Chiral pair of Fermi arcs, anomaly cancellation, and spin or valley Hall effects in Weyl metals with broken inversion symmetry

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Anomaly cancellation has been shown to occur in broken time-reversal symmetry Weyl metals, which explains the existence of a Fermi arc. We extend this result in the case of broken inversion symmetry Weyl metals. Constructing a minimal model that takes a double pair of Weyl points, we demonstrate the anomaly cancellation explicitly. This demonstration explains why a chiral pair of Fermi arcs appear in broken inversion symmetry Weyl metals. In particular, we find that this pair of Fermi arcs gives rise to either "quantized" spin Hall or valley Hall effects, which corresponds to the "quantized" version of the charge Hall effect in broken time-reversal symmetry Weyl metals.

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I. INTRODUCTION

Anomaly cancellation is the mechanism to explain the existence of a gapless surface state, topologically protected [1]. For example, the existence of a chiral edge mode in the integer quantum Hall effect is understood as follows [2]. The chiral edge state suffers gauge anomaly, which means that the U(1) current is not conserved. On the other hand, the Chern-Simons term is not invariant under the gauge transformation in the presence of a boundary. It turns out that the gauge anomaly at the boundary is canceled exactly by the gauge noninvariant term of the Chern-Simons theory in the bulk. As a result, a topological term with a gapless boundary mode consists of a topological field theory consistently.

A Weyl metal state may be regarded as a three-dimensional generalization of an integer quantum Hall phase [3–11]. The Chern-Simons term is replaced with a topological-in-origin $\mathbf{E}\cdot\mathbf{B}$ term. The "axion" θ field corresponding to the Hall conductance in the integer quantum Hall effect is proportional to the displacement from a reference point and its gradient is nothing but an applied magnetic field to describe the momentum-space distance between a pair of Weyl points in the case of time-reversal symmetry breaking. A Fermi arc state corresponds to the chiral edge mode, responsible for the existence of an anomalous Hall effect. As the gauge anomaly from the edge state must be canceled by the gauge noninvariant term from the Chern-Simons term in the integer quantum Hall state, the gauge anomaly from the Fermi arc is also canceled by a gauge noninvariant contribution at the boundary from the inhomogeneous axion term. As a result, the topological-in-origin inhomogeneous θ term with the Fermi arc state gives a consistent "topological" field theory for the broken time-reversal symmetry Weyl metal phase, where contributions from massless Weyl-fermion excitations should be taken into account, of course.

In this study, we extend the anomaly cancellation of a broken time-reversal symmetry Weyl metal state into that of a broken inversion symmetry Weyl metal phase. The minimal model of the broken time-reversal symmetry Weyl metal state is given by a pair of Weyl points, where the momentum-space distance between the pair of Weyl points is the gradient θ proportional to the applied magnetic field. On the other hand, that of the broken inversion symmetry Weyl metal phase is given by a double pair of Weyl points, where the momentum-space distance between each pair of Weyl points is determined by the strength of the inversion symmetry breaking. Based on this minimal model, we demonstrate the anomaly cancellation explicitly. This demonstration explains why a "chiral" pair of Fermi arcs instead of a Fermi arc with definite chirality appear in broken inversion symmetry Weyl metals.

One may point out that the explicit demonstration for the anomaly cancellation in the broken inversion symmetry Weyl metal phase does not give any novel conceptual aspect, compared with that in the broken time-reversal symmetry Weyl metal state. However, we claim that there are no concrete calculations to show the anomaly cancellation in the broken inversion symmetry Weyl metal state. In addition, we emphasize that there exists novel physics in the anomaly cancellation of the broken inversion symmetry Weyl metal phase. Since time-reversal symmetry is preserved, a "quantized" version of the anomalous Hall effect resulting from the Fermi arc cannot appear. Instead, we find that this pair of Fermi arcs give rise to either a "quantized" spin Hall or valley Hall effects, which may be regarded to be a "generalized" version of the two-dimensional quantum spin or valley Hall effect. In this respect, we believe that our explicit demonstration serves as a meaningful reference in understanding the "chiral" pair of Fermi arc states in various inversion symmetry-breaking Weyl metals [12–18].

II. A REVIEW ON THE ANOMALY CANCELLATION IN THE BROKEN TIME-REVERSAL SYMMETRY WEYL METAL STATE

A. An effective minimal model for broken time-reversal symmetry Weyl metals

A minimal model for abroken time-reversal symmetry Weyl metals is given by [19,20]

$$S_{\mathrm{WM}} = \int d^4x \bar{\Psi}(x)(\gamma_0\partial_0 + i\gamma^k\partial_k - \mu\gamma^0 - c_\mu\gamma^\mu\gamma^5)\Psi(x), \tag{1}$$

where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. $c^\mu = (c^0,\mathbf{c})$ is the chiral gauge field, where c^0 is the chiral chemical potential and $\mathbf{c}$ is the momentum-space distance between a pair of Weyl points. μ is the chemical potential. Here, we focus on $\mu = 0$ and $c^0 = 0$.

Introducing $\gamma^4 = -i\gamma^0$ into the above action, we have a simplified form

$$S_{\mathrm{WM}} = \int d^4x \bar{\Psi}(x) i\gamma^\mu(\partial_\mu + iA_\mu + ic_\mu\gamma^5)\Psi(x). \quad (2)$$

Here, we have $\mu = 1,2,3,4$. γ^μ is anti-Hermitian, satisfying $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$. We take into account the U(1) gauge field A_μ.

B. An axion term

The Weyl-metal action (2) suffers chiral anomaly [21], given by

$$\partial_\mu \bar{\Psi}(x) i\gamma^\mu\gamma^5\Psi(x) = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \quad (3)$$

Although it is straightforward to derive this anomaly equation based on the Fujikawa's method [22,23], we show our derivation explicitly in Appendix A in order to clarify the way of regularization. The resulting axionic action is

$$S_{ax} = -\frac{i}{16\pi^2}\int d^4x (c_\mu x^\mu)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \quad (4)$$

C. Surface states

Following Goswami and Tewari [24], we obtain a localized wave function at one surface of the Weyl metal phase. Here, the surface is defined by changing the chiral gauge field $\mathbf{c} \to \mathbf{c}\Theta(x)$, where $\Theta(x)$ is a step function in x coordinate:

$$\psi_{k_y,k_z}(x,y,z) = Ae^{ik_y y + ik_z z}\begin{pmatrix} 1 \\ i \\ -\frac{m}{\sqrt{k_z^2+m^2}-k_z} \\ i\frac{m}{\sqrt{k_z^2+m^2}-k_z} \end{pmatrix} \times e^{(-c\theta(x)+\sqrt{k_z^2+m^2})x}, \quad (5)$$

where $A = \left(\frac{(\sqrt{k_z^2+m^2}-k_z)^2\sqrt{k_z^2+m^2}(c-\sqrt{k_z^2+m^2})}{2c(k_z^2+m^2-k_z\sqrt{k_z^2+m^2})}\right)^{1/2}$ is a normalization constant and $E_{k_y} = k_y$ is an eigenvalue of this surface state. y and z define the surface coordinate and x describe the coordinate perpendicular to the surface. The chiral gauge field $\mathbf{c}$ is given along the z direction. For a state localized near the surface to exist, k_z should satisfy the following condition of $-\sqrt{c^2-m^2} < k_z < \sqrt{c^2-m^2}$. If $m^2 > c^2$ is fulfilled, there are no surface states. It is important to realize that this surface state has definite chirality, given by $\bar{\gamma}\psi_{k_y,k_z} = -\psi_{k_y,k_z}$ with the chirality operator $\bar{\gamma} = \gamma^0\gamma^2$.

D. An effective Hamiltonian for the Fermi arc

Let us now establish an effective Hamiltonian for the Fermi arc state. We introduce a surface projection operator as follows:

$$P_{\mathrm{edge}} \equiv \sum_{k_y}\sum_{-\sqrt{c^2-m^2}<k_z<\sqrt{c^2-m^2}} |\psi_{k_y,k_z}\rangle\langle\psi_{k_y,k_z}|, \quad (6)$$

where we have $\langle x,y,z|\psi_{k_y,k_z}\rangle = \psi_{k_y,k_z}(x,y,z)$. Then, we construct an effective surface Hamiltonian in the following way:

$$\begin{aligned} H_{\mathrm{eff}} &= P_{\mathrm{edge}} H P_{\mathrm{edge}} = \sum_{k_y}\sum_{\tilde{k}_z} |\psi_{k_y,k_z}\rangle k_y \langle\psi_{k_y,k_z}| \\ &= \int dx' \int dy' \int dx \int dy \sum_{k_y}\sum_{\tilde{k}_z} \\ &\quad \times |x,y\rangle\langle x,y|\psi_{k_y,k_z}\rangle k_y\langle\psi_{k_y,k_z}|x',y'\rangle\langle x',y'| \\ &\approx \int dx' \int dy' \int dx \int dy \sum_{k_y}\sum_{\tilde{k}_z} \\ &\quad \times |x,y,k_z\rangle(-i)\partial_y e^{ik_y(y-y')}\delta(x)\delta(x')\langle x',y',k_z| \\ &= \int dy \sum_{\tilde{k}_z} |x=0,y,k_z\rangle(-i)\partial_y\langle x=0,y,k_z|, \end{aligned} \quad (7)$$

where $\tilde{k}_z$ means k_z satisfying $-\sqrt{c^2-m^2} < k_z < \sqrt{c^2-m^2}$. For simplicity, we assumed that the surface wave function is localized perfectly at the surface, i.e., $\langle x,y|\psi_{k_y,k_z}\rangle \sim \delta(x)$. This expression can be translated into

$$H_{\mathrm{eff}} = \sum_{-\sqrt{c^2-m^2}<k_z<\sqrt{c^2-m^2}} \int dy \psi_{k_z}^\dagger(y)(-i)\partial_y\psi_{k_z}(y) \quad (8)$$

in the second quantization language.

E. Gauge anomaly in the Fermi arc state

$(1+1)$ dimensional Dirac theory is given by

$$S = \int d^2x \bar{\Psi}(x)(\gamma_0\partial_0 + i\gamma^1\partial_1)\Psi(x) \quad (9)$$

in the Euclidean signature. Here, we have $\gamma^0 = \sigma^1$ and $\gamma^1 = i\sigma^2$. If we set $\gamma^2 = -i\gamma^0$, we obtain

$$S = \int d^2x \bar{\Psi}(x) i\gamma^\mu\partial_\mu\Psi(x) \quad (10)$$

with $\mu = 1,2$. Here, we have $g^{\mu\nu} = -\delta^{\mu\nu}$.

Let us gauge the above theory with the chiral gauge. Then, we obtain

$$S = \int d^2x \bar{\Psi}(x) i\gamma^\mu(\partial_\mu + ieA_\mu\mathcal{P}_-)\Psi(x), \quad (11)$$

where $\mathcal{P}_- = \frac{1}{2}(1-\bar{\gamma})$ is a projection operator to select the chirality and $\bar{\gamma} = \gamma^0\gamma^1 = i\gamma^2\gamma^1 = -\sigma^3$ is the chirality matrix. Notice that we couple the U(1) gauge field only to the negative chirality sector. Recall that the edge mode in the above section has the negative chirality, i.e., $\bar{\gamma}\phi_{k_y,k_z} = -\phi_{k_y,k_z}$. This effective action is invariant under the particular or "partial" gauge transformation:

$$\begin{aligned} &A_\mu(x) \to A_\mu + \partial_\mu\theta, \\ &\Psi \to e^{i\theta\mathcal{P}_-}\Psi(x), \ \ \bar{\Psi}(x) \to \bar{\Psi}(x)e^{-i\theta\mathcal{P}_+}. \end{aligned} \quad (12)$$

The U(1) gauge current, which is a Noether current resulting from the above partial gauge symmetry, is given by

$$j^\mu = \bar{\Psi}(x)\gamma^\mu P_-\Psi(x). \quad (13)$$

Classically, i.e., in the action level this gauge current is conserved. However, it turns out that this conservation law breaks down in the partition function level because of a quantum correction, referred to as gauge anomaly. This anomaly can be understood perturbatively in the one-loop quantum correction for the gauge-field propagator. See Appendix B for more details. The result is well known [22], given by

$$\partial_\mu j^\mu(x) = \frac{i}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(x) = \frac{i}{8\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{14}$$

One can express the gauge anomaly in terms of an effective action of the U(1) gauge field as follows:

$$\frac{\delta W[A]}{\delta A_\mu} = -\langle\bar{\Psi}(x)\gamma^\mu\mathcal{P}_-\Psi(x)\rangle = -\langle j^\mu\rangle, \tag{15}$$

where the generating function is defined by $Z = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi e^{-S[\bar{\Psi},\Psi,A]} \equiv e^{-W[A]}$. Under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\eta(x)$, the generating function changes in the following way:

$$\begin{aligned}\delta_\eta W[\mathcal{A}] &\equiv W[\mathcal{A}+d\eta] - W[\mathcal{A}]\\ &= \int d^2x\,\partial_\mu\eta(x)\frac{\delta W[A]}{\delta A_\mu}\\ &= -\int d^2x\,\partial_\mu\eta(x)\langle j^\mu\rangle\\ &= \int d^2x\,\eta(x)\frac{i}{8\pi}\epsilon^{\mu\nu}F_{\mu\nu},\end{aligned} \tag{16}$$

where the gauge anomaly equation has been used.

Since $W[\mathcal{A}]$ is an effective action of only one k_z sector, we should include all k_z sectors in order to get the effective surface action of the Weyl metal phase

$$W_{\rm WM}^{\rm edge}[\mathcal{A}] = \sum_{-c<k_z<c} W[\mathcal{A}] = 2cW[\mathcal{A}]. \tag{17}$$

Here, we set $m = 0$ for simplicity. As a result, we find the gauge anomaly of the Fermi arc state in the broken time-reversal symmetry Weyl metal phase

$$\begin{aligned}\delta_\eta W_{\rm WM}^{\rm edge}[\mathcal{A}] &= \frac{ic}{4\pi}\int dtdz\,\eta(x)\epsilon^{\mu\nu}F_{\mu\nu}\\ &= \frac{ic}{2\pi}\int dtdz\,\eta(x)F_{zt}.\end{aligned} \tag{18}$$

Here, we did not take into account the role of disorder scattering for this gauge anomaly contribution. It would be quite an interesting study to investigate the role of disorder scattering for the Fermi arc state.

F. Anomaly cancellation: Callan-Harvey mechanism

Breakdown of the gauge invariance in the effective chiral surface state can be cured by anomaly inflow from the bulk effective action of the Weyl metal phase. This mechanism of anomaly cancellation is known as Callan-Harvey mechanism [1]. The Callan-Harvey mechanism has been already discussed in the broken time-reversal symmetry Weyl metal phase [24]. However, we found a subtle issue for the derivation of the anomaly cancellation. Here, we provide a rigorous derivation for the anomaly cancellation based on the original paper [1].

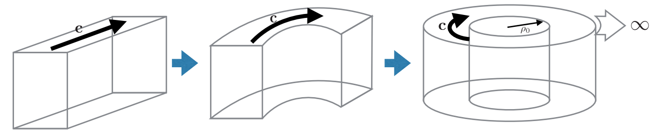

FIG. 1. Geometry of a Weyl metal sample.

First, let us point out the subtle problem. One may start from an effective axionic action Eq. (4) with setting the chiral gauge field as $\mathbf{c} = c\Theta(x_1)\hat{z}$. Here, $\Theta(x_1)$ is the step function. The axion term is

$$\begin{aligned}S_{ax}[A] &= \frac{i}{16\pi^2}\int d^4x\,\mathbf{c}\cdot\mathbf{x}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\\ &= \frac{i}{16\pi^2}\int d^4x\,cx_3\Theta(x_1)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\\ &= -\frac{i}{8\pi^2}\int d^4x\,c\epsilon^{\mu\nu\alpha\beta}[x_3A_\nu F_{\alpha\beta}\delta(x_1)\delta_{1\mu}\\ &\quad + \Theta(x_1)\delta_{3\mu}A_\nu F_{\alpha\beta}],\end{aligned} \tag{19}$$

where $\partial_x\Theta(x) = \delta(x)$ has been used. Under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\eta(x)$, the variation of the effective action ($\delta_\eta S_{ax} \equiv S_{ax}[A_\mu + \partial_\mu\eta] - S_{ax}[A_\mu]$) is given by

$$\begin{aligned}\delta_\eta S_{ax} &= -\frac{ic}{8\pi^2}\int d^4x[\epsilon^{1\nu\alpha\beta}x_3F_{\alpha\beta}\delta(x_1)\\ &\quad + \epsilon^{3\nu\alpha\beta}\Theta(x_1)F_{\alpha\beta}]\partial_\nu\eta\\ &= \frac{ic}{8\pi^2}\int d^4x[\epsilon^{1\nu\alpha\beta}F_{\alpha\beta}(\delta_{\nu3}\delta(x_1) + x_3\partial_1\delta(x_1)\delta_{\nu1})\\ &\quad + \epsilon^{3\nu\alpha\beta}\delta_{\nu1}F_{\alpha\beta}\delta(x_1)]\eta\\ &= 0.\end{aligned} \tag{20}$$

There does not exist the anomaly inflow to cancel the gauge anomaly of the Fermi arc state in this derivation.

In order to resolve this subtle point, we consider a geometry of the Weyl metal sample as shown in Fig. 1. We use the differential form since it is independent of the coordinate system and it is easier to calculate the anomaly inflow. The axion term is represented in the following way:

$$\begin{aligned}W_{\rm WM}^{\rm Bulk}[\mathcal{A},\mathcal{F}] &= -\int_{\mathcal{M}}\frac{i\theta}{4\pi^2}\mathcal{F}\wedge\mathcal{F}\\ &= -\int_{\mathcal{M}}\frac{i\theta}{4\pi^2}d(\mathcal{A}\wedge\mathcal{F})\\ &= -\frac{i}{4\pi^2}\int_{\mathcal{M}}[d(\theta\mathcal{A}\wedge\mathcal{F}) - d\theta\wedge\mathcal{A}\wedge\mathcal{F}]\\ &= \frac{i}{4\pi^2}\int_{\mathcal{M}} d\theta\wedge\mathcal{A}\wedge\mathcal{F},\end{aligned} \tag{21}$$

where $\theta \propto c_\mu x^\mu$ is an "axion" field, $\mathcal{A} = A_\mu dx^\mu$, and $\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. $\mathcal{M}$ denotes an infinite space, where the Weyl metal sample is embedded. The boundary of the Weyl metal sample is defined by the function $\theta(x)$.

Under the gauge transformation $\mathcal{A} \to \mathcal{A} + d\eta$, the variation of the effective bulk action is given by

$$\delta_\eta W_{\rm WM}^{\rm Bulk}[\mathcal{A},\mathcal{F}] = \frac{i}{4\pi^2}\int_{\mathcal{M}} d\theta \wedge d\eta \wedge \mathcal{F} = \frac{i}{4\pi^2}\int_{\mathcal{M}} [-d(\eta d\theta \wedge \mathcal{F}) + \eta d^2\theta \wedge \mathcal{F}] = \frac{i}{4\pi^2}\int_{\mathcal{M}} \eta d^2\theta \wedge \mathcal{F}. \tag{22}$$

In the cylindrical coordinate (ρ,ϕ,z), we can set $\theta(x) = -c\phi\Theta(\rho-\rho_0)$, where $\rho = \rho_0$ represents the boundary of the Weyl-metal sample (Fig. 1). We emphasize that $\theta(x)$ is not a single-valued function. As a result, $d^2\theta \neq 0$. When $\rho > \rho_0$, $\nabla\theta(x) = -\frac{1}{\rho}c\hat{\phi}$. Therefore we have $\oint_{C(\rho>\rho_0)} \nabla\theta(x)\cdot dl = -2\pi c$. However, if $\rho < \rho_0$, we have $\oint_{C(\rho<\rho_0)} \nabla\theta(x)\cdot dl = 0$. These equations are translated into $\nabla \times \nabla\theta(x) = -\frac{c}{\rho}\delta(\rho-\rho_0)\hat{z} = \hat{z}(\partial_x\partial_y - \partial_y\partial_x)\theta$.

