

Electronic sound modes and plasmons in hydrodynamic two-dimensional metalsAndrew Lucas¹ and Sankar Das Sarma²¹*Department of Physics, Stanford University, Stanford, California 94305, USA*²*Department of Physics, Condensed Matter Theory Center, Joint Quantum Institute, University of Maryland, College Park, Maryland 20742 USA*

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Using an analytically tractable kinetic model of a two-dimensional Fermi liquid of electrons, we characterize the crossovers among zero sound, first sound, and plasmons. For experimentally realized Fermi liquids in a hydrodynamic limit, both zero and first sound waves are essentially replaced by plasmons. The plasmon dispersion relation is robust against hydrodynamic effects up to acquiring the viscous-limited decay rate of a first sound wave in the hydrodynamic limit. We discuss implications for experiments in clean two-dimensional electron gases.

DOI: [10.1103/PhysRevB.97.115449](https://doi.org/10.1103/PhysRevB.97.115449)**I. INTRODUCTION**

Advances in the quality of crystal growth have led to strong experimental evidence for the hydrodynamic flows of electrons in solid-state systems [1–4]; see Ref. [5] for a recent review. Recent theoretical work has clarified the signatures of hydrodynamic behavior on correlated electron flow in a diverse set of materials [6–14], including those where the textbook Navier-Stokes equations are not applicable.

The condition for an electron liquid to obey hydrodynamics was discussed in the literature a long time ago [15–17]. The key condition is that the electron-electron (or equivalently, interparticle) collision rate must be rapid enough to bring about local thermal equilibrium. This necessary condition boils down (for an electron liquid, the subject of interest in the current paper) to the electron-electron interaction-induced scattering rate being larger than the momentum-relaxation scattering rates associated with electron-impurity and/or electron-phonon scattering rates. Leaving aside phonon scattering, which is usually important only at rather high temperatures, this implies that electron-electron scattering should be stronger than electron-impurity scattering. Since electron-electron scattering rates are typically $\propto T^2$ in a Fermi liquid, it may seem that at some finite temperatures, all metals should manifest hydrodynamic behavior. This turns out to be untrue, and typically in normal three-dimensional (3D) metals (even in the cleanest possible scenario) the electron-electron scattering rate is rather weak, and the hydrodynamic condition is never satisfied. On the other hand, graphene and ultraclean two-dimensional (2D) GaAs-based systems do satisfy the hydrodynamic condition at moderate temperatures.

An immediate consequence of hydrodynamics is the existence of an electronic sound mode (the so-called “first sound”) where the long-wavelength energy dispersion of the collective mode is linear in wave number: $\omega \propto k$. Here, we observe this collective electronic sound mode in a 2D electron liquid (either graphene or 2D semiconductor systems). Even an indirect observation of sound waves through the hydrodynamic Dyakonov-Shur instability [18] has never been cleanly observed, despite many years of experimental efforts [19,20]. Instead, the observed collective mode is the usual

long-wavelength plasmon mode with $\omega \propto \sqrt{k}$ due to the long-range nature of the Coulomb interaction. As we will see, this may be understood as an appropriate “zero sound” mode in the collisionless regime. This casts a shadow on the theoretically proposed hydrodynamic descriptions of a strongly interacting electron liquid as a fluid should have a sound mode. Our paper aims to clearly resolve this conundrum. Appropriately incorporating long-range Coulomb interactions into a hydrodynamic and kinetic model, we show that hydrodynamics in 2D metals is consistent with a long-wavelength collective mode dispersing as $\omega \propto \sqrt{k}$. For purely short-range interactions, there is indeed a linear-in-wave-number first sound mode in the hydrodynamic regime (as observed, for example, in normal He-3 [21]), but Coulomb interaction modifies this dispersion to a square root in wave-number dependence at long wavelengths.

Summary of results

In this paper, we present a simple and analytically tractable kinetic theory model for the dynamics of a two-dimensional Fermi liquid. We account for both Landau’s Fermi-liquid interaction function and the long-range nature of the Coulomb interaction as appropriate for two-dimensional metals. We observe that in a typical Fermi liquid with strong electron-electron interactions, both the first and the zero sound modes are replaced by a plasmon mode which persists (at least) until the breakdown of the kinetic theory. The main effects of hydrodynamics on this conventional plasmon mode are in the decay rate of the excitation, which is more challenging to experimentally measure. Furthermore, the higher-order wave-number corrections to the collective mode dispersions are also affected by hydrodynamics and can be captured in our model (the leading-order dispersion is fixed by the Coulomb interaction to be a plasmonic \sqrt{k}).

