Dissipation in quantum turbulence in superfluid ^{4}He above 1 K

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There are two commonly discussed forms of quantum turbulence in superfluid ^{4}He above 1 K: in one there is a random tangle of quantized vortex lines, existing in the presence of a nonturbulent normal fluid; in the second there is a coupled turbulent motion of the two fluids, often exhibiting quasiclassical characteristics on scales larger than the separation between the quantized vortex lines in the superfluid component. The decay of vortex line density, L, in the former case is often described by the equation $dL/dt = -\chi_2(\kappa/2\pi)L^2$, where κ is the quantum of circulation and χ_2 is a dimensionless parameter of order unity. The decay of total turbulent energy, E, in the second case is often characterized by an effective kinematic viscosity, ν', such that $dE/dt = -\nu'\kappa^2 L^2$. We present values of χ_2 derived from numerical simulations and from experiment, which we compare with those derived from a theory developed by Vinen and Niemela. We summarize what is presently known about the values of ν' from experiment, and we present a brief introductory discussion of the relationship between χ_2 and ν', leaving a more detailed discussion to a later paper.

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I. INTRODUCTION

Below about 2.17 K, liquid ^{4}He becomes a superfluid, in which an inviscid irrotational superfluid component coexists with a viscous normal-fluid component [1]. Any vorticity in the superfluid component is confined to quantized vortex lines, each of which carries a single quantum of circulation $\kappa = h/m$, where h is Planck's constant and m is the mass of a He atom [2]. Flow in each of the two fluids can be turbulent. Turbulence in the superfluid component (quantum turbulence) takes the form of an irregular tangle of vortex lines which interact with each other and, through a force of "mutual friction," with the normal fluid [3]. Turbulence in the normal fluid is similar to that in a classical fluid, but modified by the mutual friction. Dissipation, associated with viscosity, plays an important role in classical turbulence, notably in providing a sink where the energy flux in a high Reynolds number Richardson cascade can be absorbed at small length scales. It must play a similarly important role in quantum turbulence, although, as we shall see, dissipative mechanisms are then more complex than in the classical case.

Except at temperatures well below 1 K, where the normal fluid has disappeared, dissipation in the turbulent superfluid component is due, as we shall see, to mutual friction. If we ignore a small transverse (nondissipative) component, the force of mutual friction per unit length of the vortex line can be expressed in terms of a dimensionless parameter α [3]. Except at temperatures very close to the superfluid transition temperature, α is significantly less than unity, with the result that vortex line motion is determined largely by vortex-vortex interactions, with the mutual friction leading to only a relatively slow shrinkage in the total length, L, of vortex line per unit volume. Dissipation in the normal fluid is due to both mutual friction and viscosity.

It is the aim of this paper to discuss these forms of dissipation for two commonly studied types of quantum turbulence (QT), the dissipation being observed in the free decay of the turbulence. QT can be most easily produced by a heat current, which is carried in superfluid helium by a counterflow of the two fluids, and this is the form of QT that was first subject to detailed experimental study [4–6]. It was thought for many years that this *thermal counterflow turbulence* (TCT) involved only the superfluid component, and it took the form of a more or less random vortex tangle, for which the turbulent energy is confined to scales comparable with or less than the average spacing, $\ell = L^{-1/2}$, between the vortex lines. The corresponding energy spectrum, $E_Q(k)$, has the form

$$E_Q(k) = \frac{\rho_s \kappa^2}{4\pi\rho\ell^2 k} f\left(\frac{k\ell}{2\pi}\right), \tag{1}$$

where the function $f(x)$ depends on the precise form of the "random tangle," but it tends to unity for large x and tends rapidly to zero for $x < 1$ [7]. ρ_s/ρ is the superfluid fraction. It was suggested, on dimensional and physical grounds [6], that, when the heat current is switched off, the line density might decay as

$$\frac{dL}{dt} = -\frac{\chi_2 \kappa}{2\pi} L^2, \tag{2}$$

where χ_2 is a dimensionless parameter of order unity. Noting that the energy per unit mass associated with a random tangle of vortex lines is given by

$$E_Q = \int_0^\infty E_Q(k)dk \approx \frac{\rho_s \kappa^2}{4\pi\rho} L \ln\frac{\ell}{\xi_0}, \tag{3}$$

where ξ_0 is the vortex core parameter, we see that the turbulent energy per unit mass would then decay as

$$\frac{dE_r}{dt} = -\nu_v' \kappa^2 L^2, \quad \frac{\nu_v'}{\kappa} = \frac{\chi_2 \rho_s}{8\pi^2\rho} \ln\frac{\ell}{\xi_0}, \tag{4}$$

where ν_v' is an effective kinematic viscosity.

Recent experiments [8,9], based on the use of He_2^* excimer molecules as tracers of the normal-fluid flow, have shown that this form of QT, involving only what we shall call a *random vortex tangle*, exists in TCT only at sufficiently small heat fluxes; at larger heat fluxes, the tangle is accompanied by turbulence in both fluids on scales up to the size of the containing channel. We shall write the resulting energy spectrum as

$$E(k) = E_Q(k) + E_{Cs}(k) + E_{Cn}(k), \tag{5}$$

where $E_Q(k)$ is still given by Eq. (1), $E_{Cs}(k)$ is produced by partial polarization of the vortex lines, and $E_{Cn}(k)$ relates to the turbulent energy in the normal fluid. In the steady state this large-scale turbulence in the two fluids is partially coupled and has an energy spectrum $E(k) \propto k^{-n}$ on scales significantly larger than ℓ, where the exponent n varies with the heat flux but is always larger that the Kolmogorov value [10], 5/3 [8]; the fact that $n > 5/3$ is a reflection of the fact that coupling is incomplete, so there is dissipation on all length scales [9]. After the source of heat is turned off, the heat flux in the channel decays to zero in a time given by a thermal resistor-capacitor (RC) time constant (typically 10 ms). Then the two fluids become fully coupled in a similar time, retaining for a time the k^{-n} energy on large scales. Finally, over a further period of typically 1–10 s, the energy spectrum on large scales evolves into the form expected for a classical inertial-range Richardson cascade; i.e., a Kolmogorov spectrum, $E(k) \propto k^{-5/3}$ [10].

We emphasize three points relating to the fully coupled turbulence: (i) as long as full coupling is maintained, there is no dissipation due to mutual friction; (ii) the large-scale nondissipative motion in the superfluid component is generated by a partial polarization of the vortex lines; and (iii) large-scale motion in the normal component is nondissipative because the viscosity of the normal fluid is sufficiently small. As we shall see more clearly later, dissipation can occur in both fluids on scales comparable with or less than ℓ, which in the superfluid component is due to mutual friction (partial decoupling having occurred), and in the normal component is due a combination of viscosity and mutual friction. Because dissipation on scales of order ℓ is now much more complicated than it is if the turbulence is confined to the superfluid component and to scales of order ℓ, Eq. (4) need no longer apply.

