Lack of a thermodynamic finite-temperature spin-glass phase in the two-dimensional randomly coupled ferromagnet

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The search for problems where quantum adiabatic optimization might excel over classical optimization techniques has sparked a recent interest in inducing a finite-temperature spin-glass transition in quasiplanar topologies. We have performed large-scale finite-temperature Monte Carlo simulations of a two-dimensional square-lattice bimodal spin glass with next-nearest ferromagnetic interactions claimed to exhibit a finite-temperature spin-glass state for a particular relative strength of the next-nearest to nearest interactions [Phys. Rev. Lett. **76**, 4616 (1996)]. Our results show that the system is in a paramagnetic state in the thermodynamic limit, despite zero-temperature simulations [Phys. Rev. B **63**, 094423 (2001)] suggesting the existence of a finite-temperature spin-glass transition. Therefore, deducing the finite-temperature behavior from zero-temperature simulations can be dangerous when corrections to scaling are large.

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I. INTRODUCTION

The advent of analog quantum annealing machines [1–14] and, in particular, the D-Wave Inc. [15] D-Wave 2X quantum annealer has sparked a new interest in the study of (quasi-) planar Ising spin glasses [16–19] with finite-temperature transitions. While there have been multiple attempts to discern if the D-Wave quantum annealers display an advantage over conventional technologies [20–38], to date there are only a few "success stories" [32,39] where the analog quantum optimizers show an advantage over current conventional silicon-based computers. Recent results [26,32] suggest that problems with a more complex energy landscape are needed to discern if quantum annealers can outperform current digital computers. In particular, the search for salient features in the energy landscape [32], the careful construction of problems with particular features [32,33,38,39], as well as the attempt to induce a finite-temperature spin-glass transition for lattices restricted to the quasi-two-dimensional topologies of the quantum chips [40] have gained considerable attention. The quest for a finite-temperature spin-glass transition in quasi-two-dimensional topologies stems from the interest in creating an energy landscape that becomes more complex and rugged already at finite temperatures, such that thermal (sequential) simulated annealing [41] has a harder time in determining the optimal solution to an Ising-spin-glass-like optimization problem. On the other hand, quantum annealing should, in principle, be able to tunnel through barriers if these are thin enough. We emphasize that the comparison between simulated annealing—a well-known poor optimizer—and quantum annealing is based on the fact that both methods are sequential in nature. Comparisons to state-of-the-art optimization techniques [42] have been performed and shed a more complete light on the current situation.

Here we want to study the thermodynamic properties of a model proposed byLemke andCampbell [43]—later analyzed in much detail in Refs. [44–46]—that might have the desired finite-temperature spin-glass transition and, most importantly, be of a mostly planar topology that can easily be constructed with current superconducting flux qubits. Our results show that, unfortunately, for large enough system sizes the model is in a paramagnetic phase at finite temperatures for a parameter range where it is predicted to be a spin glass. We do note that this would have been surprising, because there is solid evidence that the lower critical dimensions of spin glasses is believed to be between two and three space dimensions [47–49]—a value below which any phase transition to a spin-glass state only occurs at zero temperature.

The paper is structured as follows: In Sec. II we describe the model and numerical details, as well as the current understanding of its properties, followed by results and concluding remarks in Sec. III.

II. MODEL AND NUMERICAL DETAILS

In their letter [43], Lemke and Campbell argue that a finite-temperature spin-glass transition can be induced in two-dimensional planar topologies with next-nearest interactions. To be precise, the model is a two-dimensional square-lattice Ising spin glass with uniform ferromagnetic next-nearest interactions of strength J, in addition to random bimodal nearest-neighbor interactions of strength $\pm\lambda J$. The Hamiltonian of the model is

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j - J \sum_{\langle\langle i,j\rangle\rangle} S_i S_j, \tag{1}$$

where in Eq. (1) $S_i \in \{\pm 1\}$ represent Ising spins on a square lattice with $N = L^2$ sites (L is the linear dimension of the lattice). $J = 1$ are ferromagnetic interactions between next-nearest neighbors (denoted by $\langle\langle i,j\rangle\rangle$) and $J_{ij} = \pm\lambda J$ are nearest-neighbor bimodally distributed spin-glass interactions (denoted by $\langle i,j\rangle$). In our simulations we set $J = 1$. Depending