Let us be more quantitative. We study a toy model of an isotropic two-dimensional Fermi liquid with a single electron-electron scattering rate γ , electron-impurity (momentum-relaxing) scattering rate $\gamma_{\text{imp}} \leq \gamma$, and effective interaction constant α . We consider spinless fermions in order to focus our discussion on the salient features of plasmon physics. The

generalization to the spinful case is straightforward. In a Fermi liquid, $\gamma \sim T^2$ (up to logarithms [22–25]). We also take the only nonvanishing Landau parameter of Fermi-liquid theory to be \mathcal{F}_0 for the sake of simplicity (although a generalization including more Landau parameters is straightforward but cumbersome). As we have assumed spinless fermions, \mathcal{F}_0 corresponds to a Landau parameter in the symmetric channel. First, let us assume the absence of long-range Coulomb interactions (which can be experimentally achieved by placing gates very close to the sample to screen out the long-range part of the interaction). If $\gamma_{\text{imp}} = 0$, then we find a simple crossover between hydrodynamic first sound waves with the dispersion relation,

$$\omega = \pm \sqrt{\frac{1 + \mathcal{F}_0}{2}} vk - i \frac{v^2}{8\gamma} k^2 + O(k^3) \quad (1)$$

valid when $|\omega| \ll \gamma$ and collisionless zero sound when $|\omega| \gg \gamma$,

$$\omega = \pm \frac{1 + \mathcal{F}_0}{\sqrt{1 + 2\mathcal{F}_0}} vk - i\gamma \frac{1 + \mathcal{F}_0}{(1 + 2\mathcal{F}_0)^2} + O\left(\frac{1}{k}\right). \quad (2)$$

Whenever $\mathcal{F}_0 > 0$, the speed of zero sound v_{s0} is always faster than the speed of first sound v_{s1} ,

$$1 \leq \frac{v_{s0}}{v_{s1}} = \sqrt{\frac{2 + 2\mathcal{F}_0}{1 + 2\mathcal{F}_0}} \leq \sqrt{2}. \quad (3)$$

So the most important difference between these two sound waves (beyond their physical interpretation) is the decay rate: $\text{Im}(\omega) \sim T^{-2}$ in the hydrodynamic limit, whereas $\text{Im}(\omega) \sim T^2$ in the collisionless limit. These results are qualitatively similar to the well-understood theory of first and zero sound waves in He-3, which were experimentally observed a long time ago [21].

If $\gamma_{\text{imp}} \neq 0$, then sound waves and plasmons are destroyed at long wavelengths: Impurity collisions act as a cutoff, overdamping these ballistically propagating modes at low enough frequencies. The two sound modes merge as $k \rightarrow 0$ and form a purely diffusive mode associated with the Ohmic diffusion of charge,

$$\omega = -i \frac{v_{s1}^2}{\gamma_{\text{imp}}} k^2, \quad (4)$$

as well as a gapped mode associated with momentum relaxation: $\omega \approx -i\gamma_{\text{imp}}$. Equation (4) defines the well-known diffusion pole of the disordered electron Green's function. When $v_{s1}k \gtrsim \gamma_{\text{imp}}$, we instead find the dispersion relation of sound waves described above but with the decay rates approximately shifted by

$$\text{Im}(\omega; \gamma_{\text{imp}} \neq 0) \approx \text{Im}(\omega; \gamma_{\text{imp}} = 0) - \frac{\gamma_{\text{imp}}}{2}. \quad (5)$$

For zero sound waves, the shift is slightly more complicated—see Eq. (47) below.

Our most important result is that, when we include long-range Coulomb interactions, we can approximately replace

$$\mathcal{F}_0 \rightarrow \mathcal{F}_0 + \frac{2\pi\alpha}{\lambda_F |k|}, \quad (6)$$

with λ_F as the Fermi wavelength. This may be construed as an effective generalization to a Landau-Silin theory applicable to 2D metals as compared with a 2D Fermi-liquid theory appropriate for neutral Fermi liquids. Equation (6) can also be understood from the random-phase approximation [26]. The $1/k$ dependence in Eq. (6) is the precise 2D Coulomb interaction behavior for the $1/r$ long-range potential. We then find that first sound (1) is replaced by

$$\omega \approx \pm \sqrt{\frac{\pi\alpha v^2}{\lambda_F}} |k| - \frac{iv^2}{8\gamma} k^2 + O(k^3), \quad (7)$$

and the collisionless zero sound is replaced by

$$\omega \approx \pm \sqrt{\frac{\pi\alpha v^2}{\lambda_F}} |k| - i\gamma \frac{\lambda_F |k|}{8\pi\alpha}. \quad (8)$$

