


## Hall viscosity and geometric response in the Chern-Simons matrix model of the Laughlin states

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We study geometric aspects of the Laughlin fractional quantum Hall (FQH) states using a description of these states in terms of a matrix quantum mechanics model known as the Chern-Simons matrix model (CSMM). This model was proposed by Polychronakos as a regularization of the noncommutative Chern-Simons theory description of the Laughlin states proposed earlier by Susskind. Both models can be understood as describing the electrons in a FQH state as forming a noncommutative fluid, i.e., a fluid occupying a noncommutative space. Here, we revisit the CSMM in light of recent work on geometric response in the FQH effect, with the goal of determining whether the CSMM captures this aspect of the physics of the Laughlin states. For this model, we compute the Hall viscosity, Hall conductance in a nonuniform electric field, and the Hall viscosity in the presence of anisotropy (or intrinsic geometry). Our calculations show that the CSMM captures the guiding center contribution to the known values of these quantities in the Laughlin states, but lacks the Landau orbit contribution. The interesting correlations in a Laughlin state are contained entirely in the guiding center part of the state/wave function, and so we conclude that the CSMM accurately describes the most important aspects of the physics of the Laughlin FQH states, including the Hall viscosity and other geometric properties of these states, which are of current interest.

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## I. INTRODUCTION

In the past few years there has been tremendous progress in the understanding of the geometric properties of quantum Hall states. The role of geometry in the quantum Hall effect first came to prominence in early work on Hall viscosity [1–3] (sometimes called *odd* viscosity), and there has been much work on Hall viscosity since then [4–13]. Recent work on geometry in the quantum Hall effect has gone in two separate directions. First, there is the study of the properties of quantum Hall states on curved spatial manifolds (Riemann surfaces) [14–21]. Second, there is the study of intrinsic geometry and anisotropy in quantum Hall states on flat space [7,8,13,22,23]. In the past year, a very interesting new theory of quantum Hall states has been proposed, known as the *bimetric* theory, and this theory promises to unify the two separate directions of research on geometry in the quantum Hall effect [24,25].

In a separate line of development, Susskind proposed in 2001 that Laughlin fractional quantum Hall (FQH) states could be described by *noncommutative Chern-Simons* (NCCS) theory [26]. This is a deformation of ordinary Chern-Simons theory in which the theory is formulated on a noncommutative analog of the flat space  $\mathbb{R}^2$  consisting of “coordinates”  $\hat{x}^1$  and  $\hat{x}^2$  obeying a nontrivial commutation relation

$$[\hat{x}^1, \hat{x}^2] = i\theta. \quad (1.1)$$

Here,  $\theta$  is a real parameter with dimensions of length squared that characterizes the degree of noncommutativity of the theory. The original motivation for this proposal was Susskind’s observation that the gauge symmetry of NCCS theory provides a discretization of the symmetry under area-preserving

diffeomorphisms that is present in a description of a FQH state as a charged fluid in a magnetic field. In particular, this discretization was argued to capture the “granularity” of a fluid constructed from electrons, and in the NCCS theory description each electron is associated with a fundamental area of size  $2\pi|\theta|$ . In addition, in the NCCS theory, a quantization rule [27] enforces

$$\theta = \ell_B^2 m, \quad m \in \mathbb{Z}, \quad (1.2)$$

where  $\ell_B$  is the magnetic length, and so one finds (for  $m > 0$ ) that the fluid described by the NCCS theory has a number density  $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$ , exactly as in the  $\nu = 1/m$  Laughlin state.

Susskind’s original proposal has been followed up by many authors [28–37]. Of all of these subsequent works, the work of Polychronakos is particularly important for this paper. In Ref. [28], Polychronakos proposed a regularization of the NCCS theory, which is known as the *Chern-Simons matrix model* (CSMM). This is a particular regularization of the NCCS theory that can be viewed as a quantum mechanics model in which the degrees of freedom are  $N \times N$  matrices (we discuss the model in detail and make this statement precise below). The quantum ground state of the CSMM having  $\theta = \ell_B^2 m$  ( $m > 0$ ) is known to describe a uniform droplet of “noncommutative fluid” with a density and area matching that of the  $\nu = 1/m$  Laughlin state. Polychronakos has also demonstrated that excitations in this model carry the appropriate fractional charge of the quasihole excitations in the Laughlin state.

Despite the successes in describing the basic properties of the Laughlin FQH states using NCCS theory and the CSMM, there have not been any attempts to study the *geometric* properties of the Laughlin states in the context of these noncommutative models. Therefore our goal in this article is

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to answer the following question: *does the CSMM accurately describe the geometric properties of the Laughlin states?*

The particular geometric properties that we are concerned with are the Hall viscosity, the Hall conductance in a nonuniform electric field, and the Hall viscosity in the presence of anisotropy (or intrinsic geometry). We compute all of these quantities in the CSMM and we find that the results in the CSMM contain only the *guiding center* contribution to the known values for these quantities in the Laughlin states. For example, the full Hall viscosity coefficient for the Laughlin  $\nu = 1/m$  state is given by [5]

$$\eta_{\text{tot}} = \frac{\hbar \rho_0 m}{4}, \quad (1.3)$$

while for the CSMM with  $\theta = \ell_B^2 m$  we find<sup>1</sup> (after regularization)

$$\eta_{\text{CSMM,reg}} = \frac{1}{2} \hbar \rho_0 \left( \frac{m-1}{2} \right), \quad (1.4)$$

which is exactly the (regularized) *guiding center Hall viscosity* of the  $\nu = 1/m$  Laughlin state [7,8,13]. The need for regularization of the guiding center part of the Hall viscosity has been discussed in Refs. [7,8,13]. In this paper, we also give a fluid interpretation of this regularization in the context of the CSMM.

Based on our calculations, we conclude quite generally that the CSMM and NCCS theory descriptions of the Laughlin FQH states capture the guiding center contribution to the geometric properties of these states, but lack the Landau orbit contribution. We argue that this is not surprising since in the fluid interpretation of the CSMM and NCCS theories, the cyclotron frequency  $\omega_c$  is sent to infinity by sending the mass of the particles in the fluid to zero. This is analogous to a projection into a Landau level (which freezes out the Landau orbit degrees of freedom), and so it makes sense that only the guiding center contribution remains. The Landau orbit contribution is often considered to be less important since the interesting correlations in a Laughlin state are contained entirely in the guiding center part of the state/wave function. Therefore we find that the CSMM description is able to capture the most important contributions to the geometric properties of the Laughlin states. We expect that our results will rekindle interest in noncommutative models of the FQH effect, as these models clearly have a role to play in the investigation of geometric properties of FQH states.

This paper is organized as follows. In Sec. II, we review the notion of Hall viscosity. In Secs. III and IV, we give a comprehensive review of the NCCS theory and CSMM, the fluid interpretation of these models, and their relation to the Laughlin states. In Sec. V, we compute the Hall viscosity in the CSMM. In Sec. VI, we compute the Hall conductance of the CSMM in a nonuniform electric field. In Sec. VII, we give a fluid interpretation of the regularization of the guiding center part of the Hall viscosity in which one subtracts the

extensive contribution to this quantity. Finally, in Sec. VIII, we present a modified version of the CSMM incorporating anisotropy, and we compute the Hall viscosity for the modified model. Section IX presents our conclusions. The paper also includes two appendices. In Appendix A, we review the form of the quantum generators of the action of the group  $U(N)$  on the fields of the CSMM, as this information is necessary for the quantization of this model, which we review in Sec. IV. In Appendix B, we present the details of the calculation of the Hall viscosity of the CSMM (which is presented in Sec. V of the main text), which involves a Kubo formula approach inspired by Ref. [12].

## II. REVIEW OF HALL VISCOSITY

In this section, we review the concept of Hall viscosity following the derivation and point of view in Ref. [13]. We also emphasize, again following Ref. [13], the separation of the Hall viscosity tensor into two parts: the *Landau orbit* contribution and the *guiding center* contribution. Finally, we review the form of both parts of the Hall viscosity tensor for typical FQH trial states including the Laughlin states. The example of the Laughlin states is of particular interest for the rest of the paper when we compare to the results obtained in the CSMM, which has been argued to describe the physics of the Laughlin states.

### A. Hall viscosity calculation

The Hall viscosity can be computed by studying the response of a FQH state to time-dependent area-preserving deformations (APDs). Before we review the calculation of the Hall viscosity, we briefly recall the setup of the quantum Hall problem. We consider  $N$  electrons on the plane, each with a charge  $-e < 0$ , in the presence of a constant background magnetic field of strength  $B > 0$  and pointing in the positive  $z$  direction. Let  $\mathbf{r}_j$  be the position coordinates of the  $N$  electrons, where  $j = 1, \dots, N$  is a particle label. We write  $r_j^a$  with  $a = 1, 2$  for the two components of the vector  $\mathbf{r}_j$  (i.e.,  $a = 1, 2$  labels the two directions of space). In this situation, the electron coordinate operators  $r_j^a$  break up into two parts as

$$r_j^a = R_j^a + \tilde{R}_j^a, \quad (2.1)$$

where  $R_j^a$  are known as the guiding center coordinates, and  $\tilde{R}_j^a$  are the Landau orbit coordinates. These coordinates obey the commutation relations

$$[R_j^a, R_k^b] = i \ell_B^2 \epsilon^{ab} \delta_{jk}, \quad (2.2a)$$

$$[\tilde{R}_j^a, \tilde{R}_k^b] = -i \ell_B^2 \epsilon^{ab} \delta_{jk}, \quad (2.2b)$$

$$[R_j^a, \tilde{R}_k^b] = 0, \quad (2.2c)$$

where  $\ell_B^2 = \hbar/(eB)$  is the square of the magnetic length  $\ell_B$ .

The Hall viscosity is defined as the response of the system (more precisely, the ground state) to time-dependent APDs of the electron coordinates. These APDs are generated by Hermitian operators  $\Lambda^{ab}$ , which are a linear combination of guiding center and Landau orbit parts,

$$\Lambda^{ab} = \Lambda^{ab} - \tilde{\Lambda}^{ab}. \quad (2.3)$$

<sup>1</sup>In the literature, the quantity  $(m-1)/2$  is referred to either as the *anisospin* (Refs. [24,25]) or minus the *guiding center spin* (Refs. [7,8,13]) of the  $\nu = 1/m$  Laughlin state.

The operators  $\Lambda^{ab}$  generate APDs of the guiding center coordinates and have the form

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{j=1}^N \{R_j^a, R_j^b\}, \quad (2.4)$$

where  $\{\cdot, \cdot\}$  denotes an anticommutator, while  $\tilde{\Lambda}^{ab}$  generates APDs of the Landau orbit coordinates, and  $\tilde{\Lambda}^{ab}$  is defined like  $\Lambda^{ab}$  but with the guiding center coordinates  $R_j^a$  replaced by the Landau orbit coordinates  $\tilde{R}_j^a$ . One can show that these generators obey the Lie algebras

$$[\Lambda^{ab}, \Lambda^{cd}] = \frac{i}{2} (\epsilon^{bc} \Lambda^{ad} + \epsilon^{bd} \Lambda^{ac} + \epsilon^{ac} \Lambda^{bd} + \epsilon^{ad} \Lambda^{bc}), \quad (2.5a)$$

$$[\tilde{\Lambda}^{ab}, \tilde{\Lambda}^{cd}] = -\frac{i}{2} (\epsilon^{bc} \tilde{\Lambda}^{ad} + \epsilon^{bd} \tilde{\Lambda}^{ac} + \epsilon^{ac} \tilde{\Lambda}^{bd} + \epsilon^{ad} \tilde{\Lambda}^{bc}). \quad (2.5b)$$

In addition, it is clear that  $[\Lambda^{ab}, \tilde{\Lambda}^{cd}] = 0$ . The generators  $\Lambda^{ab}$  (and also  $\tilde{\Lambda}^{ab}$ ) can be expressed in terms of the generators of the Lie algebra of the group  $SU(1,1)$ , and we will use this fact later.<sup>2</sup>

Finite (as opposed to infinitesimal) APDs of the electron coordinates are implemented by conjugation by the unitary operators<sup>3</sup>

$$U(\alpha) = e^{i\alpha_{ab}\Lambda^{ab}}, \quad (2.6)$$

where  $\alpha_{ab}$  is a constant, symmetric tensor with unit determinant (thus the APDs are spatially uniform since  $\alpha_{ab}$  does not depend on the spatial coordinates). For example, acting on the electron coordinates gives

$$U(\alpha)r_j^a U(\alpha)^\dagger = r_j^a + \epsilon^{ab}\alpha_{bc}r_j^c + \dots, \quad (2.7)$$

where the ellipses denote higher-order terms in  $\alpha_{ab}$ .

The APDs that we have been considering so far are closely related to strains in continuum mechanics. Suppose the vector  $\mathbf{r}$  is the location of a point in a solid or fluid before a deformation, and  $\mathbf{r}'(\mathbf{r})$  is the location of that same point after the deformation. Then for small deformations the *strain tensor*  $u_{ab}$  is defined in terms of the displacement vector  $\mathbf{u}(\mathbf{r}) = \mathbf{r}'(\mathbf{r}) - \mathbf{r}$  as

$$u_{ab} = \frac{1}{2} \left( \frac{\partial u_a}{\partial r^b} + \frac{\partial u_b}{\partial r^a} \right), \quad (2.8)$$

where  $u^a$  are the components of  $\mathbf{u}(\mathbf{r})$  and  $u_a = \delta_{ab}u^b$ . If we consider small APDs in the quantum Hall problem (i.e., we work to linear order in  $\alpha_{ab}$ ), then we find a strain tensor

$$u_{ab} = \frac{1}{2} (\delta_{ac}\epsilon^{cd}\alpha_{db} + (a \leftrightarrow b)). \quad (2.9)$$

In particular, we find for the trace  $\sum_{a=1}^2 u_{aa} = 0$ , which means that the APDs are indeed area-preserving (the trace of the

strain tensor determines the change in the area of a small element of the fluid or solid at the location  $\mathbf{r}$ ). The strain tensor is also spatially uniform since  $\alpha_{ab}$  does not depend on the spatial coordinates  $r^a$ . Therefore the APDs that we have been considering can be understood as a special case of a strain in continuum mechanics, namely, a spatially uniform and area-preserving strain. In what follows, we sometimes use the terms APD and strain interchangeably although, strictly speaking, the former is a special case of the latter.

Consider a FQH system described by a Hamiltonian  $H_0$ . Under a time-independent APD parametrized by  $\alpha_{ab}$ , the Hamiltonian is transformed to

$$H(\alpha) = U(\alpha)H_0U(\alpha)^\dagger. \quad (2.10)$$

We can define the generalized force associated with this APD as

$$F^{ab} = -\left. \frac{\partial H(\alpha)}{\partial \alpha_{ab}} \right|_{\alpha=0} = -i[\Lambda^{ab}, H_0]. \quad (2.11)$$

If we subject the system to a time-dependent APD  $\alpha_{ab}(t)$ , then we can compute the expectation value of  $F^{ab}$  in the time-dependent ground state  $|\psi(t)\rangle$  in an expansion in time derivatives of  $\alpha_{ab}(t)$ . In fact, as was argued in Ref. [12], one should actually compute the expectation value of  $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$  instead of  $F^{ab}$ . We discuss this point in more detail in the context of our Kubo formula calculation of the Hall viscosity for the CSMM in Appendix B, but just mention here that this replacement corresponds to expressing the generalized force in terms of the coordinates of the deformed system.

We now compute the expectation value of  $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$  in an expansion in time derivatives of  $\alpha_{ab}(t)$  as

$$\begin{aligned} & \langle \psi(t) | U(\alpha(t)) F^{ab} U(\alpha(t))^\dagger | \psi(t) \rangle \\ &= \langle \psi_0 | F^{ab} | \psi_0 \rangle + \Gamma^{abcd} \dot{\alpha}_{cd}(t) + \dots, \end{aligned} \quad (2.12)$$

where  $|\psi_0\rangle$  denotes the initial state of the system, the overdot on  $\alpha_{cd}(t)$  denotes a time derivative, and  $\Gamma^{abcd}$  is a four index tensor, which is independent of the parameters  $\alpha_{ab}(t)$  [in principle, there could also be an elastic term, which is proportional to  $\alpha_{ab}(t)$ , but this term is not present for a fluid state]. Park and Haldane then define the full Hall viscosity tensor  $\eta_{\text{tot}}^{abcd}$  (with all indices raised) as

$$\eta_{\text{tot}}^{abcd} = \frac{\Gamma^{abcd}}{A}, \quad (2.13)$$

where  $A$  denotes the area of the quantum Hall droplet (recall that we are working on the infinite plane, so we must assume that the quantum Hall droplet occupies a finite area  $A$ ). The intuition behind this definition is that  $\eta_{\text{tot}}^{abcd}$  encodes the linear response of the ‘‘generalized stress’’  $\frac{U(\alpha(t))F^{ab}U(\alpha(t))^\dagger}{A}$  to the ‘‘rate of strain’’ encoded by the tensor  $\dot{\alpha}_{cd}(t)$ . We also note here that for a droplet of quantum Hall fluid the area  $A$  of the droplet can be expressed as  $A = 2\pi\ell_B^2 N_\phi$ , where  $N_\phi$  is the number of fundamental flux quanta  $\Phi_0 = h/e$  piercing the droplet.

<sup>2</sup>Physicists can read about the group  $SU(1,1)$  in Ref. [38], for example.

<sup>3</sup>Here, and in the rest of the paper, we use a summation convention in which we sum over all indices, which are repeated once as a subscript and once as a superscript. All other summations will be indicated explicitly.

Using adiabatic perturbation theory, Park and Haldane showed that

$$\begin{aligned}\eta_{\text{tot}}^{abcd} &= \frac{i\hbar}{A} \langle \psi_0 | [\Lambda^{ab}, \Lambda^{cd}] | \psi_0 \rangle \\ &= \frac{i\hbar}{A} \langle \psi_0 | [\Lambda^{ab}, \Lambda^{cd}] | \psi_0 \rangle + \frac{i\hbar}{A} \langle \psi_0 | [\tilde{\Lambda}^{ab}, \tilde{\Lambda}^{cd}] | \psi_0 \rangle \\ &\equiv \eta_H^{abcd} + \tilde{\eta}_H^{abcd}.\end{aligned}\quad (2.14)$$

Thus, the full Hall viscosity tensor breaks up into two parts: the guiding center Hall viscosity tensor  $\eta_H^{abcd}$ , and the Landau orbit Hall viscosity tensor  $\tilde{\eta}_H^{abcd}$ .

The expression for the full Hall viscosity tensor can be simplified further by using the algebra of APD generators from Eq. (2.5) to find

$$\eta_{\text{tot}}^{abcd} = \frac{1}{2} (\epsilon^{ac} \eta_{\text{tot}}^{bd} + \epsilon^{ad} \eta_{\text{tot}}^{bc} + (a \leftrightarrow b)), \quad (2.15)$$

where the symmetric two-index tensor  $\eta_{\text{tot}}^{ab}$  also breaks up into guiding center and Landau orbit parts as

$$\eta_{\text{tot}}^{ab} = \eta_H^{ab} + \tilde{\eta}_H^{ab} \quad (2.16)$$

with

$$\eta_H^{ab} = -\frac{\hbar}{A} \langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle, \quad (2.17a)$$

$$\tilde{\eta}_H^{ab} = \frac{\hbar}{A} \langle \psi_0 | \tilde{\Lambda}^{ab} | \psi_0 \rangle. \quad (2.17b)$$

In what follows, we also refer to these two-index tensors as ‘‘Hall viscosity tensors.’’ Reference [13] emphasized that the guiding center contribution  $\eta_H^{ab}$  to  $\eta_{\text{tot}}^{ab}$  has a physical interpretation in terms of the intrinsic electric dipole moment along the edge of a FQH state, and in fact must be proportional to the symmetric tensor, which determines this dipole moment in order to balance the force on a FQH edge in an inhomogeneous electric field (see also Ref. [39] for a complementary discussion of this boundary dipole moment from a different point of view). We now review the form of the two parts of the Hall viscosity tensor for typical FQH trial states including the Laughlin states.

