Rapid Communications

# Thermodynamic entanglement of magnonic condensates

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Over the past decade, significant progress has been achieved to create Bose-Einstein condensates (BECs) of magnetic excitations, i.e., magnons, at room temperature, which is a novel quantum many-body system with a strong spin-spin correlation, and contains potential applications in magnonic spintronics. For quantum information science, the magnonic condensates can become an attractive source of quantum entanglement, which plays a central role in most of the quantum information processing tasks. Here we theoretically study the entanglement properties of a magnon gas above and below the condensation temperature. We show that the thermodynamic entanglement of the spins is a manifestation of the off-diagonal long-range order; the entanglement of the condensate does not vanish, even if the spins are separated by an infinitely long distance, which is fundamentally distinct from the normal magnetic ordering below the Curie temperature. In addition, the phase-transition point occurs when the derivative of the entanglement changes abruptly. These results provide a theoretical foundation for a future investigation of the magnon BEC in terms of quantum entanglement.

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*Introduction.* Magnons are the quanta of elementary excitations in magnetically ordered systems and behave like bosonic quasiparticles. Information can be encoded in the excited magnons and be transported through the magnonic spin current. This emerging field, known as magnonic spintronics or magnonics [1–3], finds attractive applications in information processing due to the zero ohmic loss of magnon current and long coherence time of magnons in low damping magnetic insulators [4]. The traditional proposals in magnonics mainly benefit from the wave nature of magnons [2,5–8], such as microwave filters, delay lines, and phase conjugators. However, the quantum nature of magnons has not been fully explored. The technological challenge in creating a magnon Bose-Einstein condensate (BEC) is that the magnon number decreases rapidly as the temperature decreases, and the magnon density in the ground state is very small. However, several important experimental progresses have been achieved over the past decade. In 2000, the BEC of dilute magnons was proposed to explain the field-induced Néel ordering in the spin-gap magnetic compound $TlCuCl_3$ [9,10]. In 2006, a magnon BEC was experimentally realized by parametrically pumping magnons into a magnetic insulator yttrium iron garnet using a microwave [11] where the magnon-magnon interaction relaxes much faster than the magnon-lattice interaction; consequently, there is a transient time of magnon number conservation, which provides a prerequisite for magnon BEC. In 2011, a low-temperature BEC of magnons was experimentally observed in gadolinium nanoparticles [12].

Besides, it has been proposed [13,14] that magnon condensates can be created through electronic pumping in a FM/normal metal bilayer where the ground state can maintain a macroscopic number of strongly correlated magnons. This system may serve as an excellent platform for studying the quantum properties of the system, such as spin-spin entanglement [15], which motivates our Rapid Communication.

Entanglement is a measure of quantum correlation between two or more quantum systems, which has attracted significant attention due to its intriguing applications in quantum information science [16–22]. The studies of entanglement in condensed-matter physics have become fruitful, covering many physical aspects. For example, the scaling behavior of entanglement in the vicinity of the transition point of a magnetic system [23,24], the entanglement area law in the superfluid phase of helium-3 [25,26], and the quantification of entanglement in cold atom many-body systems [27]. However, entanglement is usually a very fragile quantum property, which would be eliminated through decoherence, i.e., the interaction with the environment [28,29].

In this Rapid Communication, we focus on the spin-spin entanglement in a many-body magnetic system around the transition temperature of the magnon BEC. We show that the spin particles can still be entangled in a magnon condensate even if the spins are separated far apart in the condensate. This property is a manifestation of the long-range order of the magnon BEC, and it is very different from the entanglement in normal condensed systems that decays within the length scale of the Fermi wavelength [30,31].

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*Methodology.* Let us start with a group of interacting spins described by the following Hamiltonian:

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j - \sum_i \mathbf{S}_i\cdot\mathbf{B} + H_{\mathrm{ani}}, \tag{1}$$

where $\mathbf{S}_i$ is the spin operator on the $i$th site, $J$ is the exchange constant, and $\mathbf{B}$ is an external field. The first term captures the exchange energy, and the sum is taken for nearest neighbors; the second term represents the Zeeman energy, and the third term $H_{\mathrm{ani}}$ denotes the anisotropy of the system. The ground state of Hamiltonian (1) is a ferromagnetic domain where all the spins align along the direction of the magnetic field.