The quantitative prefactor of $\text{Re}(\omega)$ in *both* of the above equations is identical to the well-known plasmon dispersion relation [27–29]. The main results of this paper are twofold. First, we derive the unexpected $\text{Im}(\omega) \sim |k|$ scaling for the decay rate of plasmons in a collisionless regime. Second, we show that the crossover between the “hydrodynamic plasmon” (7) and the “collisionless plasmon” (8) is when

$$k \sim \frac{\lambda_F \gamma^2}{\alpha v^2}. \quad (9)$$

This crossover occurs at a much smaller wave number than naively expected ($k \sim \gamma/v$). The breakdown of these equations and the replacement of plasmons with conventional first or zero sound modes occur when

$$\lambda_F k \gtrsim \frac{2\pi\alpha}{\max(1, \mathcal{F}_0)}, \quad (10)$$

which is generally outside the regime of validity of kinetic theory. Since $\alpha \gtrsim 1$ is indeed required to observe hydrodynamic behavior in a two-dimensional Fermi liquid at present, it is not possible to observe either a first or a zero sound mode: The long-range Coulomb interactions will destroy conventional sound waves at all wavelengths, and the only observable long-wavelength collective mode would be the usual plasmon mode, which indeed seems to be the generic experimental situation. Thus, in contrast to the 3D neutral Fermi-liquid He-3, the hydrodynamic behavior in 2D Coulomb Fermi systems may not necessarily manifest long-wavelength sound modes.

These results clarify previous literature relating plasmons to sound waves in Fermi liquids. In particular, the hydrodynamic and collisionless plasmons are essentially the same mode at long wavelengths in a Fermi liquid, albeit with distinct decay mechanisms (and different subleading wave-number corrections). A crossover between the plasmons and the first sound will only be observable in nearly charge neutral (nondegenerate) systems, such as the Dirac fluid in graphene [24,30–32].

II. THE BOLTZMANN EQUATION

In order to derive these results, we first review the kinetic theory of a two-dimensional Fermi liquid with a circular Fermi surface. Within linear response, the most important changes to

the distribution function f occur at the Fermi surface,

$$f \approx f_{\text{eq}} + \delta f \equiv \Theta[\mu - \epsilon(\mathbf{p})] + \delta[\mu - \epsilon(\mathbf{p})]\Phi(\mathbf{x}, \theta, t). \quad (11)$$

Here $\tan \theta = p_y/p_x$ denotes the angle on the circular Fermi surface. In what follows, we will denote with p_F the Fermi momentum $\epsilon(p_F) = \mu$ and define the Fermi velocity,

$$v \equiv \epsilon'(p_F). \quad (12)$$

The quantum Boltzmann equation, neglecting spin, reads

$$\frac{\partial f}{\partial t} + \frac{\partial \epsilon}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \epsilon}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \mathcal{C}[f]. \quad (13)$$

$\mathcal{C}[\Phi]$ is the collision integral, and ϵ denotes the energy of a quasiparticle, accounting for the bare band structure $\epsilon_0(\mathbf{p})$, Landau's short-range interaction function \mathcal{E} , and the long-range part of the Coulomb interaction, arising from fluctuations in the Fermi surface,

$$\begin{aligned} \epsilon(\mathbf{x}, \mathbf{p}) &= \epsilon_0(\mathbf{p}) + \mathcal{E}[f(\mathbf{x}, \mathbf{p})] \\ &+ \frac{e^2}{4\pi\epsilon} \int \frac{d^2\mathbf{y} d^2\mathbf{q}}{(2\pi\hbar)^2} \frac{f(\mathbf{x}, \mathbf{p})[f(\mathbf{y}, \mathbf{q}) - f_{\text{eq}}]}{|\mathbf{x} - \mathbf{y}|}. \end{aligned} \quad (14)$$