### B. Values in quantum Hall trial states

In this section, we consider the form of the guiding center and Landau orbit Hall viscosity tensors  $\eta_H^{ab}$  and  $\tilde{\eta}_H^{ab}$  for typical FQH trial states including the Laughlin states. In the operator, or Heisenberg, approach (as opposed to the Schrodinger approach using wave functions) a state vector for a trial FQH state is constructed using ladder operators  $b_j$  and  $b_j^\dagger$  defined in terms of the guiding center coordinates as

$$b_j = \frac{1}{\ell_B \sqrt{2}} (R_j^1 + i R_j^2), \quad (2.18)$$

and also ladder operators  $a_j$  and  $a_j^\dagger$  defined in terms of the Landau orbit coordinates as

$$a_j = \frac{1}{\ell_B \sqrt{2}} (\tilde{R}_j^1 - i \tilde{R}_j^2). \quad (2.19)$$

We define  $|0\rangle_a$  and  $|0\rangle_b$  to be the Fock vacuum states annihilated by the  $a_j$  and  $b_j$  operators, respectively. In terms of these,

a typical FQH trial state in the  $n$ th Landau level has the form

$$|\psi_0\rangle = \left[ \prod_{j=1}^N \frac{(a_j^\dagger)^n}{\sqrt{n!}} \right] F(b_1^\dagger, \dots, b_N^\dagger) |0\rangle_a \otimes |0\rangle_b, \quad (2.20)$$

where  $F(b_1^\dagger, \dots, b_N^\dagger)$  is a homogeneous polynomial of  $N$  variables, and which is either symmetric (for bosons) or antisymmetric (for fermions) under exchange of any two variables. We use  $\text{Deg}[F]$  to denote the total degree of the polynomial function  $F$ . Then if we scale all arguments of  $F$  by a numerical factor  $\lambda$ , we have

$$F(\lambda b_1^\dagger, \dots, \lambda b_N^\dagger) = \lambda^{\text{Deg}[F]} F(b_1^\dagger, \dots, b_N^\dagger). \quad (2.21)$$

Let  $\mathcal{N}_b = \sum_{j=1}^N b_j^\dagger b_j$  be the total number operator for the  $N$  guiding center ladder operators. Then the homogeneity property of  $F$  implies that  $|\psi_0\rangle$  is an eigenvalue of  $\mathcal{N}_b$  with eigenvalue  $\text{Deg}[F]$ .

To compute  $\eta_H^{ab}$  for these trial FQH states we use a connection between the APD generators and the generators of the group  $SU(1, 1)$  (see, for example, Ref. [38]). Define the operators

$$K_0 = \frac{1}{2} \sum_{j=1}^N \left( b_j^\dagger b_j + \frac{1}{2} \right), \quad (2.22a)$$

$$K_+ = \frac{1}{2} \sum_{j=1}^N (b_j^\dagger)^2, \quad (2.22b)$$

$$K_- = \frac{1}{2} \sum_{j=1}^N (b_j)^2. \quad (2.22c)$$

These operators obey the commutation relations of the Lie algebra of the group  $SU(1, 1)$ ,

$$[K_0, K_\pm] = \pm K_\pm, \quad (2.23a)$$

$$[K_-, K_+] = 2K_0. \quad (2.23b)$$

The Fock space of the oscillators  $b_j$  forms a (reducible) representation of this algebra, and the generators  $\Lambda^{ab}$  can be expressed in terms of the  $SU(1, 1)$  generators as

$$\Lambda^{11} = K_0 + \frac{1}{2} K_+ + \frac{1}{2} K_-, \quad (2.24)$$

$$\Lambda^{22} = K_0 - \frac{1}{2} K_+ - \frac{1}{2} K_-, \quad (2.25)$$

and

$$\Lambda^{12} = \Lambda^{21} = \frac{-i}{2} (K_- - K_+). \quad (2.26)$$

It is clear that the state  $|\psi_0\rangle$  is an eigenstate of  $K_0$  with eigenvalue  $\frac{1}{2}(\text{Deg}[F] + \frac{N}{2})$ . It then follows that the expectation values  $\langle \psi_0 | K_\pm | \psi_0 \rangle$  are equal to zero as  $K_\pm | \psi_0 \rangle$  is orthogonal to  $|\psi_0\rangle$ . Then, for the trial state parametrized by the function  $F$ , we have

$$\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle = \frac{1}{2} \left[ \text{Deg}[F] + \frac{N}{2} \right] \delta^{ab}. \quad (2.27)$$

A similar computation shows that for a trial state in the  $n$ th Landau level, we have

$$\langle \psi_0 | \tilde{\Lambda}^{ab} | \psi_0 \rangle = \frac{1}{2} \left( nN + \frac{N}{2} \right) \delta^{ab}, \quad (2.28)$$

which follows since the product  $\prod_{j=1}^N \frac{(a_j^\dagger)^n}{\sqrt{n!}}$  is a homogeneous polynomial in the  $a_j^\dagger$  of total degree  $nN$ .

For the case of the  $\nu = 1/m$  Laughlin state ( $m$  a positive integer), we have

$$F(b_1^\dagger, \dots, b_N^\dagger) = \prod_{j < k} (b_j^\dagger - b_k^\dagger)^m, \quad (2.29)$$

and so

$$\text{Deg}[F] = \frac{1}{2} mN(N-1). \quad (2.30)$$

If we consider this Laughlin state in the lowest Landau level ( $n = 0$ ) then we find that

$$\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle = \frac{1}{2} \left[ \frac{1}{2} mN^2 + \left( \frac{1-m}{2} \right) N \right] \delta^{ab}, \quad (2.31a)$$

$$\langle \psi_0 | \tilde{\Lambda}^{ab} | \psi_0 \rangle = \frac{N}{4} \delta^{ab}, \quad (2.31b)$$

and so

$$\eta_H^{ab} = -\frac{\hbar}{A} \frac{1}{2} \left[ \frac{1}{2} mN^2 + \left( \frac{1-m}{2} \right) N \right] \delta^{ab}, \quad (2.32)$$

while

$$\tilde{\eta}_H^{ab} = \frac{\hbar}{4} \frac{N}{A} \delta^{ab}. \quad (2.33)$$

Both of these tensors are proportional to the identity matrix (in this rotation-invariant case), and it is convenient to denote the constants of proportionality by

$$\eta_H = -\frac{\hbar}{A} \frac{1}{2} \left[ \frac{1}{2} mN^2 + \left( \frac{1-m}{2} \right) N \right] \quad (2.34)$$

and

$$\tilde{\eta}_H = \frac{\hbar}{4} \frac{N}{A} \quad (2.35)$$

so that we can simply write  $\eta_H^{ab} = \eta_H \delta^{ab}$  and similarly for  $\tilde{\eta}_H^{ab}$ .

For a Laughlin FQH droplet with  $\nu = 1/m$ , and consisting of a large number  $N$  of particles, we have  $A \approx 2\pi \ell_B^2 mN$ . Then, in its current form, the coefficient  $\eta_H$  in the guiding center Hall viscosity tensor is the sum of an extensive (order  $N$ ) term and an intensive (order 1) term. Since  $A$  itself is proportional to  $N$ , the extensive term in  $\eta_H$  comes from the superextensive (order  $N^2$ ) term in  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$ . This term is associated with a uniform rotational motion (in fact, it is just the orbital angular momentum) of the FQH fluid, and so it has been argued that one should subtract this term when defining the guiding center Hall viscosity [8,13]. If we make this subtraction then we end up with the regularized quantities

$$\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle_{\text{reg}} = \frac{1}{2} \left[ \left( \frac{1-m}{2} \right) N \right] \delta^{ab}, \quad (2.36)$$

$$\eta_{H,\text{reg}} = -\frac{\hbar}{2} \left( \frac{1-m}{2} \right) \rho_0, \quad (2.37)$$

where  $\rho_0 = \frac{1}{2\pi \ell_B^2 m} = \frac{N}{A}$  is the density of the  $\nu = 1/m$  Laughlin FQH state at large  $N$ . We discuss the physical interpretation of this regularization scheme in the context of the CSMM in Sec. VII.

The Landau orbit contribution  $\tilde{\eta}_H$  does not require regularization as it only consists of an intensive term. In terms of the density  $\rho_0$  of the Laughlin state this coefficient has the form

$$\tilde{\eta}_H = \frac{\hbar \rho_0}{4}. \quad (2.38)$$

Then the full Hall viscosity coefficient for the  $\nu = 1/m$  Laughlin state (in the lowest Landau level and after regularization of the guiding center part) is

$$\eta_{\text{tot}} = \tilde{\eta}_H + \eta_{H,\text{reg}} = \frac{\hbar \rho_0 m}{4}, \quad (2.39)$$

as originally found by Read [5]. It is interesting to observe that since  $\rho_0 = \frac{1}{2\pi \ell_B^2 m}$ , the full Hall viscosity coefficient  $\eta_{\text{tot}}$  actually does not depend on the filling fraction of the Laughlin state (i.e., it does not depend on  $m$ ).

The coefficient  $(1-m)/2$  appearing in  $\eta_{H,\text{reg}}$  is what Haldane has termed the ‘‘guiding center spin’’ of a FQH state. This coefficient has been denoted as ‘‘ $\bar{s}$ ’’ in Ref. [8] and ‘‘ $s$ ’’ in Ref. [13]. It is also equal to minus the ‘‘anisospin’’ defined in Refs. [24,25], and denoted there by  $\zeta$ . We choose to adopt the notation of Refs. [24,25] and so we write

$$\eta_{H,\text{reg}} = \frac{\hbar}{2} \zeta \rho_0 \quad (2.40)$$

with  $\zeta = (m-1)/2$ . We see that unlike the full Hall viscosity coefficient  $\eta_{\text{tot}}$ , the guiding center contribution to the Hall viscosity has a clear dependence on  $m$ . It follows that different Laughlin states cannot be distinguished by their full Hall viscosity  $\eta_{\text{tot}}$ , but they *can* be distinguished by their guiding center Hall viscosity  $\eta_{H,\text{reg}}$  which, moreover, has been argued to be connected to the physical property of intrinsic electric dipole moment at the edge of the FQH state [13].

### III. NONCOMMUTATIVE CHERN-SIMONS THEORY

In this section, we review Susskind’s noncommutative Chern-Simons (NCCS) theory description of the Laughlin FQH states [26]. This will pave the way for the discussion of the Chern-Simons matrix model in the next section, as the Chern-Simons matrix model can be thought of as a particular regularization of the NCCS theory. To prepare the reader for this discussion in this section, we first make a few remarks about the two different formulations (‘‘operator’’ versus ‘‘star product’’ formulations) of noncommutative field theory. We then present the NCCS theory in both formulations. Finally, we discuss the NCCS theory in the limit of weak noncommutativity, and its connection with the dynamics of a fluid of charged particles in a magnetic field. From this connection one sees that the full NCCS theory should be understood as describing a fluid of charged particles in a magnetic field on a noncommutative space. Our discussion of noncommutative field theory closely follows that in Refs. [40–42]. For the fluid picture of the NCCS theory, we follow Refs. [26,43]. Readers who are already familiar with noncommutative field theory and the NCCS theory may want to skip this section.

**A. Two formulations of noncommutative field theory**

Consider a classical field theory in 2 + 1 dimensions in which the two-dimensional space is taken to be  $\mathbb{R}^2$ , and let  $\mathbf{x} = (x^1, x^2)$  denote the spatial coordinates. We denote a general field in this theory as  $\Phi(t, \mathbf{x})$ . In such a field theory the fields  $\Phi(t, \mathbf{x})$  at a fixed time  $t$  are elements of the ordinary algebra of functions on  $\mathbb{R}^2$  (the commutative algebra generated by pointwise addition and multiplication of functions of  $\mathbf{x}$ ). The noncommutative deformation of the this theory that we consider consists of replacing the ordinary space  $\mathbb{R}^2$  with a “noncommutative plane” whose two spatial coordinates do not commute with each other. The time direction will always be commutative in this article, i.e., we consider theories in two noncommutative spatial dimensions and one commutative (or ordinary) time direction.

In the noncommutative deformation of the classical field theory, the fields (again at a fixed time  $t$ ) instead take values in the algebra  $\mathbb{R}_\theta^2$  which consists of all complex linear combinations of products of position variables  $\hat{x}^a$ ,  $a = 1, 2$ , satisfying the commutation relation

$$[\hat{x}^1, \hat{x}^2] = i\theta. \tag{3.1}$$

Here,  $\theta$  is a constant real number with dimensions of length squared; it controls the “strength” of the noncommutativity of this theory. The algebra  $\mathbb{R}_\theta^2$  comes equipped with a conjugation operator “ $\dagger$ ” (which one can think of as Hermitian conjugation), and the operators  $\hat{x}^a$  are assumed to be invariant under this operation.<sup>4</sup> We see that the algebra  $\mathbb{R}_\theta^2$  is nothing but the universal enveloping algebra of the Heisenberg algebra specified by  $\hat{x}^a$  and the commutation relation of Eq. (3.1). The operators  $\hat{x}^a$  are sometimes said to be coordinates on a “noncommutative plane.” In the noncommutative theory, the notion of a point no longer makes sense, and the smallest area that one can resolve is of order  $\theta$ .

In the noncommutative field theory, the notion of integration over space is replaced with a trace in a representation of the Heisenberg algebra of the noncommutative coordinates  $\hat{x}^a$ . Usually, this representation is taken to be the Fock representation in which the ladder operators

$$\hat{a} = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 + i\hat{x}^2), \tag{3.2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 - i\hat{x}^2) \tag{3.3}$$

act on a Fock space  $\mathcal{H}_F$  generated by the action of the raising operator  $\hat{a}^\dagger$  on a vacuum state  $|0\rangle$  which is annihilated by the lowering operator  $\hat{a}$ . The action functional for the noncommutative field theory then takes the form

$$S = \int dt \text{Tr}_{\mathcal{H}_F} \{(\dots)\}, \tag{3.4}$$

where  $(\dots)$  denotes a Lagrangian written in terms of fields  $\hat{\Phi}(t)$ , which are operators on the space  $\mathcal{H}_F$ , and whose matrix elements are functions of time.

<sup>4</sup>For any complex number  $c$  and any  $y \in \mathbb{R}_\theta^2$ , we have  $(cy)^\dagger = \bar{c}y^\dagger$ , where  $\bar{c}$  is the complex conjugate of  $c$ .

It is natural to call the formulation of noncommutative field theory that we have just described the “operator formulation.” We now describe an alternative formulation, which one might call the “star-product formulation,” which may be more familiar to some readers. In this formulation one instead works with fields  $\Phi(t, \mathbf{x})$ , which are ordinary functions of the coordinates  $\mathbf{x}$  on  $\mathbb{R}^2$ , but replaces the ordinary product of functions with the *Groenewold-Moyal star product* “ $\star$ ,” which is defined as follows. For any two functions  $f(\mathbf{x})$  and  $g(\mathbf{x})$  of  $\mathbf{x}$ , we have

$$\begin{aligned} f(\mathbf{x}) \star g(\mathbf{x}) &= e^{\frac{i}{2}\theta \epsilon^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial z^a}} f(\mathbf{y})g(\mathbf{z}) \Big|_{\mathbf{y}=\mathbf{z}=\mathbf{x}} \\ &= f(\mathbf{x})g(\mathbf{x}) + \frac{i}{2}\theta \epsilon^{ab} \partial_a f(\mathbf{x}) \partial_b g(\mathbf{x}) + \dots, \end{aligned} \tag{3.5}$$

and where in the last line the ellipses denote terms of order  $\theta^2$  and higher. For two functions  $f(\mathbf{x})$  and  $g(\mathbf{x})$  which vanish at spatial infinity we have the important property that

$$\int d^2\mathbf{x} f(\mathbf{x}) \star g(\mathbf{x}) = \int d^2\mathbf{x} f(\mathbf{x})g(\mathbf{x}), \tag{3.6}$$

which follows after integration by parts on the higher derivative terms in the star product. There is no analogous result for integrals of star products of three or more functions.

These two formulations of noncommutative field theory are related by the *Wigner-Weyl* mapping of functions and operators. This mapping is as follows. Let  $f(\mathbf{x})$  be an ordinary function on  $\mathbb{R}^2$  and let

$$\tilde{f}(\mathbf{k}) = \int d^2\mathbf{x} f(\mathbf{x}) e^{-ik_a x^a} \tag{3.7}$$

be its Fourier transform. Then we can define a Weyl-ordered operator  $\hat{f}$  by taking the inverse Fourier transform but replacing  $x^a$  with  $\hat{x}^a$  in the exponential,

$$\hat{f} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tilde{f}(\mathbf{k}) e^{ik_a \hat{x}^a}. \tag{3.8}$$

One can check that this mapping satisfies the following properties which will be needed later:

$$\hat{f} \hat{g} = \widehat{f \star g}, \tag{3.9}$$

$$\text{Tr}_{\mathcal{H}_F} \{\hat{f}\} = \frac{1}{2\pi\theta} \int d^2\mathbf{x} f(\mathbf{x}). \tag{3.10}$$

To check the second property, one can express the trace over  $\mathcal{H}_F$  using a basis  $\{|x^1\rangle\}$  of eigenstates of  $\hat{x}^1$  as

$$\text{Tr}_{\mathcal{H}_F} \{\hat{f}\} = \int_{-\infty}^{\infty} dx^1 \langle x^1 | \hat{f} | x^1 \rangle \tag{3.11}$$

and then plug in the expression Eq. (3.8) for  $\hat{f}$ .

The Chern-Simons matrix model that we study below is a particular regularization of the NCCS theory in its operator formulation. Therefore, for our purposes, we generally find that the operator formulation of the NCCS theory is more convenient. However, the star product formulation is still useful for the study of the behavior of the theory near the commutative limit  $\theta \rightarrow 0$ , and so we will have occasion to use both formulations of the NCCS theory in what follows.

### B. NCCS theory in the operator formulation

We now review the operator formulation of the NCCS theory. In the operator formulation, the NCCS theory consists of three fields  $\hat{X}^a(t)$ ,  $a = 1, 2$ , and  $\hat{A}_0(t)$ . All fields should be thought of as operators on the Fock space  $\mathcal{H}_F$  whose matrix elements are functions of time. In addition, all fields are Hermitian (i.e., all fields are invariant under the “†” operation on the algebra  $\mathbb{R}_\theta^2$ ). We also consider the theory on a time interval of length  $T$  and assume periodic boundary conditions in time so that  $\hat{X}^a(0) = \hat{X}^a(T)$ , and likewise for  $\hat{A}_0(t)$ . In addition to the noncommutativity parameter  $\theta$ , the theory includes various coupling constants including  $e > 0$ , an electric charge, and  $B > 0$ , a constant magnetic field. We discuss the physical interpretation of this theory as representing a charged fluid in a magnetic field later in this section (and we will see that the charge of the particles which make up this fluid is actually  $q = -e < 0$ ).

The action for the NCCS theory in the operator formulation takes the form

$$S_{\text{NCCS}} = -\frac{eB}{2} \int_0^T dt \text{Tr}_{\mathcal{H}_F} \{ \epsilon_{ab} \hat{X}^a D_0 \hat{X}^b + 2\theta \hat{A}_0 \}, \quad (3.12)$$

where we introduced a covariant derivative

$$D_0 \hat{X}^b = \dot{\hat{X}}^b + i[\hat{X}^b, \hat{A}_0]. \quad (3.13)$$

and where the dot denotes a time derivative. The field  $\hat{A}_0$  functions as a Lagrange multiplier and its equation of motion yields the constraint

$$[\hat{X}^1, \hat{X}^2] = i\theta. \quad (3.14)$$

This constraint can only be satisfied by operators  $\hat{X}^a$  on an infinite-dimensional space. This is due to the fact that if the variables  $\hat{X}^a$  were finite-dimensional matrices, then the trace of the left-hand side of the equation is zero while the trace of the right-hand side would be proportional to the size of the matrices. The CSMM discussed in the next section is a modification of the NCCS theory, which features a modified constraint that can be satisfied by operators (matrices) on a finite-dimensional space.

If we ignore the term containing  $2\theta \hat{A}_0$  for a moment, then one can check that the action is invariant under the gauge transformation

$$\hat{X}^a \rightarrow \hat{V} \hat{X}^a \hat{V}^\dagger \quad (3.15a)$$

$$\hat{A}_0 \rightarrow \hat{V} \hat{A}_0 \hat{V}^\dagger + i \hat{V} \dot{\hat{V}}^\dagger, \quad (3.15b)$$

where  $\hat{V}(t)$  is an arbitrary time-dependent unitary operator on the Fock space  $\mathcal{H}_F$ . In particular, this follows from the fact that, under this transformation, the covariant derivative transforms as  $D_0 \hat{X}^b \rightarrow \hat{V} D_0 \hat{X}^b \hat{V}^\dagger$ . To understand these gauge transformations in the presence of the term  $2\theta \hat{A}_0$ , we need to constrain the allowed  $\hat{V}$ 's that we consider [27]. To motivate this restriction, we now briefly discuss some aspects of the geometry of the noncommutative plane.

