Nonuniform quantum turbulence in superfluids

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The problem of quantum turbulence in a channel with an inhomogeneous counterflow of superfluid turbulent helium is studied. The counterflow velocity $V_{ns}^x(y)$ along the channel is supposed to have a parabolic profile in the transverse direction y. Such statement corresponds to the recent numerical simulation by Khomenko *et al.* [Phys. Rev. B **91**, 180504 (2015)]. The authors reported about a sophisticated behavior of the vortex-line density (VLD) $\mathcal{L}(\mathbf{r},t)$, different from $\mathcal{L} \propto V_{ns}^x(y)^2$, which follows from the straightforward application of the conventional Vinen theory. It is clear that Vinen theory should be refined by taking into account transverse effects, and the way it ought to be done is the subject of active discussion in the literature. In this work, we discuss several possible mechanisms of the transverse flux of VLD $\mathcal{L}(\mathbf{r},t)$ which should be incorporated in the standard Vinen equation to describe adequately the inhomogeneous quantum turbulence. It is shown that the most effective among these mechanisms is the one that is related to the phase-slippage phenomenon. The use of this flux in the modernized Vinen equation corrects the situation with an unusual distribution of the vortex-line density, and satisfactorily describes the behavior $\mathcal{L}(\mathbf{r},t)$ both in stationary and nonstationary situations. The general problem of the phenomenological Vinen theory in the case of nonuniform and nonstationary quantum turbulence is thoroughly discussed.

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I. INTRODUCTION

The question of evolution of the vortex-line density (VLD) $\mathcal{L}(\mathbf{r},t)$ of the vortex tangle (VT) is the key issue in the macroscopic theory of quantum turbulence (QT). Although the VLD is a rough characteristic of the QT, it is responsible for many (mainly hydrodynamic) phenomena in superfluids and the knowledge of its exact dynamics is very important for an adequate interpretation of various experiments.

Long ago, Vinen [1] suggested that the rate of change of VLD $\partial\mathcal{L}(t)/\partial t$ can be described in terms of only the quantity $\mathcal{L}(t)$ itself (and also other, external parameters, such as the counterflow velocity V_{ns} and the temperature). He called this statement as a self-preservation assumption. The corresponding balance equation for the quantity $\mathcal{L}(r,t)$, the so-called Vinen equation (VE), reads as

$$\frac{\partial \mathcal{L}}{\partial t} = \alpha_V \, |\mathbf{V}_{ns}| \, \mathcal{L}^{3/2} \; - \; \beta_V \, \mathcal{L}^2. \tag{1}$$

Here, α_V and β_V are the parameters of the theory, α_V is close to the mutual friction coefficient α, and β_V is of the order of the quantum of circulation κ. Throughout its long history, the Vinen equation has undergone various improvements and modifications (see, e.g., [2–6]) although at present the form (1) is mainly used.

One of the serious problems is the application of the Vinen theory to complicated situations, in particular to inhomogeneous flows (for recent papers see, e.g., [7–12]). Each of the works cited above has their own peculiarities: their quantitative data differ from each other. However, the principal qualitative result, common for all of these papers, is that the transverse distribution of the vortex-line density $\mathcal{L}(\mathbf{r},t)$ is strikingly different from the dependence $\mathcal{L} = \gamma^2 V_{ns}^x(y)^2$ (here, $\gamma = \alpha_V/\beta_V$), which follows from the straightforward application of the conventional Vinen theory.

Thus, Khomenko *et al.* [7] found in numerical simulations that the VLD field is concentrated near the side walls. Quite similar behavior was obtained in the numerical work by Yui *et al.* [9] [see Fig. 4(a) of this paper]. In a paper by Galantucci *et al.* [12], the authors numerically studied the two-dimensional counterflow of helium II. They also obtained an unusual transverse $\mathcal{L}(y)$ distribution of the total vortex density, depicted in the middle image of Fig. 4 of paper [12]. The same observation had been made in the paper by Baggaley *et al.* [8] (see Fig. 3 of this paper). In addition, the authors of all cited works received the tail-flattened profile of the normal velocity, which was recently detected employing a technique, based on laser-induced fluorescence of metastable helium molecules (see paper by Marakov *et al.* [13]). Analyzing the obtained results, the authors of the paper [7] proposed that the first term on the right-hand side of the Vinen equation (the so-called production term) has the structure $\propto|\mathbf{V}_{ns}|^3\mathcal{L}^{1/2}$, a combination that has never been discussed before. This conclusion was the subject of a polemics between the authors of the article [7] and the author of this paper (see [14,15]).