To take a step further, let us assume that the magnon excitations are dilute, which means that the low-energy excitation can be described by the bosonic Hamiltonian: $\mathcal{H} = \sum_q \hbar\omega_q a_q^\dagger a_q$, where $\hbar$ is the Planck constant, $\omega_q$ is the magnon frequency, $a_q^\dagger$, $a_q$ are the creation and annihilation operators for magnons with wave-vector $q$; they obey the bosonic commutation relations $[a_q, a_{q'}^\dagger] = \delta_{qq'}$.

In the following, we first illustrate the theory to deal with the two-spin entanglement in a spin system containing purely thermal magnons then generalize the theory to a system with both thermal and condensed magnons.

*A. Thermal magnons.* The quantum state of two spins at the $i$th and $j$th cites of the magnetic system can be described in terms of the reduced density-matrix $\rho_{ij}$ obtained by tracing over all the other spins in the ground state. In the standard basis, $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$, the two-spin density matrix can be written as

$$\rho_{ij} = \begin{pmatrix} \langle\kappa_i^+\kappa_j^-\rangle & \langle\kappa_i^+\sigma_j^-\rangle & \langle\sigma_i^-\kappa_j^+\rangle & \langle\sigma_i^-\sigma_j^-\rangle \\ \langle\kappa_i^+\sigma_j^-\rangle & \langle\kappa_i^+\kappa_j^-\rangle & \langle\sigma_i^-\sigma_j^+\rangle & \langle\sigma_i^-\kappa_j^+\rangle \\ \langle\sigma_i^+\kappa_j^+\rangle & \langle\sigma_i^+\sigma_j^-\rangle & \langle\kappa_i^-\kappa_j^+\rangle & \langle\kappa_i^-\sigma_j^-\rangle \\ \langle\sigma_i^+\sigma_j^+\rangle & \langle\sigma_i^+\kappa_j^+\rangle & \langle\kappa_i^-\sigma_j^+\rangle & \langle\kappa_i^-\kappa_j^-\rangle \end{pmatrix}, \tag{2}$$

where $\kappa_i^\pm = \frac{1}{2}(1\pm\sigma_i^z)$, $\sigma_i^\pm = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y)$, and $\sigma_i^x,\sigma_i^y,\sigma_i^z$ are the Pauli matrices describing the $i$th spin. To simplify the density matrix, we consider the ground state that all the spins align along the direction of the external field (the $z$ axis) and the system has a rotational symmetry around the $z$ axis. Then the density matrix could be simplified as [32,33]

$$\rho_{ij} = \begin{pmatrix} \Lambda_{00} & 0 & 0 & 0 \\ 0 & \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{12} & \Lambda_{22} & 0 \\ 0 & 0 & 0 & \Lambda_{33} \end{pmatrix}, \tag{3}$$

where $\Lambda_{00},\Lambda_{11},\Lambda_{12},\Lambda_{22}$, and $\Lambda_{33}$ are the corresponding matrix elements in matrix (2). The amount of entanglement between the $i$th spin and the $j$th spin can be quantified by calculating the Wooters' concurrence defined as [33]

$$C_{ij} = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{4}$$

where $\lambda_1,\lambda_2,\lambda_3$, and $\lambda_4$ are the square root of the eigenvalues of $\rho_{ij}[(\sigma^y\otimes\sigma^y)\rho_{ij}^*(\sigma^y\otimes\sigma^y)]$ in a nonincreasing order. Here $\lambda_{1,2} = \sqrt{\Lambda_{11}\Lambda_{22}} \pm |\Lambda_{12}|$, $\lambda_{3,4} = \sqrt{\Lambda_{00}\Lambda_{33}}$, and the concurrence is given by [34]

$$C_{ij} = \frac{1}{2}\max\Big[0, \big|\langle\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\rangle\big| - \sqrt{\big(1+\langle\sigma_i^z\sigma_j^z\rangle\big)^2 - \langle\sigma_i^z + \sigma_j^z\rangle^2}\Big]. \tag{5}$$

This expression was consistent with the bispin entanglement in literature [35].