The decay of line density associated with large-scale coupled turbulence was first studied by Stalp *et al.* [11], the coupled turbulence having been generated in the wake of a moving grid. These authors showed that their experimental results could be explained in purely classical terms if it was assumed that there was at all times a Richardson-Kolmogorov cascade [$E(k) \propto k^{-5/3}$], terminated at small scales by dissipation described by the equation

$$\frac{dE_C}{dt} = -\epsilon = -\nu' \kappa^2 L^2, \tag{6}$$

where ν' is another effective kinematic viscosity; E_C is the total quasiclassical turbulent energy, given by integrating $E_{Cs}(k) + E_{Cn}(k)$ over k [the contribution of $E_Q(k)$ to the total energy is small and can be neglected]. Stalp *et al.* argued that Eq. (6) is the analog of the expression $\nu\langle\omega^2\rangle$ for dissipation in classical homogeneous turbulence, where $\langle\omega^2\rangle$ is the mean-square classical vorticity. We emphasize that, although the expressions (4) and (6) for the rate of decay of turbulent energy are similar in form, they relate to different physical situations, and in neither case has there been any truly rigorous discussion of their validity. Furthermore, as we shall discuss later, the two effective kinematic viscosities ν_v' and ν' need not have the same value. In future we shall refer to large-scale coupled turbulence of the type produced by flow through the grid, or in the decay of strongly excited TCT at large times, as *quasiclassical quantum turbulence*.

We remark here that a Kolmogorov energy spectrum can, strictly speaking, apply only to a steady state in which energy is fed in continuously at some large scale D at a rate ϵ; there is then a constant energy flux, equal to ϵ, down an inertial subrange, $2\pi/D \gg k \gg 2\pi/\ell$, within which the energy spectrum has the full Kolmogorov form $E(k) \sim \epsilon^{2/3}k^{-5/3}$ (we are ignoring the effects of intermittency). In decaying turbulence, the energy flux, ϵ, cannot be strictly independent of either time or wave number, so that the Kolmogorov dependence on wave number, $k^{-5/3}$, cannot be strictly correct. In practice, however, most of the energy is often concentrated in the largest eddies (wave numbers close to $2\pi/D$), so that ϵ is independent of k, for $k > 2\pi/D$, to a reasonable approximation, and the decay is sufficiently slow that the Kolmogorov spectrum holds with a slowly decreasing value of ϵ.

Except perhaps for a simple theoretical calculation of χ_2, reviewed later in Sec. III, there has so far been hardly any serious theoretical justification for the two forms of dissipation, and for many years even experimental justification was inadequate. Similarly, it has proved difficult to derive reliable values of the two effective kinematic viscosities from experiment. In the case of ν_v' (or equivalently χ_2), there had been no careful study of the decay of TCT at heat currents sufficiently small that there was no large-scale turbulence. In the other case, values of ν' were obtained from observations of the decay of vortex line density combined with questionable assumptions about the form of the large-scale energy spectrum as it relates to turbulence in a channel of a finite cross section. Only very recently has ν' been determined in a more satisfactory way for the case of decaying TCT [12], although the results have yet to be compared carefully with those obtained solely from the decay of vortex line density. The general aims of this paper are, as far as possible, to remedy these various shortcomings.

The results of our experiments on the decay of a random vortex tangle and our measurements of χ_2 are described in Sec. II. In Sec. III we summarize an existing theory of χ_2, assess its likely validity, and compare its predictions with experiment. In Sec. IV we describe the numerical simulations

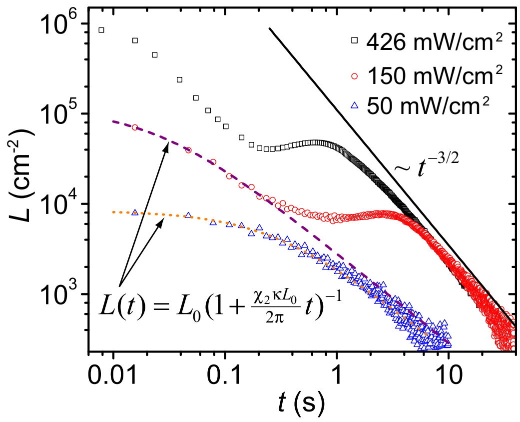

FIG. 1. Observed decays of vortex line density in decaying TCT (1.65 K).

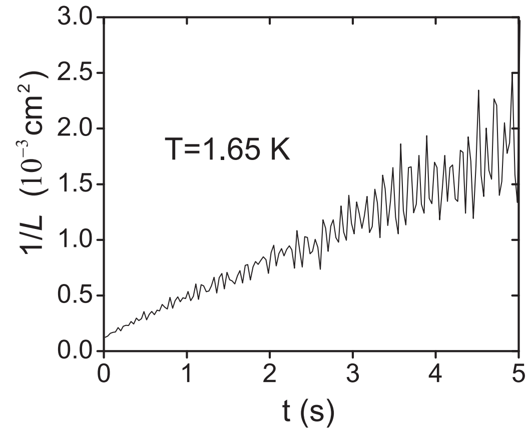

FIG. 2. Observed decay in line density from a small heat flux.

relating to a random vortex tangle, and we compare the results with the experiment and with the theory of Sec. III. In Sec. V we present a critical summary of our present knowledge of the experimental values of the effective kinematic viscosity ν', and in Sec. VI we present a brief introductory theoretical discussion of the relationship between χ_2 (or ν_v') and ν', leaving a more serious discussion of what is actually a difficult problem to a later paper. We present an overall summary of our work in Sec. VII.

II. DISSIPATION IN A RANDOM VORTEX TANGLE: THE EXPERIMENTAL MEASUREMENT OF χ_2

Our experiments on the decay of vortex line density associated with TCT have been based on the observed attenuation of second sound, using what is now a standard technique, as described in, for example, Refs. [5,13]. The actual apparatus is identical with that described in Ref. [8].

As we have explained, the form of decay of the vortex line density given by Eq. (2) can be expected to be observed in the decay of TCT only if the steady heat flux is small enough to ensure that there is no large-scale turbulence. It is evident in the decay shown by the lowest line in Fig. 1 that this is indeed the case.

In Fig. 2, we show data for a decay from a small heat flux plotted in a form, $(1/L)$ versus t, that serves to demonstrate more clearly that Eq. (2) is indeed obeyed.

Values of χ_2 deduced from decays of this type are shown as a function of temperature in Fig. 3.