The goal of my paper is to explain qualitatively this unusual behavior of the VLD profile $\mathcal{L}(y)$. For quantitative consideration and comparison, I have chosen as a reference point the paper by Khomenko *et al.* [7]. It should be borne in mind that due to the phenomenological character of the Vinen theory and the large number of factors affecting the final result, the term "quantitative consideration" implies "quantitative consideration on the order of quantity."

In short, the results of work [7] can be formulated as follows. In a rectangular channel 2×0.05 cm wide, a parabolic

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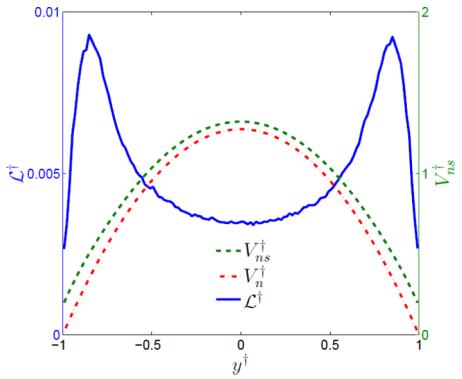

FIG. 1. Prescribed parabolic normal velocity profile V_n (dashed-dotted line), the resulting counterflow profile V_{ns} (dotted line) and the resulting profile of $\mathcal{L}(y)$ (solid line) in dimensionless unites, $T = 1.6$ K. Note that V_{ns} does not vanish on the boundary. That is, a consequence of that the resulting V_s accumulates contribution from vortices and is not equal to $-\rho_n V_n/\rho_s$. However, from comparison between V_n and V_{ns} it is seen that superfluid velocity is small, mainly because the normal density is much smaller than the superfluid densiy $\rho_n \ll \rho_s$ at $T = 1.6$ K (from the paper by Khomenko *et al.* [7]).

counterflow $V_{ns}^x(y) = V_0[1 - (y/0.05)^2]$ is applied in the x direction. The periodic conditions were assumed in all directions. The resulting distributions of the dimensional VLD, the normal and counterflow velocities $\mathcal{L}(y)$, $V_n(y)$, $V_{ns}(y)$ are presented in Fig. 1.

If one applies straightforwardly the well-known relation $\mathcal{L} = \gamma^2 V_{ns}^x(y)^2 \approx 2 \times 10^4\ V_{ns}^x(y)^2$ (here $\gamma = \alpha_V/\beta_V$), which immediately arises from Eq. (1), then the dimensionless $\mathcal{L}$ should be about $2 \times 10^{-2}\ V_{ns}^x(y)^2$, which essentially exceeds the value obtained in Ref. [7]. Another striking feature is that the profile $\mathcal{L}(y)$ is radically different from the quadratic velocity profile $\mathcal{L} \propto (V_{ns}^x(y))^2$.

In this paper, we develop an approach explaining this unusual (from the point of view of the naive use of the Vinen theory) behavior of the VLD $\mathcal{L}(y)$. In the inhomogeneous situation, the Vinen equation should be corrected to include the transverse spatial effects. In particular, we offer to incorporate into classic Vinen theory an additional space flux $\mathbf{J}(\mathbf{r},t)$ of the VLD, which redistributes the quantity $\mathcal{L}(y)$ in the y direction. It is clear that a transverse gradient of the flux $\partial J_y(y,t)/\partial y$ should be added into the balance equation (1).

In classical Vinen theory, the counterflow velocity V_{ns} is considered as an external unchanged parameter. Of course, in the reality, one should take into account the back reaction of the vortex lines on the normal fluid, that, strictly speaking, can modify the quantity V_{ns}. That problem was properly discussed in Ref. [3] . Recently, the back influence was considered in many researches, in particular, in the paper by Galantucci *et al.* [12], mentioned above. However, in this paper we consider the model situation, stated by Vinen in his pioneering work, where the relative velocity was introduced as a constant external parameter. In addition, in the work by Khomenko *et al.* [7] (with which I compared my results) the velocity of normal component was also fixed, and the back reaction of the vortex lines on the normal fluid was omitted.

In Sec. II we discuss several mechanisms of these possible fluxes, derive mathematical expressions, and compare contributions from them. In Sec. III we present numerical solutions for stationary and nonstationary cases and compare the results with the numerical data of paper [7]. In Sec. IV we discuss the problem of nonuniform and unsteady quantum turbulence and the Vinen phenomenological theory. The Conclusion is devoted to a discussion of the results and probable generalizations of the presented approach.