Inserting this equation into the above, we find the anomaly inflow from the bulk state

$$\int_{\mathcal{M}} \eta d^2\theta \wedge \mathcal{F} = \int_{\mathcal{M}} \frac{\eta}{2}(\partial_\mu\partial_\nu\theta)F_{\alpha\beta}dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta = \int_{\mathcal{M}} d^4x \frac{\eta}{4}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\theta F_{\alpha\beta} = \int d^4x\eta(\partial_x\partial_y - \partial_y\partial_x)\theta F_{zt} = -2\pi c\int dtdz\eta F_{zt}. \tag{23}$$

The variation of the effective action under the gauge transformation is

$$\delta_\eta W_{\rm WM}^{\rm Bulk}[\mathcal{A},\mathcal{F}] = -\frac{ic}{2\pi}\int dtdz\eta(x)F_{zt}. \tag{24}$$

Comparing Eq. (18) with Eq. (24), we confirm the anomaly cancellation

$$\delta_\eta W_{\rm WM}^{\rm edge} + \delta_\eta W_{\rm WM}^{\rm Bulk} = 0. \tag{25}$$

III. BROKEN INVERSION SYMMETRY WEYL METALS

Since time-reversal symmetry is preserved, Berry flux should satisfy the following constraint: $\mathcal{B}_{\rm I}(\mathbf{k}) = -\mathcal{B}_{\rm II}(-\mathbf{k})$, where band indexes I and II are used to label the bands, which are time-reversal partner to each other(spin-full case). This implies that there should be an even number of pairs of Weyl points in the broken inversion symmetry Weyl metal state. Here, we apply the Callan-Harvey mechanism to the broken inversion symmetry Weyl metal state. We find that a pair of Fermi arcs appear to give rise to a "quantized" version of either spin or valley Hall effects.

A. An effective minimal model for broken inversion symmetry Weyl metals

Following Ref. [25], we start from

$$H = \sigma^x s^z k_x - \sigma^y k_y + \left(-m_1 + m_2 k_z^2\right)\sigma^z + \alpha\sigma^x. \tag{26}$$

Parity and time-reversal transformation operators are given by $P = \sigma^z$ and $T = is^y\mathcal{K}$, respectively, where $\mathcal{K}$ perform the operation of complex conjugation. Then, it is straightforward to see the time-reversal symmetry of this effective Hamiltonian. On the other hand, the last term with the coefficient α breaks inversion symmetry. One can find that there are more terms which give rise to breaking the inversion symmetry while preserving the time-reversal symmetry: $\sigma^y s^x$, $\sigma^y s^y$, and $\sigma^y s^z$. The first and the second terms result in two nodal rings in momentum space. The third term makes a Weyl point along the k_y direction while the α term causes the Weyl point along the k_x direction. We can consider only the $\alpha\sigma^x$ term without loss of generality.

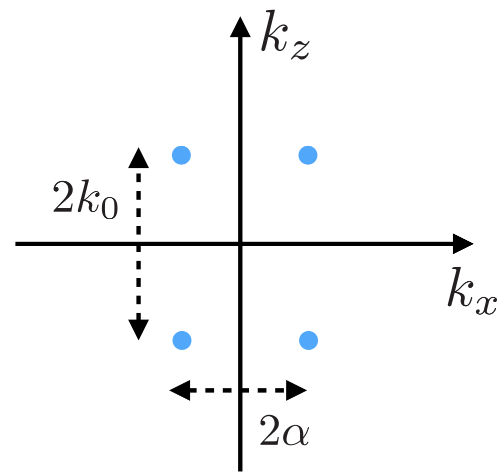

FIG. 2. Band structure of an effective Hamiltonian (26). Four blue dots denote four Weyl points in the $k_y = 0$ plane. Here, $k_0 = \sqrt{m_1/m_2}$ and α is the strength of inversion symmetry breaking.

We start to consider the inversion symmetric case with $\alpha = 0$. Since both the time-reversal and the inversion symmetry are preserved, two bands must be degenerate. Eigenvalues are given by

$$E_\pm = \pm\sqrt{k_x^2 + k_y^2 + \left(m_2k_z^2 - m_1\right)^2}. \tag{27}$$

There are two Dirac points at $(0,0,\pm\sqrt{m_1/m_2})$.

Turning on α, the band structure evolves into

$$E_{1,\pm} = \pm\sqrt{(k_x-\alpha)^2 + k_y^2 + \left(m_2k_z^2 - m_1\right)^2}, \tag{28}$$

$$E_{2\pm} = \pm\sqrt{(k_x+\alpha)^2 + k_y^2 + \left(m_2k_z^2 - m_1\right)^2}. \tag{29}$$

Each Dirac point splits into a pair of Weyl points. As a result, we have a double pair of Weyl points at $(\alpha,0,\sqrt{m_1/m_2})$, $(\alpha,0,-\sqrt{m_1/m_2})$, $(-\alpha,0,\sqrt{m_1/m_2})$, and $(-\alpha,0,-\sqrt{m_1/m_2})$ as shown in the Fig. 2. Here, the definition of "pair" will be clarified below.

B. Low-energy effective Hamiltonian with inversion symmetry breaking

In order to discuss anomaly cancellation, we write down a low-energy effective Hamiltonian near the double pair of Weyl points shown in Fig. 2. Expanding the momentum near the two Dirac points at $(0,0,\pm\sqrt{m_1/m_2}) \equiv (0,0,\pm k_0) \equiv \pm\mathbf{k}_0$, we obtain

$$\begin{aligned} H^{+}(\mathbf{k}) &= H(\mathbf{k}_0+\delta\mathbf{k}) \approx H(\mathbf{k}_0)+\delta\mathbf{k}\cdot\nabla_k H(\mathbf{k})|_{\mathbf{k}_0} \\ &= \sigma^x s^z k_x - \sigma^y k_y + \alpha\sigma^x + \sigma^z(k_z-k_0), \end{aligned} \tag{30}$$

$$\begin{aligned} H^{-}(\mathbf{k}) &= H(-\mathbf{k}_0+\delta\mathbf{k}) \approx H(-\mathbf{k}_0)+\delta\mathbf{k}\cdot\nabla_k H(\mathbf{k})|_{-\mathbf{k}_0} \\ &= \sigma^x s^z k_x - \sigma^y k_y + \alpha\sigma^x - \sigma^z(k_z+k_0), \end{aligned} \tag{31}$$

where we set $2\sqrt{m_1 m_2}=1$ for simplicity. Then, the original Hamiltonian can be approximated in the low-energy limit as follows:

$$H(\mathbf{k}) \approx H^{+}(\mathbf{k}) f(|\mathbf{k}-\mathbf{k}_0|) + H^{-}(\mathbf{k}) f(|\mathbf{k}+\mathbf{k}_0|). \tag{32}$$

Here, the function $f(x)$ is introduced to play the role of a UV cutoff for this low-energy effective Hamiltonian. Accordingly, the Bloch state is represented as

$$|\mathbf{k},\sigma,s\rangle \approx \begin{cases} |\mathbf{k},\sigma,s,+\rangle & (\mathbf{k}\sim\mathbf{k}_0) \\ |\mathbf{k},\sigma,s,-\rangle & (\mathbf{k}\sim-\mathbf{k}_0) \end{cases}. \tag{33}$$

Now, we rewrite this low-energy effective Hamiltonian as one reducible representation in the following way:

$$\begin{aligned} \tilde{H}(\mathbf{k}) &\approx \begin{pmatrix} H^{+}(\mathbf{k}) & 0 \\ 0 & H^{-}(\mathbf{k}) \end{pmatrix} \\ &= (\sigma^x s^z k_x - \sigma^y k_y + \alpha\sigma^x - \sigma^z k_0) \\ &\quad \otimes \tau^0 + \sigma^z \otimes \tau^z k_z. \end{aligned} \tag{34}$$

Accordingly, we have $|\mathbf{k},\sigma,s\rangle \to |\mathbf{k},\sigma,s,\tau\rangle$, where an additional quantum number is identified with a valley index. $\tilde{H}(\mathbf{k})$ is an eight-band Hamiltonian, which originates from the four-band one, Eq. (26), in the low-energy limit.

The inversion and time-reversal transformation operators are redefined consistently as follows:

$$\tilde{P} = \sigma^z \otimes \tau^x, \quad \tilde{T} = i s^y \otimes \tau^x \mathcal{K}. \tag{35}$$

See Appendix C for the derivation. It is easy to check out that the time-reversal symmetry is preserved for this low-energy effective Hamiltonian, i.e., $\tilde{T}\tilde{H}(\mathbf{k})\tilde{T}^{-1} = \tilde{H}(-\mathbf{k})$ while the inversion symmetry is not respected due to the α term, i.e., $\tilde{P}\tilde{H}(\mathbf{k})\tilde{P}^{-1} \neq \tilde{H}(-\mathbf{k})$.

C. Gamma matrix description

It is straightforward to write down the low-energy effective Hamiltonian with Gamma matrices. Taking into account the eight-component spinor

$$\begin{aligned} \Psi = (&\phi_{1,1,1}, \phi_{1,1,-1}, \phi_{1,-1,1}, \phi_{1,-1,-1}, \phi_{-1,1,1}, \\ &\phi_{-1,1,-1}, \phi_{-1,-1,1}, \phi_{-1,-1,-1})^T, \end{aligned} \tag{36}$$

where $\phi_{a,b,c} = \phi_{\tau^z,s^z,\sigma^z}$, we obtain the low-energy effective Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum_{\mathbf{k}} \Psi^\dagger(\mathbf{k})(\sigma^x s^z k_x - \sigma^y k_y + \alpha\sigma^x + \sigma^z\tau^z k_z - \sigma^z k_0)\Psi(\mathbf{k}) \\ &= \int d^3r\, \Psi^\dagger(r)(\sigma^x s^z i\partial_x - \sigma^y i\partial_y \\ &\quad + \sigma^z\tau^z i\partial_z + \sigma^x\alpha - \sigma^z k_0)\Psi(r). \end{aligned} \tag{37}$$

This gives rise to the following effective action:

$$\begin{aligned} S &= \int d^4x\, \Psi^\dagger(x)[\partial_0 + \sigma^x s^z i\partial_x - \sigma^y i\partial_y \\ &\quad + \sigma^z\tau^z i\partial_z + \sigma^x\alpha - \sigma^z k_0]\Psi(x) \\ &= \int d^4x\, \bar{\Psi}(x)[\Gamma^0\partial_0 + i\Gamma^1\partial_x + i\Gamma^2\partial_y + i\Gamma^3\partial_z \\ &\quad - \alpha\Gamma^1\Gamma^5\tau^z + k_0\Gamma^3\Gamma^5 s^z]\Psi(x), \end{aligned} \tag{38}$$

where $\bar{\Psi}(x) = \Psi^\dagger(x)\Gamma^0$ and Gamma matrices are given by

$$\Gamma^0\Gamma^1 = \sigma^x s^z, \quad \Gamma^0\Gamma^2 = -\sigma^y, \quad \Gamma^0\Gamma^3 = \sigma^z\tau^z \tag{39}$$

$$\Rightarrow \Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = -s^z\tau^z, \tag{40}$$

satisfying $\{\Gamma^\mu,\Gamma^\nu\} = 2g^{\mu\nu}1_{8\times 8}$ with $g^{\mu\nu} = (1,-1,-1,-1)\delta^{\mu\nu}$. We observe that there are two different representations of Γ^μ satisfying Eq. (39) with conditions: $\{\Gamma^\mu,\Gamma^\nu\} = 2g^{\mu\nu}1_{8\times 8}$ and $\{\Gamma^\mu,\Gamma^5\} = 0$.

1. $[\tau^z,\Gamma^\mu] = 0$ and $\{s^z,\Gamma^\mu\} = 0$

The first representation for Γ^μ is

$$\Gamma^0 = s^x\sigma^x, \quad \Gamma^1 = -is^y, \tag{41}$$

$$\Gamma^2 = -is^x\sigma^z, \quad \Gamma^3 = -is^x\sigma^y\tau^z. \tag{42}$$

These eight by eight gamma matrices can be rewritten as a product of four by four gamma matrices and two by two pauli matrices as follows:

$$\Gamma^0 = \gamma_v^0\tau^0, \quad \Gamma^1 = \gamma_v^1\tau^0, \quad \Gamma^2 = \gamma_v^2\tau^0, \tag{43}$$

$$\Gamma^3 = \gamma_v^3\tau^z, \quad \Gamma^5 = \gamma_v^5\tau^z, \tag{44}$$

where

$$\gamma_v^0 = s^x\sigma^x, \quad \gamma_v^1 = -is^y\sigma^0, \quad \gamma_v^2 = -is^x\sigma^z, \tag{45}$$

$$\gamma_v^3 = -is^x\sigma^y, \quad \gamma_v^5 = i\gamma_v^0\gamma_v^1\gamma_v^2\gamma_v^3 = -s^z\sigma^0. \tag{46}$$

γ_v^μ matrices are four by four matrix, which consist of σ^μ and s^μ, satisfying $\{\gamma_v^\mu,\gamma_v^\nu\} = 2g^{\mu\nu}1_{4\times 4}$.

2. $\{\tau^z,\Gamma^\mu\} = 0$ and $[s^z,\Gamma^\mu] = 0$

The other representation for Γ^μ is

$$\Gamma^0 = \sigma^z\tau^x, \quad \Gamma^1 = i\sigma^y s^z\tau^x, \tag{47}$$

$$\Gamma^2 = i\sigma^x\tau^x, \quad \Gamma^3 = -i\tau^y. \tag{48}$$

These eight by eight gamma matrices can be also rewritten as a product of four by four gamma matrices and two by two pauli matrices as follows:

$$\Gamma^0 = \gamma_s^0 s^0, \quad \Gamma^1 = \gamma_s^1 s^z, \quad \Gamma^2 = \gamma_s^2 s^0, \tag{49}$$

$$\Gamma^3 = \gamma_s^3 s^0, \quad \Gamma^5 = \gamma_s^5 s^z, \tag{50}$$

where

$$\gamma_s^0 = \sigma^z\tau^x, \quad \gamma_s^1 = i\sigma^y\tau^x, \quad \gamma_s^2 = i\sigma^x\tau^x, \tag{51}$$

$$\gamma_s^3 = -i\sigma^0\tau^y, \quad \gamma_s^5 = i\gamma_s^0\gamma_s^1\gamma_s^2\gamma_s^3 = -\sigma^0\tau^z. \tag{52}$$

γ_s^μ matrices are four by four matrix, which consist of σ^μ and τ^μ, satisfying $\{\gamma_s^\mu,\gamma_s^\nu\} = 2g^{\mu\nu}1_{4\times4}$.

As a result, we have two types of low-energy effective Hamiltonians:

$$S_{1st} = \int d^4x\bar{\Psi}(x)\big[\gamma_v^0\partial_0 + i\gamma_v^1\partial_x + i\gamma_v^2\partial_y + i\gamma_v^3\tau^z\partial_z - \alpha\gamma_v^1\gamma_v^5 - k_0\gamma_v^3\big]\Psi(x), \tag{53}$$

$$S_{2nd} = \int d^4x\bar{\Psi}(x)\big[\gamma_s^0\partial_0 + i\gamma_s^1 s^z\partial_x + i\gamma_s^2\partial_y + i\gamma_s^3\partial_z + \alpha\gamma_s^1 + k_0\gamma_s^3\gamma_s^5\big]\Psi(x). \tag{54}$$

It is not possible to find the representation satisfying both $[\tau^z,\Gamma^\mu] = 0$ and $[s^z,\Gamma^\mu] = 0$. If Γ^μ fulfills both conditions, $[\Gamma^\mu,\Gamma^5] = 0$ must be satisfied because of $\Gamma^5 = -\tau^z s^z$. This is contradictory to the condition of $\{\Gamma^\mu,\Gamma^5\} = 0$. It turns out that this property of Γ^μ is related to the Fujikawa's uncertainty principle [26], which plays an important role in the following discussion.

D. An effective axionic action for broken inversion symmetry Weyl metals

Since the total Berry flux is zero for broken inversion symmetry Weyl metals, the Hall conductivity must vanish. As a result, the conventional effective axionic action does not exist for this Weyl metal state. However, we find other types of effective axionic actions, introducing two kinds of fictitious gauge fields into the effective action: one is a spin gauge field S_μ and the other is a valley gauge field V_μ, which are coupled with a spin current $j_s^\mu = \bar{\Psi}\Gamma^\mu s^z\Psi$ and a valley current $j_v^\mu = \bar{\Psi}\Gamma^\mu\tau^z\Psi$, respectively. Both spin and valley currents are Noether currents, involved with the symmetry under $\Psi \to e^{is^z\theta}\Psi$ and $\Psi \to e^{i\tau^z\theta}\Psi$, respectively.

We start from the low-energy effective action with both spin and valley gauge fields,

$$S = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu(\partial_\mu + iA_\mu + is^zS_\mu + i\tau^zV_\mu) - \alpha\Gamma^1\Gamma^5\tau^z + k_0\Gamma^3\Gamma^5 s^z]\Psi(x), \tag{55}$$

where $\mu = 1,2,3,4$ and $\Gamma^4 = -i\Gamma^0$. It turns out that both spin and valley currents can not be conserved simultaneously when quantum corrections are taken into account. In other words, one of both symmetries related to either spin or valley current should be anomalous in the quantum level. The problem on which symmetry becomes anomalous should be determined by the UV condition. The UV condition fixes the possible representation for the low-energy effective field theory. Physically, this determines the formation of a pair of Fermi arcs.

1. $[\tau^z,\Gamma^\mu] = 0$ *and* $\{s^z,\Gamma^\mu\} = 0$

Action in the first representation Eq. (53) is symmetric under the following three kinds of transformations:

$$\Psi \to e^{i\alpha(x)}\Psi, \quad \bar{\Psi} \to \bar{\Psi}e^{-i\alpha(x)}, \tag{56}$$

$$\Psi \to e^{i\tau^z\beta(x)}\Psi, \quad \bar{\Psi} \to \bar{\Psi}e^{-i\tau^z\beta(x)}, \tag{57}$$

$$\Psi \to e^{is^z\eta(x)}\Psi, \quad \bar{\Psi} \to \bar{\Psi}e^{is^z\eta(x)} \left(\because s^z = -\gamma_v^5\right). \tag{58}$$

The first, second, and third transformations are related to the charge, valley, and spin current, respectively. We note that the third transformation related to the spin current is the chiral transformation in terms of the γ_v matrix. Therefore this low-energy effective action is not invariant under the third transformation when quantum corrections are included. In mathematical terms, the integral measure of the partition function is not invariant under the third transformation. As a result, the spin current is not a conserved current. Resorting to the Fujikawa's method, one can obtain an effective axion term as we did in the case of broken time-reversal symmetry Weyl metals. Detailed calculations are shown in Appendix D. Here, we quote the result only

$$S_{\text{eff}}^{1st} \equiv S_{\text{eff}}^{v} = -\frac{i}{4\pi^2}\int d^4x\alpha x^1\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta}, \tag{59}$$

where $F_{v,\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the field strength tensor, given by the valley gauge field V_μ.

