually use constant mean-field solution. In our case, we don't know the microscopic parameter values but for now, we will assume that the ψ(order parameter) fluctuation is very small and we can use the mean-field value of the field.

Next, we need to consider the case where we choose a specific gauge and want to calculate the effect of the gauge-field fluctuations. Again we will do it for $N = 2$. We choose the gauge such that $\psi_1 = 0$ and $\psi_2 = v$. For $N = 2$ the generators are

$$T^a = \tfrac{1}{2}\sigma^a. \tag{17}$$

We find that the mass matrix of the fields is not diagonalized. After the mass matrix diagonalization, we find that there will be three gauge fields with mass and one massless gauge field but interacting with each other. The massive fields are W^1_μ, W^2_μ with mass square, $m_1^2 = m_2^2 = (1/2)g^2v^2$ and Z_μ with mass square, $m_Z^2 = v^2(g^2 + 4y^2)/2$. We will also have a massless field B_μ. The definition of B_μ and Z_μ is

$$B_\mu = \sin\theta_W A_\mu + \cos\theta_W W^3_\mu Z_\mu = \cos\theta_W A_\mu - \sin\theta_W W^3_\mu, \tag{18}$$

where $\sin\theta_W = g/\sqrt{g^2 + 4y^2}$ and $\cos\theta_W = 2y/\sqrt{g^2 + 4y^2}$.

As mentioned before we want to calculate

$$\exp(-S(\psi)/T) = \int \mathcal{D}B\mathcal{D}Z\mathcal{D}W^i \exp[-S[\psi,\vec{A},\vec{W}^a]/T]. \tag{19}$$

For our gauge, we find that

$$\frac{dS}{dv} = 2(vol)av + 2(vol)bv^3 + (g^2/4)\langle {W^i_\mu}^2\rangle v + (g^2/4 + y^2)\langle Z_\mu^2\rangle v. \tag{20}$$

To calculate these averages, we have used only the leading order of the propagator. In this case, the W^i_μ fields and Z_μ fields are both massive. Thus their propagator will be (up to the leading order)

$$\langle W^i_\mu(r)W^{j\mu}(r)\rangle = (vol)\int \frac{d^3k}{(2\pi)^3}\frac{\delta^{ij}\delta_{\mu\nu}(\delta^{\mu\nu} - k^\mu k^\nu/k^2)}{k^2 + m_i^2}, \tag{21}$$

$$\langle Z_\mu(r)Z^\mu(r)\rangle = (vol)\int \frac{d^3k}{(2\pi)^3}\frac{\delta_{\mu\nu}(\delta^{\mu\nu} - k^\mu k^\nu/k^2)}{k^2 + m_z^2}. \tag{22}$$

Keeping our calculation to the leading order, we can exactly calculate these integrals just like the $U(1)$ case [10]:

$$\langle {W^i_\mu}^2\rangle = \frac{2(vol)}{\pi^2}\left[\Lambda - \frac{m_i\pi}{2}\right], \tag{23}$$

$$\langle {Z_\mu}^2\rangle = \frac{2(vol)}{\pi^2}\left[\Lambda - \frac{m_Z\pi}{2}\right], \tag{24}$$

where Λ is some momentum UV cutoff of the momentum.

Putting all this to Eq. (20) and integrating over v we get

$$\frac{S}{vol} = \left[\left(a + \frac{3\Lambda}{2\pi^2}\right)v^2 + \frac{b}{2}v^4 - \frac{3\left(2g + \sqrt{g^2 + 4y^2}\right)v^3}{4\sqrt{2}\pi}\right]. \tag{25}$$

This introduces a v^3 term describing a weak first-order phase transition exactly like in $U(1)$ case [10]. From this, one can calculate the size of the phase transition, etc.

III. BETA FUNCTIONS AND FIXED POINTS

The more general way to find β-function is to carry out RG calculations in $d = 4 - \epsilon$ and for general N using dimensional regularization. We define here for simplicity of the calculations $\alpha_1 = y^2$ and $\alpha_2 = g^2$ [35–37]. Thus the beta functions are [38]

$$\beta_{\alpha_1} = \epsilon\alpha_1 - \frac{\alpha_1^2 NM}{24\pi^2}, \tag{26}$$

$$\beta_{\alpha_2} = \epsilon\alpha_2 - \frac{\alpha_2^2}{48N\pi^2}(M - 22N), \tag{27}$$

$$\beta_a = a\left[2 - \frac{b(NM+1)}{8\pi^2} + \frac{3\alpha_2}{8\pi^2}\left(\frac{N^2-1}{2N}\right) + \frac{3\alpha_1}{8\pi^2}\right], \tag{28}$$

$$\beta_b = \epsilon b - \frac{b^2(NM+4)}{8\pi^2} - \frac{3\alpha_1^2}{4\pi^2} - \frac{3\alpha_2^2(N^3 + N^2 - 4N + 2)}{32\pi^2N^2} - \frac{3\alpha_1\alpha_2}{\pi^2N}(N-1) + \frac{3b\alpha_1}{4\pi^2} + \frac{3b\alpha_2}{4\pi^2}\left(\frac{N^2-1}{2N}\right). \tag{29}$$