II. TRANSVERSE FLUX OF THE VORTEX-LINE DENSITY

Let us describe various ideas on the transverse vortex-line density flux $\mathbf{J}(\mathbf{r},t)$ in inhomogeneous flows/counterflows of superfluid helium. As it was mentioned above, the first remark in this respect had been made by Vinen himself in the context of the possible influence of the channel width [1]. Unfortunately, no advanced theory had been supplemented. It is clear that the most general expression for the flux of quantity $\mathcal{L}$ is $\mathbf{J}(\mathbf{r},t) = \mathcal{L}\mathbf{V}_L$, where $\mathbf{V}_L$ is the macroscopic local velocity of the vortex tangle (see explanations in papers [3,16,17]). However, unless we do not have a general expression for $\mathbf{V}_L$ as a function (functional) of quantity $\mathcal{L}$, we can not ascertain a closure procedure, i.e., obtain a description of the vortex tangle dynamics in terms of the VLD itself. This procedure is not uniquely defined and admits different approaches.

Thus, in the cited paper [7], the authors proceeded from the following microscopic expression for the transverse flux J_{micro}:

$$J_{\text{micro}} = \frac{1}{\Omega}\int |\mathbf{V}_{ns}(y)|\mathbf{s}_z' d\xi = \frac{\alpha}{\Omega}\int |\mathbf{V}_{ns}(y)|\mathbf{s}_z' d\xi. \qquad (2)$$

Here, the integration is performed over the whole vortex-line configuration, so it should be understood as an integration along each vortex loop constituting the vortex tangle and summation over all loops, i.e.,

$$\int d\xi \to \sum_j \int_0^{L_j} d\xi_j.$$

The quantity Ω is the total volume, α is the mutual friction coefficient. The authors of work [7] calculated the quantity (2) in numerical simulation and concluded that the macroscopic expression for the transverse flux J_{Kh},

$$J_{Kh}(\mathbf{r},t) = \frac{\alpha}{2\kappa} C_{\text{flux}} \frac{\partial \mathbf{V}_{ns}^2}{\partial y}, \qquad (3)$$

best corresponds to the microscopic flux (2). The quantity C_{flux} is a constant, determined from numerical simulations.

Another mechanism, frequently discussed in the problems of nonuniform flow, is related to the diffusion flux [17,18]. That mechanism is not connected with mutual friction, and realized by the drift of vortex loops (see, e.g., [19–21]). The transverse diffusion flux can be written as follows:

$$J_{\text{dif}}(\mathbf{r},t) = D\frac{\partial^2 \mathcal{L}}{\partial y^2}. \qquad (4)$$

The value the diffusion coefficient is of the order of quantum of circulation κ. In paper [18], $D \approx 0.1 \times 10^{-3}$ cm$^2/s$, whereas in paper [21] this quantity is estimated as $D \approx 2 \times 10^{-3}$ cm$^2/s$. The latter value stemmed from the model of flying polarized loops. In the counterflowing case, however, the

polarization is mainly directed along $\mathbf{V}_{ns}$, therefore, the smaller value or the diffusion constant in the transverse direction is more truthful.

The next contribution, which we consider here, is related to the so-called phase-slippage phenomenon. This phenomenon implies appearance of additional chemical potential $\nabla\mu$ and, accordingly, the mutual friction with the crossing by the vortices of the main flow. This effect is especially important for identifying the quantization of vortices (see, e.g., [22]). We will use the corresponding technique to describe the transverse flux of VLD $J_y(y,t)$. To find an analytical expression for $J_y(y,t)$, consider the following equation (see [23–25]):

$$\mathcal{A} = \int [\dot{\mathbf{s}}(\xi) \times \mathbf{s}'(\xi)] d\xi. \tag{5}$$

The right-hand side of (5) is a net area, swept out by the motion of the line elements. Therefore, the x component of vector $\mathcal{A}$ is simply the rate of phase slippage (without the factor κ) caused by the transverse motion of the vortex lines (see [25]). It is important, however, that the sign of the x component of the vector $\mathcal{A}$ does not depend on the direction of motion of vortex-line segments (either in the positive or in the negative directions along axis y). It makes no difference in the calculation of the phase slippage and, accordingly, the additional drop in the chemical potential $\nabla\mu$, but it is essential for our purposes to determine flux $J_{ps}(y,t)$ of the VLD $\mathcal{L}$ to the side wall. To overcome this problem, we assume that all the vortex filaments are closed loops, so the averaged fluxes in both directions are equal. Therefore, the required transverse flux $J_{ps}(y,t)$ of the VLD $\mathcal{L}$ is just half of the x component of the vector $\mathcal{A}$. Taking velocity of elements $\dot{\mathbf{s}}(\xi)$ in the form of the local induction approximation (see, e.g., [2]), we arrive at the following expression:

$$J_{ps}(\mathbf{r},t) = \frac{1}{2} \int (\{\alpha \mathbf{s}' \times [\mathbf{V}_{ns} - \beta(\mathbf{s}' \times \mathbf{s}'')]\} \times \mathbf{s}'(\xi)) d\xi. \tag{6}$$

Here, the combination $\dot{\mathbf{s}}_i = \beta(\mathbf{s}' \times \mathbf{s}'')$ is the self-induced velocity of the line elements in the the local induction approximation.