Following the same method as the case of broken time-reversal symmetry Weyl metals, it is straightforward to find the valley Hall current from this axion term. Performing the integration by part as follows:

$$\begin{aligned} S_{\text{eff}}^{v} &= -\int d^4x\frac{i\alpha x^1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta} \equiv \int_M \frac{i\theta_v(x^1)}{\pi^2}\mathcal{F}_v\wedge\mathcal{F} \\ &= \int_M \frac{i}{\pi^2}[d(\theta_v\mathcal{V}\wedge\mathcal{F}) - d\theta_v\wedge\mathcal{V}\wedge\mathcal{F}] \\ &= -\frac{i}{\pi^2}\int_M d\theta_v\wedge\mathcal{V}\wedge\mathcal{F} \\ &= -\frac{i}{2\pi^2}\int d^4x\epsilon^{\alpha\beta\mu\nu}\partial_\alpha\theta_v V_\beta F_{\mu\nu}, \end{aligned} \tag{60}$$

we obtain

$$\begin{aligned} j_v^\mu &= \frac{\delta S_{\text{eff}}^v}{\delta V_\mu} = \frac{i}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}(\partial_\nu\theta_v)F_{\alpha\beta} \\ &= -\frac{i}{2\pi^2}\epsilon^{\mu1\eta\xi}\alpha F_{\eta\xi}. \end{aligned} \tag{61}$$

Since this current is evaluated in the Euclidean signature, we have to change it into the Lorentzian signature; $(v_L^1,v_L^2,v_L^3,v_L^0) = (v_E^1,v_E^2,v_E^3,iv_E^4)$. Then, we have

$$j_{L,v}^0 = \frac{\alpha}{\pi^2}F_{23}, \tag{62}$$

$$j_{L,v}^k = \frac{\alpha}{2\pi^2}\epsilon^{k1\eta\xi}F_{\eta\xi}. \tag{63}$$

This valley Hall current may be regarded as an anomaly inflow from the bulk to the pair of Fermi arcs. Performing essentially

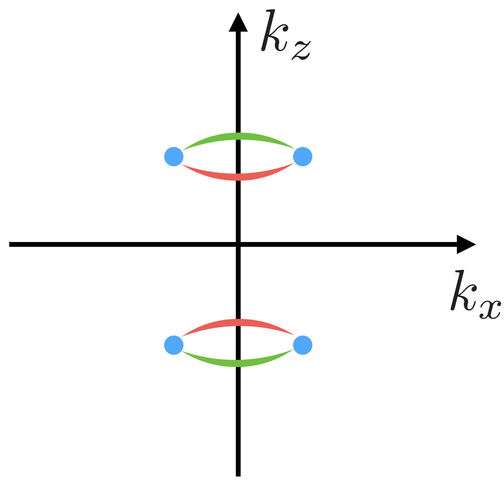

FIG. 3. A double pair of Fermi arcs (red and green lines) for the valley Hall current case; $[\tau^z,\Gamma^\mu]=0$ and $\{s^z,\Gamma^\mu\}=0$. Here, each color of Fermi arcs represents where they are located in real space (green: the upper terminated plane and red: the lower terminated plane).

the same task as the case of the broken time-reversal symmetry Weyl metal, we find that the pair of Fermi arcs is given in Fig. 3.

2. $\{\tau^z,\Gamma^\mu\}=0$ *and* $[s^z,\Gamma^\mu]=0$

The low-energy effective action Eq. (54) is symmetric under the following three types of transformations as the case of the first representation:

$$\Psi\to e^{i\alpha(x)}\Psi,\quad \bar{\Psi}\to\bar{\Psi}e^{-i\alpha(x)}, \tag{64}$$

$$\Psi\to e^{i\tau^z\beta(x)}\Psi,\quad \bar{\Psi}\to\bar{\Psi}e^{i\tau^z\beta(x)}\ \left(\because\ \tau^z=-\gamma_s^5\right), \tag{65}$$

$$\Psi\to e^{is^z\eta(x)}\Psi,\quad \bar{\Psi}\to\bar{\Psi}e^{-is^z\eta(x)}. \tag{66}$$

In this representation, the second transformation related to the valley current is the chiral transformation in terms of the γ_s^5 matrix. Therefore the valley current is not conserved because of the chiral anomaly. The corresponding axionic effective action derived from the chiral rotation is given by

$$S_{\rm eff}^{2nd}\equiv S_{\rm eff}^{s}=\frac{i}{4\pi^2}\int d^4x k_0x^3\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta}, \tag{67}$$

where $F_{s,\mu\nu}=\partial_\mu S_\nu-\partial_\nu S_\mu$ is the field strength of the spin gauge field S_μ. We refer all details to Appendix D.

It is straightforward to find the spin Hall current from this effective action, taking into account the integration by part:

$$\begin{aligned}S_{\rm eff}^{s}&=\int d^4x\frac{ik_0x^3}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta}=\int_M\frac{i\theta_s(x^1)}{\pi^2}\mathcal{F}_s\wedge\mathcal{F}\\&=\int_M\frac{i}{\pi^2}[d(\theta_s\mathcal{S}\wedge\mathcal{F})-d\theta_s\wedge\mathcal{S}\wedge\mathcal{F}]\\&=-\frac{i}{\pi^2}\int_M d\theta_s\wedge\mathcal{S}\wedge\mathcal{F}\\&=-\frac{i}{2\pi^2}\int d^4x\epsilon^{\alpha\beta\mu\nu}\partial_\alpha\theta_s S_\beta F_{\mu\nu},\end{aligned} \tag{68}$$

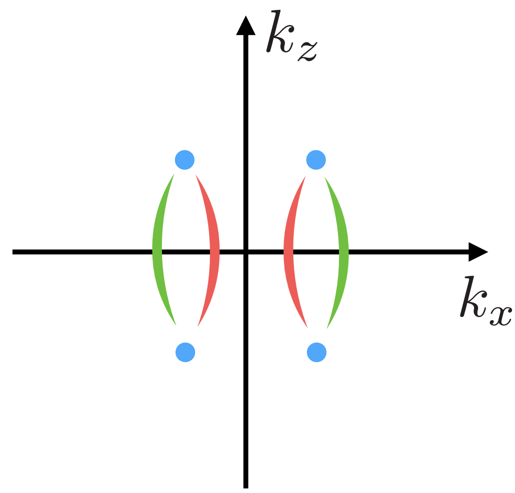

FIG. 4. A double pair of Fermi arcs (red and green lines) for the spin Hall current case; $\{\tau^z,\Gamma^\mu\}=0$ and $[s^z,\Gamma^\mu]=0$. Here, each color of Fermi arcs denotes where they are located in real space (green: the upper terminated plane and red: the lower terminated plane).

and resulting in

$$\begin{aligned}j_s^\mu&=\frac{\delta S_{\rm eff}^v}{\delta S_\mu}=\frac{i}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}(\partial_\nu\theta_s)F_{\alpha\beta}\\&=\frac{i}{2\pi^2}\epsilon^{\mu3\eta\xi}k_0F_{\eta\xi}.\end{aligned} \tag{69}$$

In the Lorentizan signature, we have

$$j_{L,s}^0=-\frac{k_0}{\pi^2}F_{12}, \tag{70}$$

$$j_{L,s}^k=-\frac{k_0}{2\pi^2}\epsilon^{k3\eta\xi}F_{\eta\xi}. \tag{71}$$

This spin Hall current may be also regarded as an anomaly inflow from the bulk to the pair of Fermi arcs. Performing essentially the same task as the case of the broken time-reversal symmetry Weyl metal, we find that the pair of Fermi arcs is given in Fig. 4.

3. *Discussion: Valley Hall effect versus spin Hall effect*

So far, we discussed that two different representations give two different physical situations. Then, what is the right physical picture? We cannot answer which is correct within only the low energy effective Hamiltonian. We need more information which should be introduced from the UV structure of the dispersion relation. However, the analysis based on the low energy effective action tells us that only one of these two different choices exists at least. These two representations take two different regularization schemes. The first representation or regularization scheme preserves the charge symmetry and the valley symmetry, but breaks the spin symmetry while the second representation or regularization scheme preserves the charge symmetry and the spin symmetry, but breaks the valley symmetry. One important thing is that there is no regularization scheme which preserves all the symmetry involved with charge, spin, and valley simultaneously as we pointed out in the Sec. III C. Although we have shown this aspect with a specific Hamiltonian, this argument can be generalized. See Appendix G for more general discussions on this point. Involved with the issue of the regularization scheme, recent studies [27,28] have shown that structure of Fermi arcs can be changed by tuning the Weyl band structure in the UV level, which does not affect the emergent symmetry of the low-energy effective Hamiltonian.

E. Callan-Harvey mechanism in broken inversion symmetry Weyl metals

1. Gauge transformation of the axion term

Effective axionic actions of $S^v_{\rm eff}[A_\mu,V_\mu]$ and $S^s_{\rm eff}[A_\mu,S_\mu]$ have anomaly with respect to the valley gauge transformation of $V_\mu \to V_\mu + \partial_\mu \eta$ and the spin gauge transformation of $S_\mu \to S_\mu + \partial_\mu \eta$, respectively. Taking into account these gauge transformations, we find

$$\delta_\eta S^v_{\rm eff}[\mathcal{A},\mathcal{V}] = \frac{2i\alpha}{\pi}\int dt dx \eta F_{xt}, \tag{72}$$

$$\delta_\eta S^s_{\rm eff}[\mathcal{A},\mathcal{S}] = -\frac{2ik^0}{\pi}\int dt dy \eta F_{zt}. \tag{73}$$

Of course, this anomaly inflow should be canceled by the anomaly of a pair of Fermi arcs, which is nothing but the Callan-Harvey mechanism.

2. Valley Hall current

Following the case of broken time-reversal symmetry Weyl metals and setting $\alpha \to \alpha\theta(z)$, one can find the edge state localized near the boundary from the low-energy effective Hamiltonian. We show all detailed calculations in Appendix E. An effective surface Hamiltonian in the $|\tau^z,k_x,k_y\rangle$ basis is given by

$$\begin{aligned} H &= \sum_{\tilde{k}_z}\sum_{k_y}\Psi^\dagger(k_x,k_y)\begin{pmatrix} k_y & 0 \\ 0 & -k_y\end{pmatrix}\Psi(k_x,k_y) \\ &= \sum_{\tilde{k}_z}\int dy \Psi^\dagger_{k_x}(y)\begin{pmatrix} -i\partial_y & 0 \\ 0 & i\partial_y\end{pmatrix}\Psi_{k_x}(y) \\ &= \sum_{\tilde{k}_z}\int dy \Psi^\dagger_{k_x}(y)(-i\partial_y\tau^3)\Psi_{k_x}(y), \end{aligned} \tag{74}$$

where $\tilde{k}_z \to -\sqrt{\alpha^2-m^2} < k_x < \sqrt{\alpha^2-m^2}$. Then, the corresponding effective surface action is

$$\begin{aligned} S &= \sum_{\tilde{k}_z}\int d\tau \int dy \Psi^\dagger_{k_x}(y)(\partial_\tau - i\tau^3\partial_y)\Psi_{k_x}(y) \\ &= \sum_{\tilde{k}_z}\int d\tau \int dy \bar{\Psi}_{k_x}(y)(\gamma^0\partial_\tau + i\gamma^1\partial_y)\Psi_{k_x}(y), \end{aligned} \tag{75}$$

where $\gamma^0 = \tau^1, \gamma^1 = i\tau^2$, and $\bar{\gamma} = \gamma^0\gamma^1 = -\tau^3$. Setting $\gamma^2 = -i\gamma^0$, we have

$$S = \sum_{-\sqrt{\alpha^2-m^2}<k_x<\sqrt{\alpha^2-m^2}}\int d^2x \bar{\Psi}_{k_x}(y) i\gamma^\mu\partial_\mu\Psi_{k_x}(y), \tag{76}$$

where $\mu = 1,2$, $\partial_0 = \partial_2$, and $\{\gamma^\mu,\gamma^\nu\} = -\delta^{\mu\nu}$.

In order to show the anomaly cancellation, we introduce both charge and valley gauge fields to the above boundary action:

$$\begin{aligned} S = &\sum_{-\sqrt{\alpha^2-m^2}<k_x<\sqrt{\alpha^2-m^2}}\int d^2x \bar{\Psi}_{k_x}(y) i\gamma^\mu \\ &\times(\partial_\mu + iA_\mu + iV_\mu\bar{\gamma})\Psi_{k_x}(y). \end{aligned} \tag{77}$$

The surface valley current is given by

$$\begin{aligned} j^\mu_v &\equiv \frac{\delta W[A,V]}{\delta V_\mu} \\ &= -\sum_{-\sqrt{\alpha^2-m^2}<k_x<\sqrt{\alpha^2-m^2}}\langle\bar{\Psi}_{k_x}\gamma^\mu\bar{\gamma}\Psi_{k_x}\rangle, \end{aligned} \tag{78}$$

where $W[A,V]$ is an effective free energy defined by $Z = e^{-W[A,V]} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{-S[\bar{\Psi},\Psi,A,V]}$.

This boundary effective action is invariant under the following valley gauge transformation:

$$\Psi_{k_x} \to e^{i\bar{\gamma}\theta}\Psi_{k_x}, \quad \bar{\Psi}_{k_x} \to \bar{\Psi}_{k_x}e^{i\bar{\gamma}\theta}, \quad V_\mu \to V_\mu + \partial_\mu\theta. \tag{79}$$

However, we find that the expectation value for the valley current j^μ_v becomes anomalous in the one-loop order. All details are shown in Appendix F. Here, we quote the result only

$$\partial_\mu\langle j^\mu_{\rm reg}(x)\rangle = \frac{2i\alpha}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(x) = \frac{i\alpha}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}(x). \tag{80}$$

This anomaly equation implies

$$\delta_\eta W^{\rm edge}_{WM,\rm valley} = -\frac{i\alpha}{\pi}\int d^2x \eta(x)\epsilon^{\mu\nu}F_{\mu\nu}(x). \tag{81}$$

$\delta_\eta W^{\rm edge}_{WM,\rm valley}$ is canceled exactly by $\delta_\eta S^v_{\rm eff}$, which is nothing but the Callan-Harvey mechanism.

3. Spin Hall current

Following the case of the valley Hall effect, we can obtain a localized surface solution for the case of the spin Hall current. See Appendix E for details. From these solutions, we can construct the surface effective action. The surface Hamiltonian in terms of the $|s^z,k_y,k_z\rangle$ basis is given by

$$\begin{aligned} H &= \sum_{\tilde{k}_z}\sum_{k_y}\Psi^\dagger(k_y,k_z)\begin{pmatrix} k_y & 0 \\ 0 & -k_y\end{pmatrix}\Psi(k_y,k_z) \\ &= \sum_{\tilde{k}_z}\int dy \Psi^\dagger_{k_z}(y)\begin{pmatrix} -i\partial_y & 0 \\ 0 & i\partial_y\end{pmatrix}\Psi_{k_z}(y) \\ &= \sum_{\tilde{k}_z}\int dy \Psi^\dagger_{k_z}(y)(-is^3\partial_y)\Psi_{k_z}(y), \end{aligned} \tag{82}$$

where $\tilde{k}_z$ is the momentum to satisfy $-\sqrt{k_0^2-m^2} < k_z < \sqrt{\alpha^2-m^2}$. Then, the corresponding effective action is

$$\begin{aligned} S &= \sum_{\tilde{k}_z}\int d\tau \int dy \Psi^\dagger_{k_z}(y)(\partial_\tau - is^3\partial_y)\Psi_{k_z}(y) \\ &= \sum_{\tilde{k}_z}\int d\tau \int dy \bar{\Psi}_{k_z}(y)(\gamma^0\partial_\tau + i\gamma^1\partial_y)\Psi_{k_z}(y), \end{aligned} \tag{83}$$

where $\gamma^0 = s^1, \gamma^1 = is^2$, and $\bar{\gamma} = \gamma^0\gamma^1 = -s^3$. If we set $\gamma^2 = -i\gamma^0$, we have

$$S = \sum_{-\sqrt{k_0^2-m^2}<k_z<\sqrt{\alpha^2-m^2}} \int d^2x \bar{\Psi}_{k_z}(y) i\gamma^\mu \partial_\mu \Psi_{k_z}(y), \quad (84)$$

where $\mu = 1,2$, $\partial_0 = \partial_2$, and $\{\gamma^\mu,\gamma^\nu\} = -\delta^{\mu\nu}$.

Following the case of the valley Hall effect, we also gauge the above surface action. While we gauged the action with charge and valley gauge fields in the valley Hall case, here we gauge the effective action with charge and spin gauge fields in the following way:

$$S = \sum_{-\sqrt{k_0^2-m^2}<k_z<\sqrt{k_0^2-m^2}} \int d^2x \bar{\Psi}_{k_z}(y) i\gamma^\mu \times (\partial_\mu + iA_\mu - iS_\mu\bar{\gamma})\Psi_{k_z}(y). \quad (85)$$

Now, the spin current is given by

$$j_s^\mu \equiv \frac{\delta W[A,S]}{\delta S_\mu} = \sum_{-\sqrt{k_0^2-m^2}<k_z<\sqrt{k_0^2-m^2}} \langle \bar{\Psi}_{k_z}\gamma^\mu\bar{\gamma}\Psi_{k_z}\rangle, \quad (86)$$

where $W[A,S]$ is an effective free energy defined by $Z = e^{-W[A,S]} = \int \mathcal{D}\bar{\Psi}\Psi e^{-S[\bar{\Psi},\Psi,A,S]}$.

This boundary effective action is invariant in the classical level under the following spin gauge transformation:

$$\Psi_{k_z} \to e^{i\bar{\gamma}\theta}\Psi_{k_z}, \quad \bar{\Psi}_{k_z} \to \bar{\Psi}_{k_z}e^{i\bar{\gamma}\theta}, \quad S_\mu \to S_\mu + \partial_\mu\theta. \quad (87)$$

On the other hand, we find that the expectation value of the spin current j_s^μ becomes anomalous in the one-loop order. All procedures are completely identical to those for the case of the valley Hall current. The variation of the effective surface action with respect to the change of the spin gauge field is

$$\delta_\eta W_{WM,\text{spin}}^{\text{edge}} = \frac{ik_0}{\pi}\int d^2x \eta(x)\epsilon^{\mu\nu}F_{\mu\nu}(x). \quad (88)$$

$\delta_\eta W_{WM,\text{spin}}^{\text{edge}}$ is also canceled perfectly by $\delta_\eta S_{\text{eff}}^s$.

IV. DISCUSSION ON THE VALLEY HALL CURRENT

A. Is the valley quantum number physical?

Up to now, we demonstrated the anomaly cancellation for the broken inversion symmetry Weyl metal phase, described by a specific model, where the spin quantum number s_z is conserved. Since s_z is conserved, it is natural to consider the spin Hall current mediated by the Fermi arc state. Actually, the spin Hall effect has been proposed in TaAs, which is an broken inversion symmetry Weyl metal material [29]. On the other hand, one may criticize the physical realization of the valley Hall current since the valley quantum number is not conserved in the UV level. Here, we argue that an emergent symmetry, which appears at low energies, plays a central role in the quantum anomaly and the anomaly cancellation, reflected in the existence of gapless surface states, even if such a symmetry does not exist in the UV level.

Generally speaking, spin is not conserved, either, because of the spin-orbit interaction. For a generic Hamiltonian of broken inversion symmetry Weyl metals, there are no conserved quantities such as spin. In this case we can introduce two kinds of "valley" indexes in the low energy effective theory to describe the broken inversion symmetry Weyl metal phase. We may apply the same argument of the anomaly cancellation as the above to this case, where s_z is replaced to another valley index. Then, we ask the following question: is the valley Hall current physical? Here, we confirm the existence of Fermi arc states even if there do not appear such symmetries in the UV lattice Hamiltonian. Then, one can guess that there are corresponding Hall currents mediated by such Fermi arc states.

B. The valley Hall effect in graphene

We start from reviewing the valley Hall effect in graphene [30]. The band structure of graphene consists of two Dirac points, referred to as valleys, K and K', respectively, where an effective Hamiltonian near each valley is given by

$$H(\mathbf{k}) = v(\tau_z k_x\sigma_x + k_y\sigma_y). \quad (89)$$

Here, k_x and k_y are the momentum measured from each Dirac point. σ_x and σ_y are Pauli matrices, involved with the pseudo-spin (A and B sublattices), and τ_z is a valley indexing matrix, the eigenvalue $+1$ (-1) of which is assigned to the K (K') valley.