Consider the occupation number basis  $\{|n\rangle\}_{n \in \mathbb{N}}$  of the Fock space  $\mathcal{H}_F$  ( $|n\rangle \propto (\hat{a}^\dagger)^n |0\rangle$ ). The radius squared operator  $\hat{R}^2 = \delta_{ab} \hat{x}^a \hat{x}^b$  is diagonal in this basis and we have  $\hat{R}^2 |n\rangle = 2\theta(n + \frac{1}{2})|n\rangle$ . Thus the occupation number  $n$  can be identified with the distance squared from the origin in the noncommutative plane.

We now restrict our attention to gauge transformations defined by unitary operators  $\hat{V}(t)$  which act as the identity on states  $|n\rangle$  with  $n$  sufficiently large, say  $n > N_0$ . The actual value of  $N_0$  is not important for the argument. This is the noncommutative analog of requiring gauge transformations in a commutative gauge theory on the space  $\mathbb{R}^2$  to tend to the identity at spatial infinity.

With this restriction on possible gauge transformations, the unitary operator  $\hat{V}(t)$  defines a map from the periodic time interval  $[0, T)$  to  $U(N_0)$ , the group of unitary matrices of size  $N_0$ . Large gauge transformations are those  $\hat{V}(t)$  that correspond to a nontrivial element of the homotopy group  $\pi_1(U(N_0)) = \mathbb{Z}$ . The full NCCS action is not invariant under these large gauge transformations because of the presence of the  $2\theta \hat{A}_0$  term. In Ref. [27], Polychronakos and Nair have shown that requiring the exponential  $e^{i \frac{\text{CSMM}}{\hbar}}$  to be invariant under these large gauge transformations enforces a quantization rule on  $\theta$ , which states that

$$eB\theta = \hbar m, \quad m \in \mathbb{Z}, \quad (3.16)$$

or

$$\theta = \ell_B^2 m, \quad m \in \mathbb{Z}, \quad (3.17)$$

where  $\ell_B^2 = \hbar/(eB)$  is the square of the magnetic length defined earlier. This quantization rule is the noncommutative analog of the level quantization, which obtains in ordinary [say  $SU(N)$ ] Chern-Simons theory on a commutative space.

### C. NCCS theory in the star product formulation

We now discuss the NCCS theory in the star product formulation. In this form the theory looks very similar to the ordinary Chern-Simons theory (i.e., Chern-Simons theory on the commutative space  $\mathbb{R}^2$ ). We proceed by deriving the star product formulation of the NCCS theory from the operator formulation by using the Wigner-Weyl mapping discussed earlier in this section. To do this, we need to know how spatial derivatives are represented in the operator formulation of the theory. Derivative operators  $\hat{\partial}_a$  in the operator formulation of noncommutative field theory are defined by

$$\hat{\partial}_1 = \frac{i\hat{x}^2}{\theta}, \quad \hat{\partial}_2 = -\frac{i\hat{x}^1}{\theta} \quad (3.18)$$

and one can check that

$$[\hat{\partial}_a, \hat{x}^b] = \delta_a^b, \quad (3.19)$$

just as one has for ordinary derivatives of functions on  $\mathbb{R}^2$ . In addition, in the Wigner-Weyl mapping one has

$$[\hat{\partial}_a, \hat{f}] = \widehat{\partial_a f}, \quad (3.20)$$

so under this mapping the ordinary derivative of a function  $f(\mathbf{x})$  with respect to  $x^a$  is mapped to the commutator of  $\hat{\partial}_a$  with  $\hat{f}$  (i.e., the *adjoint* action of  $\hat{\partial}_a$  on  $\hat{f}$ ).

The first step towards deriving the star product formulation of NCCS theory is to make a change of variables in the operator formulation by defining two new fields  $\hat{A}_a$ ,  $a = 1, 2$ , which are related to the fields  $\hat{X}^a$  by

$$\hat{X}^a = \hat{x}^a + \theta \epsilon^{ab} \hat{A}_b. \quad (3.21)$$

Under a gauge transformation the new fields transform as<sup>5</sup>

$$\hat{A}_a \rightarrow \hat{V} \hat{A}_a \hat{V}^\dagger + i \hat{V} [\hat{\partial}_a, \hat{V}^\dagger]. \quad (3.22)$$

This transformation resembles the transformation of an ordinary non-Abelian gauge field. In addition, in the new variables, the NCCS constraint of Eq. (3.14) becomes

$$\hat{F}_{12} = 0, \quad (3.23)$$

where we defined the field strength for noncommutative gauge theory as

$$\hat{F}_{ab} = [\hat{\partial}_a, \hat{A}_b] - [\hat{\partial}_b, \hat{A}_a] - i[\hat{A}_a, \hat{A}_b]. \quad (3.24)$$

Thus the constraint in NCCS theory is an exact noncommutative analog of the constraint enforced by the temporal component of the gauge field in ordinary Chern-Simons theory on a commutative space.

After tedious algebra (including many uses of the cyclic property of the trace) one can show that after performing this transformation the NCCS action takes the form

$$S_{\text{NCCS}} = -\frac{eB\theta^2}{2} \int_0^T dt \text{Tr}_{\mathcal{H}_F} \left\{ \epsilon^{ab} \hat{A}_a \dot{\hat{A}}_b - \epsilon^{ab} \hat{A}_0 [\hat{\partial}_a, \hat{A}_b] + \epsilon^{ab} \hat{A}_b [\hat{\partial}_a, \hat{A}_0] + \frac{2i}{3} \epsilon^{\mu\nu\lambda} \hat{A}_\mu \hat{A}_\nu \dot{\hat{A}}_\lambda \right\}, \quad (3.25)$$

where the greek indices  $\mu, \nu, \lambda$  run over the range 0,1,2. There is one subtle point in the derivation of this equation which involves a term which is a total time derivative. Specifically, after the transformation from the  $\hat{X}^a$  variables to the  $\hat{A}_a$  variables one finds a term

$$-\frac{eB}{2} \int_0^T dt \text{Tr}_{\mathcal{H}_F} \{-\theta \hat{x}^a \dot{\hat{A}}_a\}. \quad (3.26)$$

Since  $\hat{x}^a$  has no time dependence this term is a total derivative, and it evaluates to zero since we assumed periodic boundary conditions on all fields in the time direction.

Finally, we apply the Wigner-Weyl mapping to write the NCCS action in the star product formulation as

$$S_{\text{NCCS}} = \frac{eB\theta}{4\pi} \int_0^T dt \int d^2\mathbf{x} \epsilon^{\mu\nu\lambda} \left( A_\mu \star \partial_\nu A_\lambda - \frac{2i}{3} A_\mu \star A_\nu \star A_\lambda \right). \quad (3.27)$$

The quantization condition on  $\theta$  [Eq. (3.17)] then implies that the coefficient of the action is

$$\frac{eB\theta}{4\pi} = \frac{\hbar m}{4\pi}. \quad (3.28)$$

Then, in units where  $\hbar = 1$ , we find the NCCS action at level  $m \in \mathbb{Z}$ . If we take  $\ell_B^2 \rightarrow 0$ , which also sends  $\theta \rightarrow 0$ , then we recover the ordinary  $U(1)$  Chern-Simons theory at level  $m$  (again with  $\hbar = 1$  for now)

$$S_{\text{CS}} = \frac{m}{4\pi} \int_0^T dt \int d^2\mathbf{x} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (3.29)$$

<sup>5</sup>This is derived by requiring the gauge transformation of  $\hat{x}^a + \theta \epsilon^{ab} \hat{A}_b$  to coincide with the gauge transformation of  $\hat{X}^a$  from Eq. (3.15).

For completeness, we note here that in the star product formulation the noncommutative field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu), \quad (3.30)$$

and the equation of motion of the NCCS theory is equivalent to  $F_{\mu\nu} = 0$ , just like in ordinary Chern-Simons theory.

#### D. Fluid interpretation of the NCCS theory at small $\theta$

We now discuss the behavior of the NCCS theory in the limit of weak noncommutativity in which  $\theta$  is assumed to be small. Note that since  $\theta$  has units, and since there is no other length scale in the problem to compare  $\theta$  to, it is more accurate to say that in this section we study a *truncation* of the NCCS theory at first order in  $\theta$ . In the star product formulation of the theory this truncation simply amounts to neglecting terms of order  $\theta^2$  and higher in the star product of functions. In this limit, we will see that the NCCS theory has an interpretation as describing a fluid of charged particles in a constant magnetic field  $B$ , as was discussed by Susskind [26] (see also Refs. [43,44]).

To consider the NCCS theory in the regime of small  $\theta$  we start by using the cyclic property of the trace to write the action in the form

$$S_{\text{NCCS}} = -\frac{eB}{2} \int_0^T dt \text{Tr}_{\mathcal{H}_F} \{ \epsilon_{ab} \hat{X}^a \dot{\hat{X}}^b + 2\hat{A}_0(\theta + i[\hat{X}^1, \hat{X}^2]) \}. \quad (3.31)$$

We then use the Wigner-Weyl mapping, and keep only the terms up to order  $\theta$  in the star product, to find that in the limit of small  $\theta$

$$S_{\text{NCCS}} \rightarrow -\frac{eB}{2} \frac{1}{2\pi\theta} \int_0^T dt \int d^2\mathbf{x} (\epsilon_{ab} X^a \dot{X}^b + 2\theta A_0(1 - \epsilon^{ab} \partial_a X^1 \partial_b X^2)). \quad (3.32)$$

Susskind observed that in this limit the NCCS theory describes the dynamics of a charged fluid at constant density  $\rho_0 = \frac{1}{2\pi\theta}$  in a constant magnetic field  $B$ , and in the limit where the cyclotron frequency is sent to infinity. In fact, in Susskind's original derivation, he starts with the fluid description and then observes that it coincides with the small  $\theta$  limit of the NCCS theory. We now briefly remind the reader of this connection between the NCCS theory and fluid dynamics.

The starting point is the *Lagrange* description<sup>6</sup> of a fluid of charged particles moving on the plane  $\mathbb{R}^2$  in a background electromagnetic field. In the Lagrange description of a fluid one keeps track of the motion of the individual particles in the fluid, and measures their current position with respect to some reference configuration. In this description, we use coordinates  $\mathbf{x}$  to describe the reference configuration of the fluid and coordinates  $X^a(t, \mathbf{x})$ ,  $a = 1, 2$ , to describe the configuration of the fluid at a later time  $t$ . Without loss of generality, we may assume that  $X^a(0, \mathbf{x}) = x^a$ . Thus  $X^a(t, \mathbf{x})$  is the position, at time  $t$ , of the fluid particle which was at position  $x^a$  at  $t = 0$ . We also assign a constant density  $\rho_0$  to the fluid in the reference configuration.

<sup>6</sup>The relation between noncommutative gauge theory and the Lagrange description of a fluid is discussed in detail in Ref. [44].



The action for a Lagrange fluid made up of particles of mass  $M$  and charge  $q$  in the presence of a background electromagnetic field takes the form

$$S = \int_0^T dt \int d^2\mathbf{x} \rho_0 \left( \frac{1}{2} M \delta_{ab} \dot{X}^a \dot{X}^b + q \mathcal{A}_a(t, \mathbf{X}) \dot{X}^a - q \varphi(t, \mathbf{X}) \right), \quad (3.33)$$

where  $\mathcal{A}_a(t, \mathbf{X})$  and  $\varphi(t, \mathbf{X})$  are the vector and scalar potentials, respectively, for the external electromagnetic field. Intuitively, this action is just the sum over all particles in the fluid of the ordinary action for a massive charged particle in a background electromagnetic field. However, the discrete sum over particle labels has been replaced with an integration over the reference coordinates  $\mathbf{x}$  weighted with the density  $\rho_0$  in the reference configuration. The reference coordinates  $\mathbf{x}$  can therefore be considered as a set of continuous particle labels.

To see the connection of the fluid model to the NCCS theory, we first place the system in a uniform background magnetic field with strength  $B > 0$ . This can be accomplished by setting  $\varphi(t, \mathbf{X}) = 0$  and

$$\mathcal{A}_a(t, \mathbf{X}) = -\frac{B}{2} \epsilon_{ab} X^b, \quad (3.34)$$

where we have chosen the symmetric gauge for the vector potential. Next, we set the mass of the particles to zero,  $M = 0$ . This corresponds to taking the cyclotron frequency  $\omega_c = (eB)/M$  to infinity, which is similar to a projection into the lowest Landau level (since  $\hbar\omega_c$  is the energy gap between Landau levels). Finally, we take the charge of the particles to be  $q = -e$  with  $e > 0$ . Then at this point the action reads as

$$S = -\frac{eB}{2} \rho_0 \int_0^T dt \int d^2\mathbf{x} \epsilon_{ab} X^a \dot{X}^b. \quad (3.35)$$

Note that  $\rho_0$  can be pulled out of the integral since we assumed it was constant. We also mention here that our conventions for the direction of the magnetic field and the charge of the particles in the fluid exactly matches our conventions for the setup of the quantum Hall problem from Sec. II.

The next step is to incorporate a Lagrange multiplier which enforces the constraint that the fluid remains at the constant density  $\rho_0$  at all times. The density  $\rho(t, \mathbf{X})$  of the fluid at time  $t$  is related to the initial density  $\rho_0$  by the Jacobian  $\epsilon^{ab} \partial_a X^1 \partial_b X^2$  of the map from the reference coordinates to the fluid coordinates  $\mathbf{X}$  at time  $t$  as

$$\rho(t, \mathbf{X}) \epsilon^{ab} \partial_a X^1 \partial_b X^2 = \rho_0, \quad (3.36)$$

where we remind the reader that  $\partial_a$  is a shorthand for  $\frac{\partial}{\partial x^a}$ , i.e., a derivative with respect to the reference coordinates  $x^a$ . Then the constraint that  $\rho(t, \mathbf{X}) = \rho_0$  for all  $t$  can be written as

$$\epsilon^{ab} \partial_a X^1 \partial_b X^2 = 1. \quad (3.37)$$

We denote the Lagrange multiplier enforcing this constraint by  $A_0(t, \mathbf{x})$ , and write the action with the constraint included in the form

$$S = -\frac{eB}{2} \rho_0 \int_0^T dt \int d^2\mathbf{x} (\epsilon_{ab} X^a \dot{X}^b + 2\theta A_0(1 - \epsilon^{ab} \partial_a X^1 \partial_b X^2)), \quad (3.38)$$

where we have introduced a parameter  $\theta$  with units of (length)<sup>2</sup>. With this choice, the Lagrange multiplier field  $A_0$  has units of (time)<sup>-1</sup>.

We can now see that the small  $\theta$  limit of the NCCS action from Eq. (3.32) is exactly the action for a fluid of particles with charge  $q = -e$  at the constant density  $\rho_0 = 1/(2\pi\theta)$  in a constant background magnetic field  $B$  in the limit in which the cyclotron frequency is taken to infinity. This limit is analogous to the projection into the lowest Landau level, and it is the physical reason why this fluid theory (and the NCCS theory) is expected to describe FQH physics in the lowest Landau level [26]. In the full NCCS theory, we should then interpret the fields  $\hat{X}^a(t)$  as describing the positions of particles in a fluid on a noncommutative space, as discussed by Susskind [26] (see also Ref. [43] for a review of the physics of such *noncommutative fluids*).

#### IV. THE CHERN-SIMONS MATRIX MODEL

In this section, we discuss the Chern-Simons matrix model (CSMM), which was introduced by Polychronakos in Ref. [28]. This model can be thought of as a particular regularization of the operator formulation of the NCCS theory, in which the fields  $\hat{X}^a(t)$  (which were operators on the *infinite-dimensional* Fock space  $\mathcal{H}_F$ ) are now finite  $N \times N$  matrices  $X^a(t)$  instead. Note that we *do not* use a hatted notation for the finite size matrix variables of the NCCS theory. The parameter  $N$  serves as a regulator, which should be taken to infinity to recover the NCCS theory discussed in the previous section. The fluid interpretation of the NCCS theory carries over to the CSMM, so we still interpret the matrix variables  $X^a(t)$  as representing the coordinates of particles in a fluid on a noncommutative space, only now the fluid turns out to occupy a finite area of this space. In other words, the CSMM is a model of a finite droplet *droplet* of noncommutative fluid.

Since the CSMM can be difficult to understand, we begin this section by making a few remarks about our notation and conventions, and then discuss some subtleties of this model. We then review the quantization of this model following Refs. [28,31]. Finally, we review (following the original discussion in Ref. [28]) the calculation of the area  $A$  and density  $\rho_0$  of the droplet of noncommutative fluid represented by the ground state of the CSMM. We will then be able to identify the CSMM having  $\theta = \ell_B^2 m$  as describing the  $\nu = 1/m$  Laughlin state by comparing the results for  $\rho_0$  and  $A$  to the known answers for a droplet of FQH fluid in the  $\nu = 1/m$  Laughlin state in the limit of a large number of particles  $N$ .

##### A. Some remarks on notation

The CSMM, and especially the quantization of this model, can be quite tricky due to two separate noncommutative structures which appear. First, at the classical level the degrees of freedom in this model are Hermitian  $N \times N$  matrix variables  $X^1, X^2, A_0$ , as well as a complex vector  $\Psi$  of length  $N$ . All of these variables are functions of time. Since some of the variables are matrix variables, ordinary (i.e., classical) matrix multiplication of these variables is not commutative. Next, upon quantization of the model, the *matrix elements* of  $X^1, X^2$ , and  $A_0$  (and also the components of  $\Psi$ ) become *operators* on a

separate Hilbert space, which is unrelated to the vector space on which the classical matrix variables act. Thus, in the quantized matrix model there are two sources of noncommutativity. The first source is the fact that we are dealing with matrix variables from the start, and the second source comes from the fact that the matrix elements of the original matrix variables are now operators on a second Hilbert space, and so multiplication of individual matrix elements does not commute either, but for a different reason.

In an attempt to present this model in as clear a manner as possible, we will adhere to the following notational conventions. First, we use  $[\cdot, \cdot]_M$  to denote a matrix commutator of classical matrices, and use  $[\cdot, \cdot]$  (with no subscript) to denote the commutator of quantum operators. We also reserve the symbol  $\dagger$  to denote Hermitian conjugation of quantum operators. In all manipulations with classical matrix variables, we instead use an overline to denote complex conjugation of a matrix and a superscript “T” to denote a transpose. So if  $A$  is an  $N \times N$  matrix variable, then  $\overline{A^T}$  is its transpose conjugate, i.e., if  $A$  has matrix elements  $A_{jk}$ , then the matrix elements of  $\overline{A^T}$  are  $(\overline{A^T})_{jk} = \overline{A_{kj}}$  (and Hermitian matrices satisfy the relation  $\overline{A^T} = A$ ). As we mentioned before, in the quantum theory, the matrix elements  $A_{jk}$  are promoted to operators on a Hilbert space. We denote the Hermitian conjugate (with respect to the inner product on this Hilbert space) of the operator  $A_{jk}$  by  $A_{jk}^\dagger$ . Note that for a generic matrix variable  $A$  it is entirely possible that the operator  $A_{jk}^\dagger$  is not the same as the operator  $(\overline{A^T})_{jk}$ . In what follows, we also make every effort to avoid using “ $i$ ” as an index, and instead try to reserve it for the symbol meaning  $\sqrt{-1}$ , and occasionally for the differential geometry operation  $i_{\underline{v}}$  of interior multiplication by a vector field  $\underline{v}$ .

## B. Description of the model

In this subsection we describe the CSMM of the Laughlin quantum Hall states [28]. The degrees of freedom in this model are two  $N \times N$  matrices  $X^a(t)$ ,  $a = 1, 2$ , an  $N \times N$  matrix  $A_0(t)$ , and a complex vector  $\Psi(t)$  of length  $N$ . All degrees of freedom depend on time. The matrices  $X^a(t)$  and  $A_0(t)$  are all Hermitian and so they have real eigenvalues. The variables  $X^a$  are to be interpreted as coordinates in the Lagrange description of a fluid on the noncommutative plane, in accordance with the physical ideas of Susskind and Polychronakos [26,28] (and as we reviewed at the end of Sec. III). The number  $N$  will later be identified with the number of electrons in a Landau level. The action for the CSMM takes the form

$$S_{\text{CSMM}} = -\frac{eB}{2} \int_0^T dt \text{Tr} \{ \epsilon_{ab} X^a D_0 X^b + 2\theta A_0 + \tilde{\omega} \delta_{ab} X^a X^b \} + \int_0^T \overline{\Psi}^T (i\dot{\Psi} + A_0 \Psi), \quad (4.1)$$

where

$$\begin{aligned} D_0 X^b &:= \dot{X}^b + i[X^b, A_0]_M \\ &= \dot{X}^b - i[A_0, X^b]_M \end{aligned} \quad (4.2)$$

is a covariant derivative. Here we view  $\Psi$  as a column vector and  $\overline{\Psi}^T$  denotes the row vector whose elements are the complex conjugates of the elements of  $\Psi$ . In addition,  $e$

and  $B$  are the same charge and constant magnetic field from Sec. III,  $\tilde{\omega}$  is a frequency (the term with  $\tilde{\omega}$  is a quadratic potential for the noncommutative coordinates  $X^a$ ), and  $\theta$  is a parameter with units of length squared. We assume periodic boundary conditions on all the fields in the time direction, for example,  $X^a(0) = X^a(T)$ , so that the time direction is a circle of circumference  $T$ . Note that the action as written here differs slightly in the details (signs, etc.) from Ref. [28], but is consistent with our interpretation of this model and the NCCS theory as describing a noncommutative fluid of particles with charge  $-e < 0$ .