TABLE I. Simulation parameters and estimates of the stiffness exponent θ and breakup length ℓ for different values of λ. For both values of λ we studied different system sizes L using parallel tempering Monte Carlo. The lowest (highest) temperature simulated is $T_{\min} = 0.4$ ($T_{\max} = 2.8$) with $N_T = 50$ temperature steps. Thermalization is tested by a logarithmic binning; once the last three bins agree within error bars we deem the system to be thermalized. For all systems, this was the case after $N_{\rm sw} = 2^{22}$ Monte Carlo sweeps. Furthermore, $N_{\rm sa}$ samples were computed for each parameter combination. Note that the estimate of θ for $\lambda = 0.50$ is taken from Ref. [46], whereas the value for $\lambda = 0.75$ is estimated from the published data (see text for details).

λ	θ	ℓ	L	$N_{\rm sw}$	$T_{\min}$	$T_{\max}$	N_T	$N_{\rm sa}$
0.50	0.59(8)	45	48	2^{22}	0.4	2.8	50	10^4
			64	2^{22}	0.4	2.8	50	10^4
			96	2^{22}	0.4	2.8	50	10^4
			128	2^{22}	0.4	2.8	50	10^4
0.75	0.23(1)	9	24	2^{22}	0.4	2.8	50	10^4
			32	2^{22}	0.4	2.8	50	10^4
			48	2^{22}	0.4	2.8	50	10^4
			64	2^{22}	0.4	2.8	50	10^4

on the relative strength of the interactions, i.e., the value of λ, Ref. [43] states that a finite-temperature spin-glass transition can be induced in two space dimensions. These results were further expanded in Ref. [45]: A freezing temperature of $T_c \sim 2.1$ exists for $\lambda = 0.5$, a "slightly lower" freezing temperature for $\lambda = 0.7$, and a zero-temperature freezing for $\lambda = 1.5$. We do emphasize that these results were produced by relatively small system sizes. Extensive numerical simulations by Parisi *et al.* [44] find a *crossover* in the critical behavior for large enough system sizes. First, from a seemingly ordered state to a spin-glass-like state, followed by a second crossover to a (possibly) paramagnetic state. This means that the true thermodynamic behavior can only be observed if the system sizes exceed a certain breakup length ℓ.

However, a conclusive characterization of the critical behavior, as well as the λ dependence of the breakup length ℓ were not discussed in detail until the extensive zero-temperature study by Hartmann and Campbell [46]. By computing ground-state configurations for intermediate system sizes and estimating the stiffness exponent that describes the scaling of energy excitations when a domain is introduced into the system, they argue—based on *zero-temperature* estimates of the spin stiffness—that there should be a finite-temperature spin-glass transition for certain values of λ and linear system sizes L that fulfill $L > \ell$. In particular, they estimate that for $\lambda > \lambda_\infty = 0.27(8)$ no ferromagnetic order should be present. Because the breakup length ℓ is large for $\lambda \sim 0.5$ ($\ell \gtrsim 45$), Ref. [46] suggests studying systems with $\lambda = 0.7$ where $\ell \approx 10$. On the other hand, for $\lambda = 0.90$, the stiffness exponent $\theta = 0.09(5)$ is very close to zero. Therefore, in this work we focus on the cases where (i) we can simulate system sizes $L \gg \ell$ and (ii) the stiffness exponent θ is clearly positive, thus implying a finite-temperature phase, i.e., $\lambda = 0.50$ and 0.75. A summary of the properties of the model for these values of λ, as well as the simulation parameters are listed in Table I. The simulations were performed using parallel tempering Monte Carlo [50] combined with isoenergetic cluster updates [51,52]. Note that we determine the estimated value of θ for $\lambda = 0.75$ by performing a linear fit to the data of Ref. [46] (quality of fit ~ 0.58 [53]) and estimate $\theta(\lambda) \approx 1.083(3) - 1.12(4)\lambda$, valid in the window $\lambda \in [0.5, 1.1]$. Furthermore, by inspecting Fig. 7 in Ref. [46], we estimate that the breakup length for $\lambda = 0.75$ is approximately $\ell \approx 9$.