Let us break the inversion symmetry in this effective Hamiltonian. Then, such gapless Dirac points become gapped. When the chemical potential is located at a position inside the gap, one may expect quantized Hall responses based on the Berry curvature. Considering the existence of time-reversal symmetry, the sign of the Berry curvature is assigned to be opposite at each valley and the Hall conductivity is canceled to vanish. On the other hand, one can introduce the valley Hall conductivity, given by $\sigma_{\text{valley}} = \sigma_{\tau_z=1} - \sigma_{\tau_z=-1}$, similar to the spin Hall conductivity, where $\sigma_{\tau_z=\pm1}$ is the Hall conductivity for each valley. For the concept of the valley Hall current to be well defined, the valley quantum number should be a conserved quantity. One may wonder how the valley quantum number, which is just an index for labeling the different k-points, can be considered as a conserved physical quantity. In order to answer this question, we take into account another quantity "coupled to" each valley, which is physically meaningful. That is the pseudo-spin or chirality quantum number, given by the sublattice degree of freedom [31]. Then, the valley index can be considered as a conserved quantity in some specific cases.

For example, the existence of edge states in graphene, which depends on the shape of the edge (armchair or zigzag), can be understood with the presence or suppression of intervalley scattering [30]. Considering the conservation of the crystal-momentum component parallel with the edge and the conservation of the pseudospin vector, we can determine whether the intervalley scattering is suppressed or not [32,33]. In the case of the armchair edge, two valleys of K and K' are projected onto the same point in the surface 1D Brillouin zone. Therefore the edge-parallel component of the crystal momentum is conserved for the case when an in-going electron and an

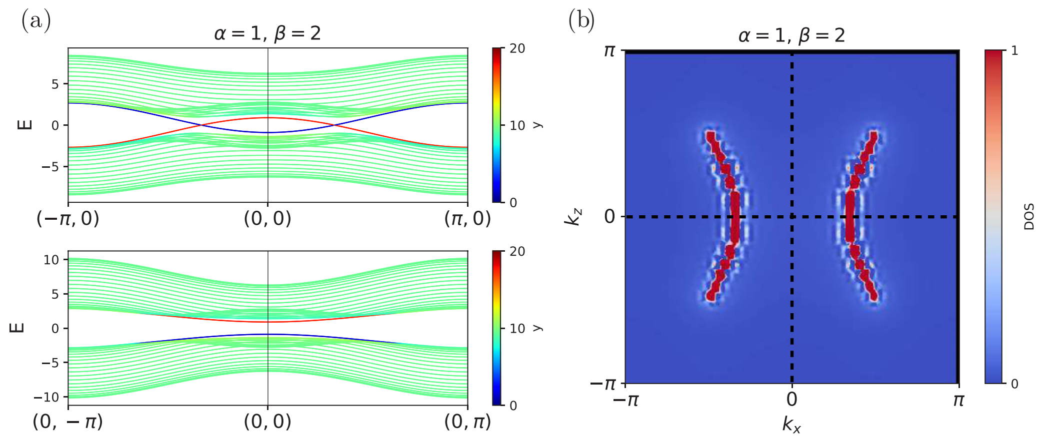

FIG. 5. Surface band structure (a) and Fermi arc structure (b) with 20 unit cells along y axis for $\alpha = 1$ and $\beta = 2$.

out-going electron belong to different valleys. In addition, the direction of the pseudospin vector for in-going and out-going fermions is the same when they belong to different valleys. As a result, scattering between different valleys is dominant for the armchair-edge case. The intervalley scattering does not protect surface states for the armchair edge. In contrast to the armchair edge, two valleys K and K' are projected onto different points in the surface 1D Brillouin zone for the zigzag-edge case. Therefore the edge-parallel component of the crystal momentum is not conserved for in-going and out-going fermions belonging to different valleys. As a result, there exist surface states in the zigzag-edge case, which is consistent with the valley Hall conductivity calculation based on the fact that the valley is a good quantum number. For details, we would like to refer to [32,33].

C. Valley Hall current in a spinless broken inversion symmetry Weyl metal phase

We apply the same argument to the case of broken inversion symmetry Weyl metals. We start from the following effective Hamiltonian for a spinless broken inversion symmetry Weyl metal:

$$H(\mathbf{k}) = t_1 \sin k_y \sigma_y + 2t_2[\cos k_x + \alpha(\cos k_y - 1)]\sigma_x + 2t_3[\cos k_z + \beta(\cos k_y - 1)]\sigma_z. \quad (90)$$

Here, σ_i are Pauli matrices related to orbital or sublattice degrees of freedom. Taking into account $t_1 = t_2 = t_3 = 1$ and $|\alpha| > 1/2, |\beta| > 1/2$, this effective Hamiltonian gives rise to four Weyl points with the linear dispersion at $(k_x,k_y,k_z) = (\pm\pi/2, 0, \pm\pi/2)$. Then, the low-energy effective Hamiltonian for the above lattice model is given by

$$H_{\text{approx}}(\mathbf{k}) = k_y\sigma_y + 2s_z\sigma_x k_x + 2\tau_z\sigma_z k_z + \pi\sigma_x + \pi\sigma_z \quad (91)$$

near the Weyl points, where s_z and τ_z are Pauli matrices involved with valley quantum numbers.

The form of this low-energy effective Hamiltonian is quite similar to Eq. (34). Repeating the same procedure performed previously, we can derive two types of valley Hall currents and the anomaly cancellation, respectively. Here, we note that there are no $\alpha-$ and $\beta-$ involved terms in this effective Hamiltonian. It means that such UV parameters as α and β do not affect the emergent symmetry of the low-energy effective theory. In this low-energy effective Hamiltonian, both valley indexes of $s_z = \pm 1$ and $\tau_z = \pm 1$ are good quantum numbers. However, they were not in the UV level. Valley indexes do not have any physical meaning intrinsically. It can have their own physical meaning by other quantities.

As discussed in graphene, the other quantity is nothing but the pseudo-spin vector, given by $\mathbf{h}(\mathbf{k})/|\mathbf{h}(\mathbf{k})|$, where $H(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \vec{\sigma}$. For our spinless broken inversion symmetry Weyl metal, we obtain $\mathbf{h}(\mathbf{k}) = (2[\cos k_x + \alpha(\cos k_y - 1)], \sin k_y, 2[\cos k_z + \beta(\cos k_y - 1)])$. Based on the same argument as that in graphene, which determines the existence of surface states, we can find that there are no surface states for xy and yz planes. Only the xz plane can host the surface state since the intervalley scattering is suppressed. This result agrees with the previous arguments [34,35] for the existence of surface states.

We confirm the above statement based on our slab-calculation for the microscopic lattice Hamiltonian, Eq. (90). More precisely, we calculated surface (xz plane) band (Fermi arc) structures for two different cases, given by Figs. 5 and 6. Subfigures (a) in Figs. 5 and 6 show the band structure of gapless surface states in the xz plane. In this respect, we can regard the valley index as a conserved quantity when we are considering the xz plane. There exist two types of gapless surface states, depending on α and β. Fermi arc states in subfigures (b) in Figs. 5 and 6 reveal these aspects more clearly. An essential point is that α and β do not affect anything on the low-energy effective Hamiltonian near the Weyl points, Eq. (91). This demonstration confirms that we can not determine which Weyl points are connected to form a pair within the low-energy effective Hamiltonian only, as discussed in Sec. III D.

Here, we have shown that the valley index in the spinless broken inversion symmetry Weyl metal phase can be considered as a physically conserved quantity, depending on the orientation of a surface. Although we have shown the existence of surface states based on the intervalley scattering argument, one can show it based on more general arguments given by Wan *et al.* [34]. Based on this argument, we can show that the same result holds for the spinfull broken inversion symmetry Weyl metal phase.

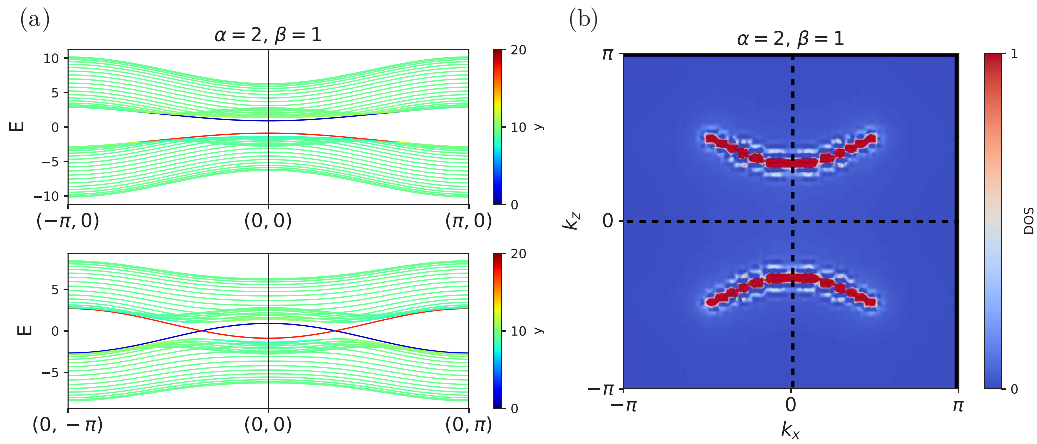

FIG. 6. Surface band structure (a) and Fermi arc structure (b) with 20 unit cell along y axis for $\alpha = 2$ and $\beta = 1$.

V. CONCLUSION

In conclusion, we applied the Callan-Harvey mechanism to the case of broken inversion symmetry Weyl metals, and found either the spin Hall effect or the valley Hall effect, depending on the UV condition. Turning on charge, spin, and valley U(1) gauge fields, we derived two types of axion terms from two kinds of low-energy effective actions, based on the Fujikawa's method. This explicit demonstration clarifies the anomaly inflow in either spin or valley currents from the Weyl metal bulk to the surface state. Solving Weyl metal equations with a surface boundary condition, we found normalizable surface zero modes, which consist of a chiral pair of Fermi arcs. Constructing the corresponding effective surface action and calculating both spin and valley surface currents, we found the gauge anomaly involved with either spin or valley U(1) gauge fields. We proved explicitly that this spin-gauge or valley-gauge anomaly at the surface is canceled exactly by the anomaly inflow from the bulk action. We would like to emphasize that our demonstration is the first concrete calculation for broken inversion symmetry Weyl metals although it is certainly expected in a conceptual point of view.

In the present study, we focused on the case that the chemical potential locates at the Weyl point. However, it is not the case in real experiments. It turns out that the chiral anomaly does not change as a function of either temperature or the presence of a chemical potential [36]. This means that "quantized" responses given by topologically protected surface states do not change even if both temperature and chemical potential are turned on. On the other hand, there would be nonquantized responses given by electrons near Fermi surfaces in the presence of Berry curvatures. Actually, the anomalous Hall effect in a Weyl metal phase with time-reversal symmetry breaking consists of contributions from not only surface states but also Fermi-surface electrons [3]. It turns out that the unbounded linear band structure of Weyl electrons does not allow the nonquantized part from Fermi-surface electrons [24]. In order to find such Fermi-surface contributions, we should take into account a more realistic dispersion of electrons, given by lattice regularization, for example. Following Sec. V of Ref. [24], one can evaluate the nonquantized part for the valley Hall effect numerically if he/she resorts to the lattice-regularized band structure of Eq. (90). However, these quantized and nonquantized responses in the presence of lattice regularization should be investigated more deeply beyond the numerical analysis in order to understand the role of the Fermi surface in the topological structure.

We believe that our explicit demonstration casts various interesting questions, involved with generalizations of the present ideal case. As far as we know, the role of disorder scattering has never been discussed clearly, particularly, in the view of anomaly cancellation. Disorder scattering gives rise to mixing between each Fermi point in the Fermi arc state, expected to spoil the present simple calculation at least when disorder strength exceeds a critical value. The role of electron correlations in the anomaly cancellation would be more important and difficult. Recently, a topological Fermi-liquid theory has been proposed to describe anomalous transport phenomena in broken time-reversal symmetry Weyl metals, where the concept of Landau's Fermi-liquid theory is generalized to incorporate both the Berry curvature and the chiral anomaly [37,38]. However, the issue on anomaly cancellation has not been discussed within such a topological Fermi-liquid theory. When inversion symmetry is broken instead of time-reversal symmetry, the situation would be much more complicated. Not only the spin current but also the valley current should be taken into account. This is somewhat analogous to the relationship between the integer quantum Hall phase and the quantum spin-Hall state in two dimensions.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF AN AXIONIC ACTION FOR BROKEN TIME-REVERSAL SYMMETRY WEYL METALS

1. Chiral transformation

We introduce the chiral rotation as follows:

$$\Psi(x) \to e^{i\alpha(x)\gamma^5}\Psi(x), \quad \bar{\Psi}(x) \to \bar{\Psi}(x)e^{i\alpha(x)\gamma^5}. \tag{A1}$$

Under this chiral transformation, the effective action (2) for a Weyl metal state changes as

$$S_{\rm WM} \to \int d^4x \bar{\Psi}(x) i\gamma^\mu[\partial_\mu + iA_\mu + i(c_\mu + ds\partial_\mu\theta(x))\gamma^5]\Psi(x). \tag{A2}$$

Here, we set $\alpha(x) = ds\theta(x)$. Multiple steps of chiral rotations result in

$$S_{\rm WM} \to \int d^4x\bar{\Psi}(x)i\gamma^\mu[\partial_\mu + iA_\mu + i(c_\mu + s\partial_\mu\theta(x))\gamma^5]\Psi(x) \equiv \int d^4x\bar{\Psi}(x)i\not{D}^{(s)}\Psi(x), \tag{A3}$$

where

$$\not{D}^{(s)} \equiv \gamma^\mu[\partial_\mu + iA_\mu + i(c_\mu + s\partial_\mu\theta(x))\gamma^5], \tag{A4}$$

$$\not{D}^{(s)\dagger} \equiv \gamma^\mu[\partial_\mu + iA_\mu - i(c_\mu + s\partial_\mu\theta(x))\gamma^5]. \tag{A5}$$

Since $\not{D}^{(s)}$ is not Hermitian because of the chiral gauge field, we choose a basis which differs from the conventional case of the chiral anomaly [22],

$$\Psi(x) = \sum_n a_n\varphi_n^{(s)}(x), \quad \bar{\Psi}(x) = \sum_n \phi_n^{(s)\dagger}(x)\bar{b}_n, \tag{A6}$$

where the eigenvectors $\varphi_n^{(s)}(x)$ and $\phi_n^{(s)\dagger}(x)$ are determined by

$$\not{D}^{(s)\dagger}\not{D}^{(s)}\varphi_n^{(s)} = \lambda_n^2\varphi_n^{(s)}, \quad \not{D}^{(s)}\not{D}^{(s)\dagger}\phi_n^{(s)} = \lambda_n^2\phi_n^{(s)}, \tag{A7}$$

$$\not{D}^{(s)}\varphi_n^{(s)} = \lambda_n\phi_n^{(s)}, \quad \not{D}^{(s)\dagger}\phi_n^{(s)}(x) = \lambda_n\varphi_n^{(s)}, \tag{A8}$$

where λ_n is an eigenvalue. Then, the path-integral measure is

$$\mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x) = [\det U]^{-1}\prod_n d\bar{b}_n da_n, \tag{A9}$$

where $[U^{-1}]_{nm} = \phi_n^{(s)\dagger}(x)\varphi_m^{(s)}(x)$.

Now, we can see how the integral measure changes under the chiral transformation. Since the wave function changes in the following way:

$$\Psi'(x) = e^{ids\theta(x)\gamma^5}\Psi(x), \quad \bar{\Psi}'(x) = \bar{\Psi}(x)e^{ids\theta(x)\gamma^5}$$
$$\Rightarrow \begin{cases} \sum_n a_n'\varphi_n(x) = \sum_n e^{ids\theta(x)\gamma^5}a_n\varphi_n(x), \\ \sum_n \phi_n^{(s)\dagger}\bar{b}_n' = \sum_n \phi_n^{(s)\dagger}(x)\bar{b}_n e^{ids\theta(x)\gamma^5}, \end{cases} \tag{A10}$$

we obtain

$$a_n' = \sum_m C_{nm}a_m, \quad \bar{b}_n' = \sum_m D_{nm}\bar{b}_m, \tag{A11}$$

$$C_{nm} = \int d^dx\varphi_n^{(s)\dagger}(x)e^{ids\theta(x)\gamma^5}\varphi_m^{(s)}(x), \tag{A12}$$

$$D_{nm} = \int d^dx\phi_m^{(s)\dagger}e^{ids\theta(x)\gamma^5}\phi_n^{(s)}(x). \tag{A13}$$

As a result, the integral measure is given by

$$\begin{aligned}\mathcal{D}\bar{\Psi}'(x)\mathcal{D}\Psi'(x) &= [\det U]^{-1}\prod_n d\bar{b}_n' da_n' \\ &= [\det U]^{-1}[\det C]^{-1}[\det D]^{-1}\prod_n d\bar{b}_n da_n \\ &= [\det C]^{-1}[\det D]^{-1}\mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x)\end{aligned} \tag{A14}$$

under the chiral transformation, where

$$[\det C]^{-1} = \exp\left[-ids\int d^dx\theta(x)\sum_n\varphi_n^{(s)\dagger}(x)\gamma^5\varphi_n^{(s)}(x)\right],$$
$$[\det D]^{-1} = \exp\left[-ids\int d^dx\theta(x)\sum_n\phi_n^{(s)\dagger}(x)\gamma^5\phi_n^{(s)}(x)\right]. \tag{A15}$$

Finally, the partition function changes as follows:

$$\begin{aligned}Z &= \int \mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x)e^{-S_{\rm WM}} \\ &\to \int\mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x)\exp\Bigg[-\int d^dx\Bigg\{\bar{\Psi}(x)i\not{D}^{(s)}\Psi(x) \\ &\quad + \int_0^s ds\ \theta(x)i\Bigg(\sum_n\varphi_n^{(s)\dagger}(x)\gamma^5\varphi^{(s)}(x) \\ &\quad + \sum_n\phi_n^{(s)\dagger}(x)\gamma^5\phi_n^{(s)}(x)\Bigg)\Bigg\}\Bigg] \\ &\equiv \int\mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x)\exp\left[-S_{\rm WM}^{(s)}\right].\end{aligned} \tag{A16}$$

2. Regularization

In order to calculate the part that changes under the chiral transformation, we follow the standard way of regularization [22], given by

$$\begin{aligned}&\sum_n\left[\varphi_n^{(s)\dagger}(x)\gamma^5\varphi_n^{(s)} + \phi^{(s)\dagger}\gamma^5\phi_n^{(s)}\right] \\ &= \lim_{M\to\infty}\sum_n\left[\varphi_n^{(s)\dagger}\gamma^5\varphi_n^{(s)} + \phi_n^{(s)\dagger}\gamma^5\phi_n^{(s)}\right]e^{-\frac{\lambda_n^2}{M^2}} \\ &= \lim_{M\to\infty}\sum_n\left[\varphi_n^{(s)\dagger}\gamma^5 e^{-\frac{\not{D}^{(s)\dagger}\not{D}^{(s)}}{M^2}}\varphi_n^{(s)} + \phi_n^{(s)\dagger}\gamma^5 e^{-\frac{\not{D}^{(s)}\not{D}^{(s)\dagger}}{M^2}}\phi_n^{(s)}\right].\end{aligned} \tag{A17}$$

One can show

$$\sum_n\phi_n^{(s)\dagger}(x)\gamma^5 e^{-\frac{\not{D}^{(s)}\not{D}^{(s)\dagger}}{M^2}}\phi_n^{(s)}(x) = \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{tr}\left[\gamma^5 e^{-\frac{\not{D}^{(s)}\not{D}^{(s)\dagger}}{M^2}}\right]e^{ik\cdot x}. \tag{A18}$$