One can easily see from the structure of the β-function that for $N = 1$, $\beta_{\alpha_1}, \beta_a, \beta_b$ completely decouples from the α_2 and one can check that it has the correct structure for $U(1)$ gauge theory with multiple scalar [10,37]. Next one can look into the fixed point structure of this theory (see Figs. 3 and 4). There are eight possible fixed points of these β-functions. Two of them are the old Gaussian and the Wilson-Fisher fixed point and fixed points where there is no $SU(N)$ or $U(1)$ charge [10]. As before, the $U(1)$-charged fixed points do not exist for $NM < 182.952$. There are four more fixed points that arise in the theory and one of them is critical, as one is completely stable in all direction except for the temperature (mass) direction.

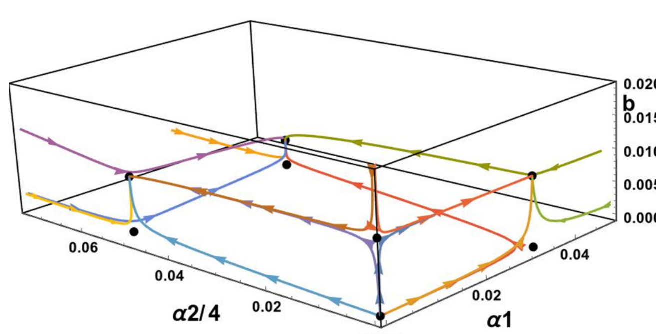

FIG. 3. RG flow diagram for the $N = 2$ and $M = 1500$ where all the attractive points exist. As we can see here, there are eight fixed points and one attractive in all directions (other than mass). That fixed point denotes the second-order phase transition of the system.

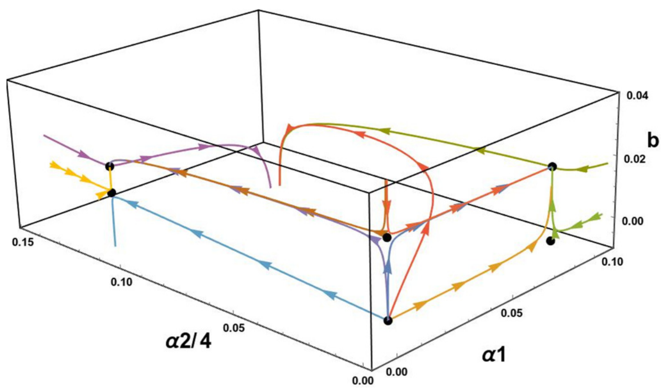

FIG. 4. RG flow diagram for $N = 2$ and $M = 1100$ where the attractive point does not exist and, as we can see, the flow does not have eight fixed points anymore. The attractive doubly charged fixed point is now gone and all flow with any nonzero initial charge flows to negative mass, denoting a first-order phase transition.

This point is doubly charged. But this fixed point does not exist for $M < M_{\rm crit}$. This $M_{\rm crit}$ is different for different values of N. For example, for $N = 2$, $M_{\rm crit} = 1277.47$. There are two singly charged ($SU(N)$ charge) fixed points also. These $SU(N)$ charged fixed points also have some critical value of M as a function of N. As previously calculated for $N = 2$, this critical value is 359 [11].

IV. CRITICAL EXPONENTS

The critical exponents of this phase transition can be easily calculated in the regular way and we can see that $\nu \to 1$ and $\eta \to 0$ as $M \to \infty$ for $\epsilon = 1$. In terms of fixed point value of the parameters $(a^* = 0, b^*, \alpha_1^*, \alpha_2^*)$ [39],

$$\frac{1}{\nu} = 2 - \frac{b^*(N+1)}{8\pi^2} + \frac{3\alpha_1^*}{8\pi^2} + \frac{3\alpha_2^*}{8\pi^2}\left(\frac{N^2-1}{2N}\right), \quad (30)$$

$$\eta = -\left[\frac{3\alpha_1^*}{4\pi^2} + \frac{3\alpha_2^*}{8\pi^2}\left(\frac{N^2-1}{2N}\right)\right]. \quad (31)$$