III. DISSIPATION IN A RANDOM VORTEX TANGLE: A THEORY OF χ_2

In this section, we shall summarize a theory of χ_2 that was proposed by Vinen and Niemela [3], and we shall compare the results with experiment. We assume that the force of mutual friction per unit length of the vortex line is given by

$$\mathbf{f} = -\rho_s\kappa\alpha\hat{\kappa}\times[\hat{\kappa}\times(\mathbf{v}_n-\mathbf{v}_L)], \tag{7}$$

where $\hat{\kappa}$ is a unit vector along the length of the vortex and $\mathbf{v}_L$ is the velocity with which the vortex moves perpendicular to its length. We have neglected any transverse component of the mutual friction. We shall further assume that during the decay described by Eq. (2), the normal fluid is at rest, apart from the local dragging by a moving vortex that is incorporated into the definition of the mutual friction parameter α [14]. Dissipation is then due entirely to mutual friction. Finally, we shall assume that the magnitude of $\mathbf{v}_L$ is given to a good enough approximation by the local induction approximation

$$v_L = \frac{\kappa}{4\pi R}\ln\left(\frac{R}{\xi_0}\right), \tag{8}$$

where R is the local radius of curvature of the vortex, and ξ_0 is the vortex core parameter. In other words, we have neglected the effect of both long-range interactions and the force of mutual friction itself on the motion of a vortex. It follows that the rate of dissipation of energy per unit mass of helium is given by

$$\frac{dE_r}{dt} = -\frac{\rho_s}{\rho}\kappa\alpha L\langle v_L^2\rangle = -\alpha\left(\frac{\rho_s\kappa^3}{16\pi^2\rho}\right)\left\langle\left[\frac{1}{R^2}\left(\ln\frac{R}{\xi_0}\right)^2\right]\right\rangle L, \tag{9}$$

where $\langle\cdots\rangle$ denotes an average over the vortex tangle. We neglect the slow variation of the logarithmic term with L, setting $R = R_0 \approx \ell$ in that term, and we follow Schwarz [15]

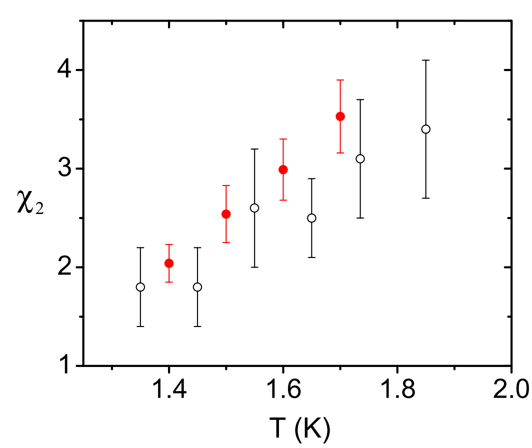

FIG. 3. Observed (open circles) and theoretical (filled circles) values of χ_2, the theoretical values being derived from Eq. (12).

by assuming that

$$\left\langle \left[\frac{1}{R^2} \right] \right\rangle = c_2^2 L, \tag{10}$$

where c_2 depends only on temperature. It follows that

$$\frac{\nu_v'}{\kappa} = \frac{\alpha c_2^2 \rho_s}{16\pi^2 \rho} \left[\ln \frac{\ell}{\xi_0} \right]^2, \tag{11}$$

and therefore

$$\chi_2 = \frac{\alpha c_2^2}{2} \ln \frac{\ell}{\xi_0}. \tag{12}$$

This derivation is based on three assumptions: (i) that, as we have mentioned, there is no motion of the normal fluid; (ii) that the vortex lines form a random tangle; and (iii) that use of the local induction approximation is justified. We shall present an argument in favor of the first assumption in Sec. VI. The other assumptions seem reasonable.

Values of χ_2 derived from Eq. (12) are included in Fig. 3. The required values of c_2 are taken from the simulations of the steady state described in Sec. IV, and values of α are taken from Ref. [16]. We see that within the error bars, there is agreement with experiment.

IV. DISSIPATION IN A RANDOM VORTEX TANGLE: SIMULATIONS RELATING TO χ_2

A brief report of simulations leading to a verification of the form of the decay of line density and to values of χ_2 at a temperature of 1.9 K has already been published [17]. Here we present the results of more detailed studies, covering a range of temperatures, first for the case of spatially uniform flows, and then for flows between solid boundaries.

A. The steady state for spatially uniform flows

For a given temperature, we must first simulate the steady-state counterflow for two reasons. It is from these states that the decays must start, and we can determine whether values of the parameter c_2, obtained for the steady state, lead via Eq. (12) to agreement with experimentally observed values of χ_2.

Our numerical simulation is based on the vortex filament model with the full Biot-Savart integral. We carry out simulations for spatially uniform flows in a cubical box, side 1 mm, with periodic boundary conditions in all directions. We replace the vortex lines by a discrete set of points with minimum spatial resolution $\Delta\xi = 8.0 \times 10^{-4}$ cm. We integrate in time with a fourth-order Runge-Kutta scheme with time resolution $\Delta t = 1.0 \times 10^{-4}$ s. The initial state is a set of randomly oriented vortex loops of radius 0.23 mm. The spatially uniform applied velocities satisfy the condition of no net mass flow $\rho_n v_n + \rho_s v_{s,a} = 0$. We have checked that any contribution to the net superflow from the evolving vortex tangle is negligible in comparison with $v_{s,a}$. The parameters used in the simulations are shown in Table I.

We run the simulations for 20 s. The vortex line density, L, is found to reach a steady average value, L_0, with fluctuations, in about 5 s. The parameter c_2, calculated from Eq. (10) and

TABLE I. Parameters used in numerical simulations.

T (K)	α	α'	v_n (mm s^{-1})
1.4	0.052	0.017	9.0
1.5	0.073	0.018	7.0
1.6	0.098	0.016	6.0
1.7	0.127	0.012	5.0

the equation

$$\left\langle \frac{1}{R^2} \right\rangle = \frac{1}{\Omega L} \int \frac{d\xi}{R^2}, \tag{13}$$

where Ω is the volume of the numerical box, together with the values of χ_2 derived from Eq. (12), are shown as a function of time for a temperature of 1.4 K in Fig. 4. As we see, they too reach steady states after a few seconds, but with significant fluctuations. We have performed similar simulations for several temperatures, the results of which are summarized in terms of time averages in Table II. The computed values of χ_2 in Fig. 3 were taken from Table II. The significant fluctuations to which we have referred have their origin, at least in part, in the relatively small computational box, compared with vortex line spacings; this is confirmed by the observed increase in the percentage fluctuation in χ_2 as the vortex line density is decreased, which is evident for the temperature of 1.4 K in Table II. We also emphasize that the theoretical/computational