As a result, we obtain

$$\begin{aligned}&\sum_n\left[\varphi_n^{(s)\dagger}(x)\gamma^5\varphi_n^{(s)}(x) + \phi_n^{(s)\dagger}\gamma^5\phi_n^{(s)}(x)\right] \\ &= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{tr}\left[\gamma^5\left(e^{-\frac{\not{D}^{(s)\dagger}\not{D}^{(s)}}{M^2}} + e^{-\frac{\not{D}^{(s)}\not{D}^{(s)\dagger}}{M^2}}\right)\right]e^{ik\cdot x}.\end{aligned} \tag{A19}$$

3. Chirality splitting

In order to perform the momentum integration in the above expression, we consider chirality splitting given by

$$\not{D}^{(s)} = (\not{\partial} + i\not{A}_+^{(s)})P_+ + (\not{\partial} + i\not{A}_-^{(s)})P_- \equiv \not{D}_+^{(s)}P_+ + \not{D}_-^{(s)}P_-, \quad \text{(A20)}$$

$$\not{D}^{(s)\dagger} = \not{D}_+^{(s)}P_- + \not{D}_-^{(s)}P_+, \quad \text{(A21)}$$

where

$$A_{\mu+}^{(s)} \equiv A_\mu + c_\mu + s\partial_\mu\theta(x), \quad \text{(A22)}$$

$$A_{\mu-}^{(s)} \equiv A_\mu - (c_\mu + s\partial_\mu\theta(x)), \quad \text{(A23)}$$

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma^5), \quad \text{(A24)}$$

$$\not{D}_\pm^{(s)} = \not{\partial} + i\not{A}_\pm^{(s)}. \quad \text{(A25)}$$

Then, we obtain

$$\not{D}^{(s)}\not{D}^{(s)\dagger} = (\not{D}_+^{(s)})^2P_- + (\not{D}_-^{(s)})^2P_+, \quad \not{D}^{(s)\dagger}\not{D}^{(s)} = (\not{D}_+^{(s)})^2P_+ + (\not{D}_-^{(s)})^2P_-, \quad \text{(A26)}$$

giving rise to

$$e^{-\frac{\not{D}^{(s)\dagger}\not{D}^{(s)}}{M^2}} = P_+e^{-\frac{(\not{D}_+^{(s)})^2}{M^2}} + P_-e^{-\frac{(\not{D}_-^{(s)})^2}{M^2}}, \quad \text{(A27)}$$

$$e^{-\frac{\not{D}^{(s)}\not{D}^{(s)\dagger}}{M^2}} = P_-e^{-\frac{(\not{D}_+^{(s)})^2}{M^2}} + P_+e^{-\frac{(\not{D}_-^{(s)})^2}{M^2}}. \quad \text{(A28)}$$

Now, it is straightforward to perform the momentum integration in the following way:

$$\begin{aligned}\sum_n\left[\varphi_n^{(s)\dagger}(x)\gamma^5\varphi_n^{(s)}(x) + \phi_n^{(s)\dagger}\gamma^5\phi_n^{(s)}(x)\right] &= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}tr\left[\gamma^5\left(e^{-\frac{(\not{D}_+^{(s)})^2}{M^2}} + e^{-\frac{(\not{D}_-^{(s)})^2}{M^2}}\right)\right]e^{ik\cdot x}\\ &= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}tr\left[\gamma^5\left(e^{-\frac{-(D_{+\mu}^{(s)}+ik_\mu)^2+\frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{+,\mu\nu}^{(s)}}{M^2}} + e^{-\frac{-(D_{-\mu}^{(s)}+ik_\mu)^2+\frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{-,\mu\nu}^{(s)}}{M^2}}\right)\right]\\ &= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}M^4e^{-k_\mu^2}tr\left[\gamma^5\left(-\frac{1}{8M^4}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta F_{+,\mu\nu}^{(s)}F_{+,\alpha\beta}^{(s)} - \frac{1}{8M^4}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta F_{-,\mu\nu}^{(s)}F_{-,\alpha\beta}^{(s)}\right)\right]\\ &= \int\frac{d^4k}{(2\pi)^4}e^{-k_\mu^2}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left[F_{+,\mu\nu}^{(s)}F_{+,\alpha\beta}^{(s)} + F_{-,\mu\nu}^{(s)}F_{-,\alpha\beta}^{(s)}\right]\\ &= \frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\left[F_{+,\mu\nu}^{(s)}F_{+,\alpha\beta}^{(s)} + F_{-,\mu\nu}^{(s)}F_{-,\alpha\beta}^{(s)}\right], \end{aligned} \quad \text{(A29)}$$

where

$$tr[\gamma^5] = tr[\gamma^5\gamma^\mu\gamma^\nu] = 0, \quad tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4\epsilon^{\mu\nu\alpha\beta} \quad \text{(A30)}$$

have been used.

Setting $\theta(x) = -c_\mu x^\mu$, we have

$$A_{\mu+}^{(s)} = A_\mu + c_\mu(1-s), \quad A_{\mu-}^{(s)} = A_\mu - c_\mu(1-s), \quad \text{(A31)}$$

$$F_{+,\mu\nu}^{(s)} = F_{-,\mu\nu}^{(s)} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \quad \text{(A32)}$$

Then, we obtain

$$\sum_n\left[\varphi_n^{(s)\dagger}(x)\gamma^5\varphi_n^{(s)}(x) + \phi_n^{(s)\dagger}\gamma^5\phi_n^{(s)}(x)\right] = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \quad \text{(A33)}$$

Finally, the effective action changes as

$$S_{\rm WM} \to S_{\rm WM}^{(s)} = \int d^4x\left[\bar{\Psi}(x)i\not{D}^{(s)}\Psi(x) - \int_0^s ds c_\mu x^\mu\frac{i}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right] \quad \text{(A34)}$$

under the chiral transformation, where $\not{D}^{(s)} = \gamma^\mu(\partial_\mu + iA_\mu + ic_\mu(1-s)\gamma^5)$. Setting $s = 1$, we obtain

$$S_{\rm WM}^{(0)} = \int d^4x[\bar{\Psi}(x)i\gamma^\mu(\partial_\mu + iA_\mu + ic_\mu\gamma^5)\Psi(x)] \quad \text{(A35)}$$

$$\Rightarrow S_{\rm WM}^{(1)} = \int d^4x\left[\bar{\Psi}(x)i\gamma^\mu(\partial_\mu + iA_\mu)\Psi(x) - \frac{ic_\mu x^\mu}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]. \quad \text{(A36)}$$

APPENDIX B: GAUGE ANOMALY OF THE U(1) SURFACE CURRENT IN BROKEN TIME-REVERSAL SYMMETRY WEYL METALS

We introduce a bosonic "spinor" $\phi(x)$ into an effective action of the surface Fermi-arc state as follows:

$$S = \int d^2x[\bar{\Psi}(x)i\gamma^\mu(\partial_\mu + ieA_\mu\mathcal{P}_-)\Psi(x) + \bar{\phi}(x)i\gamma^\mu \times (\partial_\mu + ieA_\mu\mathcal{P}_-)\phi(x) + \bar{\phi}(x)M\phi(x)], \quad \text{(B1)}$$

where M is the mass of the bosonic spinor field. Recall that $\mathcal{P}_-$ is the chirality projection operator. This is referred to as the Pauli-Villars regularization [22]. If we consider the chiral gauge transformation for $\bar{\phi}(x)$ and $\phi(x)$, they transform as $\bar{\Psi}(x)$ and $\Psi(x)$ and the mass term breaks the chiral gauge symmetry explicitly.

Performing the Fourier transformation, we obtain

$$S = \int \frac{d^2k}{(2\pi)^2}\bigg[\bar{\Psi}(k)\not{k}\Psi(k) + \bar{\phi}(k)(\not{k}+M)\phi(k) - \int \frac{d^2q}{(2\pi)^2}(\bar{\Psi}(k+q)\not{A}(q)\mathcal{P}_-\Psi(k) + \bar{\phi}(k+q)\not{A}(q)\mathcal{P}_-\phi(k))\bigg], \tag{B2}$$

where

$$\Psi(x) = \int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot x}\Psi(x),\quad \bar{\Psi}(x) = \int \frac{d^2k}{(2\pi)^2}e^{ik\cdot x}\bar{\Psi}(k), \tag{B3}$$

$$\phi(x) = \int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot x}\phi(x),\quad \bar{\phi}(x) = \int \frac{d^2k}{(2\pi)^2}e^{ik\cdot x}\bar{\phi}(k), \tag{B4}$$

$$A_\mu(q) = \int d^2x e^{iq\cdot x}A_\mu(x). \tag{B5}$$

Accordingly, Green functions are given by

$$G(k) = \langle\Psi(k)\bar{\Psi}(k)\rangle = \frac{\not{k}}{k^2},\quad \tilde{G}(k) = \langle\phi(k)\bar{\phi}(k)\rangle = \frac{\not{k}-M}{k^2-M^2}. \tag{B6}$$

The current operator is

$$j^\mu(q) = \int d^2x e^{-iq\cdot x}j^\mu(x) = \int d^2x e^{-iq\cdot x}\bar{\Psi}(x)\gamma^\mu\mathcal{P}_-\Psi(x) = \int \frac{d^2k}{(2\pi)^2}\bar{\Psi}(k+q)\gamma^\mu\mathcal{P}_-\Psi(k) \tag{B7}$$

under the Fourier transformation. Applying the Pauli-Villars regularization into the above expression, we obtain

$$j^\mu_{\rm reg}(q) = \int \frac{d^2k}{(2\pi)^2}[\bar{\Psi}(k+q)\gamma^\mu\mathcal{P}_-\Psi(k) + \bar{\phi}(k+q)\gamma^\mu\mathcal{P}_-\phi(k)]. \tag{B8}$$

Up to the one-loop order, we find

$$\begin{aligned}\langle j^\mu_{\rm reg}(q)\rangle &= \lim_{M^2\to\infty}\int \frac{d^2k}{(2\pi)^2}[-{\rm tr}(G(k)\gamma^\mu\mathcal{P}_-G(k+q)\gamma^\nu\mathcal{P}_-) + {\rm tr}(\tilde{G}(k)\gamma^\mu\mathcal{P}_-\tilde{G}(k+q)\gamma^\nu\mathcal{P}_-)]A_\nu(-q)\\ &= -\bigg[2(2q^\mu q^\nu - g^{\mu\nu}q^2) + i\epsilon^{\mu\nu}(q_\mu^2 - q_\nu^2) + 2i\sum_\alpha \delta_{\mu\nu}q_\alpha\epsilon^{\alpha\nu}q_\mu\bigg]\frac{A_\nu(-q)}{4\pi q^2}.\end{aligned} \tag{B9}$$

The difference in the sign comes from the fact whether the particle is a fermion or a boson. Here, we have used the following properties of γ^μ and the integral identity:

$${\rm tr}(\gamma^\mu) = {\rm tr}(\bar{\gamma}) = 0,\quad {\rm tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}, \tag{B10}$$

$${\rm tr}(\gamma^\mu\gamma^\nu\gamma^\alpha) = 0,\quad {\rm tr}(\gamma^\mu\gamma^\nu\bar{\gamma}) = \epsilon^{\mu\nu}2i\ (\epsilon^{zt}=1), \tag{B11}$$

$${\rm tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu) = 2(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}), \tag{B12}$$

$${\rm tr}(\gamma^\mu\bar{\gamma}) = {\rm tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\bar{\gamma}) = 0, \tag{B13}$$

$${\rm tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\bar{\gamma}) = -2i[\epsilon^{\mu\nu}\delta_{\alpha\beta}(\delta_{\beta\mu}+\delta_{\beta\nu}) + \epsilon^{\alpha\beta}\delta_{\mu\nu}(\delta_{\alpha\mu}+\delta_{\beta\mu})], \tag{B14}$$

$$\int \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2-\Delta)^n} = \frac{(-1)^n}{4\pi}\frac{1}{n-1}\frac{1}{\Delta^{n-1}}\quad (n>1). \tag{B15}$$

As a result, we find that the surface U(1) current given by the Fermi-arc state is not conserved:

$$q\cdot\langle j_{\rm reg}(q)\rangle = \frac{i\epsilon^{\mu\nu}q_\mu A_\nu(-q)}{4\pi} \Rightarrow \partial_\mu j^\mu(x) = \frac{i}{8\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{B16}$$

APPENDIX C: INVERSION AND TIME-REVERSAL TRANSFORMATION OPERATORS FOR THE LOW-ENERGY EFFECTIVE HAMILTONIAN

We point out that the inversion transformation operator is represented in the original-basis state $|\mathbf{k},\sigma,s\rangle$ as follows:

$$\begin{aligned}\mathcal{P}\mathcal{H}\mathcal{P}^{-1} &= \sum_{\mathbf{k}}\sum_{i,j}\mathcal{P}|\mathbf{k},i\rangle H_{ij}(\mathbf{k})\langle\mathbf{k},j|\mathcal{P}^{-1}\\ &= \sum_{\mathbf{k}}\sum_{i,j}|-\mathbf{k},i\rangle H_{ij}(-\mathbf{k})\langle-\mathbf{k},j| = \mathcal{H}\\ &\Rightarrow P_{ij} = \langle-\mathbf{k},i|\mathbf{k},j\rangle,\quad P_{i\alpha}H_{\alpha\beta}(\mathbf{k})P^{-1}_{\beta j} = H_{ij}(-\mathbf{k})\end{aligned} \tag{C1}$$

$$\Rightarrow P = \sigma_z : \begin{pmatrix}\mathcal{P}|\mathbf{k},\sigma=+,s\rangle = |-\mathbf{k},\sigma=+,s\rangle\\ \mathcal{P}|\mathbf{k},\sigma=-,s\rangle = -|-\mathbf{k},\sigma=-,s\rangle\end{pmatrix}. \tag{C2}$$

Note that the operator $\mathcal{P}$ relates $|\mathbf{k},i\rangle$ with $|-\mathbf{k},i\rangle$. Here, we find the representation for $\mathcal{P}$ in the enlarged Hilbert space $|\mathbf{k},\sigma,s,\tau\rangle$.

Considering

$$\mathcal{P}|\mathbf{k},\sigma,s\rangle \approx \mathcal{P}|\mathbf{k},\sigma,s,+\rangle \propto |-\mathbf{k},\sigma,s\rangle \approx |-\mathbf{k},\sigma,s,-\rangle, \tag{C3}$$

we obtain the representation of $\mathcal{P}$ in the enlarged Hilbert space, given by

$$\mathcal{P}\begin{pmatrix}|\mathbf{k},\sigma=+,s,\tau=+\rangle\\ |\mathbf{k},\sigma=-,s,\tau=+\rangle\\ |\mathbf{k},\sigma=+,s,\tau=-\rangle\\ |\mathbf{k},\sigma=-,s,\tau=-\rangle\end{pmatrix} = \begin{pmatrix}|-\mathbf{k},\sigma=+,s,\tau=-\rangle\\ -|-\mathbf{k},\sigma=-,s,\tau=-\rangle\\ |-\mathbf{k},\sigma=+,s,\tau=+\rangle\\ -|-\mathbf{k},\sigma=-,s,\tau=+\rangle\end{pmatrix}$$
$$\Rightarrow \tilde{P} = \sigma^z\otimes\tau^x. \tag{C4}$$

In the same way, we find the representation for the time-reversal transformation as well:

$$\tilde{T} = is^y\otimes\tau^x\mathcal{K}. \tag{C5}$$

APPENDIX D: DERIVATION OF AN EFFECTIVE AXIONIC ACTION FOR BROKEN INVERSION SYMMETRY WEYL METALS

1. $[\tau^z,\Gamma^\mu]=0$ and $\{s^z,\Gamma^\mu\}=0$

The procedure is quite similar to the case of broken time-reversal symmetry Weyl metals. Since any explicit calculations have not been shown for broken inversion symmetry Weyl metals as far as we know, we report all detailed steps in this appendix. Taking into account the chiral gauge transformation in this case [Eq. (38)], we obtain

$$\Psi \to e^{i\Gamma^5\tau^z\beta(x)},\quad \bar{\Psi} \to \bar{\Psi}e^{i\Gamma^5\tau^z\beta(x)}\ (\because [\tau^z,\Gamma^\mu]=0), \tag{D1}$$

$$S \to S - \int d^4x\,\bar{\Psi}(x)\Gamma^\mu\Gamma^5\tau^z\partial_\mu\beta(x)\Psi(x) \tag{D2}$$

$$\Rightarrow S' = \int d^4x\,\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - (\alpha+\partial_1\beta)\Gamma^1\Gamma^5\tau^z + k_0\Gamma^3\Gamma^5 s^z]\Psi(x), \tag{D3}$$

where we set $\partial_\mu\beta = \delta^{\mu 1}\partial_1\beta$. Considering $\beta(x)=ds\theta_v(x)$ and performing multiple steps of chiral rotations as discussed in the Appendix A, we obtain

$$S^{(s)} = \int d^4x\,\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - (\alpha+s\partial_1\theta_v)\Gamma^1\Gamma^5\tau^z + k_0\Gamma^3\Gamma^5 s^z]\Psi(x) \equiv \int d^4x\,\bar{\Psi}(x)i\not{D}^{(s)}\Psi(x), \tag{D4}$$

$$\not{D}^{(s)} = \Gamma^\mu\partial_\mu + i(\alpha+s\partial_1\theta_v)\Gamma^1\Gamma^5\tau^z - ik_0\Gamma^3\Gamma^5 s^z. \tag{D5}$$

In order to find an effective action involved with the α term, we introduce two types of gauge fields in addition to the conventional U(1) gauge field A_μ: the spin gauge field S_μ and the valley gauge field V_μ as follows:

$$\not{D}^{(s)} = \Gamma^\mu(\partial_\mu + iA_\mu + is^zS_\mu + i\tau^zV_\mu) + i(\alpha+s\partial_1\theta_v)\Gamma^1\Gamma^5\tau^z - ik_0\Gamma^3\Gamma^5 s^z. \tag{D6}$$

Following the Appendix A, we calculate the change of the integral measure under this chiral rotation and obtain an effective action

$$S^{(s)}_{\rm eff} = S^{(s)} + \int d^4x\int_0^s ds\theta_v(x)i \times \sum_n \left(\varphi_n^{(s)\dagger}\Gamma^5\tau^z\varphi_n^{(s)} + \phi_n^{(s)\dagger}\Gamma^5\tau^z\phi_n^{(s)}\right), \tag{D7}$$

where

$$\not{D}^{(s)\dagger}\not{D}^{(s)}\varphi_n^{(s)}(x) = \lambda_n^2\varphi_n^{(s)}(x),\quad \not{D}^{(s)}\not{D}^{(s)\dagger}\phi_n^{(s)}(x) = \lambda_n^2\phi_n^{(s)}(x), \tag{D8}$$

$$\not{D}^{(s)}\varphi_n^{(s)}(x) = \lambda_n\phi_n^{(s)}(x),\quad \not{D}^{(s)\dagger}\phi_n^{(s)}(x) = \lambda_n\varphi_n^{(s)}(x). \tag{D9}$$