At this point, we would like to emphasize that the frequency  $\tilde{\omega}$  appearing in the parabolic potential term of the CSMM has no relation to the cyclotron frequency  $\omega_c = (eB)/M$  in the quantum Hall problem. Indeed, as we discussed in Sec. III, the NCCS theory (and therefore the CSMM as well) describes a charged fluid in a magnetic field in the limit in which the mass  $M$  of the particles making up the fluid has been sent to zero. This sends the cyclotron frequency  $\omega_c$  to infinity. Therefore the CSMM contains no information related to the cyclotron frequency or the energy of a Landau level.

We now discuss the gauge symmetry in the CSMM. If we ignore the term with  $2\theta A_0$  for a moment, then we can see that the rest of the action is invariant under a  $U(N)$  gauge transformation:

$$X^a \rightarrow V X^a \overline{V}^T, \quad (4.3a)$$

$$A_0 \rightarrow V A_0 \overline{V}^T + iV \dot{\overline{V}}^T, \quad (4.3b)$$

$$\Psi \rightarrow V \Psi, \quad (4.3c)$$

where  $V(t)$  is an arbitrary time-dependent  $U(N)$  matrix. The presence of the term  $2\theta A_0$  in the action means that the action is not invariant under large  $U(N)$  gauge transformations, which are maps from  $[0, T) \rightarrow U(N)$ , which correspond to a nontrivial element in the homotopy group  $\pi_1(U(N)) = \mathbb{Z}$ . Since we would like  $e^{i \frac{S_{\text{CSMM}}}{\hbar}}$  to be invariant under any gauge transformation, these large gauge transformations enforce a quantization rule on  $\theta$  (the argument is identical to the argument for the full NCCS theory from Sec. III), which states that

$$eB\theta = \hbar m, \quad m \in \mathbb{Z}, \quad (4.4)$$

or

$$\theta = \ell_B^2 m, \quad m \in \mathbb{Z}. \quad (4.5)$$

The gauge field  $A_0$  can be interpreted as a matrix Lagrange multiplier. If we look at the equation of motion resulting from a variation of  $A_0$ , then we find that  $A_0$  enforces the constraint

$$ieB[X^1, X^2]_M + eB\theta \mathbb{I} - \Psi \overline{\Psi}^T = 0. \quad (4.6)$$

This constraint should be compared with Eq. (3.14) for the NCCS theory. In the NCCS case, the contribution from the vector  $\Psi$  is absent and the constraint can only be realized by infinite-dimensional matrices (i.e., operators on  $\mathcal{H}_F$ ). It is the presence of the vector  $\Psi$  which allows this constraint to be realized by finite-dimensional matrices, and this is why the CSMM can be thought of as a regularization of the NCCS theory. We refer the reader to Ref. [28] for the detailed analysis

of the constraint in the classical solution of the CSMM (which is also closely related to the Calogero model of interacting particles in one spatial dimension). In this paper our main focus is on the solution of the model in the quantum case.

We now make a few remarks and set up some notation relating to the transformation properties of the fields under the action of the group  $U(N)$ . The field  $\Psi$  transforms in the fundamental representation of  $U(N)$ . We indicate this by writing the components of  $\Psi$  with an upper Latin index,  $\Psi^j$ ,  $j = 1, \dots, N$ . Under a  $U(N)$  transformation we have

$$\Psi^j \rightarrow V^j_k \Psi^k, \quad (4.7)$$

where  $V^j_k$  are the matrix elements of a unitary matrix  $V$  in  $U(N)$ . Next, the transpose conjugate  $\bar{\Psi}^T$  transforms in the antifundamental representation of  $U(N)$ ,  $\bar{\Psi}^T \rightarrow \bar{\Psi}^T \bar{V}^T$ . We indicate this by writing the components of  $\bar{\Psi}^T$  with a lower index,  $\bar{\Psi}_j$ ,  $j = 1, \dots, N$  (and recall that the components of  $\bar{\Psi}^T$  are just the complex conjugates of the components of  $\Psi$ ). In components we have

$$\bar{\Psi}_j \rightarrow \bar{\Psi}_k (\bar{V}^T)^k_j. \quad (4.8)$$

Finally, the matrix variables  $X^a$  transform in the adjoint representation of  $U(N)$ ,  $X^a \rightarrow V X^a \bar{V}^T$ . Thus the index structure of  $X^a$  is such that it has one upper and one lower index,  $(X^a)^j_k$ ,  $j, k = 1, \dots, N$ . The component form of the  $U(N)$  transformation is then

$$(X^a)^j_k \rightarrow V^j_\ell (X^a)^\ell_m (\bar{V}^T)^m_k. \quad (4.9)$$

These conventions will be extremely useful later when we try to write down quantum states that respect the constraint of the CSMM.

We already mentioned that the matrix variables  $X^a$  are Hermitian matrices. Thus their matrix elements  $(X^a)^j_k$  are generically complex numbers. For the quantization of this system, it will be more convenient to parametrize  $X^a$  in terms of scalar variables, which are manifestly real. Then, when we quantize the theory, these real variables will be promoted to Hermitian operators on the quantum Hilbert space. Our choice of parametrization is as follows. First, let  $T^A$ ,  $A = 1, \dots, N^2 - 1$ , be the  $N \times N$  generators, in the fundamental representation, of the Lie algebra of  $SU(N)$ . The matrices  $T^A$  are all Hermitian and traceless, and can be normalized to obey the relations

$$\text{Tr}\{T^A T^B\} = \delta^{AB}, \quad (4.10a)$$

$$[T^A, T^B]_M = i \sum_{C=1}^{N^2-1} f^{ABC} T^C, \quad (4.10b)$$

where  $f^{ABC}$  are the structure constants for  $SU(N)$ . These structure constants have a very important property, which is that they are antisymmetric under exchange of *any* two indices  $A, B$ , or  $C$  (typically one only expects antisymmetry under  $A \leftrightarrow B$ ). We will take advantage of this property later

on. Using the generators  $T^A$ , we can parametrize  $X^a$  (for  $a = 1, 2$ ) as

$$X^a(t) = x_0^a(t) \frac{\mathbb{I}}{\sqrt{N}} + \sum_{A=1}^{N^2-1} x_A^a(t) T^A, \quad (4.11)$$

where  $x_0^a(t)$  and  $x_A^a(t)$ ,  $A = 1, \dots, N^2 - 1$ , are  $N^2$  real scalar variables. In the quantum theory, these variables will be promoted to Hermitian operators. The factor of  $\sqrt{N}$  on the identity matrix term was chosen for convenience.

The Poisson brackets for this system can be obtained from the corresponding symplectic form, which can in turn be read off from the action (which is first order in time derivatives). The full symplectic form on the phase space for this system is

$$\Omega = \Omega_X + \Omega_\Psi \quad (4.12)$$

with

$$\Omega_X = -eB \sum_{A=0}^{N^2-1} dx_A^1 \wedge dx_A^2 \quad (4.13)$$

and

$$\Omega_\Psi = -i d\Psi^j \wedge d\bar{\Psi}_j. \quad (4.14)$$

Our conventions for Poisson brackets are as follows. To any function  $f$  on phase space, we associate a vector field  $\underline{v}_f$  defined as the solution to the equation  $df = -i_{\underline{v}_f} \Omega$ . Then the Poisson bracket of any two functions  $f$  and  $g$  is given by  $\{f, g\} = i_{\underline{v}_f} i_{\underline{v}_g} \Omega$ . Using this convention, we obtain the classical Poisson brackets (with  $A, B = 0, \dots, N^2 - 1$  now)<sup>7</sup>

$$\{x_A^1, x_B^2\} = \frac{1}{eB} \delta_{AB}, \quad (4.15a)$$

$$\{\Psi^j, \bar{\Psi}_k\} = -i \delta_k^j. \quad (4.15b)$$

Upon quantization, in which we replace Poisson brackets with commutators as  $\{f, g\} \rightarrow -\frac{i}{\hbar} [f, g]$ , we find the commutation relations in the quantum CSMM to be

$$[x_A^1, x_B^2] = i \ell_B^2 \delta_{AB}, \quad (4.16a)$$

$$[\Psi^j, \bar{\Psi}_k] = \hbar \delta_k^j, \quad (4.16b)$$

where  $\ell_B^2 = \hbar/(eB)$  is the magnetic length.

Finally, when the gauge field  $A_0$  is set to zero, the Hamiltonian for this system is given by

$$H_{\text{CSMM}} = \frac{eB\tilde{\omega}}{2} \text{Tr}\{\delta_{ab} X^a X^b\}. \quad (4.17)$$

All of the energy in the system is associated with the harmonic trap, and the only energy scale is associated with frequency  $\tilde{\omega}$  of the harmonic trap. We now review the quantization of this model.

<sup>7</sup>The reader should beware that the symbol  $B$  is now being used for two purposes. It is the strength of the magnetic field felt by the noncommutative fluid described by the CSMM and NCCS theory, and it is also (along with the capital latin letters  $A, C, \dots$ ) an index on the  $SU(N)$  generators  $T^A$  and the variables  $x_A$ . It should be clear from the context whether  $B$  represents the magnetic field strength or an index.

### C. Quantization of the CSMM

We now discuss the quantization of the CSMM. Instead of trying to solve the constraint before quantization, we follow previous approaches to this model and first quantize, then impose the constraint on quantum states, i.e., physical states should be annihilated by the constraint operator. As we discussed above, upon quantization, the matrix elements of  $X^1$  and  $X^2$  and the components of  $\Psi$  obey the quantum commutation relations from Eq. (4.16). In what follows, we instead work with the oscillator variables

$$b^j = \frac{1}{\sqrt{\hbar}} \Psi^j, \quad (4.18)$$

with  $b_j^\dagger = \frac{1}{\sqrt{\hbar}} \bar{\Psi}_j$ , and

$$a_A = \frac{1}{\ell_B \sqrt{2}} (x_A^1 + i x_A^2), \quad (4.19)$$

with  $a_A^\dagger = \frac{1}{\ell_B \sqrt{2}} (x_A^1 - i x_A^2)$ . These variables obey the commutation relations

$$[a_A, a_B^\dagger] = \delta_{AB}, \quad (4.20)$$

$$[b^j, b_k^\dagger] = \delta_k^j. \quad (4.21)$$

The Hamiltonian for this system has the form

$$\begin{aligned} H_{\text{CSMM}} &= \frac{eB\tilde{\omega}}{2} \delta_{ab} (X^a)^j{}_k (X^b)^k{}_j \\ &= \frac{eB\tilde{\omega}}{2} \sum_{A=0}^{N^2-1} \delta_{ab} x_A^a x_A^b. \end{aligned} \quad (4.22)$$

In terms of the oscillator variables  $a_A$  and  $a_A^\dagger$  this becomes

$$H_{\text{CSMM}} = \hbar\tilde{\omega} \frac{N^2}{2} + \hbar\tilde{\omega} \sum_{A=0}^{N^2-1} a_A^\dagger a_A. \quad (4.23)$$

Note that the first term represents the zero point energy of  $N^2$  harmonic oscillators.

Next, we turn to an analysis of the constraint. Classically, and in terms of the variables  $x_A^a$ , the constraint from Eq. (4.6) takes the form

$$-eB \sum_{A,B,C=1}^{N^2-1} x_A^1 x_B^2 f^{ABC} T^C + eB\theta \mathbb{I} - \Psi \bar{\Psi}^T = 0. \quad (4.24)$$

To interpret the constraint in the quantum theory, we study its  $j, k$  matrix element

$$-eB \sum_{A,B,C=1}^{N^2-1} x_A^1 x_B^2 f^{ABC} (T^C)^j{}_k + eB\theta \delta_k^j - \Psi^j \bar{\Psi}_k = 0. \quad (4.25)$$

In terms of the oscillator variables, one can show that this matrix element of the constraint takes the form

$$\begin{aligned} i \frac{\hbar}{2} \sum_{A,B,C=1}^{N^2-1} (a_A^\dagger a_B + a_B a_A^\dagger) f^{ABC} (T^C)^j{}_k \\ + eB\theta \delta_k^j - \hbar b^j b_k^\dagger = 0. \end{aligned} \quad (4.26)$$

Note that in deriving this expression we needed to use the antisymmetry of the structure constants  $f^{ABC}$  under exchange of its indices. Finally, we use the commutation relations of the oscillator variables to rewrite this as

$$i \hbar \sum_{A,B,C=1}^{N^2-1} a_A^\dagger a_B f^{ABC} (T^C)^j{}_k + (eB\theta - \hbar) \delta_k^j - \hbar b_k^\dagger b^j = 0, \quad (4.27)$$

where we used the fact that  $\sum_{A,B=1}^{N^2-1} \delta_{AB} f^{ABC} = 0$ . Note the shift in the coefficient of the  $\delta_k^j$  term which resulted from this manipulation.<sup>8</sup> Finally, we define  $G^j{}_k$  to be the  $j, k$  matrix element of the constraint, but divided by a factor of  $\hbar$  for convenience,

$$G^j{}_k = i \sum_{A,B,C=1}^{N^2-1} a_A^\dagger a_B f^{ABC} (T^C)^j{}_k + \left( \frac{\theta}{\ell_B^2} - 1 \right) \delta_k^j - b_k^\dagger b^j. \quad (4.28)$$

In the quantum theory physical states  $|\psi\rangle$  will be those states, which satisfy

$$G^j{}_k |\psi\rangle = 0, \quad \forall j, k. \quad (4.29)$$

To understand the form of the physical states  $|\psi\rangle$  we now analyze the constraint. First set  $j = k$  and sum over all  $j$ . Then the constraint implies that

$$b_j^\dagger b^j |\psi\rangle = N \left( \frac{\theta}{\ell_B^2} - 1 \right) |\psi\rangle. \quad (4.30)$$

Now we already know that  $\theta$  is quantized as an integer,  $\theta = \ell_B^2 m$ ,  $m \in \mathbb{Z}$ . If we take  $m > 0$ , then this equation reads as

$$b_j^\dagger b^j |\psi\rangle = N(m-1) |\psi\rangle. \quad (4.31)$$

Thus we find that the total number of  $b^j$  quanta in physical states must be equal to  $N(m-1)$ .

Next, we consider the off-diagonal components of the constraint. For this, it is convenient to instead consider

$$G^A := G^j{}_k (T^A)^k{}_j, \quad (4.32)$$

which is the trace of the product of the constraint matrix (with elements  $G^j{}_k$ ) and a generator  $T^A$  of  $SU(N)$ . We find that these operators take the form

$$G^A = -i(\mathcal{O}_X(T^A) + \mathcal{O}_\Psi(T^A)), \quad (4.33)$$

<sup>8</sup>In Ref. [28], Polychronakos instead performs normal-ordering of the constraint by making the replacement  $b^j b_k^\dagger \rightarrow b_k^\dagger b^j$ . There is then no shift of the coefficient of the  $\delta_k^j$  term. This difference between normal ordering the constraint vs treating it as is completely accounts for the fact that Polychronakos found that the CSMM with  $\theta = \ell_B^2 m$  describes the  $\nu = \frac{1}{m+1}$  Laughlin state, while we will find that it describes the  $\nu = 1/m$  Laughlin state (if we treated the constraint like Polychronakos then this would result in a trivial replacement of  $m \rightarrow m+1$  in all results in this article). Our treatment of the constraint is also identical to the treatment in Ref. [45], which discusses new Chern-Simons matrix models which can describe non-Abelian FQH states (our  $m$  is equal to their  $k+1$  for their model with  $p=1$ ).

where  $\mathcal{O}_X(T^A)$  and  $\mathcal{O}_\Psi(T^A)$  are the quantum operators which generate the action of the  $SU(N)$  generator  $T^A$  on the  $X^a$  and  $\Psi$  variables, respectively. We define these operators and demonstrate their properties in Appendix A. Thus the set of constraints

$$G^A|\psi\rangle = 0, \quad A = 1, \dots, N^2 - 1 \quad (4.34)$$

simply expresses the fact that physical states must be singlets under the total  $SU(N)$  action, as originally noted by Polychronakos [28].

To summarize, we find that the constraint in the CSMM breaks up into two separate parts. The first is associated with the  $U(1)$  part of the total  $U(N)$  action and states that physical states  $|\psi\rangle$  obey Eq. (4.31). The second part is associated with the  $SU(N)$  part of  $U(N)$  and states that physical states should be singlets under the  $SU(N)$  action. Now that we understand the constraint, we can write down a basis of physical states satisfying this constraint. To this end, we introduce the matrix-valued operator<sup>9</sup>

$$A^\dagger = a_0^\dagger \frac{\mathbb{I}}{\sqrt{N}} + \sum_{B=1}^{N^2-1} a_B^\dagger T^B \quad (4.35)$$

with matrix elements

$$(A^\dagger)^j_k = a_0^\dagger \frac{1}{\sqrt{N}} \delta_k^j + \sum_{B=1}^{N^2-1} a_B^\dagger (T^B)^j_k. \quad (4.36)$$

Then, as was shown by Hellerman and Van Raamsdonk in Ref. [31], one possible basis for all physical states is given by states of the form

$$\{|c_1, \dots, c_N\rangle\} = \text{Tr}[(A^\dagger)^N]^{c_N} \dots \text{Tr}[A^\dagger]^{c_1} |\psi_0\rangle, \quad (4.37)$$

where each  $c_j \in \mathbb{N}$  for  $j = 1, \dots, N$ , and

$$|\psi_0\rangle = [\epsilon^{j_1 \dots j_N} b_{j_1}^\dagger (b^\dagger A^\dagger)_{j_2} \dots (b^\dagger (A^\dagger)^{N-1})_{j_N}]^{(m-1)} |0\rangle. \quad (4.38)$$

Note that all  $U(N)$  indices  $j, k$ , etc. are contracted in these expressions, and so every operator present is a singlet under the  $SU(N)$  action. The overall power of  $m - 1$  in  $|\psi_0\rangle$  is required to satisfy the  $U(1)$  part of the constraint coming from Eq. (4.31).

Since the Hamiltonian of the CSMM just counts the total number of  $a_A$  quanta in a state, we find that  $|\psi_0\rangle$  is the unique ground state of the CSMM, and that it has an energy

$$\begin{aligned} E_0 &= \hbar\tilde{\omega} \left[ \frac{N^2}{2} + \frac{1}{2}(m-1)N(N-1) \right] \\ &= \hbar\tilde{\omega} \left[ \frac{1}{2}mN^2 + \left( \frac{1-m}{2} \right) N \right]. \end{aligned} \quad (4.39)$$

<sup>9</sup>Perhaps a more precise notation for this operator would be  $A^\dagger = a_0^\dagger \otimes \frac{\mathbb{I}}{\sqrt{N}} + \sum_{B=1}^{N^2-1} a_B^\dagger \otimes T^B$ , which expresses the fact that  $A^\dagger$  acts on the tensor product  $\mathcal{H}_Q \otimes \mathcal{H}_N$  of an infinite-dimensional Hilbert space  $\mathcal{H}_Q$ , which arises upon quantization of the model, and an  $N$ -dimensional vector space  $\mathcal{H}_N$  on which the classical matrix variables  $X^a$  act.

The excited states  $\{|c_1, \dots, c_N\rangle\}$  then have an energy

$$E(\{c_1, \dots, c_N\}) = E_0 + \hbar\tilde{\omega} \sum_{j=1}^N c_j j. \quad (4.40)$$

It follows that the partition function of the CSMM at an inverse temperature  $\beta$  is just

$$Z = \text{Tr}_Q[e^{-\beta H_{\text{CSMM}}}] = q^{\frac{1}{2}mN^2 + (\frac{1-m}{2})N} \prod_{j=1}^N \frac{1}{1-q^j}, \quad (4.41)$$

where  $\text{Tr}_Q[\cdot]$  denotes a trace over the quantum Hilbert space (consisting of physical states obeying the constraint of the CSMM), and where we defined  $q = e^{-\beta\hbar\tilde{\omega}}$ . As  $N \rightarrow \infty$ , the product  $\prod_{j=1}^N \frac{1}{1-q^j}$  becomes the partition function for the oscillator modes of a single chiral boson, which we know is the edge theory of a Laughlin fractional quantum Hall state.