To detect the existence of a spin-glass transition, we measure the Binder cumulant g [54] of the spin-glass order parameter q via

$$g_{\rm q} = \frac{1}{2}\left(3 - \frac{[\langle q^4\rangle]_{\rm av}}{[\langle q^2\rangle]_{\rm av}^2}\right). \tag{2}$$

In Eq. (2), $\langle\cdots\rangle$ represents a thermal average over Monte Carlo steps and $[\cdots]_{\rm av}$ an average over $N_{\rm sa}$ realizations of the disorder (see Table I for details). The spin-glass order parameter q is given by

$$q = \frac{1}{N}\sum_{i=1}^{N} S_i^\alpha S_i^\beta, \tag{3}$$

where "α" and "β" represent two copies of the system with the same disorder. The Binder cumulant is dimensionless and scales as $g_{\rm q} = G[L^{1/\nu}(T - T_c)]$. Therefore, if $T = T_c$, data for different system sizes cross. If, however, there is no transition, data for different system sizes do not cross. To rule out a transition at a temperature not simulated, a finite-size scaling of the data can be used. Finally, we also measure the average of the square of the magnetization $m^2 \equiv [\langle m^2\rangle]_{\rm av}$ with

$$m = \frac{1}{N}\sum_{i=1}^{N} S_i^\alpha. \tag{4}$$

Note that we measure the square of the magnetization because, on average, $m \equiv [\langle m\rangle]_{\rm av} = 0$. Furthermore, the magnetic susceptibility $\chi_{\rm m}$ is related to m^2 via $\chi_{\rm m} = Nm^2$.

III. RESULTS AND CONCLUSIONS

We have performed large-scale Monte Carlo simulations of the Hamiltonian in Eq. (1) for system sizes $L \gg \ell$ and $\lambda = 0.50$ and 0.75. Our results for the Binder cumulant—which should display a crossing if there is a finite-temperature transition—are summarized in Fig. 1. The Binder cumulant for the spin-glass order parameter $g_{\rm q}$ does not show a crossing down to low temperatures for both values of λ studied. In addition, a finite-size scaling of the data for $\lambda = 0.75$ shown in Fig. 2 strongly suggests that $T_c = 0$. Furthermore, the magnetization m^2 as a function of the temperature T decreases with increasing system sizes for both values of λ studied (see Fig. 1). Based on these results, we conclude that the system is in a *paramagnetic* state for both $\lambda = 0.50$ and 0.75 in the thermodynamic limit.

Our results show that the model introduced in Ref. [43] and studied in detail in subsequent publications [44–46] does not exhibit a finite-temperature spin-glass transition in the thermodynamic limit for values of the parameter λ where it is expected to show such behavior. In agreement with the results of Ref. [44], however for larger system sizes and and better statistics, we show that, indeed, the thermodynamic limit is a