As we have seen, these beta functions have a very interesting structure of fixed points (we have $M > N$). There are eight fixed points, but not all of them exist at every value of M and N. The M and N comes from the microscopic theory [1]. For $N = 1$, the theory contains only the Abelian gauge field. The question one needs to ask is for what values of N and M there exist a doubly charged critical point. We can easily find out the relation between N and $M_{\rm cric}$. That relation is quadratic:

$$M_{\rm crit} = 607.765 + 174.594N + 106.058N^2. \quad (32)$$

The region on the $N - M$ plane for which the theory has a critical point is in the shaded region of Fig. 5.

The critical exponents can also be calculated in fixed dimension ($d = 3$) in the large M limit, where the coupling constants are $b \sim O(1/M), y \sim O(1/\sqrt{M}), g \sim O(1/\sqrt{M})$. This method is similar to what is described by Ma [40]. From this calculation, we get for M complex fields in fundamental representation of $SU(N)$,

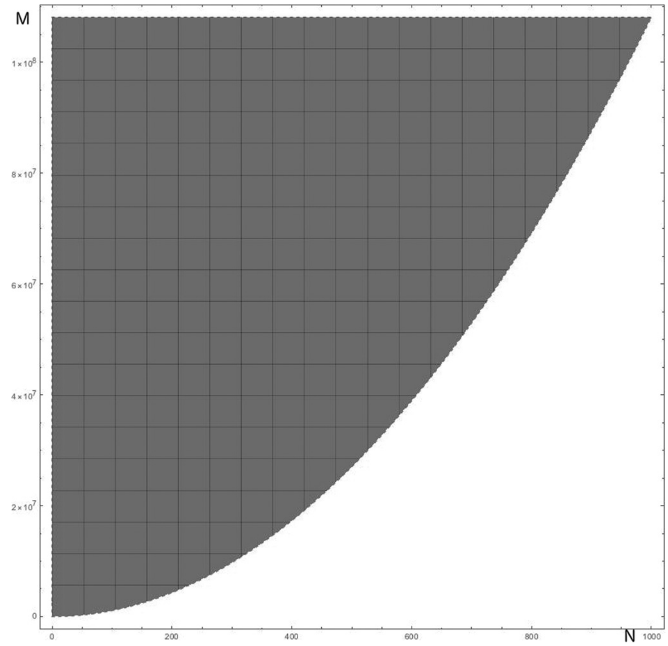

FIG. 5. Shaded region on $N - M$ plane for which the theory has a critical point.

$$\eta = -\frac{1}{NM}[2.0264 + 2.1615(N^2 - 1)], \quad (33)$$

$$\nu = 1 - \frac{4.86}{NM} - \frac{4.32}{NM}(N^2 - 1). \quad (34)$$

This result matches with the already known results for $N = 1$ [10,41].

V. DISCUSSION AND CONCLUSION

From this analysis, we found that for $SU(M)$ antiferromagnets, there is a temperature-driven phase transition for a very large M compared to N (representation of the spin as defined in the introduction). This critical value $M_{\rm crit}$ can be calculated for as a function of N.

The critical exponents of this second-order phase transition are calculated in both ϵ expansion and in the large-M expansion. The next question is what the phases are that lie on either side of the phase transition. But we should be careful in attaching more deep meaning to this method, as mentioned in the introduction, that it has been observed that higher order corrections in ϵ can alter the qualitative structure of the RG flow for more simpler gauge group $U(1)$ [11,19]. Also, as one expects that beta functions are probably asymptotic in nature, thus one needs to do some analytic continuation for correct results [20]. It might be insightful to do an RG for $d = 2 + \varepsilon$ dimensions expanding in order of ε, as it already gives a very interesting result for $U(1)$ case [23]. All this we expect to do in the future as they have potential to give very interesting results and correction to current understanding.

As discussed in the introduction, FRG is a nonperturbative method which produces a qualitatively different result than the ϵ-expansion for $U(1)$-gauge theory with many complex scalar fields [25–27,30]. It will be interesting to analyze our theory using FRG in future.

It has already been discovered numerically that for $M = 1$ there is no electroweak phase transition at all for large value of b/g^2 [12–14]. For large M, there is a phase transition. This phase transition corresponds to the breaking of the leftover symmetry ($U(M)$ flavor symmetry) [15,16]. It is known that those phases are connected to conventional Higgs and confinement phases [15]. The lattice limit of the order parameters still remains an open question.