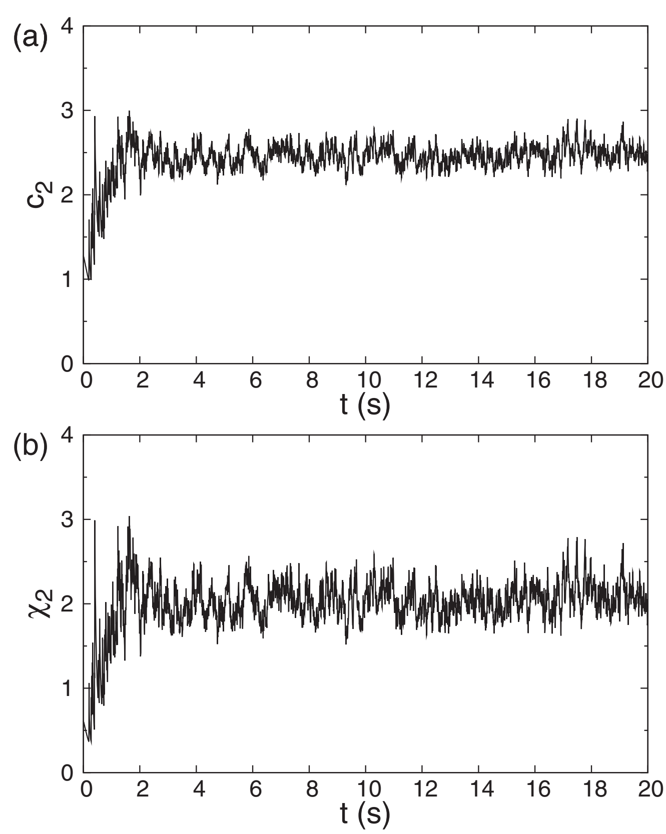

FIG. 4. Value of c_2 derived from simulations of the approach of counterflow to a steady state, and the corresponding value of χ_2 derived from Eq. (12).

TABLE II. Statistically steady values of the vortex line density, L_0, the parameter c_2, the mean radius of curvature, R_0, and the corresponding values of χ_2 derived from Eq. (12).

T (K)	v_n (cm/s)	L_0 (10^3 cm^{-2})	c_2	R_0 (10^{-3} cm)	χ_2
1.4	0.9	6.54 ± 0.30	2.47 ± 0.12	5.03 ± 0.25	2.04 ± 0.19
1.5	0.7	5.80 ± 0.24	2.31 ± 0.13	5.71 ± 0.32	2.54 ± 0.29
1.6	0.6	6.14 ± 0.25	2.16 ± 0.12	5.93 ± 0.31	2.99 ± 0.31
1.7	0.5	6.34 ± 0.30	2.06 ± 0.11	6.12 ± 0.33	3.53 ± 0.37
1.4	0.7	3.59 ± 0.29	2.47 ± 0.21	6.82 ± 0.59	2.10 ± 0.34
1.4	1.1	10.0 ± 0.27	2.44 ± 0.08	4.10 ± 0.13	1.97 ± 0.13

values of χ_2 plotted in Fig. 3 were derived from Table II; the agreement with experiment was therefore evidence that Eq. (12) is at least approximately valid if the values of c_2 are taken from numerical simulations of the steady state. We must now turn to numerical simulations of the decaying turbulence to check whether the simulated decays obey Eq. (2) with values of χ_2 that agree with those in Table II.

B. Decays from spatially uniform flows

In these simulations the applied velocities, v_n and $v_{s,a}$, are turned off at time $t = 0$, and the way in which the line density decays with t is determined. Data are averaged over 30 decays at each temperature.

Figure 5 shows the way in which the simulated line density decays with time at 1.4 K, in the form of a plot of $1/L$ against time.

We see that, in contrast to the corresponding experimental decay (Fig. 2), Eq. (2) is apparently not obeyed; the slope of the plotted line, which ought to be proportional to the constant χ_2, increases markedly with time (the values of χ_2 are also too large). The increase of at times greater than about 1 s may be due in part to the vortex line density becoming too small (the ratio of the line spacing to the spatial period has become greater than about 0.3), and in part to the effect of the logarithmic factor in Eq. (12). A possible explanation of the discrepancy at smaller times is that the parameter c_2 in Eq. (12) changes during the simulated decay. Figure 6(a) shows that c_2 does indeed change during the simulated decay. Furthermore, as we see from Fig. 6(b), this changing c_2 leads via Eq. (12) to a changing value of χ_2 that would lead, at least qualitatively, to a decay curve with the shape shown in Fig. 5. Similar results emerge from simulations at other temperatures.

The fact that the variation with time of the slope of the line in Fig. 5 is indeed due to the variation with time of the parameter c_2 is shown more strikingly in Fig. 7, where we compare on the same graph the time dependence of the value χ_2 derived both by differentiating $1/L$ in Fig. 5 with respect to time and by substituting the value of c_2 from Fig. 6(a) into Eq. (12). Even the random fluctuations of c_2 are reflected to a significant degree in fluctuations in χ_2 derived from Fig. 5. The situation at other temperatures is similar. We conclude then that the theory underlying Eq. (12) is in reasonably good agreement with the results of the simulations, but not, to a significant extent, with experiment. This suggests strongly

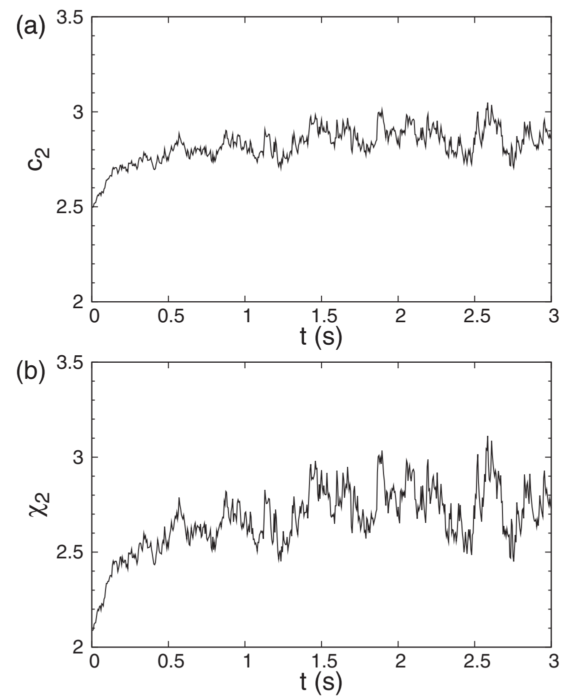

FIG. 6. (a) The variation of the parameter c_2 with time from simulations of the decaying line density at $T = 1.4$ K and $v_n = 9$ mm s^{-1}. (b) The variation of χ_2 with time, obtained by substituting c_2 from Fig. 6(a) into Eq. (12).