We have treated Coulomb interactions here in the conventional self-consistent Vlasov approximation. The interaction function is

$$\mathcal{E}[\Phi] \approx \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \mathcal{F}(\theta - \phi)\Phi(\phi), \quad (15)$$

with \mathcal{F} as an even function. Combining Eqs. (13) and (14), we obtain

$$\frac{\partial \delta f}{\partial t} + \frac{\partial \epsilon_0}{\partial \mathbf{p}} \cdot \frac{\partial \delta f}{\partial \mathbf{x}} - \frac{\partial f_{\text{eq}}}{\partial \mathbf{p}} \cdot \frac{\partial \epsilon}{\partial \mathbf{x}} = \mathcal{C}[f], \quad (16)$$

which leads to

$$\frac{\partial \Phi}{\partial t} + \mathbf{v} \cdot \left[\frac{\partial \Phi}{\partial \mathbf{x}} + \frac{\partial \epsilon}{\partial \mathbf{x}} \right] = \delta\mathcal{C}[\Phi], \quad (17)$$

where $\mathbf{v} = \partial\epsilon_0/\partial\mathbf{p}$ and $\delta\mathcal{C}[\Phi]$ denotes a linearized collision integral, which will be a local in \mathbf{x} but a nonlocal in the θ linear expression in Φ .

In this paper we are interested in studying propagating waves. So without loss of generality we may look for solutions to (17) of the form

$$\Phi(\mathbf{x}, \theta, t) = \sum_{n \in \mathbb{Z}} a_n e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + n\theta)}. \quad (18)$$

Writing

$$\mathcal{F}(\phi) = \sum_{k \in \mathbb{Z}} \mathcal{F}_n e^{in\phi}, \quad (19)$$

with $\mathcal{F}_n = \mathcal{F}_{-n}$, we find

$$\begin{aligned} \frac{\partial \mathcal{E}[\Phi]}{\partial \mathbf{x}} &= i\mathbf{k} \sum_{m,n} \int_0^{2\pi} \frac{d\phi}{2\pi} \mathcal{F}_m e^{im(\theta-\phi)} a_n e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + n\phi)} \\ &= i\mathbf{k} \sum_n \mathcal{F}_n a_n e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + n\phi)}. \end{aligned} \quad (20)$$

Taking the spatial Fourier transform of the long-range

Coulomb interaction, we find

$$\begin{aligned} &\frac{e^2}{4\pi\epsilon} \int d^2\mathbf{x} \frac{\partial}{\partial \mathbf{x}} \int \frac{d^2\mathbf{y} d^2\mathbf{q}}{(2\pi\hbar)^2} \frac{\delta[\mu - \epsilon(\mathbf{q})]\Phi(\mathbf{y}, \theta, t)}{|\mathbf{x} - \mathbf{y}|} \\ &= i\mathbf{k} \frac{\alpha}{\lambda_F |\mathbf{k}|} a_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \end{aligned} \quad (21)$$

where

$$\alpha \equiv \frac{e^2}{4\pi\epsilon\hbar v_F} \quad (22)$$

is the effective fine-structure constant and $\lambda_F = 2\pi\hbar/p_F$ is the Fermi wavelength. Finally, similar to (20), rotational invariance demands that

$$\delta\mathcal{C}[\Phi] = - \sum_n \gamma_n a_n e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + n\phi)}, \quad (23)$$

with $\gamma_n \geq 0$ also required by the second law of thermodynamics. Combining (17), (20), (21), and (23) and choosing $\mathbf{k} = k\hat{\mathbf{x}}$, we obtain

$$\begin{aligned} -i\omega a_n + \frac{ikv}{2}(a_{n+1} + a_{n-1}) + \frac{ikv}{2}(\mathcal{F}_{n+1}a_{n+1} + \mathcal{F}_{n-1}a_{n-1}) \\ + ik \frac{\pi v \alpha}{\lambda_F |k|} a_0 \delta_{|n|,1} = -\gamma_n a_n. \end{aligned} \quad (24)$$

This infinite set of algebraic equations governs the normal modes of our kinetic theory.

A key observation is that the replacement (6) “removes” the third term of (24). In other words, long-range Coulomb interactions are a k -dependent \mathcal{F}_0 . Since k is not dynamical in (24), we can therefore solve (24) without explicitly accounting for long-range Coulomb interactions and then include them at the end of the calculation through (6). Indeed, in the section that follows, we will set $\alpha = 0$ and characterize the zero-to-first sound crossover of this model, which is akin to solving for the collective modes in the short-range interacting 2D Fermi liquids.