We analyzed this theory with no topological terms. The critical exponents can also be calculated with a topological term; the $U(1)$ case has been calculated recently [41] but $SU(N) \times U(1)$ case still remains open. I plan to study in future the effect of the topological term in this Lagrangian.

ACKNOWLEDGMENTS

I thank Dr. Ganpathy Murthy for his help and valuable discussion to understand a number of subtle issues about this problem. I am very thankful to Dr. R. Kaul for introducing me to this area of physics. I am very grateful to Dr. M. Eides for his very helpful comments and discussions. I regard Dr. P. Arnold for very helpful discussions. I thank NSF-DMR 1306897 and the University of Kentucky for supporting this project.

[1] N. Read and S. Sachdev, Phys. Rev. B **42**, 4568 (1990).
[2] T. Grover and T. Senthil, Phys. Rev. Lett. **98**, 247202 (2007).
[3] T. Grover and T. Senthil, Phys. Rev. Lett. **107**, 077203 (2011).
[4] P. E. Lammert, D. S. Rokhsar, and J. Toner, Phys. Rev. E **52**, 1778 (1995).
[5] S.-S. Lee and P. A. Lee, Phys. Rev. Lett. **95**, 036403 (2005).
[6] Q. Huo, Y. Jiang, R. Z. Wang, and H. Yan, Europhys. Lett. **101**, 27001 (2013).
[7] M. Blasone, P. Jizba, and G. Vitiello, *Quantum Field Theory and its Macroscopic Manifestations: Boson Condensation, Ordered Patterns and Topological Defects* (Imperial College Press, London, 2011).
[8] L. P. Gor'kov, Sov. Phys. JETP **36**, 6 (1959).
[9] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[10] B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974).
[11] P. Arnold and L. G. Yaffe, Phys. Rev. D **49**, 3003 (1994).
[12] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Phys. Rev. Lett. **77**, 2887 (1996).
[13] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. B **503**, 357 (1997).
[14] K. Kajantie, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. B **407**, 356 (1993).
[15] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979).
[16] P. Arnold and D. Wright, Phys. Rev. D **55**, 6274 (1997).
[17] D. P. Arovas and A. Auerbach, Phys. Rev. B **38**, 316 (1988).
[18] A. Auerbach and D. P. Arovas, Phys. Rev. Lett. **61**, 617 (1988).
[19] S. Kolnberger and R. Folk, Phys. Rev. B **41**, 4083 (1990).
[20] R. Folk and Y. Holovatch, J. Phys. A **29**, 3409 (1996).
[21] I. D. Lawrie, Nucl. Phys. B **200**, 1 (1982).
[22] I. D. Lawrie, J. Phys. C **15**, L879 (1982).
[23] I. D. Lawrie and C. Athorne, J. Phys. A **16**, L587 (1983).
[24] H. Gies, in *Renormalization Group and Effective Field Theory Approaches to Many-Body Systems*, edited by A. Schwenk and J. Polonyi, Lecture Notes in Physics Vol. 852 (Springer, Berlin, Heidelberg, 2012).
[25] B. Bergerhoff, F. Freire, D. F. Litim, S. Lola, and C. Wetterich, Phys. Rev. B **53**, 5734 (1996).
[26] N. Tetradis, Nucl. Phys. B **488**, 92 (1997).
[27] L. Bartosch, Phys. Rev. B **88**, 195140 (2013).
[28] M. Reuter and C. Wetterrich, Nucl. Phys. B **408**, 91 (1993).
[29] C. Wetterich, Nucl. Phys. B **352**, 529 (1991).
[30] B. Bergerhoff, D. Litim, S. Lola, and C. Wetterich, Int. J. Mod. Phys. A **11**, 4273 (1996).
[31] M. Reuter and C. Wetterich, Nucl. Phys. B **427**, 291 (1994).
[32] M. Reuter and C. Wetterich, Nucl. Phys. B **417**, 181 (1994).
[33] N. Kawashima and Y. Tanabe, Phys. Rev. Lett. **98**, 057202 (2007).
[34] K. S. D. Beach, F. Alet, M. Mambrini, and S. Capponi, Phys. Rev. B **80**, 184401 (2009).
[35] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, 1995).
[36] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2007).
[37] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, UK, 2007).
[38] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.97.214429 for details of the calculation of beta functions and critical exponents.
[39] M. Kiometzis, H. Kleinert, and A. M. J. Schakel, Phys. Rev. Lett. **73**, 1975 (1994).
[40] S. Ma, Phys. Rev. A **7**, 2172 (1973).
[41] S. Sakhi, Phys. Rev. D **90**, 045028 (2014).