Now, applying the Holstein-Primakoff transformation (HPT) for converting spin operators to bosonic operators [36], $S_i^+ = \sqrt{2S - a_i^\dagger a_i}\, a_i$, $S_i^- = a_i^\dagger\sqrt{2S - a_i^\dagger a_i}$, $S_i^z = S - a_i^\dagger a_i$, where $S_i^\pm = \sigma_i^\pm/2$, $S_i^z = \sigma_i^z/2$ and $a_i, a_i^\dagger$ are the magnon creation and annihilation operators in real space that obey bosonic commutation relations, the concurrence can be reduced to the one-particle reduced density matrix (1-RDM) of the system [37,38], i.e., $C_{ij} = 2\langle a_i^\dagger a_j\rangle$ (see the Appendix for details). Here we have assumed that the average magnon density is very small ($\langle a_i^\dagger a_i\rangle \ll 2S$) such that it is reasonable to expand the HPT to the linear order of the magnon creation (annihilation) operator. Since this approximation is valid for a magnetic system well below the Curie temperature ($T_c$), our results presented below are applicable at this regime. However, for systems with a strong anisotropy and/or strong applied fields, the spin-wave gap becomes sufficiently large, limiting the number of thermal magnons; our results can become applicable at higher temperatures.

Through a Fourier transform of the magnon operators, i.e., $a_i = 1/\sqrt{N_s}\sum_q e^{-i\mathbf{q}\cdot\mathbf{R}_i} a_q$, the concurrence is recast in the following form:

$$C_{ij} = \frac{2}{N_s}\left|\sum_q e^{i\mathbf{q}\cdot\mathbf{R}_{ij}} n_q\right|, \tag{6}$$

where $N_s$ is the total number of spins in the system, $n_q = \langle a_q^\dagger a_q\rangle$ is the density of magnons with wave-vector $q$, and $\mathbf{R}_{ij} = \mathbf{R}_j - \mathbf{R}_i$ is the relative distance between the $i$th spin and the $j$th spin. Translational symmetry of the system is used in the derivation such that the concurrence only depends on the relative distance between the two spins. If only one-magnon excitation is considered $n_q = 1$, then the sum gives zero entanglement of two different spins in the thermal dynamic limit, which is qualitatively consistent with the literature [39]. Generally, the magnon distribution obeys the Bose-Einstein statistics $n_q = 1/(e^{\hbar\omega_q/k_BT} - 1)$ where the dispersion relation $\omega_q/\gamma = H_{\mathrm{ex}}q^2 + B + H_{\mathrm{an}}$, $H_{\mathrm{ex}}$ is the exchange field, $H_{\mathrm{an}}$ is the anisotropy field, $\gamma$ is the gyromagnetic ratio, and $k_B$ is the Boltzmann constant.

In the continuum limit, the sum in Eq. (6) could be replaced by integral in the momentum space, i.e., $\sum_q \to V\int 4\pi q^2 dq$. Given that $n_q$ is an even function of $q$ due to the $q^2$ terms in $\omega_q$, the concurrence becomes

$$C_r = \frac{1}{\pi^2 r}\left|\int_0^\infty \frac{p\sin rp}{z_0 e^{bp^2} - 1}dp\right|, \tag{7}$$

where $p = qd$, $d$ is a lattice constant, $r = |\mathbf{R}_{ij}|$, $z_0 = \exp[\hbar\gamma(B + H_{an})/(k_BT)]$, $b = \gamma H_{\mathrm{ex}}/(k_BT) = T_c/T$, and $T_c = \gamma H_{\mathrm{ex}}/k_B$ is the approximated Curie temperature. For small fields and weak anisotropy $z_0 \approx 1$, the integral can be evaluated analytically by only considering the excitation of long-wavelength magnons such that $e^{bp^2} \approx bp^2 + b^2p^4/2$,