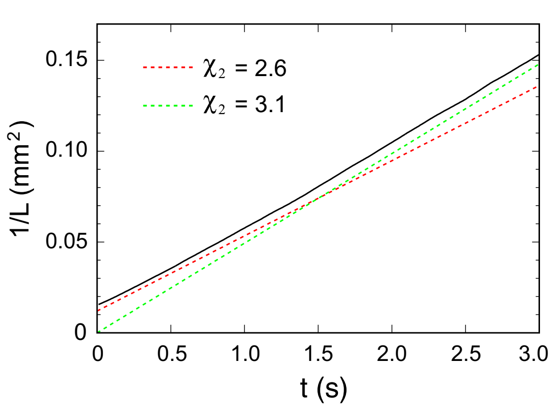

FIG. 5. $(1/L)$ plotted against time from simulations at $T = 1.4$ K and $v_n = 9$ mm s^{-1}.

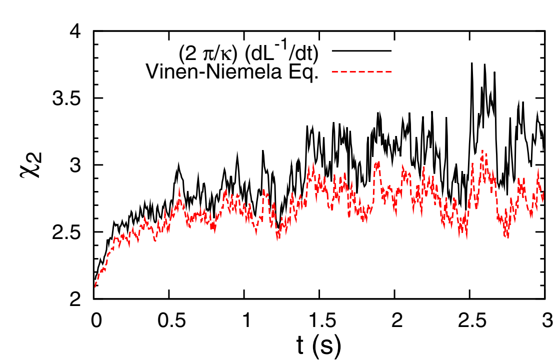

FIG. 7. Plots of χ_2 against time derived as explained in the text.

TABLE III. Parameters analogous to those in Table II for flow between parallel plates.

T (K)	v_n (cm s^{-1})	L_0 (10^3 cm^{-2})	c_2	R_0 (10^{-3} cm)	χ_2
1.4	1.1	5.94 ± 0.64	2.17 ± 0.14	6.03 ± 0.37	1.60 ± 0.20
1.5	0.9	5.67 ± 0.43	2.02 ± 0.14	6.61 ± 0.41	1.97 ± 0.26
1.6	0.8	6.58 ± 0.70	1.84 ± 0.12	6.76 ± 0.39	2.19 ± 0.28
1.7	0.7	6.74 ± 0.67	1.79 ± 0.11	6.86 ± 0.41	2.68 ± 0.33

that some factor relevant to the experiments is missing from both the theory and the simulations. A possible candidate for this factor is the fact that, in contrast to the theory and the simulations, the experiments relate to flow in a channel of a finite cross section. We investigate this possibility in the next subsection.

C. Decays from flows in channels of a finite cross section

Simulations relating to decaying counterflow in two types of channel have been carried out: one is formed between two parallel solid boundaries, separated by 1 mm; the other is a channel with a square (1 mm × 1 mm) cross section, again with solid boundaries. The conditions at the solid boundaries are that the normal fluid velocity vanishes, and that the normal component of the superfluid velocity vanishes. Otherwise, periodic boundary conditions are applied. In the steady state, a parabolic velocity profile in the normal fluid is prescribed. Here we shall present only the results for two parallel boundaries; the results for the channel with a square cross section are broadly similar, but are less clear-cut because of large fluctuations in the line density in the steady state. Data relating to the steady states in the case of the parallel plates are displayed in Table III.

Before we proceed further, we recall that the presence of solid boundaries in the steady state is known from simulations to lead to severe spatial inhomogeneity in the vortex line density [18–21]; the vortex line density is greatly enhanced near the boundary (values of L_0, and other parameters, in Table III are spatial averages). We must therefore enquire whether there is also inhomogeneity in the value of c_2. That there is indeed such inhomogeneity is shown in Figs. 8 and 9, derived from the simulations. We see that the parameter c_2 is strongly reduced in regions where the vortex line density

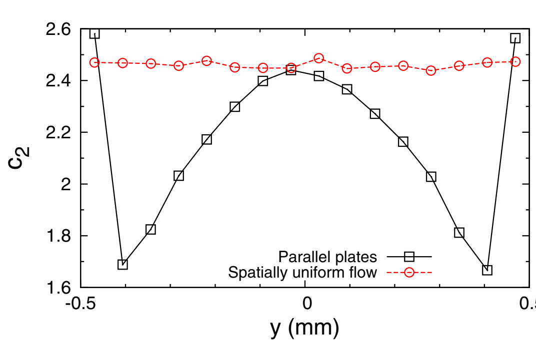

FIG. 8. Plots showing how c_2, averaged over time in the steady state, varies with position across the channel. Blue line: flow between parallel plates; red line: spatially uniform flow. Temperature = 1.4 K.

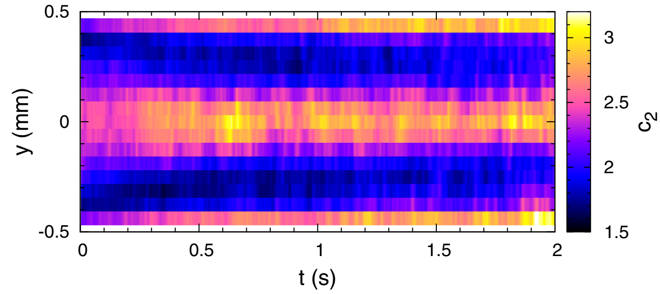

FIG. 9. Diagram showing how c_2 varies with position across the channel and with time during a decay. Temperature = 1.4 K.

is increased, and that this reduction persists in time during a decay.

The experimental observations of χ_2, reported in Sec. II, relate to the decay of spatially averaged vortex line densities. Our simulations of the decays between parallel plates lead to the corresponding values of χ_2 that are displayed in Fig. 10, where they are compared with the predictions of Eq. (12), in which we have substituted spatially averaged values of the parameter c_2 taken from our simulations. We see that the agreement between the simulations and the predictions of Eq. (12) is still good and provides further confirmation that the theory of Sec. III is valid. Furthermore, for times less than 1 s, the variation with time of χ_2 has largely disappeared, and the actual values of χ_2 are in better agreement with experiment. This improved agreement with experiment is comforting and suggests that boundary effects are important in determining values of c_2 and therefore the precise form of the decays. However, reservations must be emphasized. It is now clear that values of c_2 are quite sensitive to the precise form of the flows, and our simulations still relate to flows that are not exactly the same as those in our experiments. The experiments [8] use wider channels; in practice, the velocity profile of the normal fluid differs generally from the Poiseuille form [8], and the vortex lines in the superfluid component are likely to suffer drag or pinning at the solid boundaries. Simulations that take these differences into account are starting to be practicable [22], and they could eventually allow more satisfactory comparison with experiment.