The change of the integral measure can be evaluated in the following way:

$$\begin{aligned}
&\sum_n\left[\varphi_n^{(s)\dagger}(x)\gamma_v^5\varphi_n^{(s)}(x) + \phi_n^{(s)\dagger}\gamma_v^5\phi_n^{(s)}(x)\right]\\
&= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{tr}\left[\gamma_v^5\left(e^{-\frac{(\not{D}_+^{(s)})^2}{M^2}} + e^{-\frac{(\not{D}_-^{(s)})^2}{M^2}}\right)\right]e^{ik\cdot x}\\
&= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-\frac{k_\mu^2}{M^2}}\mathrm{tr}\left[\gamma_v^5\left(e^{-\frac{-(D_{+\mu}^{(s)})^2-2ik_\mu D_{+\mu}^{(s)}+\frac{i}{4}[\Gamma^\mu,\Gamma^\nu]F^{(s)}_{+,\mu\nu}}{M^2}} + e^{-\frac{-(D_{-\mu}^{(s)})^2-2ik_\mu D_{-\mu}^{(s)}+\frac{i}{4}[\Gamma^\mu,\Gamma^\nu]F^{(s)}_{-,\mu\nu}}{M^2}}\right)\right]\\
&= \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-k_\mu^2}\mathrm{tr}\left[-\frac{1}{16}\gamma_v^5([\Gamma^\mu,\Gamma^\nu](F_{\mu\nu}+\tau^zF_{v,\mu\nu}))^2\right]\\
&= -\frac{1}{4}\lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{-k_\mu^2}\mathrm{tr}\left[\gamma_v^5\Gamma^\mu\Gamma^\nu\Gamma^\alpha\Gamma^\beta\tau^z\left(F_{v,\mu\nu}F_{\mu\nu}+F_{\mu\nu}F_{v,\alpha\beta}\right)\right] = \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta},
\end{aligned} \tag{D10}$$

where

$$\not{D}^{(s)} = \not{D}_+^{(s)}P_{v,+} + \not{D}_-^{(s)}P_{v,-}, \tag{D11}$$

$$\not{D}_\pm^{(s)} = \Gamma^\mu\left(\partial_\mu + iA^{(s)}_{\pm,\mu}\right), \tag{D12}$$

$$A^{(s)}_{\pm,\mu} = A_\mu + \tau^zV_\mu + k_0\delta^{\mu 3}\tau^z \pm [-S_\mu + \delta^{\mu 1}(\alpha+s\partial_1\theta_v)], \tag{D13}$$

$$F^{(s)}_{\pm,\mu\nu} = \partial_\mu A^{(s)}_{\pm,\nu} - \partial_\nu A^{(s)}_{\pm,\mu} = F_{\mu\nu} + \tau^zF_{v,\mu\nu} \mp F_{s,\mu\nu}, \tag{D14}$$

$$P_{v,\pm} = \frac{1\pm\gamma_v^5}{2}. \tag{D15}$$

$F_{\mu\nu}$ is the field strength tensor of the U(1) gauge field and $F_{v,\mu\nu}$ is that of the valley gauge field. Here, we have used the representation of Γ^μ matrices in terms of γ^μ and τ^μ matrices ($\gamma_v^4 = -i\gamma_v^0$) since it is more convenient for calculations. We also used

$$\mathrm{tr}\big[\gamma_v^5\big] = \mathrm{tr}\big[\gamma_v^5\gamma_v^\mu\gamma_v^\nu\big] = 0, \tag{D16}$$

$$\mathrm{tr}\big[\gamma_v^5\gamma_v^\mu\gamma_v^\nu\gamma_v^\alpha\gamma_v^\beta\big] = -4\epsilon^{\mu\nu\alpha\beta}, \tag{D17}$$

$$\mathrm{tr}[A\otimes B] = \mathrm{tr}[A]\mathrm{tr}[B]. \tag{D18}$$

Finally, we reach the following expression after the chiral transformation:

$$\therefore S_{\mathrm{eff}}^{(s)} = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - (\alpha + s\partial_1\theta_v)\Gamma^1\Gamma^5\tau^z + k_0\Gamma^3\Gamma^5 s^z]\Psi(x) + i\int d^4x\int_0^s ds\frac{\theta_v}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta}. \tag{D19}$$

If we set $\theta_v = -\alpha x^1$ with $s = 1$, we obtain

$$S_{\mathrm{eff}}^{(1)} = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu + k_0\Gamma^3\Gamma^5 s^z]\Psi(x) - i\int d^4x\frac{\alpha x^1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta}. \tag{D20}$$

In other words, we find

$$S_{\mathrm{eff}}^v \equiv -\int d^4x\frac{i\alpha x^1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{v,\mu\nu}F_{\alpha\beta}, \tag{D21}$$

where both charge and valley gauge fields are involved. Note that the coefficient of the effective action is four times larger than that of the broken time-reversal symmetry Weyl metal.

2. $\{\tau^z,\Gamma^\mu\} = 0$ and $[s^z,\Gamma^\mu] = 0$

Now, we consider the other chiral rotation and obtain

$$\Psi \to e^{i\Gamma^5 s^z\beta(x)},\quad \bar{\Psi} \to \bar{\Psi}e^{i\Gamma^5 s^z\beta(x)}\ (\because [\tau^z,\Gamma^\mu] = 0), \tag{D22}$$

$$S \to S - \int d^4x\bar{\Psi}(x)\Gamma^\mu\Gamma^5 s^z\partial_\mu\beta(x)\Psi(x) \tag{D23}$$

$$\Rightarrow S' = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - \alpha\Gamma^1\Gamma^5\tau^z + (k_0 - \partial_3\beta)\Gamma^3\Gamma^5 s^z]\Psi(x), \tag{D24}$$

where we set $\partial_\mu\beta = \delta^{\mu 3}\partial_3\beta$. Taking $\beta(x) = ds\theta_s(x)$ and performing essentially the same steps of chiral rotations before, we obtain

$$S^{(s)} = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - \alpha\Gamma^1\Gamma^5\tau^z + (k_0 - s\partial_3\theta_s)\Gamma^3\Gamma^5 s^z]\Psi(x) \equiv \int d^4x\bar{\Psi}(x)i\not{D}^{(s)}\Psi(x), \tag{D25}$$

$$\not{D}^{(s)} = \Gamma^\mu\partial_\mu + i\alpha\Gamma^1\Gamma^5\tau^z - i(k_0 - s\partial_3\theta_v)\Gamma^3\Gamma^5 s^z. \tag{D26}$$

We also introduce the whole set of U(1) gauge fields, given by

$$\not{D}^{(s)} = \Gamma^\mu(\partial_\mu + iA_\mu + is^z S_\mu + i\tau^z V_\mu) + i\alpha\Gamma^1\Gamma^5\tau^z - i(k_0 - s\partial_3\theta_s)\Gamma^3\Gamma^5 s^z. \tag{D27}$$

Considering the change of the integral measure under this chiral rotation, we find the following effective action:

$$S_{\mathrm{eff}}^{(s)} = S^{(s)} + \int d^4x\int_0^s ds\theta_v(x)i\sum_n\big(\varphi_n^{(s)\dagger}\Gamma^5 s^z\varphi_n^{(s)} + \phi_n^{(s)\dagger}\Gamma^5 s^z\phi_n^{(s)}\big), \tag{D28}$$

where

$$\not{D}^{(s)\dagger}\not{D}^{(s)}\varphi_n^{(s)}(x) = \lambda_n^2\varphi_n^{(s)}(x),\quad \not{D}^{(s)}\not{D}^{(s)\dagger}\phi_n^{(s)}(x) = \lambda_n^2\phi_n^{(s)}(x), \tag{D29}$$

$$\not{D}^{(s)}\varphi_n^{(s)}(x) = \lambda_n\phi_n^{(s)}(x),\quad \not{D}^{(s)\dagger}\phi_n^{(s)}(x) = \lambda_n\varphi_n^{(s)}(x). \tag{D30}$$

It is essentially the same procedure to evaluate the change of the integral measure as follows:

$$\begin{aligned}
&\sum_n \left[\varphi_n^{(s)\dagger}(x)\gamma_s^5\varphi_n^{(s)}(x) + \phi_n^{(s)\dagger}\gamma_s^5\phi_n^{(s)}(x)\right] \\
&= \lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\mathrm{tr}\left[\gamma_s^5\left(e^{-\frac{(\not{D}_+^{(s)})^2}{M^2}} + e^{-\frac{(\not{D}_-^{(s)})^2}{M^2}}\right)\right]e^{ik\cdot x} \\
&= \lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4} e^{-\frac{k_\mu^2}{M^2}}\mathrm{tr}\left[\gamma_s^5\left(e^{-\frac{-(D_{+\mu}^{(s)})^2 - 2ik_\mu D_{+\mu}^{(s)} + \frac{i}{4}[\Gamma^\mu,\Gamma^\nu]F_{+,\mu\nu}^{(s)}}{M^2}} + e^{-\frac{-(D_{-\mu}^{(s)})^2 - 2ik_\mu D_{-\mu}^{(s)} + \frac{i}{4}[\Gamma^\mu,\Gamma^\nu]F_{-,\mu\nu}^{(s)}}{M^2}}\right)\right] \\
&= \lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4} e^{-k_\mu^2}\mathrm{tr}\left[-\frac{1}{16}\gamma_s^5([\Gamma^\mu,\Gamma^\nu](F_{\mu\nu} + s^zF_{s,\mu\nu}))^2\right] \\
&= -\frac{1}{4}\lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4} e^{-k_\mu^2}\mathrm{tr}\left[\gamma_s^5\Gamma^\mu\Gamma^\nu\Gamma^\alpha\Gamma^\beta s^z\left(F_{s,\mu\nu}F_{\mu\nu} + F_{\mu\nu}F_{s,\alpha\beta}\right)\right] = \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta},
\end{aligned} \tag{D31}$$

where

$$\not{D}^{(s)} = \not{D}_+^{(s)}P_{s,+} + \not{D}_-^{(s)}P_{s,-}, \tag{D32}$$

$$\not{D}_\pm^{(s)} = \Gamma^\mu\left(\partial_\mu + iA_{\pm,\mu}^{(s)}\right), \tag{D33}$$

$$A_{\pm,\mu}^{(s)} = A_\mu + s^z(S_\mu - \alpha\delta^{\mu 1}) \mp [V_\mu + (k_0 - s\partial_3\theta_v)\delta^{\mu 3}], \tag{D34}$$

$$F_{\pm,\mu\nu}^{(s)} = \partial_\mu A_{\pm,\nu}^{(s)} - \partial_\nu A_{\pm,\mu}^{(s)} = F_{\mu\nu} + s^zF_{s,\mu\nu} \mp F_{v,\mu\nu}, \tag{D35}$$

$$P_{s,\pm} = \frac{1\pm\gamma_s^5}{2}. \tag{D36}$$

Here, $F_{s,\mu\nu}$ is the field strength tensor of the spin gauge field.

As a result, we find

$$\begin{aligned}
S_{\text{eff}}^{(s)} &= \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - \alpha\Gamma^1\Gamma^5\tau^z + (k_0 - s\partial_3\theta_s)\Gamma^3\Gamma^5 s^z]\Psi(x) \\
&\quad + i\int d^4x\int_0^s ds\frac{\theta_s}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta}.
\end{aligned} \tag{D37}$$

Setting $\theta_s = k_0x^3$ and $s = 1$, we obtain

$$S_{\text{eff}}^{(1)} = \int d^4x\bar{\Psi}(x)[i\Gamma^\mu\partial_\mu - \alpha\Gamma^1\Gamma^5\tau^z]\Psi(x) + i\int d^4x\int_0^s ds\frac{k_0x^3}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta}, \tag{D38}$$

where the topological-in-origin θ term is given by

$$S_{\text{eff}}^{s} \equiv \int d^4x\frac{ik_0x^3}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{s,\mu\nu}F_{\alpha\beta}. \tag{D39}$$

APPENDIX E: DERIVATION OF A PAIR OF FERMI-ARC SURFACE STATES FOR INVERSION SYMMETRY-BREAKING WEYL METALS

1. Valley Hall current

If we express the Hamiltonian in terms of γ_v^μ matrices, we have

$$\begin{aligned}
H_{\text{WM}} &= \int d^3x\bar{\Psi}(x)\left[i\gamma_v^1\partial_1 + i\gamma_v^2\partial_2 + i\gamma_v^3\tau^z\partial_3 - \alpha\gamma_v^1\gamma_v^5\right. \\
&\quad \left. - k_0\gamma_v^3 + m\tau^z\right]\Psi(x) \\
&= \int d^3x\Psi^\dagger(x)\left[i\gamma_v^0\gamma_v^1\partial_1 + i\gamma_v^0\gamma_v^2\partial_2 + i\gamma_v^0\gamma_v^3\tau^z\partial_3\right. \\
&\quad \left. - \alpha\gamma_v^0\gamma_v^1\gamma_v^5 - k_0\gamma_v^0\gamma_v^3 + m\gamma_v^0\tau^z\right]\Psi(x),
\end{aligned} \tag{E1}$$

where

$$\begin{aligned}
&\gamma_v^0 = s^x\sigma^x,\quad \gamma_v^1 = -is^y,\quad \gamma_v^2 = -is^x\sigma^z, \\
&\gamma_v^3 = -is^x\sigma^y,\quad \gamma_v^5 = -s^z.
\end{aligned} \tag{E2}$$

Here, we introduced a term $m\tau^z\gamma_v^0\Psi^\dagger\Psi$ that gives a mass to each valley. Since this mass term preserves the time-reversal symmetry, it is allowed.

The above Hamiltonian looks quite similar to the Hamiltonian of the broken time-reversal symmetry Weyl metal except for the representation of gamma matrices. In this respect, we may use the boundary solution of the broken time-reversal symmetry Weyl metal in order to find that of the broken inversion symmetry Weyl metal. Unfortunately, it is not much straightforward to apply the case of the time-reversal symmetry breaking to that of the inversion symmetry breaking directly since the representations of gamma matrices are different from each other. Therefore we perform the canonical transformation to change the representation into Weyl one. Since the canonical transformation in the particle-number conserving system is nothing but the unitary transformation, we have

$$\mathcal{H} = \Psi^\dagger H \Psi = \Psi^\dagger U^\dagger U H U^\dagger U \Psi \equiv \Psi'^\dagger H' \Psi', \quad \text{(E3)}$$

$$\Psi' = U\Psi, \quad H' = UHU^\dagger. \quad \text{(E4)}$$

Here, unitary matrix U should satisfy the following relations:

$$U\gamma_v^0\gamma_v^1 U^\dagger = \gamma^0\gamma^1, \quad U\gamma_v^0\gamma_v^2 U^\dagger = \gamma^0\gamma^2, \quad \text{(E5)}$$

$$U\gamma_v^0\gamma_v^3 U^\dagger = \gamma^0\gamma^3, \quad U\gamma_v^0\gamma_v^5 U^\dagger = \gamma^0\gamma^5, \quad \text{(E6)}$$

where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then, the resulting unitary matrix U is given by

$$U = \eta\sigma^2 \otimes \frac{1+s^3}{2} + \xi\sigma^3 \otimes \frac{1-s^3}{2} = \begin{pmatrix} \eta\sigma^2 & 0 \\ 0 & \xi\sigma^3 \end{pmatrix}, \quad \begin{cases} \eta^*\eta = 1 \\ \xi^*\xi = 1 \end{cases}, \quad \text{(E7)}$$

where $\eta\xi^* = i$ ans $U\gamma_v^i U^\dagger = \gamma^i$ are satisfied.

Under this canonical transformation, we note that both time-reversal symmetry and inversion symmetry operators are changed as well:

$$\tilde{P} = \sigma^z \otimes \tau^x \rightarrow \bar{P} = U\sigma^z \otimes \tau^x U^\dagger = -\sigma^z \otimes s^z \otimes \tau^z, \quad \text{(E8)}$$

$$\tilde{T} = is^y \otimes \tau^x \mathcal{K} \rightarrow \bar{T} = Uis^y \otimes \tau^x \mathcal{K} U^\dagger = -\eta\xi\sigma^x \otimes s^y \otimes \tau^x \mathcal{K}. \quad \text{(E9)}$$

Rewriting the Hamiltonian in terms of this Weyl representation of gamma matrices, we obtain

$$H_{\rm WM} = \int d^3x \Psi'^\dagger(x)[i\gamma^0\gamma^1\partial_1 + i\gamma^0\gamma^2\partial_2 + i\gamma^0\gamma^3\tau^z\partial_3 - \alpha\gamma^0\gamma^1\gamma^5 - k_0\gamma^0\gamma^3 + m\gamma^0\tau^z]\Psi'(x), \quad \text{(E10)}$$

where $\Psi'(x) = U\Psi(x)$. Since we do not take into account any scattering terms between different valleys ($\tau^z = 1$ and $\tau^z = -1$), τ^z must be a good quantum number. As a result, we can divide the above Hamiltonian into two sectors of $\tau^z = 1$ and $\tau^z = -1$, given by

$$H_{\rm WM} = H_{{\rm WM},\tau^z=1} + H_{{\rm WM},\tau^z=-1}, \quad \text{(E11)}$$

$$H_{{\rm WM},\tau^z=\pm1} = \int d^3x \Psi'^\dagger_\pm[i\gamma^0\gamma^1\partial_1 + i\gamma^0\gamma^2\partial_2 \pm i\gamma^0\gamma^3\partial_3 - \alpha\theta(z)\gamma^0\gamma^1\gamma^5 - k_0\gamma^0\gamma^3 \pm m\gamma^0]\Psi'_\pm, \quad \text{(E12)}$$

$$\Psi'_\pm = \Psi'_{\tau^z=\pm1}, \quad \Psi' = \begin{pmatrix} \Psi'_{\tau^z=+1} \\ \Psi'_{\tau^z=-1} \end{pmatrix}. \quad \text{(E13)}$$

In the above, we set $\alpha = \alpha\theta(z)$ to get a boundary solution at the $z = 0$ plane. Then, the Dirac equation is also separated and given by

$$H_{\rm surf}\psi(x,y,z) = E\psi(x,y,z), \quad H_{\rm surf} = H_{{\rm surf},\tau^z=1} \oplus H_{{\rm surf},\tau^z=-1} \quad \text{(E14)}$$

$$\Rightarrow \psi(x,y,z) = \begin{pmatrix} \psi_{\tau^z=1} \\ \psi_{\tau^z=-1} \end{pmatrix}, \quad \begin{cases} H_{{\rm surf},\tau^z=1}\psi_{\tau^z=1} = E_+\psi_{\tau^z=1}, \\ H_{{\rm surf},\tau^z=-1}\psi_{\tau^z=-1} = E_-\psi_{\tau^z=-1}. \end{cases} \quad \text{(E15)}$$

First, we consider
(i) $\tau^z = 1$

$$H_{{\rm surf},\tau^z=1}\psi_{\tau^z=1} = E_+\psi_{\tau^z=1}, \quad \text{(E16)}$$

$$\begin{pmatrix} -i\vec{\sigma}\cdot\nabla - \alpha\theta(z)\sigma^1 + k_0\sigma^3 & m \\ m & i\vec{\sigma}\cdot\nabla - \alpha\theta(z)\sigma^1 - k_0\sigma^3 \end{pmatrix} \times \psi_{\tau^z=1}(x,y,z) = E_+\psi_{\tau^z=1}(x,y,z). \quad \text{(E17)}$$

Since there are translational symmetries along the x and y axes, we set $\psi_{\tau^z=1}(x,y,z) = e^{ik_x x + ik_y y}\phi_{\tau^z=1,k_x,k_y}(z)$ and obtain

$$\begin{pmatrix} \sigma^1(k_x - \alpha\theta(z)) + \sigma^2 k_y + \sigma^3(-i\partial_z + k_0) & m \\ m & -\sigma^1(k_x + \alpha\theta(z)) - \sigma^2 k_y + \sigma^3(i\partial_z - k_0) \end{pmatrix}\phi_{\tau^z=1,k_x,k_y}(z) = E_+\phi_{\tau^z=1,k_x,k_y}(z). \quad \text{(E18)}$$

In order to solve this equation, we use the following ansatz:

$$\phi_{+,k_x,k_y}(z) = u_+(z)\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} + v_+(z)\begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}. \quad \text{(E19)}$$

We note that both eigenstates have the eigenvalue -1 for $\gamma^0\gamma^2$. Then, we obtain

$$(ik_x - (i\alpha\theta(z) + i\partial_z - k_0))u_+(z) + mv_+(z) = 0, \quad \text{(E20)}$$

$$mu_+(z) + (ik_x + (i\alpha\theta(z) + i\partial_z - k_0))v_+(z) = 0, \quad \text{(E21)}$$

$$E_+ = k_y \quad \text{(E22)}$$

$$\Rightarrow \begin{cases} i(k_x - \partial_z)\tilde{u}_+(z) + m\tilde{v}_+(z) = 0 \\ m\tilde{u}_+(z) + i(k_x + \partial_z)\tilde{v}_+(z) = 0 \end{cases},$$

$$\begin{cases} u_+(z) = e^{-ik_0 z - \alpha\theta(z)z}\tilde{u}_+(z) \\ v_+(z) = e^{-ik_0 z - \alpha\theta(z)z}\tilde{v}_+(z) \end{cases} \quad \text{(E23)}$$

$$\Rightarrow \left(\partial_z^2 - \left(k_x^2 + m^2\right)\right)\tilde{u}_+(z)/\tilde{v}_+(z) = 0 \quad \text{(E24)}$$

$$\Rightarrow \begin{cases} \tilde{u}_+(z) = A_+^1 e^{-\sqrt{k_x^2+m^2}z} + A_+^2 e^{\sqrt{k_x^2+m^2}z} \\ \tilde{v}_+(z) = B_+^1 e^{-\sqrt{k_x^2+m^2}z} + B_+^2 e^{\sqrt{k_x^2+m^2}z} \end{cases}. \quad \text{(E25)}$$

Boundary conditions for $\tilde{u}_+$ and $\tilde{v}_+$ are given as follows:

$$\lim_{\epsilon\to 0^+} \tilde{u}_+(\epsilon) = \lim_{\epsilon\to 0^-} \tilde{u}_+(\epsilon), \quad \lim_{\epsilon\to 0^+} \tilde{v}_+(\epsilon) = \lim_{\epsilon\to 0^-} \tilde{v}_+(\epsilon), \tag{E26}$$

$$\lim_{\epsilon\to 0^+} \tilde{u}'_+(\epsilon) = \lim_{\epsilon\to 0^-} \tilde{u}'_+(\epsilon), \quad \lim_{\epsilon\to 0^+} \tilde{v}'_+(\epsilon) = \lim_{\epsilon\to 0^-} \tilde{v}'_+(\epsilon), \tag{E27}$$

where the prime symbol represents a derivative with resect to the argument.