#### D. Density of the droplet

Here, we review the calculation of the density of the FQH droplet described by the CSMM in the large  $N$  limit. We will see from this calculation that the CSMM with  $\theta = \ell_B^2 m$  corresponds to the Laughlin state at filling fraction  $\nu = 1/m$ . We do not find  $\nu = \frac{1}{m+1}$  as we treated the constraint of Eq. (4.6) as is instead of normal-ordering it as in Polychronakos' original paper [28].

We compute the density of the droplet following the reasoning outlined by Polychronakos [28]. The key is to examine the eigenvalue of the operator

$$\text{Tr}\{\delta_{ab} X^a X^b\} = \sum_{A=0}^{N^2-1} \delta_{ab} x_A^a x_A^b \quad (4.42)$$

in the ground state  $|\psi_0\rangle$  of the CSMM (the trace here is a matrix trace). Since this operator is proportional to  $H_{\text{CSMM}}$ , we have  $\text{Tr}\{\delta_{ab} X^a X^b\}|\psi_0\rangle = R^2|\psi_0\rangle$ , where the eigenvalue  $R^2$  is given by

$$R^2 = 2\ell_B^2 \left( m \frac{N(N-1)}{2} + \frac{N}{2} \right). \quad (4.43)$$

We interpret this eigenvalue as a sum of contributions from  $N$  different particles at different radial positions by writing it as

$$R^2 = \sum_{j=1}^N R_j^2, \quad (4.44)$$

where

$$R_j^2 = 2\ell_B^2 \left( m(j-1) + \frac{1}{2} \right). \quad (4.45)$$

Indeed, the  $R_j^2$  can be thought of as the eigenvalues of the classical matrix  $\delta_{ab} X^a X^b$ , since the operator  $R^2$  is equal to the trace of this matrix. Thus we think of the ground state of the droplet as containing  $N$  particles at definite radial positions  $R_j$  but with complete uncertainty in their angular position. In addition, since  $R_j^2$  is linear in  $j$ , the area  $\pi(R_j^2 - R_{j-1}^2) = 2\pi\ell_B^2 m$  of the annulus between consecutive particles is independent of  $j$ . This implies that the particles are distributed uniformly, i.e., the density is a constant within the droplet.

The size of the droplet is given by the largest value of  $R_j^2$ , which is

$$R_N^2 = 2\ell_B^2(m(N-1) + \frac{1}{2}) \approx 2\ell_B^2 mN \quad (4.46)$$

for large  $N$ . Then at large  $N$  we compute the density as being that of  $N$  particles evenly spread out over a disk of radius  $R_N^2 \approx 2\ell_B^2 mN$ , and so

$$\rho_0 = \frac{N}{\pi R_N^2} \approx \frac{1}{2\pi \ell_B^2 m}, \quad (4.47)$$

which is exactly the density of the Laughlin state with filling fraction  $\nu = 1/m$  (in the limit of a large number  $N$  of electrons).

## V. HALL VISCOSITY OF THE CSMM

We now compute the Hall viscosity in the CSMM following the calculation of Park and Haldane [13] (which we reviewed in Sec. II). We find that the Hall viscosity tensor contains only a *single* contribution, and that this contribution is equal to the guiding center Hall viscosity of the Laughlin state. In other words, the CSMM lacks the Landau orbit contribution to the Hall viscosity, but does contain the (physically important) guiding center contribution.

To compute the Hall viscosity in this system, we recall that in the fluid interpretation of the NCCS theory and the CSMM (which we reviewed at the end of Sec. III), the variables  $X^a$  represent a noncommutative analog of fluid coordinates in a Lagrange description of a fluid [26,28,43]. In the case of the CSMM, this is a finite droplet of noncommutative fluid. Thus, to compute the Hall viscosity, we first need to identify the quantum operators  $\Lambda^{ab}$ , which generate APDs (or strains) of the noncommutative fluid coordinates  $X^a$ . Since we expand the noncommutative coordinates in terms of the scalar variables  $x_A^a$ ,  $A = 0, \dots, N^2 - 1$ , we can instead search for operators which implement APDs of these variables. These operators will then automatically implement the correct transformations of the  $X^a$  coordinates, as the operators do not act on the matrix indices of the  $X^a$  variables.

Since the commutation relations of the variables  $x_A^a$  are identical to the commutation relations of the guiding center coordinates in the quantum Hall problem, we immediately see that the desired operators are given by

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{A=0}^{N^2-1} \{x_A^a, x_A^b\}. \quad (5.1)$$

These operators obey the same algebra as in Eq. (2.5a). It follows that the unitary operators which implement the APDs are  $U(\alpha) = e^{i\alpha_{ab}\Lambda^{ab}}$ , with  $\alpha_{ab}$  a constant symmetric matrix. To first order in  $\alpha_{ab}$ , we have (for all  $A = 0, \dots, N^2 - 1$ )

$$U(\alpha)x_A^a U(\alpha)^\dagger = x_A^a + \epsilon^{ab}\alpha_{bc}x_A^c + \dots, \quad (5.2)$$

which implies (for all  $j, k$ )

$$U(\alpha)(X^a)^j_k U(\alpha)^\dagger = (X^a)^j_k + \epsilon^{ab}\alpha_{bc}(X^c)^j_k + \dots \quad (5.3)$$

It is important to note that the APD generators  $\Lambda^{ab}$  act only on the physical position indices  $a$  of the variables  $X^a$ . There is no action at all on the  $U(N)$  indices  $j, k$  of the matrix elements  $(X^a)^j_k$ . Thus the generators  $\Lambda^{ab}$  act identically on all matrix

elements of  $X^a$ , and so they are indeed the correct quantum generators of APDs of the noncommutative fluid coordinates  $X^a$  (which we recall are actually  $N \times N$  Hermitian matrices in the classical theory).

Now we want to compute the Hall viscosity in the ground state  $|\psi_0\rangle$  of the CSMM. We compute this using a Kubo formula approach similar to that of Ref. [12]. We present the Kubo formula calculation of the Hall viscosity in Appendix B. Our result is that the Hall viscosity tensor in this model takes the form ( $A$  is the area of the droplet)

$$\eta_{\text{CSMM}}^{abcd} = \frac{i\hbar}{A} \langle \psi_0 | [\Lambda^{ab}, \Lambda^{cd}] | \psi_0 \rangle. \quad (5.4)$$

We note that the tensor  $\eta_{\text{CSMM}}^{abcd}$  contains only a single contribution, as opposed to the two separate terms (guiding center and Landau orbit contributions) appearing in the discussion of the Hall viscosity tensor from Sec. II. Note that in deriving this result it was crucial that the CSMM has a unique ground state and a finite energy gap set by the frequency  $\tilde{\omega}$  of the harmonic trap.

Due to the commutation relations of the generators  $\Lambda^{ab}$  [which are the same as Eq. (2.5a)], the four index tensor  $\eta_{\text{CSMM}}^{abcd}$  can again be expressed in terms of a symmetric two-index tensor:

$$\eta_{\text{CSMM}}^{ab} = -\frac{\hbar}{A} \langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle. \quad (5.5)$$

Therefore, to compute the Hall viscosity tensor of the CSMM, we just need to compute the expectation values  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$ . To compute these, we first note that the CSMM Hamiltonian can be written as

$$H_{\text{CSMM}} = \hbar\tilde{\omega}\delta_{ab}\Lambda^{ab} = \hbar\tilde{\omega}(\Lambda^{11} + \Lambda^{22}). \quad (5.6)$$

From this we can already deduce that

$$\langle \psi_0 | \delta_{ab}\Lambda^{ab} | \psi_0 \rangle = \frac{E_0}{\hbar\tilde{\omega}} = \frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N. \quad (5.7)$$

We can go further and compute the individual expectation values of  $\Lambda^{11}$  and  $\Lambda^{22}$  by deriving a Virial theorem for the CSMM. To derive this theorem, consider the operator

$$Q = \sum_{A=0}^{N^2-1} x_A^1 x_A^2. \quad (5.8)$$

A short computation shows that

$$[Q, \delta_{ab}\Lambda^{ab}] = 2i\ell_B^2(-\Lambda^{11} + \Lambda^{22}). \quad (5.9)$$

If we take the expectation value of this equation in the state  $|\psi_0\rangle$  (or any eigenstate of  $\delta_{ab}\Lambda^{ab}$ ), then we find that

$$\langle \psi_0 | \Lambda^{11} | \psi_0 \rangle = \langle \psi_0 | \Lambda^{22} | \psi_0 \rangle. \quad (5.10)$$

Combining this result with Eq. (5.7) gives the result that

$$\langle \psi_0 | \Lambda^{11} | \psi_0 \rangle = \langle \psi_0 | \Lambda^{22} | \psi_0 \rangle = \frac{1}{2} \left[ \frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N \right]. \quad (5.11)$$

Finally, it remains to compute the expectation value of the off-diagonal generator  $\Lambda^{12} = \Lambda^{21}$ . In terms of the oscillator

variables  $a_A$  and  $a_A^\dagger$  this operator takes the form

$$\Lambda^{12} = \frac{1}{4i} \sum_{A=0}^{N^2-1} (a_A a_A - a_A^\dagger a_A^\dagger). \quad (5.12)$$

Now all eigenstates of  $H_{\text{CSMM}}$  are eigenstates of the total number operator for the  $a_A$  oscillators. Since  $\Lambda^{12}$  clearly does not commute with the total number operator, we immediately conclude that the expectation value of  $\Lambda^{12}$  in any eigenstate of  $H_{\text{CSMM}}$  is zero.

Therefore our final result for the expectation value of the APD generators  $\Lambda^{ab}$  in the CSMM ground state is

$$\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle = \frac{1}{2} \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right] \delta^{ab}. \quad (5.13)$$

This means that we can write  $\eta_{\text{CSMM}}^{ab} = \eta_{\text{CSMM}} \delta^{ab}$ , where the coefficient  $\eta_{\text{CSMM}}$  of Hall viscosity in this model is equal to

$$\eta_{\text{CSMM}} = -\frac{\hbar}{A} \frac{1}{2} \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right]. \quad (5.14)$$

Now since  $A = \pi R_N^2 \approx 2\pi \ell_B^2 m N$  for the CSMM at large  $N$ , this exactly matches the result (before regularization) for the *guiding center Hall viscosity*  $\eta_H$  of the  $\nu = 1/m$  Laughlin state. The Landau orbit contribution  $\tilde{\eta}_H$  is absent in the CSMM. Finally, as was the case for the ordinary Laughlin state, this result can be regularized by subtracting off the extensive term in  $\eta_{\text{CSMM}}$  (or the superextensive term in  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$ ). We discuss a fluid interpretation of this regularization of the Hall viscosity later in Sec. VII.

## VI. HALL CONDUCTANCE OF THE CSMM IN A NONUNIFORM ELECTRIC FIELD

In this section, we study the Hall conductance of the CSMM when it is subjected to a nonuniform electric field. Our motivation for studying this setup is the well-known result of Hoyos and Son, which shows that in a quantum Hall state the Hall conductance  $\sigma_H(\mathbf{k})$  at finite wave vector  $\mathbf{k}$  has a universal contribution at order  $k^2$  ( $k^2 = \delta^{ab} k_a k_b$ ), which is related to the Hall viscosity [11] (see also Ref. [12] for a Kubo formula approach to this relation). We find a similar contribution in the CSMM, but depending only on the guiding center Hall viscosity as opposed to the full Hall viscosity. Again, this is not surprising as we only expect the CSMM to describe the dynamics of the guiding center degrees of freedom in a FQH state.

In this section, we first review the result of Ref. [11] on the Hall conductance at finite wave vector. We then warm up by calculating the Hall conductance of the CSMM subjected to a *uniform* electric field. The reason for this is that there are several subtle points associated with the computation of the Hall conductance in the CSMM that we want to explain clearly. Finally, we compute the Hall conductance of the CSMM in a nonuniform electric field, where we find a result which resembles the result of Hoyos and Son [11], but with the full Hall viscosity replaced by the guiding center Hall viscosity. We note here that the Hall conductance of the NCCS theory in a uniform electric field was computed previously in Refs. [34,36] at the classical level by solving the equations of motion for

the NCCS theory in a uniform electric field. We therefore emphasize that our treatment in this section deals directly with the quantized CSMM theory as opposed to the classical NCCS theory.

### A. The result of Hoyos and Son

We start by reviewing the result of Ref. [11]. Consider a quantum Hall system in a nonuniform electric field  $\mathbf{E} = (E(\mathbf{x}), 0)$  pointing in the  $x^1$  direction, and where the spatial dependence is only on the  $x^1$  coordinate, so that  $\partial_2 E(\mathbf{x}) = 0$ . The Hall conductance  $\sigma_H(\mathbf{k})$  at finite wave vector is defined by the relation

$$j^2(\mathbf{k}) = \sigma_H(\mathbf{k}) E(\mathbf{k}), \quad (6.1)$$

where  $j^2(\mathbf{k})$  is the Fourier transform of the charge current in the  $x^2$  direction, and  $E(\mathbf{k})$  is the Fourier transform of  $E(\mathbf{x})$ . The result of Ref. [11] is that (recall that  $E(\mathbf{x})$  is a function of only  $x^1$ )

$$\frac{\sigma_H(\mathbf{k})}{\sigma_H(\mathbf{0})} = 1 + C_2 (k_1 \ell_B)^2 + \dots, \quad (6.2)$$

where the Hall conductance at zero wave vector is simply ( $\nu$  is the filling fraction)

$$\sigma_H(\mathbf{0}) = \nu \frac{e^2}{h}. \quad (6.3)$$

The coefficient  $C_2$  is given by

$$C_2 = \frac{\eta_{\text{tot}}}{\hbar \rho_0} - \frac{2\pi}{\nu} \frac{\ell_B^2}{\hbar \omega_c} B^2 \mathcal{E}''(B), \quad (6.4)$$

where  $\eta_{\text{tot}}$  denotes the full Hall viscosity of the quantum Hall state (as opposed to just the guiding center part),  $\mathcal{E}(B)$  is the energy density of the quantum Hall state viewed as a function of the external field  $B$ , and  $\mathcal{E}''(B)$  denotes the second derivative of  $\mathcal{E}(B)$  with respect to  $B$ . In addition,  $\rho_0$  denotes the number density of the quantum Hall state, and  $\omega_c = (eB)/M$  is the cyclotron frequency, where  $M$  is the mass of the particles making up the quantum Hall state. As an example, for a quantum Hall state consisting of  $N$  electrons in the lowest Landau level and occupying an area  $A$ , we have  $\mathcal{E}(B) = \frac{\hbar \omega_c}{2} \frac{N}{A} = \frac{\hbar \omega_c}{2} \rho_0$ , and for a Laughlin  $\nu = 1/m$  state this gives  $\mathcal{E}(B) = \frac{\hbar \omega_c}{4\pi \ell_B^2 m}$ .

In the context of the CSMM, the quantity that we actually compute is the current at the location of the center of mass of the droplet (we explain the reason for this in the next section). Therefore we need to Fourier transform the result of Hoyos and Son back to real space in order to compare with our calculation in the CSMM later in this section. In real space, we find that

$$j^2(\mathbf{x}) = \nu \frac{e^2}{h} (E(\mathbf{x}) - C_2 \ell_B^2 \partial_1^2 E(\mathbf{x}) + \dots). \quad (6.5)$$

In particular, at the origin  $\mathbf{x} = \mathbf{0}$  (where the center of mass of a uniform droplet would be located) we have

$$j^2(\mathbf{x} = \mathbf{0}) = \nu \frac{e^2}{h} (E^{(0)} - C_2 \ell_B^2 E^{(2)} + \dots), \quad (6.6)$$

where  $E^{(0)}$  and  $E^{(2)}$  are the coefficients in the Taylor series expansion of  $E(\mathbf{x})$  about the origin,

$$E(\mathbf{x}) = E^{(0)} + E^{(1)}x^1 + \frac{1}{2!}E^{(2)}(x^1)^2 + \dots, \quad (6.7)$$

and where we again remind the reader that we assumed that  $E(\mathbf{x})$  has no  $x^2$  dependence.

### B. Uniform electric field

We now compute the Hall conductance of the CSMM in a uniform electric field. Our reason for treating this simple case first is to highlight a few subtleties in the calculation of the Hall conductance of the CSMM. The first subtlety is associated with the fact that one cannot resolve individual points in space in the CSMM, since the spatial coordinates are actually the noncommuting matrices  $X^1$  and  $X^2$ . However, in the CSMM one can still define a notion of the center of mass coordinate of the FQH droplet, and the expectation value of this center of mass coordinate can be computed in any state  $|\psi\rangle$  of the quantized CSMM. We define the center of mass coordinates  $X_{\text{COM}}^a$  as

$$X_{\text{COM}}^a = \frac{1}{N} \text{Tr}\{X^a\} = \frac{x_0^a}{\sqrt{N}}, \quad (6.8)$$

where in the second equality we evaluated the trace and found that  $X_{\text{COM}}^a$  is proportional to the variable  $x_0^a$  introduced in Eq. (4.11) of Sec. IV. To motivate this definition we simply note that if the  $X^a$  were diagonal matrices, then their diagonal elements could be interpreted as the positions of  $N$  particles, and then  $\frac{1}{N} \text{Tr}\{X^a\}$  would agree with the usual definition of the center of mass coordinate of  $N$  particles (assuming all particles have equal masses).

Our strategy to compute the Hall conductance in the CSMM is to compute the drift velocity  $\mathbf{v}_{\text{drift}}$  of the center of mass coordinate when the system is placed in an electric field  $\mathbf{E}$ . We can then use the fact that the CSMM describes a droplet of particles with charge  $-e$  and density  $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$  (computed in Sec. IV) to compute the charge current  $\mathbf{j}_{\text{COM}}$  at the center of mass as

$$\mathbf{j}_{\text{COM}} = -e\rho_0\mathbf{v}_{\text{drift}}. \quad (6.9)$$

The result can then be compared with the result of Hoyos and Son for the current at the origin (location of the center of mass) as expressed in Eq. (6.6).

Next, we need to discuss the issue of how to couple the CSMM to an external electric field. This can be done using the fluid interpretation of this theory from Sec. III. First, recall from Sec. III that an ordinary charged fluid on commutative flat space  $\mathbb{R}^2$  can be coupled to a background electromagnetic field by including vector and scalar potentials  $\mathcal{A}_a(t, \mathbf{X})$  and  $\varphi(t, \mathbf{X})$ , respectively, in the action for the Lagrange description of this fluid, Eq. (3.33). In our case, we are only interested in adding a scalar potential  $\varphi(t, \mathbf{X})$  for the external electric field. Using the fluid interpretation, we can incorporate this potential into the NCCS theory by adding a term to the NCCS action of the form

$$S_{EM} = e \int_0^T dt \text{Tr}_{\mathcal{H}_F} \{\hat{\varphi}(\hat{\mathbf{X}}, t)\}, \quad (6.10)$$

where the operator  $\hat{\varphi}(\hat{\mathbf{X}}, t)$  is the operator representing the scalar potential for the external electromagnetic field (and recall that the charge of the particles is  $q = -e$ ).

In defining the operator  $\hat{\varphi}(\hat{\mathbf{X}}, t)$  we encounter an ordering ambiguity. For example, if the scalar potential for the electric field configuration on a commutative space is  $\varphi(t, \mathbf{X}) = X^1 X^2$ , then we could define  $\hat{\varphi}(\hat{\mathbf{X}}, t) = \hat{X}^1 \hat{X}^2$ ,  $\hat{\varphi}(\hat{\mathbf{X}}, t) = \hat{X}^2 \hat{X}^1$ , or the symmetric Weyl ordering  $\hat{\varphi}(\hat{\mathbf{X}}, t) = \frac{1}{2}(\hat{X}^1 \hat{X}^2 + \hat{X}^2 \hat{X}^1)$ , for example. We choose to use Weyl ordering since this is consistent with our use of Weyl ordering to go between star product and operator formulations of noncommutative field theory [recall Eq. (3.8)], however, in the examples of this section we do not actually encounter this ordering ambiguity. Weyl-ordering for the external field was also adopted by the authors of Ref. [34], who also considered the NCCS theory in the presence of external fields.