It should be emphasized that, although there can be no *net* mass flow in thermal counterflow, the velocity profiles of the two fluids in a channel of finite cross section can be such that there is a finite mass flow locally. When the heat flux is turned off, this local mass flow may persist and may give rise to large-scale circulations. We see no evidence that such circulations have any significant effect on the form of the decay, although they may be affecting the value of c_2 during the decay.

V. DISSIPATION IN QUASICLASSICAL QUANTUM TURBULENCE: EXPERIMENTAL VALUES OF ν'

We turn now to the decay of large-scale coupled turbulence, as observed in the wake of flow through a grid and in the decay of TCT when the steady heat flux is large. We shall not be concerned with the early stages in these decays. In the case of grid turbulence, it has been supposed [11] that a Kolmogorov spectrum is established quickly, with energy-containing eddies

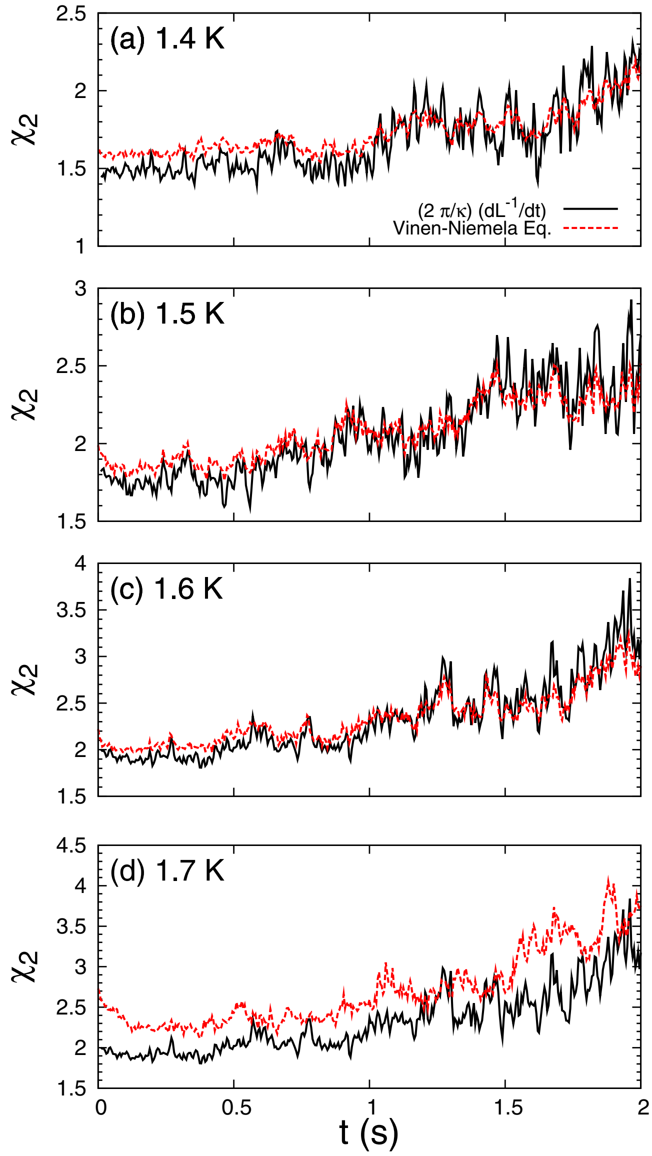

FIG. 10. Plots of χ_2 against time for flow between parallel plates.

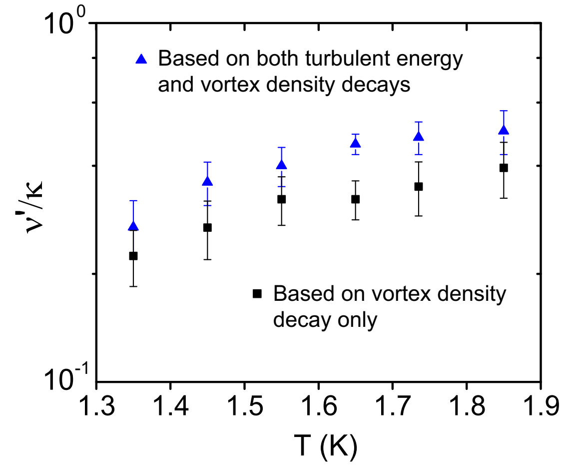

FIG. 11. Values of ν' for decaying TCT derived from measurements of both the decaying turbulent energy and the decaying vortex line density. Values of ν' derived from the decay of line density alone, based on Eq. (14), are included for comparison.

having a size significantly smaller than the channel width; then the energy-containing eddies grow in size, essentially by a classical process (see Ref. [23], p. 347), until their size saturates at a value comparable with the width of the channel. Recent experiments have cast doubt on the supposed details of this evolution of the energy-containing eddies, but, as we shall see, there seems now to be little doubt that eventually the turbulence settles down to a quasisteady state in which the energy-containing eddies have a fixed size, determined by the channel width, and in which there is an inertial subrange characterized by a Kolmogorov energy spectrum, terminated by dissipation described by Eq. (6). In the case of the decay of TCT when the steady heat flux is large, the initial stages are complicated, as we outlined in our Introduction, but again there is little doubt that eventually the turbulence settles down to a state similar to that seen in the decay of grid turbulence.

As was shown first by Stalp *et al.* [11], the decay of vortex line density in the state to which the turbulence settles down is given by

$$L(t) = \frac{(3C)^{3/2} D}{2\pi \kappa \nu'^{1/2}} (t - t_0)^{-3/2}, \tag{14}$$

where C is the Kolmogorov constant, D is the (time-independent) linear size of the energy-containing eddies, and t_0 is a constant. Equation (14) is based on an assumed energy spectrum that has the Kolmogorov form with a simple cutoff for wave numbers less than $2\pi/D$. Until recently, all measurements of the effective kinematic viscosity, ν', have been based on observations of $L(t)$ and the assumption that D is exactly equal to the width of the channel in which the flow is taking place. The questionable assumptions underlying this work meant that the values of ν' were, at best, uncertain to within a factor of perhaps 2 or 3. Furthermore, since the effective size of the energy-containing eddies could depend on the precise way in which the turbulence was generated, values of ν' from different experiments might not agree.