Let us also stress that the regime of validity of the Boltzmann equation is $k\lambda_F \ll 1$. Therefore, unless α is parametrically small, the long-range Coulomb interactions will dominate the dynamics of the ± 1 harmonics. This is why, in the Introduction, we stated that plasmons destroy sound waves in typical Fermi liquids.

III. A SOLVABLE TOY MODEL

To find the exact solutions advertised in the Introduction, we must now choose a simple model for \mathcal{F}_n and γ_n . A simple solvable model is the relaxation time model of [10–12]

$$\gamma_n = \begin{cases} 0, & n = 0, \\ \gamma_{\text{imp}}, & |n| = 1, \\ \gamma, & |n| \geq 2. \end{cases} \quad (25)$$

Although this model is not microscopically realistic when only two-body collisions are important [33,34], it captures many nontrivial features of the hydrodynamic-to-ballistic crossover, and for our purposes this will suffice. We will also consider the simplest possible nontrivial choice of \mathcal{F}_n ,

$$\mathcal{F}_n = \begin{cases} 0, & n \neq 0, \\ \mathcal{F}_0, & n = 0. \end{cases} \quad (26)$$

These assumptions can be relaxed at the expense of losing some of the analytic tractability. For a more microscopic model of \mathcal{F}_n and γ_n , it would be straightforward to numerically compute the normal modes of (24).

The solvability of (24), in the models (25) and (26), comes from the following observation. For $|n| \geq 2$, we have the generic equations:

$$(\gamma - i\omega)a_n + \frac{ikv}{2}(a_{n-1} + a_{n+1}) = 0. \quad (27)$$

Let us look for solutions of the form $a_n = \lambda^{n-1}a_1$ for $n > 1$. Equation (27) implies

$$\lambda^2 + \frac{2(\gamma - i\omega)}{ikv}\lambda + 1 = 0. \quad (28)$$

This is solved by

$$\lambda = -i\frac{\gamma - i\omega}{kv} \left[\pm \sqrt{1 + \left(\frac{kv}{\gamma - i\omega}\right)^2} - 1 \right]. \quad (29)$$

It is important to keep in mind that $|\lambda| \leq 1$ is required in order for a normal mode to exist (be normalizable).

Now, we look for the spectrum of normal modes. Equations (24) are analogous to the solution of tight-binding models in one dimension with scattering off of a defect near the origin. Looking for solutions with $\lambda = e^{-i\phi}$, which heuristically correspond to a δ -function fluctuation of the Fermi surface: $\Phi(\theta) \sim \delta(\theta - \phi)$, we expect a continuum of normal modes with

$$\omega = -i\gamma + kv \cos \phi \quad (30)$$

for (almost) any real $0 < \phi \leq 2\pi$. It remains to satisfy the boundary conditions $a_{\pm 1} = 0$ ($a_0 = 0$ then trivially follows). This can be performed as follows. With the exception of $\phi = 0, \pi$, there are two ϕ 's which have the same value of ω but different λ 's. We add these two modes together with suitably chosen constant prefactors so that $a_{\pm 1}$ vanishes. The final result is

$$a_n = \begin{cases} e^{i(n-1)\phi} - e^{i(1-n)\phi}, & n > 1, \\ 0, & n \leq 1, \end{cases} \quad (31)$$

and a similar mode with $a_n \neq 0$ only for $n < -1$.

Now let us look for the remaining modes, which do not obey (30). These are analogous to the ‘‘bound states’’ of the tight-binding model and will have a_0 and/or $a_{\pm 1}$ nonvanishing. The equations for a_0 and $a_{\pm 1}$ are

$$-i\omega a_0 + \frac{ikv}{2}(a_1 + a_{-1}) = 0, \quad (32a)$$

$$(\gamma_{\text{imp}} - i\omega)a_{\pm 1} + \frac{ikv}{2}(a_0 + a_{\pm 2}) + \frac{ikv\mathcal{F}_0}{2}a_0 = 0. \quad (32b)$$

First, we look for a mode which has $a_0 = 0$ but $a_{\pm 1} \neq 0$. Equation (32a) implies that $a_1 = -a_{-1}$. So let us solve just for the modes with $n > 0$. Making the ansatz $a_n = \lambda^{n-1}a_1$, we find

$$i\left(\omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv\right)a_1 = 0. \quad (33)$$

We find that (29) and (33) are exactly solved by

$$\omega = -i\gamma_{\text{imp}} - i\frac{v^2}{4(\gamma - \gamma_{\text{imp}})}k^2. \quad (34)$$