To move further, we have to introduce the closure procedure and to express the right-hand side of Eq. (6) via quantities $\mathcal{L}$ and $\mathbf{V}_{ns}$. It corresponds to the self-preservation assumption expressed by Vinen that the macroscopic dynamics of the vortex tangle depends only on the VLD $\mathcal{L}(t)$. The other, more subtle, characteristics of the vortex structure, different from $\mathcal{L}$, must adjust to it. In particular, the first contribution, containing the external counterflow velocity, can be written as $\alpha I_{\parallel} \mathcal{L} |\mathbf{V}_{ns}|$, where $I_{\parallel}$ is the structure parameter of the vortex tangle, introduced by Schwarz [2]. The last term in Eq. (6) with the self-induced velocity can be expressed as $\alpha\beta\mathcal{L}(I_l \mathcal{L}^{1/2})$. where I_l is another structure parameter. Usually, at this point the substitution $\mathcal{L}^{1/2} = \gamma |\mathbf{V}_{ns}|$ is used, and both contributions are reduced to a combination

$$J_{ps,1}(\mathbf{r},t) = \tfrac{1}{2}\alpha(I_{\parallel} - \gamma\beta I_l)\mathcal{L}|\mathbf{V}_{ns}|. \tag{7}$$

Being multiplied by $\rho_s\kappa$, this expression (up to a factor $\frac{1}{2}$) coincides with the formula for mutual friction. This is not surprising because it is well known from the vortex dynamics that a vortex crossing the channel transfers the momentum to the main flow (see [26]). Therefore, the final expression should be proportional to $\mathbf{V}_{ns}$ and the whole scheme becomes self-consistent. But, this above consideration concerns only homogeneous or near-homogeneous cases. In the highly inhomogeneous situation, which we are interested in here, the simple relations such as $\mathcal{L}^{1/2} = \gamma |\mathbf{V}_{ns}|$ do not work and the question of determining the transverse flux remains open. A very similar problem of using the structure parameters of the vortex tangle also arises for nonstationary situations (see a related discussion in the review [3]). This problem is very intriguing, and we decided to explore yet another version of the closure procedure, which leads to the following formula for the transverse flux:

$$J_{ps,2}(y,t) = \alpha I_{\parallel} \mathcal{L} |\mathbf{V}_{ns}| - \alpha\beta I_l \mathcal{L}^{3/2}. \tag{8}$$

Thus, we have obtained two forms for the transverse flux associated with the phase-slippage mechanism. They are identical in case of a uniform flow, when $\mathcal{L}^{1/2} = \gamma |\mathbf{V}_{ns}|$, however, in inhomogeneous situations they differ and can result in different results.

Our further goal is to analyze the results on the nonuniform quantum turbulence obtained in the numerical work by Khomenko *et al.* [7], basing on supposition of the transverse flux of VLD $\mathcal{L}(y)$.

Let us discuss all of the transverse fluxes, expressed by the equations (3), (4), (7), and (8). The first, proposed in the paper [7], was discussed in details in the paper [14] (Sec. IV). Briefly, it does not include the VLD $\mathcal{L}(y)$ itself, which is very unusual. This results in that the master equation (9) for the profile $\mathcal{L}(y)$ is simply an algebraic (not a differential) equation. The solution of this equation gives the quantity $\mathcal{L}(y)$, which is very different from the one observed in Ref. [7]. Further, I do not consider this term.

Using the conditions of numerical modeling [7] and taking that $|\mathbf{V}_{ns}| \sim 1$ cm/s, $\mathcal{L} \sim 10^4$ 1/cm^2, $\alpha \sim 0.1$, $\partial/\partial y \sim 1/0.05$, $I_{\parallel} \sim 0.3$, $D \sim 0.1 \times 10^{-3}$ we conclude that the phase-slippage terms are estimated to be about $10\,000.01/\mathrm{cm}^2$ s, whereas the diffusion term is estimated as about $400.01/\mathrm{cm}^2$ s. Thus, formally, the diffusion contribution can be neglected. Here, however, there appears the problem of a small term with the highest derivative in the nonlinear differential equation. This is a complex numerical problem, which is beyond the scope of this paper. The impact of the diffusion flux was studied in a recent work by Saluto *et al.* [10]. The authors observed that the influence of vortex diffusion is focused on local values of $\mathcal{L}(y)$ in the region with large gradients, rather than on the form of the spatial distribution VLD.

Thus, the diffusion term (4) is small for our particular problem, although, being a second-order derivative, it would be essential for other situations. In this paper, I omit this term.