This uncertainty can be circumvented if a measurement of $L(t)$ is combined with a measurement of the way the total turbulent energy decays, since this decay of total energy yields the value of the energy flux, ϵ, in Eq. (6). The time dependence of the total energy can be deduced from the recently developed visualization technique based on the use of He_2^* excimer molecules as tracers, provided that it is assumed that the turbulence is isotropic. The first such study, on the decay of TCT, was reported recently [12], and the resulting values of ν' are displayed in Fig. 11, along with values of ν' derived from the same measurements of the decay of line density, but using Eq. (14) [all these measurements relate to a channel with a square cross section, 9.5 mm × 9.5 mm, and D in Eq. (14) was taken to be 9.5 mm]. We see that the measurements based on our technique are systematically slightly larger than those based on Eq. (14), but only by a factor that is barely outside the experimental error.

In Fig. 12 we collect together the results of measurements of ν' for various types of decaying coupled quantum turbulence, as described in the caption to the figure. Most of these data

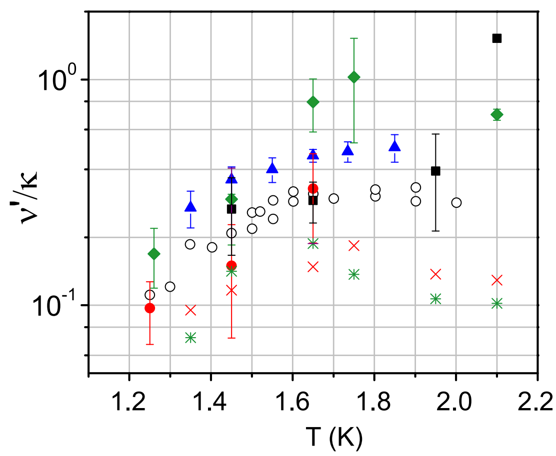

FIG. 12. Values of ν' for various types of decaying coupled turbulence. ▲, Ref. [12]; ○, Ref [11]; ♦, Ref. [13] decay of superflow in channel D7; •, Ref. [13] decay of superflow in channel D10; ■, Ref. [13] decay of counterflow in channel D10; ×, Ref. [24] with no grid; *, Ref. [24] with grid.

were derived from measurements on line density only, and for this reason they are subject to some uncertainty. There is a hint that the value of ν' may depend a little on the type of flow, but the relatively large experimental errors make it hard to be sure. All we can say is that ν'/κ lies in the range 0.1–1, its value increasing as the temperature increases from 1.3 to 2.1 K.

These measurements of ν' have all been based on the decay of the quantum turbulence. Some information about ν' has also been obtained from observations of vortex line density in the steady flow of superfluid ^{4}He through a channel or through a grid [24]. In essence, it was tentatively assumed that the steady flow led to the generation of large eddies, size D and characteristic velocity U. The velocity U is assumed to be proportional to the average steady flow velocity U_0 ($U = \zeta U_0$, where the constant ζ is a little less than unity), and D is assumed to be independent of U_0. The large eddies are assumed to decay through a cascade at a rate determined by the turnover time D/U, the energy being ultimately dissipated at a rate given by Eq. (6). These assumptions lead then to a steady vortex line density given by

$$L = \zeta^{3/2} \frac{1}{(\nu' \kappa D)^{1/2}} U_0^{3/2}. \qquad (15)$$

It is confirmed by experiment that L is proportional to $U_0^{3/2}$. Equation (15) can then be used to estimate ν', subject to reasonable guesses about the values of ζ and D. The results are not inconsistent with those described above, demonstrating that the concept of an effective kinematic viscosity is applicable to dissipation in both steady and decaying turbulence; however, reliable absolute values of ν' cannot be deduced.

VI. DISSIPATION IN QUASICLASSICAL QUANTUM TURBULENCE: THE RELATION BETWEEN ν'_v AND ν'

A. Introduction

We devote this section to an introductory discussion of the relation between ν'_v, derived from our values of χ_2, and ν'. We have already emphasized that these two kinematic viscosities relate to different physical situations, and that they may not therefore be equal.

In the case of ν'_v, we are dealing with a situation in which turbulent energy in the superfluid component is confined to scales of order ℓ or less, in the form of a random vortex tangle, and we assumed in our earlier discussion that there was no turbulent motion of the normal fluid. As we have seen, turbulent energy is then being dissipated by mutual friction, at a rate that is given to a reasonable approximation by the prediction of Eq. (12). In the case of ν', there is again turbulent energy in the superfluid component on scales of order ℓ or less, but this is accompanied by turbulent energy in both fluids at larger scales. On sufficiently large scales, there is strong coupling between the two fluids, and viscosity in the normal fluid can be neglected. There is then a Kolmogorov (inertial range) energy spectrum in this coupled motion, leading to constant fluxes of energy in k space in both the superfluid and normal components (ϵ_s and ϵ_n). We must discuss how this situation changes as the scale of the turbulence moves toward the scale ℓ, i.e., how the energy spectra for the two fluids behave as the wave number approaches $2\pi/\ell$. In connection with dissipation, we need ultimately to answer several questions. How, and at what wave numbers, is turbulent energy in the normal fluid dissipated, remembering that such dissipation can be due to both viscosity and mutual friction? Is there significant dissipation in the superfluid component due to mutual friction at a wave number smaller than $2\pi/\ell$? And how is dissipation in the superfluid component modified, in comparison with that for a random vortex tangle, for wave numbers of order or greater than $2\pi/\ell$, by any motion on those scales of the normal fluid or by the polarization of the tangle required to generate the large-scale turbulence.

B. Guidance from the calculations of Boué *et al.*

These questions can be answered to some degree by appealing to the work of Boué *et al.* [25]. These authors used a two-fluid Sabra shell model, based on modified HVBK equations, to calculate the energy spectra for both the superfluid and normal components. The HVBK equations are course-grained (continuum) equations of motion for the two fluids, and Boué *et al.* modify them by the addition of an effective kinematic viscosity, equal to our ν', to the equation for the superfluid component. Our ν' is indeed an effective kinematic viscosity in the sense that the associated dissipation, equal to $\nu'\kappa^2 L^2$, appears to be analogous to the classical dissipation $\nu\langle\omega^2\rangle$, where $\langle\omega^2\rangle$ is the mean-square classical vorticity. However, this analogy is misleading because our ν' is actually due, at least in part, to mutual friction, so that its effect ought not to be represented by a term of the form $\nu'\nabla^2 v_s$, as assumed by Boué *et al.* Leaving aside this questionable aspect of the analysis by Boué *et al.*, there is still the assumption that course-grained equations of motion can be used. This assumption is probably justified in describing turbulence on scales that are large compared with the vortex line spacing, ℓ, but Boué *et al.* use it on scales as small as the vortex line spacing. It must fail on such small scales, although the precise scale below which it fails noticeably is not clear. We shall return to this point later.