$$C_r = \frac{1}{2\pi r}\frac{T}{T_c}(1 - e^{-r/\xi_e}), \tag{8}$$

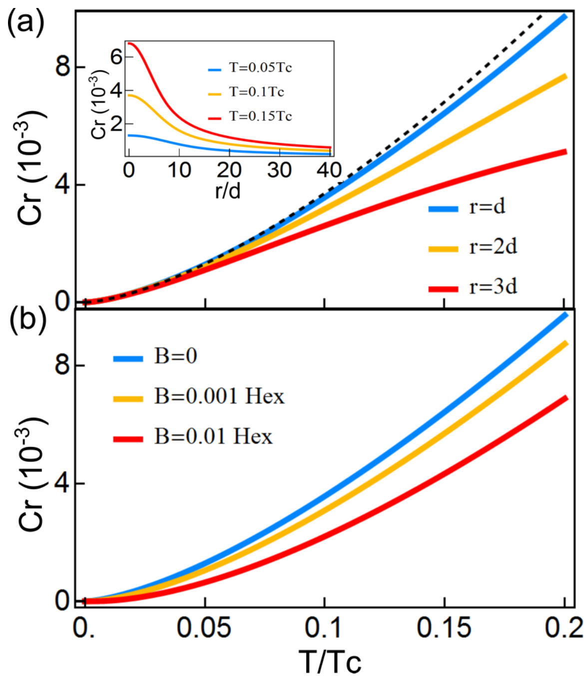


FIG. 1. Concurrence as a function of temperature for $r = d$ (the blue line), $2d$ (the yellow line), and $3d$ (the red line), respectively. The dashed line represents the reduced number of magnons as a function of $T$. The inset shows the concurrence as a function of the distance between the two spins for $T = 0.05T_c$ (the blue line), $0.10T_c$ (the yellow line), and $0.15T_c$ (the red line), respectively. (b) Concurrence of nearest spins as a function of temperature for external fields $B = 0$ (the blue line), $0.001H_{ex}$ (the yellow line), and $0.01H_{ex}$ (the red line), respectively.

where $\xi_e = d\sqrt{T_c/2T}$ is defined as the entanglement length of the system.

Figure 1 visualizes the temperature dependence of concurrence for $r = d$, $2d$, and $3d$, respectively. Regardless of the distance between the two spins, the concurrence is zero at 0 K and then increases monotonically as the temperature increases. This is because the spins are perfectly aligned along the $z$ axis and the bispin state is a separable state $|\uparrow\uparrow\rangle$ when $T = 0$ K. At finite temperatures, the magnons are excited, and the weighted sum of magnon density gives the amount of two-spin entanglement as indicated by Eq. (6). According to the Bose-Einstein statistics, the magnon number decays exponentially with the magnon energy, then the higher-energy terms gives a negligible contribution to the concurrence. For low-energy magnons, the cosine factor in Eq. (6) is almost equal to 1, and the concurrence is equivalent to the magnon density. The higher the temperature, the larger the density of low-energy magnons, hence the larger the two-spin entanglement. This argument naturally gives an estimate of the concurrence $C_{ij} \approx 2/N\sum_q n_q$ as shown by a dashed line in Fig. 1(a). This line roughly captures the temperature dependence of the entanglement between two nearest spins.

For external fields comparable with $k_BT/\gamma$, $z_0 > 1$, the temperature dependence of concurrence could be derived by calculating the integral Eq. (7) numerically as shown in Fig. 1(b). The general trend of $C_r$ vs $T$ is not qualitatively altered by the finite fields. Nevertheless, as the field increases, the entanglement of the two spins decreases and finally approaches zero for infinitely large fields. This is because the magnon gap ($\gamma B$) keeps increasing with the field and it becomes harder to excite magnons for larger fields, then the two-spin state approaches a separable state $|\uparrow\uparrow\rangle$ with zero entanglement.