III. SOLUTIONS

Thus, we introduced and discussed several mechanisms for the transverse flux of VLD and concluded that the most effective of them is associated with the phase-slippage mechanism. A microscopic equation for this flux is given by Eq. (6), its macroscopic closure variants are given by the formulas (7) and (8). Our goal now is to incorporate these terms into the

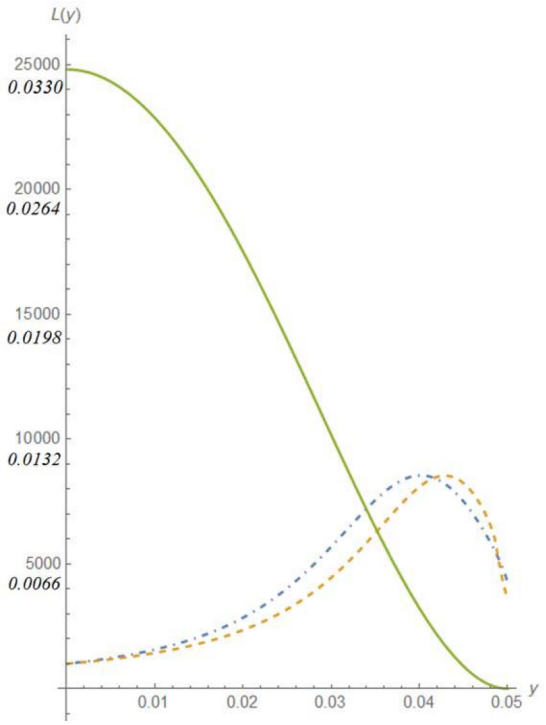

FIG. 2. Profiles of VLD $\mathcal{L}(y)$ obtained in numerical solution of Eq. (9) without the term $\partial\mathcal{L}/\partial t$ with the different expressions for transverse flux (7) (dashed line) and (8) (dotted-dashed line). We also inserted the curve satisfying the standard Vinen relation $\mathcal{L}(y) = \gamma^2|\mathbf{V}_{ns}|^2$. Additionally, for quantitative comparison with Fig. 1, we plotted (in italics) on the vertical axis the values corresponding to dimensionless variables $L(y)$, used in the paper by Khomenko *et al.* [7] .

Vinen equation (1),

$$\frac{\partial\mathcal{L}}{\partial t} + \frac{\partial J_{ps}(y,t)}{\partial y} = \alpha_V\,|\mathbf{V}_{ns}|\,\mathcal{L}^{3/2} - \beta_V\,\mathcal{L}^2, \qquad (9)$$

and to study its solutions under the conditions that are identical to those studied in the work by Khomenko *et al.* [7]. Namely, we have selected the temperature of system, the geometry and size of the of the channel, and parabolic counterflow velocity $\mathbf{V}_{ns}(y)$ coinciding with the ones accepted in their work. We study two cases: a stationary situation and a completely unsteady problem.

A. Stationary case: Profile of VLD $\mathcal{L}(y)$

In Fig. 2 we displayed profiles of the VLD $\mathcal{L}(y)$, obtained through the numerical solution of Eq. (9) without the term $\partial\mathcal{L}/\partial t$. These two curves correspond to different expressions for transverse fluxes (7) and (8), which, in turn, correspond to different alternative variants of the structure parameter (see Sec. II). We have chosen the system temperature $T = 1.6$ K, the channel size 2×0.05 cm, the parabolic counterflow velocity $\mathbf{V}_{ns}(y) = 1.19 \times [1 - (y/0.05)^2]$ cm/s, coinciding with the conditions adopted in the work [7]. Additionally, only half of the channel width is considered, namely, $0 \leqslant y \leqslant 0.05$ cm. The boundary condition $\mathcal{L}(y = 0) = 10001/\text{cm}^2$ had been taken from the result of paper [7] and from the solution of the fully nonstationary problem (see further). It is noteworthy that they are very close to each other. Additionally, for quantitative comparison with Fig. 1, we plotted (in italics) on the vertical axis the values corresponding to dimensionless variables $L(y)$, used in paper by Khomenko *et al.* [7].

The most important (albeit expected) result is that the VLD profile does not really satisfy the standard Vinen relation $\mathcal{L}(y) = \gamma^2|\mathbf{V}_{ns}|^2$, which is shown with use of a solid curve. On the contrary, the vortex tangle is concentrated in the region closer to the side wall (but not directly on the wall). This behavior can be understood qualitatively from the following considerations. The structure of fluxes expressed by the formulas (7) and (8) is that their maximal values are at the central parts ($y = 0$) of the channel (due to the large value of the counterflow velocity V_{ns}) and the VLD $\mathcal{L}$ is intensively removed from this region. On the contrary, because of the vanishing of the counterflow velocity V_{ns} on the side walls ($y = 0.05$), the flux is almost extinguished, and $\mathcal{L}$ does not penetrate into this region. Clearly, to support a stationary solution in the regions where $\mathcal{L}(y) \neq \gamma^2|\mathbf{V}_{ns}|^2$, either the production or the decay (second) term on the right-hand side of Eq. (9) should prevail.