In spite of these shortcomings, it is interesting to examine the conclusions to be drawn from the analysis of Boué *et al.*, especially as they relate to the effect of the finite viscosity of the normal fluid in the range of temperatures in which we are interested [Fig. 1(b) in Ref. [25]]. We find that, at temperatures less than about 1.5 K, the normal fluid is brought to rest by its viscosity on a length scale significantly larger than ℓ, but that the superfluid is then brought to rest only on significantly smaller scales, comparable with ℓ. At first glance this is surprising, because it might be thought that with the normal fluid at rest the superfluid would also be brought to rest by mutual friction. There is, however, a simple explanation. On scales larger than ℓ, there is according to Boué *et al.* a flow of energy in the turbulent superfluid to higher wave numbers in a Richardson cascade. If the normal fluid is at rest, this cascade has associated with it two characteristic times: the turnover time for eddies centered on wave number k, which is of order $\tau_t = (ku)^{-1}$ (where u is the characteristic velocity in these eddies), and the time taken for the energy in these eddies to be dissipated by mutual friction, which is of order $\tau_\gamma = \ell^2/\alpha\kappa$. If $\tau_t \ll \tau_\gamma$, then the mutual friction has little effect. It is easy to show that this condition is indeed satisfied in the cases we are considering.

At temperatures above about 1.5 K, Boué *et al.* show that energy is lost from both the normal component and the superfluid component only on length scales comparable with ℓ. It follows then that at all temperatures relevant to the present study, turbulent energy is lost from the superfluid component only on length scales comparable with the vortex line spacing, ℓ. We emphasize that this conclusion is dependent on the questionable assumption that turbulence in the superfluid component is behaving quasiclassically on scales down to a value close to the vortex line spacing ℓ.

C. Dissipation in the superfluid component

If we accept this assumption, then we can conclude that, even in quasiclassical quantum turbulence of the type we are considering, energy is dissipated in the superfluid component only on length scales comparable with the vortex line spacing ℓ, as is the case when we have only a random vortex tangle. It is therefore tempting to conclude that the dissipation in the superfluid component is still given by the theory of Sec. III. However, two questions must still be addressed. The first relates to the fact that, according to Boué *et al.*, and in contrast to the assumptions in Sec. III, there is motion of the normal fluid on scales of order ℓ, at least at the higher temperatures. But it seems reasonable to assume that on this scale the vortex line velocity given by the local induction approximation, which is dominated by quantum effects, is uncorrelated with the velocity field of the normal fluid, which is essentially classical. In this case, the theory of Sec. III still holds. The second relates to the fact that in our quasiclassical quantum turbulence, the vortex lines must be moving at a velocity that includes a quasiclassical component, corresponding to a large-scale quasiclassical velocity field arising from a partial polarization of the lines. This component is associated with the long-range, nonlocal, interaction of the vortex lines, and the large-scale coupling between the two fluids ensures that this component is not subject to any dissipation by mutual friction. But the fact that the argument of Sec. III is based on the local induction approximation ensures that this component is automatically excluded from any contribution to the dissipation (although the existence of the large-scale motion may influence the value of c_2).

We conclude then that the dissipation in the superfluid component in quasiclassical quantum turbulence may still be given correctly by the theory of Sec. III, subject, of course, to the assumption implicit in the work of Boué *et al.* that turbulence in the superfluid component can be regarded as quasiclassical on all scales larger than ℓ.

D. The total energy dissipation

To obtain the total energy dissipation, we must add the energy dissipated in the normal fluid. We note that at small wave numbers, within the inertial range, where there is complete coupling between the two fluids, the energy fluxes in the normal and superfluid components must be given, respectively, by $\epsilon_n = (\rho_n/\rho)\epsilon$ and $\epsilon_s = (\rho_s/\rho)\epsilon$, where ϵ is the total energy flux. It follows that the effective kinematic viscosity ν', describing the total dissipation, is given by

$$\frac{\nu'}{\kappa} = \frac{\alpha c_2^2}{16\pi^2}\left[\ln\frac{\ell}{\xi_0}\right]^2 = \frac{\chi_2}{8\pi^2}\ln\frac{\ell}{\xi_0}. \tag{16}$$

We emphasize that, as is the case with ν_v', there is a strong dependence on the parameter c_2, to which we shall return.

E. Comparison with experiment

In principle, Eq. (16) serves to predict both the value of ν' and the relation between ν' and χ_2 (or ν_v'). The latest experimental data displayed in Figs. 3 and 11 are, within large experimental errors, more or less consistent with the predicted relation between ν' and χ_2. However, such agreement has in practice little real significance, because all three dissipative coefficients depend on the parameter c_2, the precise value of which depends on the details of the flow concerned. Strictly speaking, our demonstration that c_2 depends on these details has been established by simulations that relate only to particular flows in which the normal fluid is not turbulent, and for which the density of the vortex line is small. These flows rarely correspond to reality, especially when we are dealing with flows at high velocities that involve turbulence in both fluids and high densities of the vortex line. Although the development of simulations that relate to these more general conditions has started, we can be fairly certain that full development will take many years. In the meantime, we must assume that the dependence of c_2 on the details of the flow is quite general. The consequences are particularly serious for the value of ν', since the flows to which ν' is applicable are as yet the least accessible to realistic simulation.

In comparing experiment and theory relating to quasiclassical quantum turbulence, we must also recognize, as we have already emphasized, that the theory is based on the questionable assumption made by Boué *et al.* that the turbulence in the superfluid component behaves classically (in effect that the discrete vortex structure is unimportant) even when the length scale is comparably with ℓ. Perhaps *fully* classical behavior may not be required, but at least there must still be something equivalent to a Richardson cascade, with "eddies" that have lifetimes sufficiently small that they are not damped significantly by mutual friction with a stationary normal fluid. We guess that justification of this assumption can come only from suitable numerical simulations. It is our plan to attempt such simulations in the near future.

VII. SUMMARY AND CONCLUSIONS

We have summarized what we know from experiment about dissipation in quantum turbulence in superfluid ^{4}He above 1 K, the dissipation being described by either the parameter χ_2, applicable to turbulence existing in the superfluid component only on scales comparable with the spacing between the quantized vortex lines ("random tangles"), or the effective kinematic viscosity ν', applicable to quasiclassical quantum turbulence, such as that generated by flow through a grid. Theoretical predictions for these two parameters are discussed, and it is argued that both depend on the dissipative effects of mutual friction, which are in turn dependent on the dimensionless parameter c_2 that relates the mean-square curvature of the vortices to their mean-square separation. This parameter can in principle be obtained from simulations, but it is argued that simulations that are sufficiently realistic are for the most part not yet practicable. To this extent, our understanding of dissipation in quantum turbulence in ^{4}He above 1 K remains incomplete.

We have also drawn attention to the need to investigate more carefully than hitherto the extent to which turbulence in the superfluid component can be treated classically on length scales larger than, but comparable with, the spacing between the vortex lines.

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