This is the hydrodynamic shear diffusion mode, which has become gapped by momentum relaxation. In order for the diffusion constant to be positive and for the theory to be stable, we require that $\gamma > \gamma_{\text{imp}}$. Also, note that this mode disappears into the continuum of normal modes once $|\lambda| = 1$. We find that this occurs when $\omega = -i\gamma$ or when $|k|v = 2(\gamma - \gamma_{\text{imp}})$. Finally, observe that, if $\gamma_{\text{imp}} = 0$, (34) is nothing more than the conventional hydrodynamic diffusive mode associated with transverse momentum. The diffusion constant and the attenuation constant of first sound (1) are related by viscous hydrodynamics [10,11].

The final mode has $a_0 \neq 0$ and $a_1 = a_{-1}$. We find the equations,

$$\omega a_0 = kva_1, \quad (35a)$$

$$\left(\omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv\right)a_1 = \frac{k(v + \mathcal{F}_0)}{2}a_0, \quad (35b)$$

or

$$\omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv = \frac{k^2v^2(1 + \mathcal{F}_0)}{2\omega}. \quad (36)$$

Equations (29) and (36) do not have a simple analytic solution for all ω 's. Let us instead focus on the two asymptotic limits of interest for large and small k 's. We start with the hydrodynamic limit of small k where we find from (29) that

$$\lambda \approx -\frac{ikv}{2[\gamma - i\omega(k=0)]}. \quad (37)$$

In the special case of $\gamma_{\text{imp}} = 0$, then $\omega(k=0) = 0$, and we find (1). This is the dispersion relation for a first sound wave. Using the fact that the dynamical viscosity of this toy model is [10–12]

$$\nu = \frac{v^2}{4\gamma}, \quad (38)$$

we see that the decay rate of sound is consistent with the hydrodynamic coefficients of this model.

Upon setting $\gamma \sim T^2$, one then finds from (38) that the viscosity of the Fermi liquid is proportional to T^{-2} . In fact, the viscosity of a Fermi liquid is modified by factors of $\ln(T/T_F)$ [23]. These additional logarithmic factors are associated with the breakdown of the relaxation-time approximation in a two-dimensional Fermi liquid. More generally, $\nu = v^2/4\gamma_2$ with γ_2 defined in (23), and the logarithms observed in the viscosity are due to γ_2 . Further discussion of the breakdown of the relaxation-time approximation can be found in Refs. [33,34].

In the case of $\gamma_{\text{imp}} \neq 0$, a bit more care is required. We expect that, in the $k \rightarrow 0$ limit, there is one gapped mode with $\omega = -i\gamma_{\text{imp}} + \dots$ and one diffusive gapless mode. If $\gamma_{\text{imp}} \ll \gamma$, then $\gamma - i\omega(k=0) \approx \gamma$ in both cases, and (1) generalizes

to

$$\omega \approx \pm \sqrt{\frac{v(v + \mathcal{F}_0)}{2} k^2 - \left(\frac{\gamma_{\text{imp}}}{2} + \frac{v^2}{8\gamma} k^2 \right)^2} - i \left(\frac{\gamma_{\text{imp}}}{2} + \frac{v^2}{8\gamma} k^2 \right) + \dots \quad (39)$$

If γ_{imp} is comparable to γ , then (39) cannot be trusted. There will be no appreciable first sound mode, and instead we find the following low- k expansion of a diffusive mode and a gapped mode:

$$\omega = -i \frac{v^2(1 + \mathcal{F}_0)}{2\gamma_{\text{imp}}} k^2 + O(k^4), \quad (40a)$$

$$\omega = -i\gamma_{\text{imp}} - i \frac{v^2}{2} \left[\frac{1}{\gamma - \gamma_{\text{imp}}} - \frac{1 + \mathcal{F}_0}{\gamma_{\text{imp}}} \right] k^2 + O(k^4). \quad (40b)$$

At large k , we find zero sound waves. Making the ansatz that

$$\omega \approx v_{s0}k + \zeta + O(k^{-1}), \quad (41)$$

we find that

$$\lambda \approx \frac{v_{s0} - \sqrt{v_{s0}^2 - v^2}}{v} \left[1 - \frac{i\gamma + \alpha}{\sqrt{v_{s0}^2 - v^2}k} + O\left(\frac{1}{k^2}\right) \right]. \quad (42)$$