Considering the boundary conditions and normalizability of the wave functions, we find

$$-\sqrt{\alpha^2 - m^2} < k_x < \sqrt{\alpha^2 - m^2} \tag{E28}$$

$$\phi_{+,k_x,k_y}(z) = A_+^2 e^{-ik_0 z} e^{-\alpha\theta(z)z} e^{\sqrt{k_x^2+m^2}z} \begin{pmatrix} 1 \\ i \\ i\frac{m}{\sqrt{k_x^2+m^2}+k_x} \\ \frac{m}{\sqrt{k_x^2+m^2}+k_x} \end{pmatrix} \tag{E29}$$

$$|A_+^2|^2 = \frac{m^2}{m^2 + \left(\sqrt{k_x^2+m^2} - k_x\right)^2} \frac{\sqrt{k_x^2+m^2}\left(\alpha - \sqrt{k_x^2+m^2}\right)}{\alpha}, \tag{E30}$$

$$E_+ = k_y. \tag{E31}$$

Next, we consider

(ii) $\tau^z = -1$

$$H_{\mathrm{surf},\tau^z=-1}\psi_{\tau^z=-1} = E_-\psi_{\tau^z=-1} \tag{E32}$$

$$\begin{pmatrix} -i\vec{\sigma}\cdot\nabla_\perp + i\sigma^3\partial_z - \alpha\theta(z)\sigma^1 + k_0\sigma^3 & -m \\ -m & i\vec{\sigma}\cdot\nabla_\perp - i\sigma^3\partial_z - \alpha\theta(z)\sigma^1 - k_0\sigma^3 \end{pmatrix}\psi_{\tau^z=-1}(x,y,z) = E_-\psi_{\tau^z=-1}(x,y,z). \tag{E33}$$

Setting $\psi_{\tau^z=-1}(x,y,z) = e^{ik_x x + ik_y y}\phi_{\tau^z=-1,k_x,k_y}(z)$, we have

$$\begin{pmatrix} \sigma^1(k_x - \alpha\theta(z)) + \sigma^2 k_y + \sigma^3(i\partial_z + k_0) & -m \\ -m & -\sigma^1(k_x + \alpha\theta(z)) - \sigma^2 k_y + \sigma^3(-i\partial_z - k_0) \end{pmatrix} \times\phi_{\tau^z=-1,k_x,k_y}(z) = E_-\phi_{\tau^z=-1,k_x,k_y}(z). \tag{E34}$$

Now, we consider the ansatz of

$$\phi_{\tau^z=-1,k_x,k_y}(z) = u_-(z)\begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} + v_-(z)\begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}, \tag{E35}$$

where both eigenstates have the eigenvalue 1 for $\gamma^0\gamma^2$. Then, we obtain

$$(i\partial_z + k_0 + i\alpha\theta(z) - ik_x)u_-(z) - mv_-(z) = 0, \tag{E36}$$

$$mu_-(z) + (i\partial_z + k_0 + i\alpha\theta(z) + ik_x)v_-(z) = 0, \tag{E37}$$

$$E_- = -k_y \tag{E38}$$

$$\Rightarrow \begin{cases} i(\partial_z - k_x)\tilde{u}_-(z) - m\tilde{v}_-(z) = 0 \\ m\tilde{u}_- + i(\partial_z + k_x)\tilde{v}_-(z) = 0 \end{cases},$$

$$\begin{cases} u_(z) = e^{ik_0 z - \alpha\theta(z)z}\tilde{u}_-(z) \\ v_(z) = e^{ik_0 z - \alpha\theta(z)z}\tilde{v}_-(z) \end{cases} \tag{E39}$$

$$\Rightarrow \left(\partial_z^2 - \left(k_x^2 + m^2\right)\right)\tilde{u}_-(z)/\tilde{v}_-(z) = 0 \tag{E40}$$

$$\Rightarrow \begin{cases} \tilde{u}_-(z) = A_-^1 e^{-\sqrt{k_x^2+m^2}z} + A_-^2 e^{\sqrt{k_x^2+m^2}z} \\ \tilde{v}_-(z) = B_-^1 e^{-\sqrt{k_x^2+m^2}z} + B_-^2 e^{\sqrt{k_x^2+m^2}z} \end{cases}. \tag{E41}$$

Considering the boundary conditions and normalizability of the wave functions, we obtain

$$-\sqrt{\alpha^2 - m^2} < k_x < \sqrt{\alpha^2 - m^2}, \tag{E42}$$

$$\phi_{-,k_x,k_y}(z) = A_-^2 e^{ik_0 z} e^{-\alpha\theta(z)z} e^{\sqrt{k_x^2+m^2}z} \begin{pmatrix} 1 \\ -i \\ i\frac{m}{\sqrt{k_x^2+m^2}+k_x} \\ -\frac{m}{\sqrt{k_x^2+m^2}+k_x} \end{pmatrix}, \tag{E43}$$

$$|A_-^2|^2 = \frac{m^2}{m^2 + \left(\sqrt{k_x^2+m^2} - k_x\right)^2} \times \frac{\sqrt{k_x^2+m^2}\left(\alpha - \sqrt{k_x^2+m^2}\right)}{\alpha}, \tag{E44}$$

$$E_- = -k_y. \tag{E45}$$

In summary, we find a pair of Fermi-arc surface states:

$$-\sqrt{\alpha^2 - m^2} < k_x < \sqrt{\alpha^2 - m^2}, \tag{E46}$$

$$\psi_{\tau^z=1,k_x,k_y} = A(k_x)e^{ik_x x}e^{ik_y y}e^{-ik_0 z}e^{-\alpha\theta(z)z}e^{\sqrt{k_x^2+m^2}z} \times \begin{pmatrix} 1 \\ i \\ C(k_x) \\ -iC(k_x) \end{pmatrix}, \quad E_+ = k_y, \tag{E47}$$

$$\psi_{\tau^z=-1,k_x,k_y} = A(k_x)e^{ik_x x}e^{ik_y y}e^{ik_0 z}e^{-\alpha\theta(z)z}e^{\sqrt{k_x^2+m^2}z} \times \begin{pmatrix} 1 \\ -i \\ C(k_x) \\ iC(k_x) \end{pmatrix}, \quad E_- = -k_y, \tag{E48}$$

$$|A(k_x)|^2 = \frac{m^2}{m^2 + \left(\sqrt{k_x^2+m^2} - k_x\right)^2} \times \frac{\sqrt{k_x^2+m^2}\left(\alpha - \sqrt{k_x^2+m^2}\right)}{\alpha}, \tag{E49}$$

$$C(k_x) = i\frac{m}{\sqrt{k_x^2+m^2}+k_x}, \tag{E50}$$

where they are characterized by opposite chirality quantum numbers given by

$$\gamma^0\gamma^2\psi_{\tau^z=1,k_x,k_y} = -\psi_{\tau^z=1,k_x,k_y}, \quad \gamma^0\gamma^2\psi_{\tau^z=-1,k_x,k_y} = +\psi_{\tau^z=-1,k_x,k_y}. \tag{E51}$$

2. Spin Hall current

The effective Hamiltonian in the γ_s^μ representation is

$$H_{\rm WM} = \int d^3x \Psi^\dagger(x)\big[i\gamma_s^0\gamma_s^1 s^z\partial_x + i\gamma_s^0\gamma_s^2\partial_y + i\gamma_s^0\gamma_s^3\partial_z + \alpha\gamma_s^0\gamma_s^1 + k_0\gamma_s^0\gamma_s^3\gamma_s^5 + m\gamma_s^0\big]\Psi(x), \tag{E52}$$

where

$$\gamma_s^0 = \sigma^z\tau^x,\quad \gamma_s^1 = i\sigma^y\tau^x,\quad \gamma_s^2 = i\sigma^x\tau^x, \quad \gamma_s^3 = -i\tau^y,\quad \gamma_s^5 = -\tau^z. \tag{E53}$$

Here, we also introduced a term $m\gamma_s^0\Psi^\dagger\Psi$ that gives a mass to to each spin sector. This mass term also respects the time-reversal symmetry.

Following the previous section, we find the canonical transformation

$$U = \eta\sigma^2 \otimes \frac{1+\tau^3}{2} + \xi\sigma^1 \otimes \frac{1-\tau^3}{2} = \begin{pmatrix} \eta\sigma^2 & 0 \\ 0 & \xi\sigma^1 \end{pmatrix}, \quad \eta\eta^* = \xi\xi^* = 1, \tag{E54}$$

where $\eta^*\xi = i$ and $U\gamma_s^i U^\dagger = \gamma^i$ are satisfied.

Under this canonical transformation, both the time-reversal symmetry and inversion symmetry operators are also changed as follows:

$$\tilde{P} = \sigma^z \otimes \tau^x \to \bar{P} = U\sigma^z \otimes \tau^x U^\dagger = \tau^x, \tag{E55}$$

$$\tilde{T} = is^y \otimes \tau^x\mathcal{K} \to \bar{T} = Uis^y \otimes \tau^x\mathcal{K}U^\dagger = \eta\xi s^y \otimes \sigma^z \otimes \tau^x\mathcal{K}. \tag{E56}$$

Now, we start from

$$H_{\rm WM} = \int d^3x \Psi'^\dagger(x)[i\gamma^0\gamma^1 s^z\partial_x + i\gamma^0\gamma^2\partial_y + i\gamma^0\gamma^3\partial_z + \alpha\gamma^0\gamma^1 + k_0\gamma^0\gamma^3\gamma^5 + m\gamma^0]\Psi'(x), \tag{E57}$$

where $\Psi'(x) = U\Psi(x)$. Since we do not consider any scattering terms between different spin sections ($s^z = 1$ and $s^z = -1$), s^z is a good quantum number. As a result, the above Hamiltonian is separated into two spin sectors with $s^z = 1$ and $s^z = -1$:

$$H_{\rm WM} = H_{{\rm WM},s^z=1} + H_{{\rm WM},s^z=-1}, \tag{E58}$$

$$H_{{\rm WM},s^z=\pm1} = \int d^3x \Psi_\pm'^\dagger[\pm i\gamma^0\gamma^1\partial_x + i\gamma^0\gamma^2\partial_y + i\gamma^0\gamma^3\partial_z + \alpha\gamma^0\gamma^1 + k_0\theta(x)\gamma^0\gamma^3\gamma^5 + m\gamma^0]\Psi'_\pm, \tag{E59}$$

$$\Psi'_\pm = \Psi'_{s^z=\pm1},\quad \Psi' = \begin{pmatrix} \Psi'_{s^z=+1} \\ \Psi'_{s^z=-1} \end{pmatrix}. \tag{E60}$$

Here, we set $k_0 = k_0\theta(x)$ to get a boundary solution at the $x = 0$ plane. Accordingly, the Dirac equation is

$$H_{\rm surf}\psi(x,y,z) = E\psi(x,y,z), \quad H_{\rm surf} = H_{{\rm surf},s^z=1} \oplus H_{{\rm surf},s^z=-1} \tag{E61}$$

$$\Rightarrow \psi(x,y,z) = \begin{pmatrix} \psi_{s^z=1} \\ \psi_{s^z=-1} \end{pmatrix}, \quad \begin{cases} H_{{\rm surf},s^z=1}\psi_{s^z=1} = E_+\psi_{s^z=1}, \\ H_{{\rm surf},s^z=-1}\psi_{s^z=-1} = E_-\psi_{s^z=-1} \end{cases}. \tag{E62}$$

First, we consider

(i) $s^z = 1$.

$$H_{{\rm surf},s^z=1}\psi_{s^z=1} = E_+\psi_{s^z=1}, \tag{E63}$$

$$\begin{pmatrix} -i\vec{\sigma}\cdot\nabla - \alpha\sigma^1 + k_0\theta(x)\sigma^3 & m \\ m & i\vec{\sigma}\cdot\nabla + \alpha\sigma^1 + k_0\theta(x)\sigma^3 \end{pmatrix}\psi_{s^z=1}(x,y,z) = E_+\psi_{s^z=1}(x,y,z). \tag{E64}$$

Since there are translational symmetries along the y and z axes, we set $\psi_{s^z=1}(x,y,z) = e^{ik_y y + ik_z z}\phi_{s^z=1,k_y,k_z}(x)$ and obtain

$$\begin{pmatrix} \sigma^1(-i\partial_x - \alpha) + \sigma^2 k_y + \sigma^3(k_z + k_0\theta(x)) & m \\ m & \sigma^1(i\partial_x + \alpha) - \sigma^2 k_y - \sigma^3(k_z - k_0\theta(x)) \end{pmatrix}\phi_{s^z=1,k_y,k_z}(x) = E_+\phi_{s^z=1,k_y,k_z}(x). \tag{E65}$$

Following the previous section, we use the ansatz of

$$\phi_{+,k_y,k_z}(x) = u_+(x)\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} + v_+(x)\begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}. \tag{E66}$$

Here, both eigenstates have the eigenvalue $\gamma^0\gamma^2 = -1$. Then, we obtain

$$(\partial_x - i\alpha + k_0\theta(x) + k_z)u_+(x) + mv_+(x) = 0, \tag{E67}$$

$$mu_+(x) + (\partial_x - i\alpha + k_0\theta(x) - k_z)v_+(x) = 0, \tag{E68}$$

$$E_+ = k_y \tag{E69}$$

$$\Rightarrow \begin{cases}(\partial_x + k_z)\tilde{u}_+(x) + m\tilde{v}_+(x) = 0 \\ m\tilde{u}_+(x) + (\partial_x - k_z)\tilde{v}_+(x) = 0\end{cases}, \quad \begin{cases}u_+(x) = e^{i\alpha x - k_0\theta(x)x}\tilde{u}_+(x) \\ v_+(x) = e^{i\alpha x - k_0\theta(x)x}\tilde{v}_+(x)\end{cases} \tag{E70}$$

$$\Rightarrow \left(\partial_x^2 - \left(k_z^2 + m^2\right)\right)\tilde{u}_+(x)/\tilde{v}_+(x) = 0 \tag{E71}$$

$$\Rightarrow \begin{cases}\tilde{u}_+(x) = A_+^1 e^{-\sqrt{k_z^2+m^2}x} + A_+^2 e^{\sqrt{k_z^2+m^2}x} \\ \tilde{v}_+(x) = B_+^1 e^{-\sqrt{k_z^2+m^2}x} + B_+^2 e^{\sqrt{k_z^2+m^2}x}\end{cases}. \tag{E72}$$

Considering the boundary conditions and normalizability of the wave functions, we find

$$-\sqrt{k_0^2 - m^2} < k_z < \sqrt{k_0^2 - m^2}, \tag{E73}$$

$$\phi_{+,k_y,k_z} = A_+^2 e^{i\alpha x} e^{-k_0\theta(x)x} e^{\sqrt{k_z^2+m^2}} \begin{pmatrix} 1 \\ i \\ -\frac{m}{\sqrt{k_z^2+m^2}-k_z} \\ \frac{im}{\sqrt{k_z^2+m^2}-k_z}\end{pmatrix}, \tag{E74}$$

$$E_+ = k_y. \tag{E75}$$

Here, A_+^2 is the same as that of the previous section.