As for the quantitative data such as the maximum value of VLD $\mathcal{L}(y)$, width of the curve, the value on the side wall, they agree with the result of the paper [7] in limits of 20%–30%. This can be considered as a satisfactory agreement, taking into account that the whole approach was based on the conventional (uniform) Vinen theory, with numerical parameters proposed by Vinen [1] and Schwarz [2].

One more important result concerns the fundamental question of the use of the Schwarz's relations for the structure parameters of the inhomogeneous quantum turbulence. It is easy to see that both solutions of Eq. (9) with the transverse fluxes expressed by Eqs. (7) and (8) are very close. This fact confirms the widespread view that the Vinen equation based on Feynman's scenario is a fairly robust construction.

B. Nonstationary case: Development of quantum turbulence in the inhomogeneous counterflow

The rather elegant results are obtained when solving the full equation (9), with the term $\partial\mathcal{L}/\partial t$. This procedure faces the standard problem of initial conditions, typical for the Vinen theory. Equation (9) is a balance relation between the growth and the disappearance of vortex lines. The mechanism of spontaneous appearance of vortices in the helium flow has not been built into this equation.

At present, there are various theories of the initial appearance of vortex filament, which can be divided into two groups. The first group offers the different mechanisms (tunneling, fluctuation growth, etc.) of initial generation of vortices. Another group is based on the idea that in helium permanently exists a background of remnant vortices. From the point of view of the phenomenological theory, the former group can be taken into account by introducing the initiating term into the Vinen equation. In turn, the latter group should lead to some initial value of VLD [$\mathcal{L}(t = 0) = \mathcal{L}_{\text{back}}$] in the Vinen equation. The better agreement between experimental data on the propagation of intense heat pulses (generating vortices and interacting with these "own" vortices) and the corresponding numerical solution, was obtained when assuming the existence

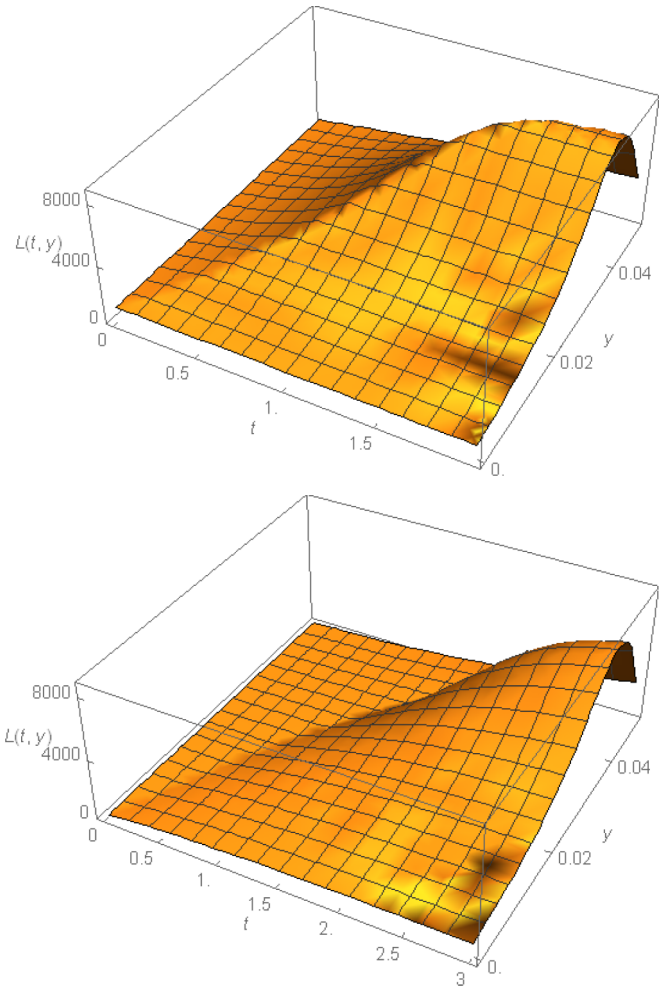

FIG. 3. The spatiotemporal behavior of VLD $\mathcal{L}(t,y)$ with the obtained in numerical solution of Eq. (9) with the expression (7) for transverse flux. The upper and lower pictures correspond to different values for the level of the remnant vorticity $\mathcal{L}_{\rm back}$. Namely, in the upper graphics the quantity $\mathcal{L}_{\rm back}$ was equal to 1000 $1/{\rm cm}^2$, whereas for lower image $\mathcal{L}_{\rm back}$ was equal to 100 $1/{\rm cm}^2$.