Finally, to couple the CSMM to the external electromagnetic field, we use the same action  $S_{EM}$  as above but replace the operators  $\hat{X}^a$  on the infinite-dimensional space  $\mathcal{H}_F$  with the finite  $N \times N$  matrix variables of the CSMM. From this action, we can then read off the new Hamiltonian for the CSMM coupled to the external electric field.

There is one more subtlety with the calculation of the Hall conductance of the CSMM that we need to address before we can proceed. The issue is that the parabolic potential in the CSMM competes with the applied electric field to determine the long time behavior of the CSMM in the presence of the electric field. This is best illustrated for the case of the CSMM in a constant electric field  $E^{(0)}$  pointing in the  $x^1$  direction. The Hamiltonian describing this system is

$$\begin{aligned} H' &= H_{\text{CSMM}} + eE^{(0)}\text{Tr}\{X^1\} \\ &= H_{\text{CSMM}} + eE^{(0)}NX_{\text{COM}}^1, \end{aligned} \quad (6.11)$$

and where the trace is a classical matrix trace. To derive this Hamiltonian, we used the fluid interpretation of the CSMM theory and incorporated a scalar potential  $\varphi(t, \mathbf{X}) = -E^{(0)}X^1$  to describe the coupling to a constant electric field in the  $x^1$  direction. This Hamiltonian can be immediately diagonalized by noting that

$$H' = T(\mathbf{R})H_{\text{CSMM}}T(\mathbf{R})^\dagger - \frac{eN(E^{(0)})^2}{2B\bar{\omega}}, \quad (6.12)$$

where  $T(\mathbf{R})$  is a unitary translation operator<sup>10</sup> (similar to a magnetic translation) of the form

$$T(\mathbf{R}) = \exp\left(-\frac{i\epsilon_{ab}NX_{\text{COM}}^a R^b}{\ell_B^2}\right), \quad (6.13)$$

and where in this case

$$\mathbf{R} = \left(\frac{E^{(0)}}{B\bar{\omega}}, 0\right). \quad (6.14)$$

The ground state of this Hamiltonian is  $|\psi'_0\rangle = T(\mathbf{R})|\psi_0\rangle$  and represents a stationary state with  $\langle\psi'_0|X_{\text{COM}}^1|\psi'_0\rangle = -\frac{E^{(0)}}{B\bar{\omega}}$

<sup>10</sup>We have  $[X_{\text{COM}}^a, X_{\text{COM}}^b] = \frac{i\ell_B^2}{N}\epsilon^{ab}$  and  $T(\mathbf{R})X_{\text{COM}}^aT(\mathbf{R})^\dagger = X_{\text{COM}}^a + R^a$ .



and  $\langle \psi'_0 | X_{\text{COM}}^2 | \psi'_0 \rangle = 0$ , which corresponds to the equilibrium position in the total potential

$$V = \frac{eBN\tilde{\omega}}{2} \delta_{ab} X_{\text{COM}}^a X_{\text{COM}}^b + eE^{(0)} N X_{\text{COM}}^1 \quad (6.15)$$

felt by the center of mass.

We see that if we simply diagonalize the Hamiltonian  $H'$  for the CSMM in the presence of the external field, we find no time dependence and, in the ground state, the center of mass of the droplet just sits at its equilibrium position  $(-\frac{E^{(0)}}{B\tilde{\omega}}, 0)$  under the influence of the combined forces of the parabolic potential and the applied electric field.

To compute the Hall conductance of this model, we instead need to consider a nonequilibrium situation in which we start with the system in the ground state  $|\psi_0\rangle$  of the unperturbed CSMM (which we will now assume is properly normalized) and then suddenly turn on the electric field. We then study the time evolution of the center of mass coordinate at small times  $t \ll \frac{1}{\tilde{\omega}}$ , where  $\frac{1}{\tilde{\omega}}$  is the time scale set by the parabolic potential. Therefore we consider the ‘‘quantum quench’’ problem in which the state of the system at time  $t$  is given by

$$|\psi(t)\rangle = e^{-i\frac{H't}{\hbar}} |\psi_0\rangle, \quad (6.16)$$

where  $|\psi_0\rangle$  is the ground state of the unperturbed CSMM Hamiltonian  $H_{\text{CSMM}}$ , and  $H'$  is the perturbed CSMM Hamiltonian including the applied electric field. We then compute

$$\begin{aligned} \langle \psi(t) | X_{\text{COM}}^a | \psi(t) \rangle &= \langle \psi_0 | X_{\text{COM}}^a | \psi_0 \rangle \\ &+ \frac{it}{\hbar} \langle \psi_0 | [H', X_{\text{COM}}^a] | \psi_0 \rangle + \dots \end{aligned} \quad (6.17)$$

and identify the drift velocity  $\mathbf{v}_{\text{drift}}$  of the center of mass with the term linear in  $t$  in this expansion,

$$v_{\text{drift}}^a = \frac{i}{\hbar} \langle \psi_0 | [H', X_{\text{COM}}^a] | \psi_0 \rangle. \quad (6.18)$$

We now consider the case of a uniform electric field  $E^{(0)}$  pointing in the  $x^1$  direction so that  $H'$  takes the form shown in Eq. (6.11). In this case, the drift velocity evaluates to

$$\mathbf{v}_{\text{drift}} = \left( 0, -\frac{E^{(0)}}{B} \right). \quad (6.19)$$

Then the nonzero part of the charge current at the center of mass of the droplet, at times  $t \ll \frac{1}{\tilde{\omega}}$ , is

$$j_{\text{COM}}^2 = e\rho_0 \frac{E^{(0)}}{B} = v \frac{e^2}{h} E^{(0)}, \quad (6.20)$$

with  $v = 1/m$ , and where we used  $\rho_0 = \frac{1}{2\pi\ell_B^2 m}$ . Therefore we find that the Hall conductance of the CSMM with  $\theta = \ell_B^2 m$  is given by

$$\sigma_H = \frac{1}{m} \frac{e^2}{h}, \quad (6.21)$$

exactly as in the  $\nu = 1/m$  Laughlin state.

For the case of a uniform electric field we can actually go further and compute the full time dependence of the center of

mass coordinate. We find that

$$\langle \psi(t) | X_{\text{COM}}^1 | \psi(t) \rangle = \frac{E^{(0)}}{B\tilde{\omega}} (-1 + \cos(\tilde{\omega}t)), \quad (6.22)$$

$$\langle \psi(t) | X_{\text{COM}}^2 | \psi(t) \rangle = -\frac{E^{(0)}}{B\tilde{\omega}} \sin(\tilde{\omega}t). \quad (6.23)$$

We see that the center of mass moves in a large circle about its equilibrium position  $(-\frac{E^{(0)}}{B\tilde{\omega}}, 0)$ , but that at early times  $t \ll \frac{1}{\tilde{\omega}}$  the droplet drifts in the  $x^2$  direction with velocity vector  $\mathbf{v}_{\text{drift}} = (0, -\frac{E^{(0)}}{B})$ .

### C. Nonuniform electric field

We now compute the Hall conductance of the CSMM in a nonuniform electric field. We consider an electric field, which points in the  $x^1$  direction and which depends only on the  $x^1$  coordinate. Since we are interested in contributions to the current, which depend on the second derivative of the electric field, it is sufficient to consider an electric field, which depends at most quadratically on the  $x^1$  coordinate. Thus, for an ordinary classical charged fluid described by the action of Eq. (3.33), we would add a scalar potential of the form

$$\varphi(t, \mathbf{X}) = -E^{(0)} X^1 - \frac{1}{2} E^{(1)} (X^1)^2 - \frac{1}{3!} E^{(2)} (X^1)^3, \quad (6.24)$$

which corresponds, after computing minus the spatial gradient, to an electric field  $\mathbf{E} = (E(\mathbf{X}), 0)$  with

$$E(\mathbf{X}) = E^{(0)} + E^{(1)} X^1 + \frac{1}{2} E^{(2)} (X^1)^2. \quad (6.25)$$

The coefficients  $E^{(j)}$ ,  $j = 0, 1, 2$  in this expression (which are all fixed real numbers) can be understood as the coefficients in the Taylor expansion of  $E(\mathbf{X})$  about the origin.

This form of the scalar potential for the ordinary classical fluid, combined with the considerations from earlier in this section on how to couple the CSMM to external fields, leads to a Hamiltonian

$$H' = H_{\text{CSMM}} + H_1 \quad (6.26)$$

with

$$H_1 = e\text{Tr} \left\{ E^{(0)} X^1 + \frac{1}{2} E^{(1)} (X^1)^2 + \frac{1}{3!} E^{(2)} (X^1)^3 \right\}, \quad (6.27)$$

where the trace denotes a matrix trace. This Hamiltonian then describes the CSMM in the presence of a nonuniform electric field in the  $x^1$  direction. To compute the Hall conductance, we again consider a time-dependent problem where the state at time  $t$  is given by  $|\psi(t)\rangle = e^{-i\frac{H't}{\hbar}} |\psi_0\rangle$  with  $|\psi_0\rangle$  the ground state of  $H_{\text{CSMM}}$ . The drift velocity is again given by Eq. (6.18) and since  $\langle \psi_0 | [H_{\text{CSMM}}, X_{\text{COM}}^a] | \psi_0 \rangle = 0$  (since  $|\psi_0\rangle$  is an eigenstate of  $H_{\text{CSMM}}$ ), this reduces to

$$v_{\text{drift}}^a = \frac{i}{\hbar} \langle \psi_0 | [H_1, X_{\text{COM}}^a] | \psi_0 \rangle. \quad (6.28)$$

It remains to actually compute the matrix element  $\langle \psi_0 | [H_1, X_{\text{COM}}^a] | \psi_0 \rangle$ .

To compute this matrix element, we first note that we already know the answer for the term in  $H_1$  proportional to  $E^{(0)}$  from the previous subsection. Next, we can immediately see that the term proportional to  $E^{(1)}$  will vanish since the

commutator of  $\text{Tr}\{(X^1)^2\}$  with  $X_{\text{COM}}^a$  is linear in the center of mass coordinate and we know that  $\langle \psi_0 | X_{\text{COM}}^a | \psi_0 \rangle = 0$  in the unperturbed ground state of the CSMM. To handle the term proportional to  $E^{(2)}$ , we use Eq. (4.11) to find that

$$\begin{aligned} \text{Tr}\{(X^1)^3\} &= \frac{(x_0^1)^3}{\sqrt{N}} + \frac{3}{\sqrt{N}} x_0^1 \sum_{A=1}^{N^2-1} x_A^1 x_A^1 \\ &+ \sum_{A,B,C=1}^{N^2-1} x_A^1 x_B^1 x_C^1 \text{Tr}\{T^A T^B T^C\}. \end{aligned} \quad (6.29)$$

Then we have  $[\text{Tr}\{(X^1)^3\}, X_{\text{COM}}^1] = 0$  and

$$[\text{Tr}\{(X^1)^3\}, X_{\text{COM}}^2] = \frac{3i\ell_B^2}{N} \sum_{A=0}^{N^2-1} x_A^1 x_A^1. \quad (6.30)$$

We find that  $v_{\text{drift}}^1 = 0$ , while

$$\begin{aligned} v_{\text{drift}}^2 &= -\frac{E^{(0)}}{B} + \frac{i}{\hbar} \left( e \frac{E^{(2)}}{3!} \right) \frac{3i\ell_B^2}{N} \langle \psi_0 | \sum_{A=0}^{N^2-1} x_A^1 x_A^1 | \psi_0 \rangle \\ &= -\frac{E^{(0)}}{B} - \frac{eE^{(2)}\ell_B^4}{\hbar N} \langle \psi_0 | \Lambda^{11} | \psi_0 \rangle \\ &= -\frac{E^{(0)}}{B} + \frac{E^{(2)}\ell_B^2}{B} \frac{\eta_{\text{CSMM}}}{\hbar\rho_0}, \end{aligned} \quad (6.31)$$

where we used the fact that  $\langle \psi_0 | \Lambda^{11} | \psi_0 \rangle = -\frac{A}{\hbar} \eta_{\text{CSMM}}$  and  $\rho_0 = \frac{N}{A}$ . If we now compute  $j_{\text{COM}}^2 = -e\rho_0 v_{\text{drift}}^2$  then we find that

$$\begin{aligned} j_{\text{COM}}^2 &= v \frac{e^2}{h} \left( E^{(0)} - E^{(2)} \ell_B^2 \frac{\eta_{\text{CSMM}}}{\hbar\rho_0} \right) \\ &= v \frac{e^2}{h} \left( E^{(0)} - E^{(2)} \ell_B^2 \frac{\eta_H}{\hbar\rho_0} \right), \end{aligned} \quad (6.32)$$

where the second line follows from the fact that  $\eta_{\text{CSMM}} = \eta_H$ , where  $\eta_H$  was the guiding center Hall viscosity for the Laughlin state. Finally, we should regularize this expression to obtain a finite answer for the current in the  $N \rightarrow \infty$  limit. This just amounts to the replacement  $\eta_H \rightarrow \eta_{H,\text{reg}}$  in the final expression (we discuss the physical interpretation of this regularization in Sec. VII). Therefore our final expression for the center of mass current in a nonuniform electric field is

$$j_{\text{COM}}^2 = v \frac{e^2}{h} \left( E^{(0)} - E^{(2)} \ell_B^2 \frac{\eta_{H,\text{reg}}}{\hbar\rho_0} \right). \quad (6.33)$$

Equation (6.33) is the main result of this section.

It is interesting to compare Eq. (6.33) with the result of Hoyos and Son, Eq. (6.6), where the coefficient  $C_2$  was given in Eq. (6.4). We see that the CSMM result contains a contribution like the first term in  $C_2$ , but with the total Hall viscosity  $\eta_{\text{tot}}$  replaced with the guiding center Hall viscosity  $\eta_{H,\text{reg}}$ . As we remarked earlier, this makes sense because we only expect the CSMM to describe the dynamics of the guiding center degrees of freedom in the quantum Hall problem. We also find that the CSMM result does not contain any contribution resembling the second term in  $C_2$  which is proportional to  $\mathcal{E}''(B)$ . This is also not surprising since the CSMM itself does not contain any information about the energy associated with electrons filling

a Landau level. Indeed, we can see from the fluid interpretation of the NCCS theory from Sec. III that the NCCS theory (and therefore the CSMM theory, which is a regularization of it), is obtained by sending the energy scale  $\hbar\omega_c$  to infinity. Therefore we find that the CSMM accurately captures the *guiding center contribution* to the response of a FQH state to a nonuniform electric field.

## VII. $N \rightarrow \infty$ LIMIT, REGULARIZATION OF THE HALL VISCOSITY, AND FLUID INTERPRETATION

In Ref. [13], Park and Haldane argued that one should regularize the guiding center Hall viscosity by subtracting the extensive term in  $\eta_H = -\frac{\hbar}{A^2} [\frac{1}{2} m N^2 + (\frac{1-m}{2}) N]$ , which amounts to subtracting the term  $\frac{1}{2} m N^2$  from

$$\frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N. \quad (7.1)$$

In the quantum Hall problem, this regularization (or something similar to it) is necessary to obtain a finite value for the guiding center Hall viscosity in the thermodynamic limit  $N \rightarrow \infty$ .

In this section we give an interpretation of this regularization scheme in the context of the fluid interpretation (reviewed in the last subsection of Sec. III) of the NCCS theory and CSMM. Our starting point is to note that the expectation value  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$  in the CSMM is actually proportional to the total angular momentum of the state  $|\psi_0\rangle$ . The fact that the Hall viscosity is related to angular momentum has been discussed extensively in Ref. [9], so this is not a new observation. However, this observation will allow us to understand the origin of the superextensive term  $\frac{1}{2} m N^2$  in  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$ , and to explain why it should be subtracted when computing the Hall viscosity of the CSMM.

We start by deriving an expression for the angular momentum in the CSMM theory. To do this we use the fluid interpretation of the NCCS theory and CSMM from the last part of Sec. III. Our derivation of the expression for the angular momentum consists of several steps. First, we derive an expression for the angular momentum of a classical fluid of charged particles on a commutative space  $\mathbb{R}^2$  and in the presence of a constant background magnetic field. Next, we take the limit in which the mass of the particles making up the fluid goes to zero. We then perform the noncommutative deformation of the expression for the angular momentum to obtain an expression for the angular momentum in NCCS theory. Finally, the expression for the angular momentum in NCCS theory can also be used for the CSMM, after we replace the infinite-dimensional operator variables in the NCCS theory with the  $N \times N$  matrix variables of the CSMM.

We start with the action for a fluid of particles of mass  $M$ , charge  $q = -e$ , and constant (initial) density  $\rho_0$  in a constant magnetic field  $B$  (see the discussion in the last subsection of Sec. III),

$$S = \int_0^T dt \int d^2\mathbf{x} \rho_0 \left( \frac{1}{2} M \delta_{ab} \dot{X}^a \dot{X}^b - \frac{eB}{2} \epsilon_{ab} X^a \dot{X}^b \right), \quad (7.2)$$

where we remind the reader that for the classical fluid the fields  $X^a(t, \mathbf{x})$  are ordinary functions of time  $t$  and spatial coordinates

$\mathbf{x} \in \mathbb{R}^2$ . For now we omit the Lagrange multiplier field  $A_0(t, \mathbf{x})$  which keeps the density fixed to  $\rho_0$  at all times, as this term plays no role in the definition of the angular momentum of the theory. The momentum variables  $P_a(t, \mathbf{x})$  canonically conjugate to  $X^a(t, \mathbf{x})$  are obtained by differentiating the Lagrangian<sup>11</sup> with respect to  $\dot{X}^a$ , and we have

$$P_1 = M\dot{X}^1 + \frac{eB}{2}X^2, \quad (7.3)$$

$$P_2 = M\dot{X}^2 - \frac{eB}{2}X^1. \quad (7.4)$$

The expression for the angular momentum of this fluid is then

$$\begin{aligned} L_z &= \int d^2\mathbf{x} \rho_0 (X^1 P_2 - X^2 P_1) \\ &= \int d^2\mathbf{x} \rho_0 \left\{ M\epsilon_{ab} X^a \dot{X}^b - \frac{eB}{2} \delta_{ab} X^a X^b \right\}, \end{aligned} \quad (7.5)$$

and the limit  $M \rightarrow 0$  gives

$$L_z = - \int d^2\mathbf{x} \rho_0 \frac{eB}{2} \delta_{ab} X^a X^b. \quad (7.6)$$

Next, we set  $\rho_0 = \frac{1}{2\pi\theta}$  as is appropriate for the fluid interpretation of NCCS theory, and we perform the noncommutative deformation of this expression (see Sec. III) by replacing  $\frac{1}{2\pi\theta} \int d^2\mathbf{x} (\dots) \rightarrow \text{Tr}_{\mathcal{H}_F} \{ \dots \}$  and  $X^a(t, \mathbf{x}) \rightarrow \hat{X}^a(t)$ . This gives an expression for the angular momentum in NCCS theory,

$$L_z = - \frac{eB}{2} \text{Tr}_{\mathcal{H}_F} \{ \delta_{ab} \hat{X}^a \hat{X}^b \}. \quad (7.7)$$

Finally, we obtain an expression for the angular momentum of the CSMM by replacing the operators  $\hat{X}^a(t)$  with the  $N \times N$  matrix variables  $X^a(t)$  of the CSMM, and by replacing the trace over the infinite-dimensional space  $\mathcal{H}_F$  by the trace over  $N \times N$  matrices,

$$L_{z, \text{CSMM}} = - \frac{eB}{2} \text{Tr} \{ \delta_{ab} X^a X^b \}. \quad (7.8)$$

We now compute the angular momentum in the quantum ground state  $|\psi_0\rangle$  of the CSMM. We first use the expansion of Eq. (4.11) to write  $L_{z, \text{CSMM}}$  as

$$L_{z, \text{CSMM}} = - \frac{eB}{2} \sum_{A=0}^{N^2-1} \delta_{ab} x_A^a x_A^b = -\hbar \delta_{ab} \Lambda^{ab}, \quad (7.9)$$

where  $\Lambda^{ab}$  are the strain generators for the CSMM introduced in Sec. V. We see that our derivation of the angular momentum for the CSMM theory makes sense since  $\delta_{ab} \Lambda^{ab}$  is exactly the operator which generates rotations of the noncommutative coordinates  $X^a$  in the CSMM.