To solve for the speed of zero sound waves v_{s0} , we can neglect ζ . Equation (36) gives

$$v_{s0}(v_{s0} + \sqrt{v_{s0}^2 - v^2}) = v^2(1 + \mathcal{F}_0), \quad (43)$$

which is solved by

$$v_{s0} = \pm v \frac{1 + \mathcal{F}_0}{\sqrt{1 + 2\mathcal{F}_0}}. \quad (44)$$

Next, we can perturbatively solve (36) for ζ , and we find

$$\zeta = -i \frac{(\gamma + 2\mathcal{F}_0\gamma_{\text{imp}})(1 + \mathcal{F}_0)}{(1 + 2\mathcal{F}_0)^2}. \quad (45)$$

The decay rate of zero sound modes is thus approximately set by γ but is smaller whenever $\mathcal{F}_0 \neq 0$. Combining (44) and (45) we obtain (2) when $\gamma_{\text{imp}} = 0$.

We can qualitatively estimate the crossover between the zero and the first sound waves by asking when $|\omega| \sim \gamma$. This occurs when

$$k \sim \frac{\gamma}{v\sqrt{1 + \mathcal{F}_0}}. \quad (46)$$

As emphasized previously, we are most interested in theories where (the effective) $\mathcal{F}_0 \gg 1$. In such systems, we find that the hydrodynamic regime—as measured by the presence of the first sound—is pushed to much longer length scales than naively anticipated.

If $\gamma_{\text{imp}} \neq 0$, then we find that the zero sound decay rate is approximately shifted by

$$\text{Im}(\omega; \gamma_{\text{imp}} \neq 0) \approx \text{Im}(\omega; \gamma_{\text{imp}} = 0) - \gamma_{\text{imp}} \frac{2\mathcal{F}_0(1 + \mathcal{F}_0)}{(1 + 2\mathcal{F}_0)^2}. \quad (47)$$

In the limit $\mathcal{F}_0 \gg 1$, the zero sound decay rate remains approximately offset by (5), analogous to the first sound waves. When \mathcal{F}_0 is small, the decay rate of the first sound waves is, interestingly, nearly independent of momentum relaxation. We emphasize that this is a theoretical point of interest, as in a real electronic Fermi liquid, the long-range Coulomb interactions always make the effective $\mathcal{F}_0 \gg 1$.

IV. CONCLUSION

We have described an analytically solvable toy model for the crossover between sound waves and plasmons in a strongly interacting two-dimensional Fermi liquid. At long wavelengths, the conventional plasmon mode arising from the long-range Coulomb interactions overtakes both the first sound and the zero sound modes of simple two-dimensional Fermi liquids. We have further computed the decay rates of plasmons at both short and long wavelengths and shown that signatures of hydrodynamics in the plasmonic decay are severely limited. The transition out of the hydrodynamic regime occurs (for plasmons) at a parametrically long length scale (9). Given that the plasmonic decay rate is also affected by impurities, this means that the electron-impurity scattering rate needs not simply obey $\gamma_{\text{imp}} \ll \gamma$ but

$$\gamma_{\text{imp}} \ll \gamma^3 \left(\frac{\lambda_F}{v\alpha} \right)^2, \quad (48)$$

in order for impurity scattering to not dominate the decay of the plasmons. Although $\gamma_{\text{imp}} \ll \gamma$ is achievable in two-dimensional electron gases, this stronger constraint may not be.

The absence of a soundlike collective mode in 2D electron liquids is completely consistent with the system being in the hydrodynamic regime since, as we show explicitly, the long-range Coulomb interaction modifies all collective modes (both zero and first sound modes of neutral short-range interacting models) to being plasmonlike with $\omega \propto \sqrt{k}$ at long wavelengths. Hydrodynamic effects do manifest themselves in the effective plasmon decay rate and in the higher-order wave-number corrections to the mode dispersion, but these are not easy effects to measure experimentally. We have thus resolved the conundrum of why strongly interacting and ultraclean 2D Fermi liquids (e.g., graphene, high-mobility 2D GaAs systems) only manifest plasma-type long-wavelength collective mode dispersion. This is an essential effect of the long-range nature of Coulomb interaction—the sound modes are no longer linear in the wave number in a Coulomb system. The same behavior should also apply to 3D Coulomb Fermi liquids where the hydrodynamic zero and first sound modes will in fact develop a mass gap because of the long-range nature of the 3D Coulomb interaction, but we know of no examples of 3D Coulomb Fermi liquids crossing over to the hydrodynamic regime (3D metals are always in the collisionless regime). Several 2D Coulomb Fermi-liquid systems, however, are expected to be in the hydrodynamic regime at moderate temperatures (e.g., monolayer and bilayer graphene, high-mobility 2D GaAs systems), and we predict that the only observable collective modes in these systems will always manifest the square-root in wave-number long-wavelength 2D plasmon dispersion, no matter how deep they are in the hydrodynamic regime (i.e., no