of an initial level of VLD $\mathcal{L}_{\rm back}$, whereas the introduction of the initiating term led to an unsatisfactory correlation with the experimental observations. Thus, it may be surmised that this is an argument in favor of the theory of remnant vortices. The according activity was described in the review paper [3]; the level of the remnant vorticity $\mathcal{L}_{\rm back}$ used in the according numerical simulation is estimated approximately as 10^2–10^3 $1/{\rm cm}^2$. This values agrees with qualitative estimation, made in the paper by Awschalom and Schwarz [27]. This estimation says that the density of remnant vortices $\mathcal{L}_{\rm back}$ is of the order $\mathcal{L}_{\rm back} \leqslant 2\ln(d/a_0)/d^2$, where $d = 0.1$ cm is size of a container and a_0 is the core radius.

The spatiotemporal behavior of VLD $\mathcal{L}(y,t)$ obtained in the numerical solution of Eq. (9) with the nonstationary term $\partial\mathcal{L}/\partial t$ is shown in Fig. 3. The upper and lower images correspond to the different values for the level of the remnant vorticity $\mathcal{L}_{\rm back}$. We again have chosen all conditions of work [7].

It is a very remarkable fact that the choice of initial level is not very crucial for long time behavior, rather it is responsible for the time of formation of a vortex tangle. This fact was well known for the homogenous vortex tangle, but, to our knowledge, there was no confirmation in the nonuniform situation.

On a time slices $t_{\rm sat}$ of about 2 s for larger value of $\mathcal{L}_{\rm back}$ and of about 3 s for the smaller one, when the saturation and crossover to the steady-state regime occurs, the solutions $\mathcal{L}(y,t = t_{\rm sat})$ are consistent with the stationary solutions, found above, as it should be for regular (not singular) solution of differential equations. Additionally, the obtained pictures demonstrate how the according scenario develops in time.

IV. NONUNIFORM QUANTUM TURBULENCE AND THE VINEN PHENOMENOLOGICAL THEORY

In Sec. II, we described the problems of the closure procedure for the microscopic equation for the flux (6) and questions of the choice of the form for the structure parameters. Bearing in mind that to compare various possibilities we have chosen two variants, leading to different expressions (7) and (8). In this regard, it seems appropriate to return to the basics of Vinen's phenomenological theory as applied to the complex nonstationary and inhomogeneous situations.

The main idea of the Vinen approach was the assumption of self-preservation, i.e., the suggestion that the macroscopic vortex dynamics can be described in terms of the quantity $\mathcal{L}(t)$ only. Selecting a set of variables to describe the macroscopic dynamics of statistical systems is, in general, a difficult and delicate step. For instance, the usual gas dynamics variables, such as density, momentum, and energy (per unit volume) are just the first moments of the distribution function of the Boltzmann's kinetic theory. Higher moments relax to approach equilibrium much faster than do the first listed variables. This circumstance allows one to truncate an infinite hierarchy of the moment equations and obtain a closed description using the listed quantities.

Unfortunately, in the case of quantum turbulence, the assumption of self-preservation is not motivated, the restriction to the only variable $\mathcal{L}(t)$ is not justified, and, in general, the Vinen equation is not valid. Indeed, let us consider a very simple counterexample. Assume that the velocity $\mathbf{V}_{ns}(\mathbf{s},t)$ changes instantly to the opposite. Since the Vinen-type equation includes the absolute value of relative velocity $|\mathbf{V}_{ns}(\mathbf{s},t)|$ magnitude, then formally the system remains unaffected by the change. This is wrong, of course. The structure of the VT, mean curvature, anisotropy, and polarization parameters will become reorganized. That implies the violation of the self-preservation assumption, and dynamics of the VLD $\mathcal{L}(t)$ depends on other, more subtle, characteristics of the vortex structure, different from $\mathcal{L}(t)$.