For the ground state of the CSMM, we have  $L_{z, \text{CSMM}} |\psi_0\rangle = L_0 |\psi_0\rangle$  with

$$L_0 = -\hbar \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right], \quad (7.10)$$

and our previous results for  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$  and  $\eta_{\text{CSMM}}$  can be rewritten in the form

$$\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle = - \frac{1}{2\hbar} L_0 \delta^{ab} \quad (7.11)$$

$$\eta_{\text{CSMM}} = \frac{1}{2} \frac{L_0}{A}. \quad (7.12)$$

Thus we see that the Hall viscosity coefficient  $\eta_{\text{CSMM}}$  (before regularization) is equal to one half the angular momentum density  $\frac{L_0}{A}$  in the ground state of the CSMM (compare with the angular momentum interpretation of the Hall viscosity from Ref. [9]). Finally, we also note that  $L_0$  is exactly the guiding center part of the angular momentum of the Laughlin  $\nu = 1/m$  state. In the lowest Landau level the Landau orbit contribution to the angular momentum is simply  $\hbar \frac{N}{2}$ , which leads to the total angular momentum of the Laughlin state  $L_{z, \nu=1/m} = \hbar [ -\frac{1}{2} m N^2 + m \frac{N}{2} ]$ .

We now give a fluid interpretation of the superextensive (order  $N^2$ ) term in  $L_0$ , which is equal to  $-\frac{1}{2} \hbar m N^2$ . This can be rewritten in terms of the density  $\rho_0 = \frac{1}{2\pi \ell_B^2 m}$  and radius  $R_N^2 \approx 2\ell_B^2 m N$  of the droplet described by the CSMM as

$$-\frac{\pi}{4} eB \rho_0 R_N^4. \quad (7.13)$$

This is exactly the angular momentum of a droplet of radius  $R_N$  of the classical fluid described by the small  $\theta$  limit of the NCCS action in the presence of an additional parabolic potential, as we now describe.

Recall that in the small  $\theta$  limit the NCCS theory is described by the fluid action of Eq. (3.32). Let us add to this action a parabolic potential term which is the commutative analog of the potential term in the CSMM action,

$$S_{\text{para}} = - \frac{eB\tilde{\omega}}{2} \rho_0 \int_0^T dt \int d^2\mathbf{x} \delta_{ab} X^a X^b, \quad (7.14)$$

where  $\rho_0 = \frac{1}{2\pi\theta}$ . The equations of motion which result from Eq. (3.32) plus  $S_{\text{para}}$  are  $\dot{X}^1 = \tilde{\omega} X^2$  and  $\dot{X}^2 = -\tilde{\omega} X^1$ , as well as the constant density constraint enforced by  $A_0$ . For the initial condition  $X^a(0, \mathbf{x}) = x^a$  the solution to these equations can be expressed concisely as

$$X^1(t, \mathbf{x}) + i X^2(t, \mathbf{x}) = (x^1 + i x^2) e^{-i\tilde{\omega} t}. \quad (7.15)$$

Finally, using Eq. (7.6) for the angular momentum, we find that a droplet of radius  $\mathcal{R}$  has angular momentum

$$\begin{aligned} L_{\text{orb}} &= - \int_{|\mathbf{x}| \leq \mathcal{R}} d^2\mathbf{x} \rho_0 \frac{eB}{2} \delta_{ab} X^a X^b \\ &= - \frac{\pi}{4} eB \rho_0 \mathcal{R}^4, \end{aligned} \quad (7.16)$$

where ‘‘orb’’ stands for ‘‘orbital’’ since this angular momentum is associated with an overall rotation of the fluid.

We see that the superextensive term in  $L_0$  is exactly the orbital angular momentum of a classical fluid on a commutative space in a magnetic field undergoing uniform rotational motion. Based on this observation, and using the *anisospin*  $\varsigma = (m-1)/2$  defined earlier, the full angular momentum in the ground state of the CSMM can be written as

$$L_0 = L_{\text{orb}} + \hbar \varsigma N. \quad (7.17)$$

<sup>11</sup>We define the Lagrangian  $\mathcal{L}$  by  $S = \int dt \int d^2\mathbf{x} \rho_0 \mathcal{L}$ .

Now that we have identified the orbital contribution to the total angular momentum the remaining extensive term, which has a coefficient  $\zeta$ , can be interpreted as a spin angular momentum for the  $N$  particles in the fluid, in keeping with the interpretations of Hall viscosity of Refs. [5,7–9,13].

Now that we understand the connection between the expectation value  $\langle \psi_0 | \Lambda^{ab} | \psi_0 \rangle$  and the total angular momentum  $L_0$  of the state  $|\psi_0\rangle$ , we can give a fluid interpretation of the regularization scheme for the guiding center Hall viscosity proposed in Ref. [13]. Specifically, the regularization scheme of Ref. [13] corresponds to subtracting the orbital contribution to  $L_0$ ,

$$\eta_{\text{CSMM,reg}} = \frac{1}{2} \left( \frac{L_0 - L_{\text{orb}}}{A} \right) = \frac{1}{2} \hbar \zeta \rho_0. \quad (7.18)$$

This can be justified by noting that the classical charged fluid in a constant magnetic field and on ordinary commutative space does not exhibit a Hall viscosity,<sup>12</sup> and so the Hall viscosity in the fluid described by the CSMM must only be due to the remaining terms in  $L_0$ , which do not have an interpretation in terms of the classical fluid on a commutative space.

### VIII. HALL VISCOSITY IN THE PRESENCE OF ANISOTROPY

In this section, we introduce a simple modification of the CSMM which incorporates a constant unimodular metric  $g_{ab}$  (i.e., a constant metric with determinant equal to 1). This metric parametrizes an anisotropy or intrinsic geometry of a FQH state, as discussed in the works of Haldane and collaborators [7,8,13,23]. As emphasized by Haldane [7,8], introducing a unimodular metric  $g_{ab}$  into the guiding center part of a FQH state enables one to see the clear separation of the full Hall viscosity tensor  $\eta_{\text{tot}}^{abcd}$  into Landau orbit and guiding center contributions. When such a metric is used in the construction of the guiding center part of a FQH state, the guiding center Hall viscosity tensor  $\eta_H^{ab}$  is modified to be proportional to  $g^{ab}$  (the inverse metric of  $g_{ab}$  with  $g^{ab}g_{bc} = \delta_c^a$ ) instead of  $\delta^{ab}$ . In this section we show that for our modified CSMM, the two-index Hall viscosity tensor  $\eta_{\text{CSMM}}^{ab}$  is also modified to be proportional to  $g^{ab}$ . This confirms that our modification of the CSMM does indeed correspond to incorporating a nontrivial metric  $g_{ab}$  into the definition of the guiding center part of a FQH state. We also note here that the introduction of a second metric (in addition to the metric of space) into the quantum Hall problem is exactly the starting point for the construction of the bi-metric theory of FQH states of Refs. [24,25].

The action for our modified CSMM takes the form

$$S_{\text{CSMM}} = -\frac{eB}{2} \int_0^T dt \text{Tr} \{ \epsilon_{ab} X^a D_0 X^b + 2\theta A_0 + \tilde{\omega} g_{ab} X^a X^b \} + \int_0^T \bar{\Psi}^T (i\dot{\Psi} + A_0 \Psi). \quad (8.1)$$

<sup>12</sup>This can be seen directly by writing down the equations of motion for this classical fluid in the Euler description (i.e., in terms of mass density and velocity fields), and then noting that no viscosity term is present. The Euler equations for a charged fluid in a magnetic field and a general external potential appear, for example, in Eqs. (46) and (47) of Ref. [46].

Note that the only change is the replacement of  $\delta_{ab}$  with  $g_{ab}$  in the quadratic potential term. This is the only part of the action which could conceivably depend on a metric, since the time derivative term already uses the epsilon symbol  $\epsilon_{ab}$  to contract indices. To quantize this system, we make a change to a new set of variables  $\tilde{X}^{\tilde{a}}$ , which diagonalize the potential term but, crucially, obey the same commutation relations as the original variables. In other words, the symplectic form on the phase space of this model takes the same form in the new variables as in the old ones. Therefore the Poisson brackets and quantum commutation relations of the new variables will be identical to those for the old variables.

To describe this change of variables, we decompose the metric and inverse metric in terms of coframes  $e_{\tilde{a}}^a$  and frames  $E_a^{\tilde{a}}$  as

$$g_{ab} = e_{\tilde{a}\tilde{b}}^a e_b^{\tilde{b}}, \quad (8.2a)$$

$$g^{ab} = E_a^{\tilde{a}} \delta^{\tilde{a}\tilde{b}} E_b^{\tilde{b}}. \quad (8.2b)$$

Note that we use new indices  $\tilde{a}, \tilde{b} = 1, 2$  for the internal indices of the frames and coframes. The frames and coframes satisfy the relations  $E_a^{\tilde{a}} e_b^{\tilde{a}} = \delta_b^{\tilde{a}}$  and  $E_a^{\tilde{a}} e_a^{\tilde{b}} = \delta_{\tilde{a}}^{\tilde{b}}$ , which just express the fact that the matrices  $e$  and  $E$  (with entries  $e_{\tilde{a}}^a$  and  $E_a^{\tilde{a}}$ , respectively) are inverses of each other. In addition, it is possible to choose  $\det(e) = \det(E) = 1$ . This can be seen as follows. First, note that the relation between  $g_{ab}$  and  $e_{\tilde{a}}^a$  can be expressed in matrix form as  $g = e^T e$ , where  $g$  is the matrix with entries  $g_{ab}$ . This implies that  $\det(e)^2 = \det(g) = 1$ , so that  $\det(e) = \pm 1$ . However, the parametrization of  $g$  in terms of  $e$  is invariant under the transformation  $e \rightarrow Se$  for any matrix  $S \in O(2)$ , i.e., any  $S$  such that  $S^T S = \mathbb{I}$ . Then if for some reason we found a decomposition of  $g$  with  $\det(e) = -1$ , we can always switch to a new parametrization with  $\det(e) = 1$  by replacing  $e$  with  $Se$  for any  $S \in O(2)$  with  $\det(S) = -1$ . Then, since  $E = e^{-1}$  as matrices, we also guarantee that  $\det(E) = 1$ .

Using the frames and coframes we introduce new matrix variables  $\tilde{X}^{\tilde{a}}$  as

$$\tilde{X}^{\tilde{a}} = e_{\tilde{a}}^a X^a, \quad (8.3a)$$

$$X^a = E_a^{\tilde{a}} \tilde{X}^{\tilde{a}}. \quad (8.3b)$$

In terms of these variables, we have

$$g_{ab} X^a X^b = \delta_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} \tilde{X}^{\tilde{b}} \quad (8.4)$$

and, crucially,

$$\begin{aligned} \epsilon_{ab} X^a D_0 X^b &= \epsilon_{ab} E_a^{\tilde{a}} E_b^{\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}} \\ &= \det(E) \epsilon_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}} \\ &= \epsilon_{\tilde{a}\tilde{b}} \tilde{X}^{\tilde{a}} D_0 \tilde{X}^{\tilde{b}}. \end{aligned} \quad (8.5)$$

We can then carry out the quantization of this modified CSMM using the  $\tilde{X}^{\tilde{a}}$  variables in exactly the same way that we quantized the original CSMM in Sec. IV. For example, we would start by expanding the  $\tilde{X}^{\tilde{a}}$  in terms of a new set of real scalar variables  $\tilde{x}_A^{\tilde{a}}$  ( $A = 0, \dots, N^2 - 1$ ) exactly as in Eq. (4.11). This procedure results in a new ground state  $|\tilde{\psi}_0\rangle$  for the modified CSMM depending on the unimodular metric  $g_{ab}$ .

We can now calculate the Hall viscosity in this modified CSMM. The setup for this calculation is the same as in Sec. V and, in particular, we still apply an APD (or strain) to the physical position variables  $X^a$  and not the new variables  $\tilde{X}^{\tilde{a}}$ . The final expression for the two-index Hall viscosity tensor  $\eta_{\text{CSMM}}^{ab}$  is now proportional to the expectation value of the strain generators  $\Lambda^{ab}$  in the ground state  $|\tilde{\psi}_0\rangle$  of the modified CSMM,

$$\eta_{\text{CSMM}}^{ab} = -\frac{\hbar}{A} \langle \tilde{\psi}_0 | \Lambda^{ab} | \tilde{\psi}_0 \rangle. \quad (8.6)$$

The expectation value  $\langle \tilde{\psi}_0 | \Lambda^{ab} | \tilde{\psi}_0 \rangle$  is easily computed by writing  $\Lambda^{ab} = E_a^a E_b^b \tilde{\Lambda}^{\tilde{a}\tilde{b}}$ , where

$$\tilde{\Lambda}^{\tilde{a}\tilde{b}} = \frac{1}{4\ell_B^2} \sum_{A=0}^{N^2-1} \{ \tilde{x}_A^{\tilde{a}}, \tilde{x}_A^{\tilde{b}} \} \quad (8.7)$$

are the strain generators for the new variables, and by noting that

$$\langle \tilde{\psi}_0 | \tilde{\Lambda}^{\tilde{a}\tilde{b}} | \tilde{\psi}_0 \rangle = \frac{1}{2} \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right] \delta^{\tilde{a}\tilde{b}}, \quad (8.8)$$

which follows since all quantities here are in terms of the new ‘‘tilde’’ variables. Then the original expectation value of interest evaluates to

$$\begin{aligned} \langle \tilde{\psi}_0 | \Lambda^{ab} | \tilde{\psi}_0 \rangle &= \frac{1}{2} \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right] \delta^{\tilde{a}\tilde{b}} E_a^a E_b^b \\ &= \frac{1}{2} \left[ \frac{1}{2} m N^2 + \left( \frac{1-m}{2} \right) N \right] g^{ab}. \end{aligned} \quad (8.9)$$

After regularization, which consists of subtracting off the order  $N^2$  term in this expectation value, the Hall viscosity tensor for the modified CSMM takes the form

$$\eta_{\text{CSMM,reg}}^{ab} = -\frac{\hbar}{A} \frac{1}{2} \left( \frac{1-m}{2} \right) N g^{ab} = \eta_{\text{CSMM,reg}} g^{ab}, \quad (8.10)$$

where  $\eta_{\text{CSMM,reg}} = \frac{1}{2} \hbar \epsilon \rho_0$  as before, and where we defined  $\rho_0 = \frac{N}{A}$ . We find that the Hall viscosity tensor for the modified CSMM is exactly the guiding center part of the Hall viscosity tensor of the Laughlin state with nontrivial guiding center metric  $g_{ab}$  [8,13].

We close this section by calculating the area  $A$  and the shape of the droplet of fluid described by the ground state  $|\tilde{\psi}_0\rangle$  of the modified CSMM. To do this, we follow the method from the end of Sec. IV and consider the eigenvalue of  $\text{Tr}\{g_{ab} X^a X^b\}$  when acting on the state  $|\tilde{\psi}_0\rangle$ . We again find that  $\text{Tr}\{g_{ab} X^a X^b\} |\tilde{\psi}_0\rangle = R^2 |\tilde{\psi}_0\rangle$  with the same eigenvalue  $R^2$  from Eq. (4.43), and we can again interpret  $R^2$  as a sum of contributions from  $N$  particles,  $R^2 = \sum_{j=1}^N R_j^2$  with  $R_j^2 = 2\ell_B^2 (m(j-1) + \frac{1}{2})$ . However, the interpretation of the shape of the droplet is different now since  $g_{ab} X^a X^b$  is a general quadratic form of the noncommutative position coordinates. In the simple case where  $g_{ab} = \delta_{ab}$ , we argued that the droplet was circular, with the  $j$ th particle located somewhere on a circle of radius  $R_j$ . In this case, we will argue that the droplet has the shape of an ellipse, with the particular geometry of the ellipse determined by the eigenvectors and eigenvalues of the metric  $g_{ab}$  considered as a matrix, and where the  $j$ th particle is now

located somewhere on an ellipse whose size is determined by  $R_j$ .

To facilitate this analysis, we use a convenient parametrization [23] of the unimodular metric  $g_{ab}$  in terms of a single complex parameter  $\gamma \in \mathbb{C}$ ,  $|\gamma| < 1$ , and write

$$g = \frac{1}{1-|\gamma|^2} \begin{pmatrix} (1+\gamma)(1+\bar{\gamma}) & i(\gamma-\bar{\gamma}) \\ i(\gamma-\bar{\gamma}) & (1-\gamma)(1-\bar{\gamma}) \end{pmatrix}. \quad (8.11)$$

If we also write  $\gamma = \tanh(\frac{\alpha}{2})e^{i\beta}$  for real  $\alpha > 0$  and a real phase  $\beta$ , then we find that the matrix  $g$  has the decomposition

$$g = SDS^T \quad (8.12)$$

with

$$S = \begin{pmatrix} \cos(\frac{\beta}{2}) & \sin(\frac{\beta}{2}) \\ -\sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2}) \end{pmatrix} \quad (8.13)$$

and

$$D = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix}. \quad (8.14)$$

Here,  $e^{\pm\alpha}$  are the eigenvalues of  $g$  and the columns of the matrix  $S$  are the normalized eigenvectors of  $g$ . In component form we can also write

$$g_{ab} = S_a^{\tilde{a}} D_{\tilde{a}\tilde{b}} S_b^{\tilde{b}}, \quad (8.15)$$

where for  $S_a^{\tilde{a}}$ ,  $a$  indexes the rows of the matrix  $S$  and  $\tilde{a}$  indexes the columns.

We now introduce new noncommutative coordinates (i.e., matrices)  $Y^{\tilde{a}}$  defined as

$$Y^{\tilde{a}} = S_a^{\tilde{a}} X^a, \quad (8.16)$$

and in terms of these, we have

$$\begin{aligned} g_{ab} X^a X^b &= D_{\tilde{a}\tilde{b}} Y^{\tilde{a}} Y^{\tilde{b}} \\ &= e^\alpha (Y^1)^2 + e^{-\alpha} (Y^2)^2. \end{aligned} \quad (8.17)$$

We now see that in the modified CSMM with metric  $g_{ab}$ , we can interpret the  $j$ th particle as residing on an ellipse with the lengths of the minor and major axes of that ellipse given by  $r_{1,j} = e^{-\frac{\alpha}{2}} R_j$  and  $r_{2,j} = e^{\frac{\alpha}{2}} R_j$ <sup>13</sup>. Furthermore, this ellipse has its minor and major axes lined up with the axes of the  $Y^{\tilde{a}}$  coordinate system, which is rotated from the  $X^a$  coordinate system by an angle of  $\frac{\beta}{2}$  as shown in Fig. 1. The area of the ellipse where the  $j$ th particle is located is  $\pi r_{1,j} r_{2,j} = \pi R_j^2$ , and since  $R_j^2$  is linear in  $j$ , we again find that the particle density is constant inside the droplet. Finally, the area of the droplet is equal to the area of the ellipse for particle  $N$ , which is  $A = \pi R_N^2 \approx 2\pi \ell_B^2 m N$ , just as in the ordinary CSMM.

We conclude that the modified CSMM incorporating the unimodular metric  $g_{ab}$  describes an elliptical droplet of fluid with the same area  $A$  and constant density  $\rho_0$  as the ordinary CSMM, and where the details of the shape of the ellipse are determined by the eigenvalues and eigenvectors of the metric

<sup>13</sup>Recall that the equation  $a^2 x^2 + b^2 y^2 = R^2$  describes an ellipse in the  $(x, y)$  plane with the lengths of the two axes of the ellipse given by  $\frac{R}{a}$  and  $\frac{R}{b}$ .

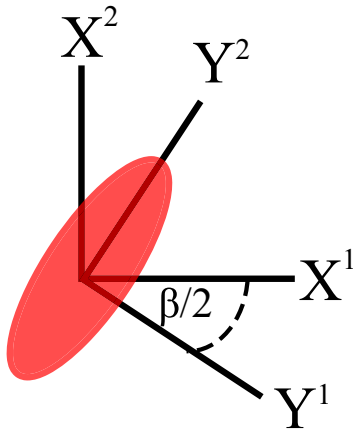


FIG. 1. The shape and orientation of the droplet of fluid, which is described by the ground state  $|\tilde{\psi}_0\rangle$  of the modified CSMM incorporating the unimodular spatial metric  $g_{ab}$ .

$g_{ab}$ . In addition, since the density  $\rho_0$  is the same as for the original CSMM, we find that the coefficient  $\eta_{\text{CSMM,reg}} = \frac{1}{2}\hbar\zeta\rho_0$  of Hall viscosity for the CSMM with  $g_{ab} \neq \delta_{ab}$  is numerically equal to the coefficient for the case where  $g_{ab} = \delta_{ab}$ . The only difference between these two cases is the structure of the Hall viscosity tensor, since for  $g_{ab} \neq \delta_{ab}$  the two index tensor  $\eta_{\text{CSMM,reg}}^{ab}$  is proportional to  $g^{ab}$  instead of  $\delta^{ab}$ .