Next, we consider

(ii) $s^z = -1$.

$$H_{\text{surf},s^z=-1}\psi_{s^z=-1} = E_-\psi_{s^z=-1}, \tag{E76}$$

$$\begin{pmatrix} i\sigma^1\partial_x - i\vec{\sigma}\cdot\nabla_\perp - \alpha\sigma^1 + k_0\theta(x)\sigma^3 & m \\ m & -i\sigma^1\partial_x + i\vec{\sigma}\cdot\nabla_\perp + \alpha\sigma^1 + k_0\theta(x)\sigma^3\end{pmatrix}\psi_{s^z=-1}(x,y,z) = E_-\psi_{s^z=-1}(x,y,z). \tag{E77}$$

Taking $\psi_{s^z=-1}(x,y,z) = e^{ik_y y + ik_z z}\phi_{s^z=-1,k_y,k_z}$, we have

$$\begin{pmatrix} \sigma^1(i\partial_x - \alpha) + \sigma^2 k_y + \sigma^3(k_z + k_0\theta(x)) & m \\ m & \sigma^1(-i\partial_x + \alpha) - \sigma^2 k_y - \sigma^3(k_z - k_0\theta(x))\end{pmatrix}\phi_{s^z=-1,k_y,k_z}(x) = E_-\phi_{s^z=-1,k_y,k_z}(x). \tag{E78}$$

Now, the ansatz is

$$\phi_{s^z=-1,k_y,k_z}(x) = u_-(x)\begin{pmatrix}1\\-i\\0\\0\end{pmatrix} + v_-(x)\begin{pmatrix}0\\0\\1\\i\end{pmatrix}, \tag{E79}$$

where both eigenstates have the eigenvalue $\gamma^0\gamma^2 = 1$. As a result, we obtain

$$(\partial_x + i\alpha + k_0\theta(x) + k_z)u_-(x) + mv_-(x) = 0, \tag{E80}$$

$$mu_-(x) + (\partial_x + i\alpha + k_0\theta(x) - k_z) = 0, \tag{E81}$$

$$E_- = -k_y \tag{E82}$$

$$\Rightarrow \begin{cases}(\partial_x + k_z)\tilde{u}_-(x) + m\tilde{v}_-(x) = 0 \\ m\tilde{u}_-(x) + (\partial_x - k_z)\tilde{v}_-(x) = 0\end{cases}, \quad \begin{cases}u_-(x) = e^{-i\alpha x - k_0\theta(x)x}\tilde{u}_-(x) \\ v_-(x) = e^{-i\alpha x - k_0\theta(x)x}\tilde{v}_-(x)\end{cases} \tag{E83}$$

$$\Rightarrow \left(\partial_x^2 - \left(k_z^2 + m^2\right)\right)\tilde{u}_-(x)/\tilde{v}_-(x) = 0 \tag{E84}$$

$$\Rightarrow \begin{cases}\tilde{u}_-(x) = A_-^1 e^{-\sqrt{k_z^2+m^2}x} + A_-^2 e^{\sqrt{k_z^2+m^2}x} \\ \tilde{v}_-(x) = B_-^1 e^{-\sqrt{k_z^2+m^2}x} + B_-^2 e^{\sqrt{k_z^2+m^2}x}\end{cases}. \tag{E85}$$

Considering the boundary conditions and normalizability of the wave functions, we find

$$-\sqrt{k_0^2 - m^2} < k_z < \sqrt{k_0^2 - m^2}, \tag{E86}$$

$$\phi_{-,k_y,k_z} = A_-^2 e^{i\alpha x} e^{-k_0\theta(x)x} e^{\sqrt{k_z^2+m^2}x} \begin{pmatrix} 1 \\ -i \\ -\frac{m}{\sqrt{k_z^2+m^2}-k_z} \\ -\frac{im}{\sqrt{k_z^2+m^2}-k_z} \end{pmatrix}. \tag{E87}$$

Here, A_-^2 is the same as that of the previous section.
In summary, we find a pair of Fermi-arc surface states

$$-\sqrt{k_0^2 - m^2} < k_z < \sqrt{k_0^2 - m^2}, \tag{E88}$$

$$\psi_{s^z=1,k_y,k_z} = A(k_z) e^{ik_y y + ik_z z} e^{i\alpha x} e^{-k_0\theta(x)x} e^{\sqrt{k_z^2+m^2}x} \begin{pmatrix} 1 \\ i \\ C(k_z) \\ -iC(k_z) \end{pmatrix}, \quad E_+ = k_y, \tag{E89}$$

$$\psi_{s^z=-1,k_y,k_z} = A(k_z) e^{ik_y y + ik_z z} e^{i\alpha x} e^{-k_0\theta(x)x} e^{\sqrt{k_z^2+m^2}x} \begin{pmatrix} 1 \\ -i \\ -C(k_z) \\ -iC(k_z) \end{pmatrix}, \quad E_- = -k_y, \tag{E90}$$

$$C(k_z) = \frac{m}{\sqrt{k_z^2 + m^2} - k_z}, \tag{E91}$$

where they are characterized by opposite chirality quantum numbers given by

$$\gamma^0\gamma^2\psi_{s^z=1,k_y,k_z} = -\psi_{s^z=1,k_y,k_z}, \quad \gamma^0\gamma^2\psi_{s^z=-1,k_y,k_z} = +\psi_{s^z=-1,k_y,k_z}. \tag{E92}$$

APPENDIX F: CALCULATION OF THE ONE-LOOP QUANTUM CORRECTION TO THE U(1) SURFACE CURRENT IN THE CASE OF INVERSION SYMMETRY-BREAKING WEYL METALS

1. Valley Hall current

Since we do not take into account any interactions between fields with different k_x momentum, we will not include the summation over k_x from now on. Introducing the Pauli-Villars regularization field into the effective action, we start from the following surface action

$$S = \int d^2x[\bar{\Psi}(x)i\gamma^\mu(\partial_\mu + iA_\mu + iV_\mu\bar{\gamma})\Psi(x) + \bar{\phi}(x)i\gamma^\mu(\partial_\mu + iA_\mu + iV_\mu\bar{\gamma})\phi(x) + \bar{\phi}(x)M\phi(x)]. \tag{F1}$$

If we consider the valley gauge transformation of $\bar{\phi}(x)$ and $\phi(x)$ like that of $\bar{\Psi}(x)$ and $\Psi(x)$, the mass term breaks the valley gauge symmetry explicitly.

Under the Fourier transformations

$$\Psi(x) = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x}\Psi(x), \quad \bar{\Psi}(x) = \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot x}\bar{\Psi}(k), \tag{F2}$$

$$\phi(x) = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x}\phi(x), \quad \bar{\phi}(x) = \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot x}\bar{\phi}(k), \tag{F3}$$

$$A_\mu(q) = \int d^2x e^{iq\cdot x}A_\mu(x), \quad V_\mu(q) = \int d^2x e^{iq\cdot x}A_\mu(x), \tag{F4}$$

we obtain

$$S = \int \frac{d^2k}{(2\pi)^2}\bigg[\bar{\Psi}(k)\not{k}\Psi(k) + \bar{\phi}(k)(\not{k} + M)\phi(k) - \int \frac{d^2q}{(2\pi)^2}(\bar{\Psi}(k+q)\not{A}(q)\mathcal{P}_+\Psi(k) + \bar{\phi}(k+q)\not{A}(q)\mathcal{P}_+\phi(k))$$
$$- \int \frac{d^2q}{(2\pi)^2}(\bar{\Psi}(k+q)\not{V}(q)\bar{\gamma}\Psi(k) + \bar{\phi}(k+q)\not{V}(q)\bar{\gamma}\phi(k))\bigg], \tag{F5}$$

where both Green's functions are

$$G(k) = \langle \Psi(k)\bar{\Psi}(k)\rangle = \frac{\not{k}}{k^2}, \quad \tilde{G}(k) = \langle \phi(k)\bar{\phi}(k)\rangle = \frac{\not{k} - M}{k^2 - M^2}. \tag{F6}$$

The valley current of

$$j_v^\mu(q) = \int d^2x e^{-iq\cdot x} j_v^\mu(x) = \int d^2x e^{-iq\cdot x}\bar{\Psi}(x)\gamma^\mu\bar{\gamma}\Psi(x) = \int \frac{d^2k}{(2\pi)^2}\bar{\Psi}(k+q)\gamma^\mu\bar{\gamma}\Psi(k) \tag{F7}$$

is regularized as

$$j_{\text{reg}}^\mu(q) = \int \frac{d^2k}{(2\pi)^2}[\bar{\Psi}(k+q)\gamma^\mu\bar{\gamma}\Psi(k) + \bar{\phi}(k+q)\gamma^\mu\bar{\gamma}\phi(k)]. \tag{F8}$$

Up to the one-loop order, there are two contributions from A_μ and V_μ, respectively, in the following way:

$$\begin{aligned}\langle j_{\text{reg}}^\mu(q)\rangle = &\int \frac{d^2k}{(2\pi)^2}[-\text{tr}(G(k)\gamma^\mu\bar{\gamma}G(k+q)\gamma^\nu) + \text{tr}(\tilde{G}(k)\gamma^\mu\bar{\gamma}\tilde{G}(k+q)\gamma^\nu)]A_\nu(-q)\\ &+ \int \frac{d^2k}{(2\pi)^2}[\text{tr}(G(k)\gamma^\mu\bar{\gamma}G(k+q)\gamma^\nu\bar{\gamma}) - \text{tr}(\tilde{G}(k)\gamma^\mu\bar{\gamma}\tilde{G}(k+q)\gamma^\nu\bar{\gamma})]V_\nu(-q).\end{aligned} \tag{F9}$$

As a result, we find

$$\therefore \langle j_{\text{reg}}^\mu(q)\rangle = \sum_\nu\left[\frac{i\left(\epsilon^{\mu\nu}\left(q_\mu^2 - q_\nu^2\right) + 2\delta_{\mu\nu}\sum_\alpha \epsilon^{\alpha\mu}q_\mu q_\alpha\right)}{2\pi q^2} - \frac{i}{2\pi}\epsilon^{\mu\nu}\right]A_\nu(-q) + \sum_\nu \frac{q^\mu q^\nu}{\pi q^2}V_\nu(-q) \tag{F10}$$

$$\begin{aligned}\Rightarrow q_\mu\langle j_{\text{reg}}^\mu(q)\rangle &= \sum_{\mu,\nu}\left[\frac{i}{2\pi q^2}\left(\epsilon^{\mu\nu}q_\mu\left(q_\mu^2 - q_\nu^2\right) + 2q_\mu\delta_{\mu\nu}\sum_\alpha \epsilon^{\alpha\mu}q_\mu q_\alpha\right) - \frac{i}{2\pi}\epsilon^{\mu\nu}q_\mu\right]A_\nu(-q) + \sum_\nu \frac{1}{\pi}q^\nu V_\nu(-q)\\ &= \frac{i}{2\pi}\sum_{\mu,\nu}\left[\epsilon^{\mu\nu}q_\mu\left(q_\mu^2 + q_\nu^2\right)/q^2 - \epsilon^{\mu\nu}q_\mu\right]A_\nu(-q) + \sum_\nu \frac{2}{\pi}q^\nu V_\nu(-q) = -\frac{i}{\pi}\sum_{\mu,\nu}\epsilon^{\mu\nu}q_\mu A_\nu(-q) + \sum_\nu \frac{1}{\pi}q^\nu V_\nu(-q).\end{aligned} \tag{F11}$$

If we ignore the contribution from the V_ν field considering that it is a fictitious gauge field, we obtain the anomaly

$$q_\mu\langle j_{\text{reg}}^\mu(q)\rangle = -\frac{i}{\pi}\sum_{\mu,\nu}\epsilon^{\mu\nu}q_\mu A_\nu(-q) \tag{F12}$$

$$\Rightarrow \partial_\mu\langle j_{\text{reg}}^\mu(x)\rangle = -\frac{i}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(x) = -\frac{i}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}(x). \tag{F13}$$

One can show that this anomaly is canceled by the anomaly inflow of the bulk, considering

$$\begin{aligned}\delta_\eta W[V,A] \equiv W[\mathcal{V} + d\eta,\mathcal{A}] - W[\mathcal{V},\mathcal{A}] &= \int d^2x\partial_\mu\eta(x)\frac{\delta W}{\delta V_\mu} = \int d^2x\partial_\mu\eta(x)j_v^\mu(x)\\ &= -\int d^2x\eta(x)\partial_\mu j_v^\mu(x) = \frac{i}{2\pi}\int d^2x\eta(x)\epsilon^{\mu\nu}F_{\mu\nu}(x).\end{aligned} \tag{F14}$$

2. Spin Hall current

Since we do not take into account any interactions between fields with different k_z momentum, we will not include the summation over k_z from now on. Introducing the Pauli-Villars regularization field into the effective action, we start from the following surface action:

$$S = \int d^2x[\bar{\Psi}(x)i\gamma^\mu(\partial_\mu + iA_\mu - iS_\mu\bar{\gamma})\Psi(x) + \bar{\phi}(x)i\gamma^\mu(\partial_\mu + iA_\mu - iS_\mu\bar{\gamma})\phi(x) + \bar{\phi}(x)M\phi(x)]. \tag{F15}$$

If we consider the spin gauge transformation of $\bar{\phi}(x)$ and $\phi(x)$ like that of $\bar{\Psi}(x)$ and $\Psi(x)$, the mass term breaks the spin gauge symmetry explicitly. Since the form of action is completely same as that of valley Hall case except for that V_μ is changed to S_μ, all calculation is same. Therefore we give only the result here:

$$\delta_\eta W[S,A] = -\frac{i}{2\pi}\int d^2x\eta(x)\epsilon^{\mu\nu}F_{\mu\nu}(x). \tag{F16}$$

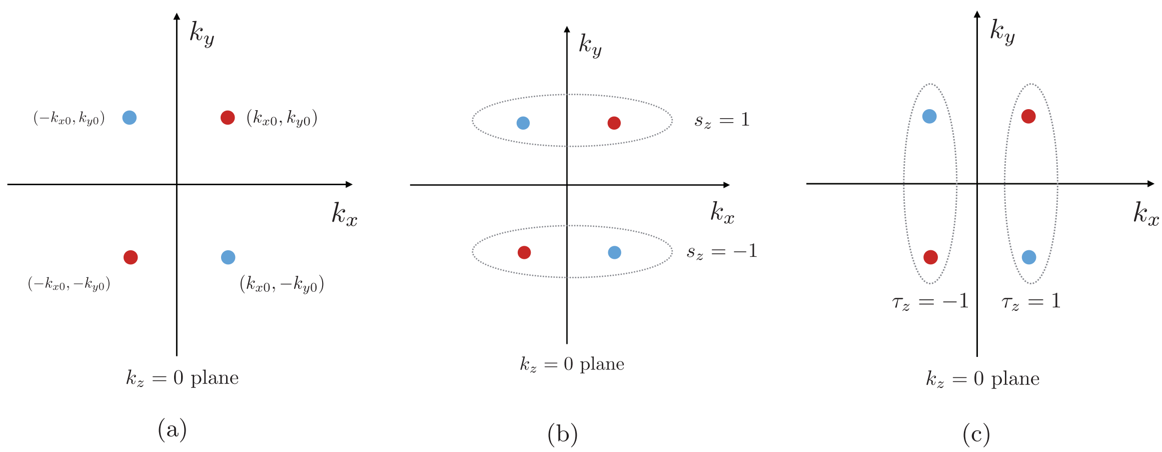

FIG. 7. (a) Position of four Weyl points in $k_x - k_y$ plane; color (red, blue) refers the chiral charge (+1,−1) of each Weyl fermion, (b) Two pairs of Weyl points classified according to eigen value of s_z, (c) Two pairs of Weyl point classified according to eigen value of τ_z.

APPENDIX G: GENERALIZATION FOR THE EXISTENCE OF TWO DIFFERENT REGULARIZATION SCHEMES IN BROKEN INVERSION SYMMETRY WEYL METALS

Consider a Hamiltonian that has four Weyl points, shown in Fig. 7, for example. In the diagonalized basis, we obtain

$$\begin{aligned}\mathcal{H} &= \sum_k \Psi^\dagger(k) diag(-E_{k_{x0},k_{y0}}, E_{k_{x0},k_{y0}}, -E_{-k_{x0},k_{y0}}, E_{-k_{x0},k_{y0}}, -E_{k_{x0},-k_{y0}}, E_{k_{x0},-k_{y0}}, -E_{-k_{x0},-k_{y0}}, E_{-k_{x0},-k_{y0}})\Psi(k)\\ &\equiv \sum_k \Psi^\dagger(k)H(k)\Psi(k),\end{aligned} \tag{G1}$$

where $E_{\pm k_{x0},\pm k_{y0}} = \sqrt{(k_x \mp k_{x0})^2 + (k_y \mp k_{y0})^2 + k_z^2}$. This diagonalized Hamiltonian commutes with two matrices; $\mathcal{O}_{s_z}$ and $\mathcal{O}_{\tau_z}$ given by

$$\mathcal{O}_{s_z} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{O}_{\tau_z} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{G2}$$

Here, τ and s do not mean the valley and spin necessarily but denote two different quantum numbers.

One can find that these different two observables turn out to commute with the Hamiltonian for any broken inversion symmetry Weyl metals. If we classify the energy bands according to the eigenvalues of $\mathcal{O}_{\tau_z}$ and $\mathcal{O}_{s_z}$, we find

$$\begin{aligned} s_z = 1:&\quad \text{diag}(-E_{k_{x0},k_{y0}}, E_{k_{x0},k_{y0}}, -E_{-k_{x0},k_{y0}}, E_{-k_{x0},k_{y0}})\\ s_z = -1:&\quad \text{diag}(-E_{k_{x0},-k_{y0}}, E_{k_{x0},-k_{y0}}, -E_{-k_{x0},-k_{y0}}, E_{-k_{x0},-k_{y0}}),\end{aligned} \tag{G3}$$

$$\begin{aligned} \tau_z = 1:&\quad \text{diag}(-E_{k_{x0},k_{y0}}, E_{k_{x0},k_{y0}}, -E_{k_{x0},-k_{y0}}, E_{k_{x0},-k_{y0}})\\ \tau_z = -1:&\quad \text{diag}(-E_{-k_{x0},k_{y0}}, E_{-k_{x0},k_{y0}}, -E_{-k_{x0},-k_{y0}}, E_{-k_{x0},-k_{y0}}).\end{aligned} \tag{G4}$$

Each sector is composed of two Weyl points.

For one s_z sector, one can apply the unitary transformation to the Hamiltonian which gives the following transformed one:

$$\tilde{H}(k) = \begin{pmatrix} \tilde{H}_{s_z=1}(k) & 0 \\ 0 & \tilde{H}_{s_z=-1} \end{pmatrix} = U_{s_z}(k)H(k)U^{\dagger}_{s_z}(k), \quad U_{s_z} = \begin{pmatrix} U_{s_z=1}|_{4\times4} & 0 \\ 0 & U_{s_z=-1}|_{4\times4} \end{pmatrix}, \tag{G5}$$

$$\tilde{H}_{s_z}(k) = -i\gamma^0(\gamma^1 k_x + \gamma^2(k_y - s_z k_{y0}) + \gamma^3 k_z - s_z\gamma^1\gamma^5 k_{x0}), \tag{G6}$$

where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = i\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$ with $i = 1,2,3$, which satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}1_{4\times4}$ and $\gamma^5 = -\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here, σ^i represent Pauli matrices. The representation of gamma matrices is the Weyl one in the Euclidean signature. One can always find the unitary transformation U_{s_z} which gives gamma matrices in this Weyl representation. Since we changed the basis by the unitary transformation, we need to change the representation of $\mathcal{O}_{s_z}$ and $\mathcal{O}_{\tau_z}$ coherently, $\tilde{\mathcal{O}}_i = U_{s_z}\mathcal{O}_i U^{\dagger}_{s_z}$. Recall that $\mathcal{O}_{s_z}$ commutes with U_{s_z}, and thus the representation of $\mathcal{O}_{s_z}$ is not changed, $\tilde{\mathcal{O}}_{s_z} = \mathcal{O}_{s_z}$. How about the representation of $\mathcal{O}_{\tau_z}$? Interestingly, $\tilde{\mathcal{O}}_{\tau_z}$ is also the same as $\mathcal{O}_{\tau_z}$; $\tilde{\mathcal{O}}_{\tau_z} = \mathcal{O}_{\tau_z}$. This can be verified easily. An important point is that $\tilde{\mathcal{O}}_{\tau_z} = \begin{pmatrix} \gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} = \gamma^5 \otimes 1_\tau$. As a result, the transformation $e^{i\tilde{\mathcal{O}}_{\tau_z}\theta}$ becomes anomalous in this representation (chiral anomaly).

There can exist other representations of gamma matrices even though the Hamiltonian $\tilde{H}(k)$ is the same. If we denote another representation of gamma matrices as $\bar{\gamma}^\mu$, it should satisfy the following properties:

$$\gamma^0\gamma^i = \bar{\gamma}^0\bar{\gamma}^i \ (i = 1,2,3), \quad \{\bar{\gamma}^\mu,\bar{\gamma}^\nu\} = 2\delta^{\mu\nu}1_{4\times4}, \tag{G7}$$

$$\bar{\gamma}^5 = -\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3. \tag{G8}$$

From the properties of $\gamma^0\gamma^i = \bar{\gamma}^0\bar{\gamma}^i$ and $\{\bar{\gamma}^\mu,\bar{\gamma}^\nu\} = 2\delta^{\mu\nu}1_{4\times4}$, one can find the following identity: $\bar{\gamma}^5 = \gamma^5$. Therefore we obtain $\tilde{\mathcal{O}}_{\tau_z} = \gamma^5 \otimes 1_\tau = \bar{\gamma}^5 \otimes 1_\tau$. As a result, any $\tilde{\mathcal{O}}_{\tau_z}$ related transformation is still anomalous in any other representations. One may consider additional unitary transformations such as $\tilde{U}_{s_z}(k)$; $\tilde{\tilde{H}}(k) = \tilde{U}_{s_z}(k)\tilde{H}(k)\tilde{U}_{s_z}(k)$. Even in this case, $\tilde{\tilde{\mathcal{O}}}_{\tau_z}$ is still anomalous since antiunitary relation is preserved under the unitary transformation. Therefore we conclude that if we choose gamma matrices which commute with $\mathcal{O}_{s_z}$, $\mathcal{O}_{\tau_z}$ should be proportional γ^5. This is also true for the opposite case: if we choose γ matrices commuting with $\mathcal{O}_{\tau_z}$, then $\mathcal{O}_{s_z}$ is proportional to γ^5, which should be anomalous. In our specific model for broken inversion symmetry Weyl metals, τ and s correspond to valley and spin, respectively.

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