To clarify the situation, let us consider a way of derivation of VE from the dynamics of vortex filaments in the local induction approximation (see, e.g. [28]). It will suffice for illustration purposes. Integrating an equation for the change of the length of line element over ξ inside a volume Ω, Schwarz concluded that in the counterflowing helium II the quantity $\mathcal{L}(t)$ obeys the

equation (see [2])

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\alpha \mathbf{V}_{ns}}{\Omega} \int \langle \mathbf{s}' \times \mathbf{s}'' \rangle \, d\xi \; - \frac{\alpha\beta}{\Omega} \int \langle |\mathbf{s}''|^2 \rangle \, d\xi. \qquad (10)$$

The quantity $\mathcal{L}(t)$ is related to the first derivative $\mathbf{s}'$ of the function $\mathbf{s}(\xi)$ since $\mathcal{L}(t) \propto \int |\mathbf{s}'| d\xi$. The rate of change of $\mathcal{L}(t)$ includes quantities involving the higher-order derivative $\mathbf{s}''$, namely, $\langle \mathbf{s}' \times \mathbf{s}'' \rangle$ and $\langle |\mathbf{s}''|^2 \rangle$. In a steady state, these higher-order quantities are directly expressed via the VLD $\mathcal{L}$ as $\langle \mathbf{s}' \times \mathbf{s}'' \rangle \propto \; I_l \mathcal{L}^{1/2}$ and $\langle |\mathbf{s}''|^2 \rangle \propto c_2^2(T)\mathcal{L}$. Here, the $I_l, c_2(T)$ are temperature-dependent parameters introduced by Schwarz [2]. But, in the nonstationary situation, $\mathbf{s}''$ is a new independent variable, and one needs a new independent equation for it and for other quantities, related to curvature of line. This new equation, in turn, will involve higher derivatives $\mathbf{s}'''$, $\mathbf{s}^{IV}$, and so on. This infinite hierarchy can be truncated if, for some reasons, the higher-order derivatives relax faster than the low-order derivatives, and take their "equilibrium" values (with respect to the moments of low order).

Strictly speaking, there are no theoretical grounds for assuming that the relaxation of higher moments is faster than that of the quantity $\mathcal{L}(t)$. Thus, in general, no equation of the type $\partial \mathcal{L}(t)/\partial t = \mathcal{F}(\mathcal{L})$ exists! At the same time, in some (unclear) conditions, and with the use of additional arguments (see [1]), the required equation can be written. The attempt was successful as this theory explained a large number of hydrodynamic experiments, including the main experiment by Gorter and Mellink [29] (see, for details, the review [30]). It concerned, however, only stationary or near-stationary situations. In a strongly unsteady case, the region of applicability of this equation is unclear (see the above counterexample with a sudden inversion of the counterflow velocity).

Meanwhile, it seems intuitively plausible that for slow changes (both in space and time) the assumption of self-preservation is valid. That was the starting point in the construction of the so-called hydrodynamics of superfluid turbulence (HST), which was the unification of the Vinen equation and the classical two-fluid hydrodynamics (see, e.g., [16,31,32]). The HST equations have been applied to study a large number of hydrodynamic and thermal problems, including heat transfer and boiling in He II (see, e.g., [33–37]). The numerical and analytic results were in very good agreement with numerous experimental data. This fact pointed out that the Vinen equation is robust and is, in general, quite suitable for the unsteady hydrodynamic problems.

It follows from the results of this work that the situation with inhomogeneous flow is quite similar. This is confirmed by the curves depicted on the upper and lower images in Figs. 2 and 3. In these images we display the results obtained from solutions of the Vinen equation (9) with different expressions (7) and (8) for the transverse flux. The qualitative similarity and closeness of the quantitative solutions indicates again that the Vinen equation is rather insensitive to a particular choice of the transverse flux and is robust to study various inhomogeneous situations.

V. CONCLUSION

We conclude by saying that the study of the inhomogeneous flow/counterflow of superfluids in the channel on the basis of the Vinen equation (1) requires the introduction of additional terms describing the transverse flux of the VLD $\mathcal{L}$ towards the side walls. The analysis demonstrated that the most efficient mechanism is related to the phase-slippage mechanism. The corresponding solutions of the Vinen equation with the additional term in both stationary and nonstationary cases agree with observations obtained earlier in numerical simulations. They showed that the VLD $\mathcal{L}(y,t)$, as a function of y, is concentrated in the domain near the side walls. The reason for this behavior is the special structure of the transverse flux. This construction forces the vortex filaments to escape from the central part, at the same time it does not allow them to touch the walls.

One of our results, important for the macroscopic theory of quantum turbulence, concerns the structure functions of the vortex tangle, such as the parameters of anisotropy and polarization. Just like in the unsteady situation, the use of such parameters in the usual form, introduced by Schwarz, can only be done approximately and with reservations. This fact confirms the widespread view that the Vinen equation can be used to explore the rough, engineering problems (although the corresponding studies may require some fitting parameters), but it is not suitable for the description of the fine structure of the vortex tangle. At the same time, the Vinen-Feynman phenomenological theory is an excellent illustration of how the rough macroscopic approach allowed us to clarify fine and complicated phenomena of the statistical physics of stringlike objects.

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