## IX. CONCLUSION

In this paper, we investigated the geometric properties of the Laughlin FQH states within the CSMM description of these states which, roughly speaking, models these states as a charged fluid in a magnetic field and propagating on a noncommutative space. We focused our attention on the specific properties of Hall viscosity, Hall conductance in a nonuniform electric field, and the Hall viscosity in the presence of anisotropy. We found that the answers for these quantities calculated from the CSMM description contain only the *guiding center* contribution to the known answers for these quantities in the Laughlin states.

These results lead us to the general conclusion that the CSMM description of the Laughlin FQH states accurately captures the guiding center contribution to the geometric properties of these states, but lacks the Landau orbit contribution. As we remarked in Introduction, the Landau orbit contribution is often considered to be a trivial contribution since the interesting correlations in the Laughlin state are contained in the guiding center part of its wave function/state vector. Therefore we find that the CSMM description captures the most important contribution, namely the guiding center contribution, to the physics of the Laughlin FQH states. However, any attempt to completely describe the Laughlin states using the CSMM or NCCS theory must also include some auxiliary degrees of freedom which account for the missing Landau orbit contributions to the geometric properties of these states.

There are several possible directions for future work in this area. One direction would be to continue to develop the fluid interpretation of the CSMM. One goal of this work would be to find an appropriate definition of a density operator

$\rho(\mathbf{x})$ , which is a function of a commutative two-dimensional coordinate  $\mathbf{x} \in \mathbb{R}^2$  and which is defined on length scales much larger than the scale set by  $\theta$  in the noncommutative theory. One could then check whether this density operator satisfies the Girvin-Macdonald-Platzman algebra, and also attempt to compute the static structure factor and compare to the known answer for the Laughlin states [47]. Another goal of this work would be to connect the CSMM description of the Laughlin states with a different fluid description of these states, which is Wiegmann's vortex fluid description [48]. In this description, the Laughlin FQH state with  $N$  electrons is modeled as a rotating incompressible fluid containing  $N$  point vortices each carrying a quantized circulation  $\Gamma$ , which depends on the filling fraction of the Laughlin state. On this topic we note that Bettelheim has recently introduced a method for defining density and velocity fields in the CSMM, which are functions of a commutative coordinate  $\mathbf{x}$  in Ref. [49], and it would be interesting to develop his approach further and to use it to connect with Wiegmann's vortex fluid description. We also note that the problem of defining density operators in NCCS theory and the CSMM has been considered before in Refs. [34,36].

A second direction for future work would be to investigate the Hall viscosity and other geometric response properties in matrix models which describe other more complicated FQH states. For example, a matrix model for the Jain states [50] has been proposed in Ref. [51]. More recently, the authors of Ref. [45] proposed a class of matrix models for the *Blok-Wen* series of non-Abelian FQH states [52]. It would also be interesting to search for new matrix models which can describe other FQH states of interest.

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## APPENDIX A: QUANTUM GENERATORS OF THE $U(N)$ ACTION

In this appendix, we consider the form of the quantum generators of the  $U(N)$  transformations of the matrix model variables  $X^a$  and  $\Psi$ . We use this result in Sec. IV to show that the constraint of Eq. (4.6) simply forces physical states in the CSMM to be singlets under the  $SU(N)$  action, and to carry a certain total charge under the  $U(1)$  action. This information is sufficient to write down a basis of physical states (states respecting the constraint) for the model following Ref. [31].

We start with the generators for the  $U(N)$  transformation of the complex vector variable  $\Psi$ . Under a  $U(N)$  transformation by a matrix  $V$  we have  $\Psi \rightarrow V\Psi$  or in components

$$\Psi^j \rightarrow V^j_k \Psi^k. \quad (\text{A1})$$

We are interested in the infinitesimal form of this transformation, so we take  $V = e^{iT}$  for a Hermitian matrix  $T$  [the Lie algebra of the group  $U(N)$  consists of the  $N \times N$  Hermitian matrices]. Then, to first order in  $T$ , we have  $\Psi \rightarrow \Psi + iT\Psi$ . In components, the first-order change in  $\Psi^j$  generated by  $T$  is

$$\delta_T \Psi^j = iT^j_k \Psi^k. \quad (\text{A2})$$

We now look for a quantum operator  $\mathcal{O}_\Psi(T)$  such that

$$[\mathcal{O}_\Psi(T), \Psi^j] = iT^j_k \Psi^k, \quad (\text{A3})$$

i.e., the quantum commutator of  $\mathcal{O}_\Psi(T)$  with  $\Psi^j$  implements the infinitesimal  $U(N)$  action generated by  $T$  (this is what we mean when we say that a quantum operator *generates* the  $U(N)$  action). The correct operator is (in terms of  $b^j$  instead of  $\Psi^j$ )

$$\mathcal{O}_\Psi(T) = -ib_j^\dagger T^j_k b^k. \quad (\text{A4})$$

Thus  $\mathcal{O}_\Psi(T)$  is the quantum operator which generates the  $U(N)$  transformation  $V = e^{iT}$  acting on  $\Psi$ . One can also check that the operators  $\mathcal{O}_\Psi(T)$  obey the Lie algebra of  $U(N)$ . To check this it is sufficient to check that the map  $T \mapsto \mathcal{O}_\Psi(T)$  is a Lie algebra homomorphism, i.e., that

$$[\mathcal{O}_\Psi(T_1), \mathcal{O}_\Psi(T_2)] = \mathcal{O}_\Psi(-i[T_1, T_2]_M), \quad (\text{A5})$$

and it is straightforward to verify that this relation holds for our generators  $\mathcal{O}_\Psi(T)$ .

Next, we consider the matrix variables  $X^a$ . Under a  $U(N)$  transformation, we have  $X^a \rightarrow V X^a \bar{V}^T$ . Writing  $V = e^{iT}$  as before, we find that to first order in  $T$  we have  $X^a \rightarrow X^a + i[T, X^a]_M$ . Note that for  $T = \alpha \mathbb{I}$ , i.e., for  $U(1)$  transformations, the matrix variables  $X^a$  are invariant. Therefore we can restrict our attention to  $SU(N)$  transformations for the  $X^a$  variables. We then choose  $T$  to be one of the generators  $T^A$  of  $SU(N)$ , and examine the infinitesimal action of  $V = e^{iT^A}$  on the scalar variables  $x_0^a$  and  $x_A^a$ ,  $A = 1, \dots, N^2 - 1$ , which appear in the expansion of  $X^a$  from Eq. (4.11). We have

$$\begin{aligned} \delta_{T^A} X^a &= i[T^A, X^a]_M = i \sum_{B=1}^{N^2-1} x_B^a [T^A, T^B]_M \\ &= - \sum_{B,C=1}^{N^2-1} x_B^a f^{ABC} T^C. \end{aligned} \quad (\text{A6})$$

From this we read off that  $\delta_{T^A} x_0^a = 0$  (reflecting the invariance under  $U(1)$  transformations), and

$$\delta_{T^A} x_B^a = - \sum_{C=1}^{N^2-1} x_C^a f^{ACB}, \quad B = 1, \dots, N^2 - 1. \quad (\text{A7})$$

We now look for a quantum operator  $\mathcal{O}_X(T^A)$ , which generates this action on the variables  $x_A^a$  ( $A = 1, \dots, N^2 - 1$ ), i.e., an operator which commutes with  $x_0^a$  and satisfies

$$[\mathcal{O}_X(T^A), x_B^a] = - \sum_{C=1}^{N^2-1} x_C^a f^{ACB} \quad (\text{A8})$$

for  $B = 1, \dots, N^2 - 1$ . One can check that the correct operator is (in terms of the oscillator variables  $a_A$ )

$$\mathcal{O}_X(T^A) = \sum_{B,C=1}^{N^2-1} f^{ACB} a_B^\dagger a_C. \quad (\text{A9})$$

This completes the construction of the quantum generators of the  $U(N)$  action on the  $X^a$  and  $\Psi$  variables in the CSMM. This is all the information which is needed to analyze the  $j \neq k$  elements of the CSMM constraint  $G^j_k$  from Eq. (4.28).

## APPENDIX B: KUBO FORMULA APPROACH TO HALL VISCOSITY IN THE CSMM

In this appendix, we use a Kubo formula approach inspired by Ref. [12] to compute the Hall viscosity in the ground state of the CSMM. For this computation we subject the CSMM to a time-dependent APD (or strain) parametrized by  $\alpha_{ab}(t)$  such that the dynamics of the system is described by the time-dependent Hamiltonian

$$H(\alpha(t)) = U(\alpha(t)) H_{\text{CSMM}} U(\alpha(t))^\dagger. \quad (\text{B1})$$

Here the operator  $U(\alpha(t))$  is the APD generator for the CSMM, which we derive in Sec. V of the main text. We also assume that at the time  $t_0$  we have  $\alpha_{ab}(t_0) = 0$  so that  $|\psi(t_0)\rangle = |\psi_0\rangle$ , which is the ground state of the CSMM from Eq. (4.38). As we discussed in Sec. II, the generalized force associated with the APD parametrized by the coefficients  $\alpha_{ab}$  is

$$F^{ab} = - \left. \frac{\partial H(\alpha)}{\partial \alpha_{ab}} \right|_{\alpha=0} = -i[\Lambda^{ab}, H_{\text{CSMM}}]. \quad (\text{B2})$$

To calculate the Hall viscosity, we need to compute the expectation value of the generalized force  $F^{ab}$  in the state  $|\psi(t)\rangle$  of the system, where  $|\psi(t)\rangle$  is the solution to the time-dependent Schrodinger equation

$$H(\alpha(t))|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle. \quad (\text{B3})$$

We now discuss the details of this computation.

First, to set up this problem in a form which is amenable to perturbation theory and the Kubo formula, we make a time-dependent change of states by writing

$$|\psi(t)\rangle = U(\alpha(t))|\phi(t)\rangle. \quad (\text{B4})$$

The state  $|\phi(t)\rangle$  is then the solution to a time-dependent Schrodinger equation with a new Hamiltonian  $H'(t)$  given by

$$H'(t) = H_{\text{CSMM}} + V(t) \quad (\text{B5})$$

with

$$\begin{aligned} V(t) &= -i\hbar U(\alpha(t))^\dagger \frac{\partial U(\alpha(t))}{\partial t} \\ &\approx \hbar \frac{\partial \alpha_{ab}(t)}{\partial t} \Lambda^{ab} + \dots, \end{aligned} \quad (\text{B6})$$

where in the second line we expanded the perturbation  $V(t)$  to first order in  $\alpha_{ab}(t)$ . The new Hamiltonian  $H'(t)$  is now expressed as a time-independent term  $H_{\text{CSMM}}$  plus a

time-dependent perturbation  $V(t)$ , and is therefore in a form<sup>14</sup> which is amenable to an application of standard linear response theory.

To compute the Hall viscosity, we naively want to compute the expectation value of  $F^{ab}$  in the state  $|\psi(t)\rangle$ . However, in Ref. [12], the authors argued that one should instead compute the expectation value of  $U(\alpha(t))F^{ab}U(\alpha(t))^\dagger$ , which is equivalent to expressing the generalized force  $F^{ab}$  in terms of the strained coordinates  $U(\alpha(t))x_A^a U(\alpha(t))^\dagger$  instead of the original coordinates  $x_A^a$  of the CSMM (in the language of Ref. [12], we express the generalized force in terms of the “ $\mathbf{X}$ ” variables as opposed to the unstrained “ $\mathbf{x}$ ” variables). The reason for this is as follows. We view the APD parametrized by  $\alpha_{ab}(t)$  as an active transformation (i.e., we physically deform the fluid/CSMM), and so in the computation of the response to this APD we should use the generalized force expressed in terms of the coordinates of the deformed system. Now we have

$$\langle \psi(t) | U(\alpha(t)) F^{ab} U(\alpha(t))^\dagger | \psi(t) \rangle = \langle \phi(t) | F^{ab} | \phi(t) \rangle, \quad (\text{B7})$$

and so it remains to compute the expectation value  $\langle \phi(t) | F^{ab} | \phi(t) \rangle$ .

In interaction picture perturbation theory in the strength of the potential  $V(t)$ , the expectation value of any time-independent operator  $A$  in the state  $|\phi(t)\rangle$  is given by the standard Kubo formula as

$$\begin{aligned} & \langle \phi(t) | A | \phi(t) \rangle - \langle \phi(t_0) | A | \phi(t_0) \rangle \\ &= -\frac{i}{\hbar} \int_{t_0}^t dt' \langle \phi(t_0) | [A_I(t), V_I(t')] | \phi(t_0) \rangle + \dots, \end{aligned} \quad (\text{B8})$$

where  $A_I(t) = e^{i\frac{H_{\text{CSMM}}(t-t_0)}{\hbar}} A e^{-i\frac{H_{\text{CSMM}}(t-t_0)}{\hbar}}$  is in the interaction picture defined by evolution with  $H_{\text{CSMM}}$ , and likewise for  $V_I(t') = e^{i\frac{H_{\text{CSMM}}(t'-t_0)}{\hbar}} V(t') e^{-i\frac{H_{\text{CSMM}}(t'-t_0)}{\hbar}}$ . Note also that for any time-independent  $A$  we have  $A_I(t_0) = A$ , and we also have  $|\phi(t_0)\rangle = |\psi(t_0)\rangle = |\psi_0\rangle$ .

For the application to the calculation of the Hall viscosity, we set  $A = F^{ab}$  and keep only the term in  $V(t)$ , which is linear in the parameters  $\alpha_{ab}(t)$ . This yields the expression

$$\langle F^{ab} \rangle_t - \langle F^{ab} \rangle_{t_0} = -i \int_{t_0}^t dt' \langle [F_I^{ab}(t), \Lambda_I^{cd}(t')] \rangle_{t_0} \frac{\partial \alpha_{cd}(t')}{\partial t'}, \quad (\text{B9})$$

where we used the shorthand notation  $\langle F^{ab} \rangle_t \equiv \langle \phi(t) | F^{ab} | \phi(t) \rangle$ , etc. Next, since  $\langle [F_I^{ab}(t), \Lambda_I^{cd}(t')] \rangle_{t_0} = \langle [F_I^{ab}(t-t'+t_0), \Lambda_I^{cd}(t_0)] \rangle_{t_0}$ , this can be rewritten as

$$\langle F^{ab} \rangle_t - \langle F^{ab} \rangle_{t_0} = - \int_{-\infty}^{\infty} dt' \mathcal{X}^{abcd}(t-t') \frac{\partial \alpha_{cd}(t')}{\partial t'}, \quad (\text{B10})$$

where we defined the response function

$$\mathcal{X}^{abcd}(t) = \lim_{\epsilon \rightarrow 0^+} i \Theta(t) \langle [F_I^{ab}(t+t_0), \Lambda_I^{cd}(t_0)] \rangle_{t_0} e^{-\epsilon t}, \quad (\text{B11})$$

<sup>14</sup>The change of basis from  $|\psi(t)\rangle$  to  $|\phi(t)\rangle$  is equivalent to the change from the “ $\mathbf{x}$ ” to the “ $\mathbf{X}$ ” variables in Ref. [12]. We thank Barry Bradlyn for helpful discussions on this point.

and where we also sent  $t_0 \rightarrow -\infty$  in Eq. (B10). Note that in Eq. (B10) the Heaviside function  $\Theta(t-t')$  allows us to extend the upper limit of the integral over  $t'$  to  $+\infty$ , while the presence of the factor  $e^{-\epsilon(t-t')}$  allows us to send  $t_0 \rightarrow -\infty$ .

Next, we perform a Fourier transform<sup>15</sup> and consider the frequency-dependent response function

$$\begin{aligned} \mathcal{X}^{abcd}(\omega) &= \int_{-\infty}^{\infty} dt \mathcal{X}^{abcd}(t) e^{i\omega t} \\ &= \lim_{\epsilon \rightarrow 0^+} i \int_0^{\infty} dt e^{i\omega_+ t} \langle [F_I^{ab}(t+t_0), \Lambda_I^{cd}(t_0)] \rangle_{t_0}, \end{aligned} \quad (\text{B12})$$

where  $\omega_+ = \omega + i\epsilon$ . Now we note that

$$F_I^{ab}(t+t_0) = -i [\Lambda_I^{ab}(t+t_0), H_{\text{CSMM}}] = \hbar \frac{d\Lambda_I^{ab}(t+t_0)}{dt}, \quad (\text{B13})$$

where we used the equation of motion for  $\Lambda_I^{ab}(t+t_0)$  in the interaction picture. Then an integration by parts with respect to  $t$  in the expression for  $\mathcal{X}^{abcd}(\omega)$  yields a “strain-strain” form of the response function  $\mathcal{X}^{abcd}(\omega)$  analogous to Eq. (3.5) of Ref. [12],

$$\begin{aligned} \mathcal{X}^{abcd}(\omega) &= -i\hbar \langle [\Lambda^{ab}(t_0), \Lambda^{cd}(t_0)] \rangle_{t_0} \\ &+ \lim_{\epsilon \rightarrow 0^+} \hbar \omega_+ \int_0^{\infty} dt e^{i\omega_+ t} \langle [\Lambda^{ab}(t+t_0), \Lambda^{cd}(t_0)] \rangle_{t_0}. \end{aligned} \quad (\text{B14})$$

In the case where the unperturbed Hamiltonian has a unique ground state and a finite energy gap, one finds that

$$\begin{aligned} \lim_{\omega \rightarrow 0} \mathcal{X}^{abcd}(\omega) &= -i\hbar \langle [\Lambda^{ab}(t_0), \Lambda^{cd}(t_0)] \rangle_{t_0} \\ &= -i\hbar \langle \psi_0 | [\Lambda^{ab}, \Lambda^{cd}] | \psi_0 \rangle, \end{aligned} \quad (\text{B15})$$

i.e., the first term in Eq. (B14) gives the full response at  $\omega = 0$  [12]. These assumptions (unique ground state and finite energy gap) hold for the CSMM for any finite value of  $\tilde{\omega}$ , and so this formula for the response at  $\omega = 0$  can be applied to the CSMM.<sup>16</sup> We note that this form of the response at  $\omega = 0$  is what one obtains from a Hall viscosity calculation using adiabatic perturbation theory [1,5,9,13].

Finally, we can complete the calculation of  $\langle F^{ab} \rangle_t \equiv \langle \phi(t) | F^{ab} | \phi(t) \rangle$  to lowest order in time derivatives of  $\alpha_{cd}(t)$ . First, after a Fourier transformation (taking  $t_0 \rightarrow -\infty$  in order to do the integration over  $t'$ ), we can write

$$\langle F^{ab} \rangle_t - \langle F^{ab} \rangle_{t_0} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} i\omega \mathcal{X}^{abcd}(\omega) \alpha_{cd}(\omega) e^{-i\omega t}. \quad (\text{B16})$$

<sup>15</sup>Our convention for Fourier transforms is  $f(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$ ,  $f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t}$ .

<sup>16</sup>One should not confuse  $\omega$ , the frequency appearing in the Fourier transform of the response function, with  $\tilde{\omega}$ , which sets the strength of the parabolic potential in the CSMM.



Next, we expand  $\mathcal{X}^{abcd}(\omega)$  about  $\omega = 0$  as

$$\mathcal{X}^{abcd}(\omega) = -i\hbar\langle\psi_0|[\Lambda^{ab}, \Lambda^{cd}]|\psi_0\rangle + \dots \quad (\text{B17})$$

and invert the Fourier transformation to find

$$\langle F^{ab}\rangle_t - \langle F^{ab}\rangle_{t_0} = i\hbar\langle\psi_0|[\Lambda^{ab}, \Lambda^{cd}]|\psi_0\rangle\frac{\partial\alpha_{cd}(t)}{\partial t} + \dots \quad (\text{B18})$$

For a system with an area  $A$  ( $A = 2\pi\ell_B^2 mN$  for the CSMM with  $\theta = \ell_B^2 m$ ), we then find that the Hall viscosity tensor is given by

$$\eta_{\text{CSMM}}^{abcd} = \frac{i\hbar}{A}\langle\psi_0|[\Lambda^{ab}, \Lambda^{cd}]|\psi_0\rangle, \quad (\text{B19})$$

and this tensor encodes the linear response of the ‘‘generalized stress’’  $\frac{F^{ab}}{A}$  to the ‘‘rate of strain’’ given by  $\frac{\partial\alpha_{cd}(t)}{\partial t}$ .

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*Correction:* The sans serif capital Greek letter Lambda did not render properly and has been fixed.