matter how strong the quasiparticle collision rate is). Our paper also indicates that the easiest way to see the hydrodynamic sound mode in a 2D system would be to screen out the long-range part of the interparticle Coulomb interaction by putting a parallel metal gate close to the 2D layer. In the hydrodynamic regime this will lead to a linearly dispersing sound wave whose speed of sound is sensitive to the properties of the gate. Sound waves in 2D can also be observed in systems with explicitly short-ranged interactions, such as thin films of normal He-3.

Many previous works have attempted to include the interplay of both electron-electron interactions [35–37] and impurities [38] in the plasmon dispersion relation of a weakly interacting metal. Our paper provides a simple model, treating impurity scattering and interactions on an equal footing all the way from the collisionless to the hydrodynamic regimes. In particular, the viscous plasmonic decay rate given in (7) may be much larger than the plasmon decay rate due to electron-phonon scattering [37]. More recent work on plasmons in the hydrodynamic regime includes Refs. [39–41]. Our key findings, including the novel collisionless dispersion relation (8) and the unexpectedly early crossover from hydrodynamic to collisionless plasmons on the length scale (9), are not contained in these earlier works. Our paper also explains why the collective modes of 2D Fermi liquids, which are expected to be in a hydrodynamic regime, including doped graphene [42] and 2D semiconductor systems, have exhibited no hints of hydrodynamics.

We should also make some comments on the implication of our paper in the context of the widely used “hydrodynamic dielectric function” approach to collective modes in metals and semiconductors, particularly in calculating the response of finite systems, including surfaces, interfaces, and inhomogeneous electron systems, to external electromagnetic fields. Some representative, but by no means exhaustive, references are Refs. [43–54]. This theory is extensively used in the emergent field of nanoplasmonics where collective electronic response properties of ultrasmall systems are studied for potential technological applications. This heuristic approach, dating back to Bloch in 1933 [43], treats the metal (or doped semiconductor) as a hydrodynamic fluid rather uncritically without any consideration for whether the system is or is not in the collision-dominated regime. In fact, most 3D metals and doped semiconductors are not in the collision-dominated regime since the electron-electron scattering rate is invariably weaker than

electron-impurity and electron-phonon scattering rates at low and high temperatures, respectively. But the simplicity of the hydrodynamic approach makes it particularly attractive for response calculations as a dynamical generalization of the static Thomas-Fermi approximation with the fluid pressure term approximated by a parametrized ansatz so as to give the correct plasma dispersion (e.g., as obtained by the random phase approximation) in the bulk up to second order in the wave number. Such a hydrodynamic response theory has been used extensively for calculating surface and interface plasma modes of finite electron systems in many situations mainly because of its manifest simplicity and tractability: The bulk hydrodynamic dielectric function has a simple finite frequency pole with a well-defined analytic form for the wave-number dependence of the pole. Our paper shows that such hydrodynamic response theories, although being easy to implement numerically, are not rigorous from a fundamental microscopic perspective. In particular, a simple ansatz for the fluid pressure applicable for all frequencies in the response calculation is not justifiable. The dissipative response of the system changes qualitatively between the collision-dominated regimes at low frequencies (where hydrodynamics is valid) to a collisionless ballistic theory at high frequencies where hydrodynamics manifestly fails. Our paper also shows that, although the leading-order collective mode is always defined by the standard plasma frequency in Coulomb systems (independent of the collision-dominated hydrodynamic or collisionless ballistic regime), all higher-order dispersion corrections depend explicitly on whether the system is or is not in the hydrodynamic regime (analogous to the difference between zero sound and first sound). Since most of these hydrodynamic theories of electron response are focused on obtaining the collective mode dispersion beyond the long-wavelength limit (i.e., the so-called “nonlocal” effects), we caution against taking the quantitative predictions of these theories too literally. In addition, our paper shows that the nature of the collective mode damping depends crucially on whether the system is the collision-dominated hydrodynamic or collisionless ballistic regime, another subtlety not considered in these hydrodynamic response theories.

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