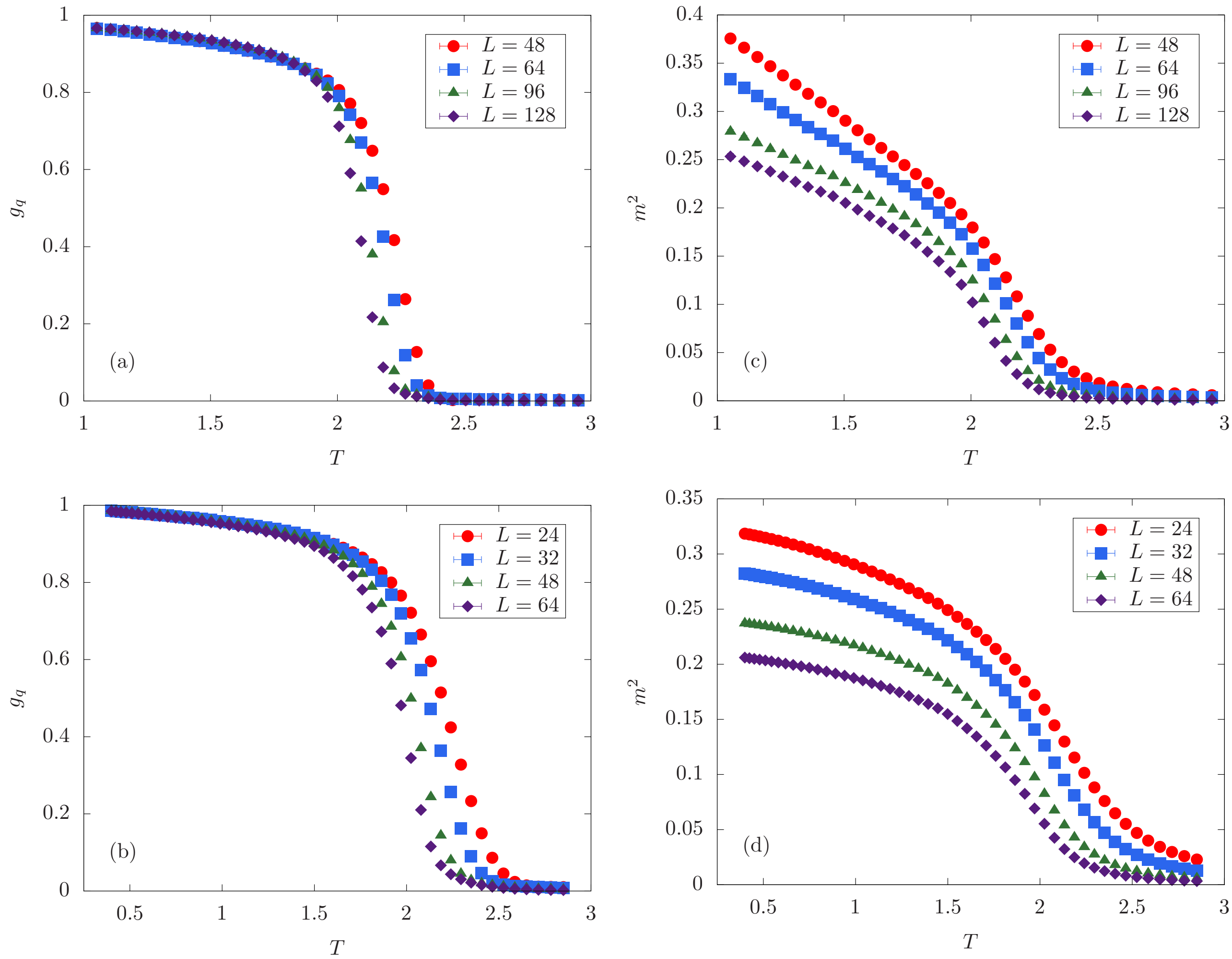

FIG. 1. Binder cumulant g_q for the spin-glass order parameter as a function of the temperature T for the model described in Ref. [43] with $\lambda = 0.50$ (a) and $\lambda = 0.75$ (b) and system sizes $L > \ell$. In both cases the data show no crossing at any finite temperature studied, thus suggesting that there is no finite-temperature spin-glass phase. Square of the magnetization m^2 as a function of T for different system sizes for $\lambda = 0.50$ (c) and $\lambda = 0.75$ (d). The data decreases with increasing system size, i.e., the system is likely in a paramagnetic phase.

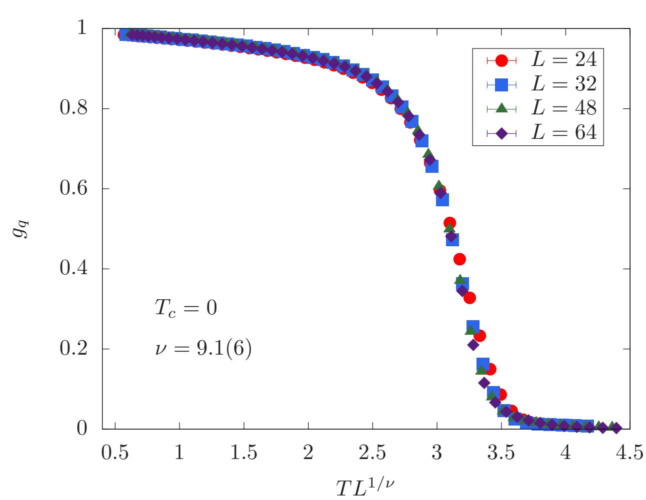

FIG. 2. Finite-size scaling of the data shown in Fig. 1(b) for $\lambda = 0.75$ with $T_c = 0$.

paramagnetic phase at finite temperature. This also means that deducing a finite-temperature behavior from zero-temperature simulations can be dangerous when the system sizes are not in the thermodynamic limit [46]. Given recent interest in inducing finite-temperature spin-glass transitions in quasiplanar topologies [26], we conjecture that adding any set of interactions that do not grow with the system size to a nearest-neighbor lattice will likely not result in a finite-temperature spin-glass transition.

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