*B. Condensed magnons.* Up to this stage, our theory is restricted to the magnon's gas without magnon number conservation. To achieve magnon BEC, it is essential to produce a magnon gas with conserved number of magnons, at least, temporarily. This may be realized by tuning the magnetic field, lowering the temperature in some special materials [9,10], and parametric pumping [11]. To describe a magnon gas with a fixed magnon number, the Hamiltonian should be modified as $\mathcal{H} = \sum_q(\hbar\omega_q - \mu)a_q^\dagger a_q = \sum_q(\hbar\gamma H_{ex}q^2 - \mu_{eff})a_q^\dagger a_q$, where $\mu$ is the chemical potential and $\mu_{eff} \equiv \mu - \hbar\gamma(B + H_{an})$ is the effective potential that subtracts the bottom of the magnon band. Similar to the BEC of atoms, the number of magnons in the spin system can be written as the sum of the magnon number in the ground state and the excited state, i.e. [40],

$$N = N_0 + (4\pi b)^{-3/2}N_s g_{3/2}(z), \tag{9}$$

where $N_0$ is the magnon number in the ground state, $z = \exp(-\mu_{eff})$ is the fugacity, and $g_{3/2}(z) = \sum_{k=1}^\infty z^k k^{-3/2}$ is the Bose-Einstein integral.

Above the critical temperature $T_b$, the magnon population in the ground state is neglectable such that $N_0 \ll N$, then $N/N_s = (4\pi b)^{-3/2}g_{3/2}(z)$. Below $T_b$, the effective chemical potential is zero ($z = 1$), and the occupancy of the ground state is comparable with the total number of magnons where the condensation temperature is determined by the equality $N/N_s = (4\pi b_{T=T_b})^{-3/2}g_{3/2}(1)$. Combining the two limits, the chemical potential can be readily determined from a set of self-consistent equations,

$$\mu_{eff} = k_BT \ln z, \quad T = T_b[g_{3/2}(1)/g_{3/2}(z)]^{2/3}, \tag{10}$$

where the first equation is the definition of fugacity and the second relation is obtained from the consistent calculation of magnon density below and above $T_b$. Once the chemical potential is determined, the entanglement of two spins could be calculated through the integral similar to Eq. (7) for $T > T_b$,

$$C_r = \frac{1}{\pi^2 r}\left|\int_0^\infty \frac{p \sin rp}{z^{-1}e^{bp^2} - 1}dp\right|. \tag{11}$$

Below $T_b$, this integral fails since significant numbers of spins are in the ground state and they have to be considered separately from the magnons in the excited state. By separating the magnon number in the ground state from Eq. (11), we obtain

$$C_r = \left|\frac{2N_0}{N_s} + \frac{1}{\pi^2 r}\int_{0^+}^\infty \frac{p \sin rp}{e^{bp^2} - 1}dp\right|, \tag{12}$$

where the magnon number in the ground state is $N_0 = N[1 - (T/T_b)^{3/2}]$. Figure 2 shows the concurrence as a function of temperature in a system with $T_b/T_c = 0.08$ (the blue line) and 0.1 (the yellow line), respectively. The concurrence keeps decreasing as the temperature increases, and this is a result of the interplay between thermal and condensed magnons around the critical temperature and it is quite different from the magnon gas with unconserved numbers as shown in Fig. 1. For $T < T_b$, the condensed magnon density $N_0/V$ decreases

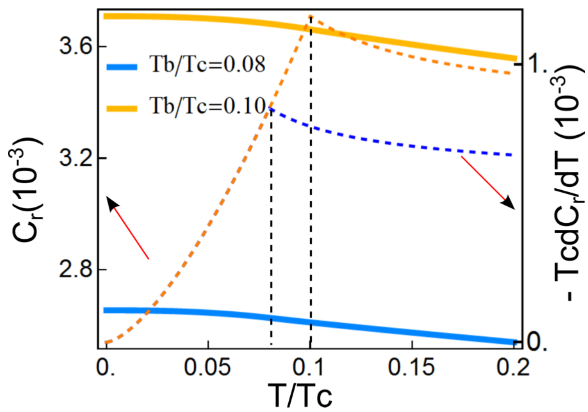


FIG. 2. (a) Concurrence as a function of temperature (the left axis) for $T_b/T_c = 0.08$ (the blue line), 0.1 (the yellow line), and the corresponding slopes of the curve are indicated as dashed lines. The black dashed lines indicate the position of condensation temperature ($T_b$).

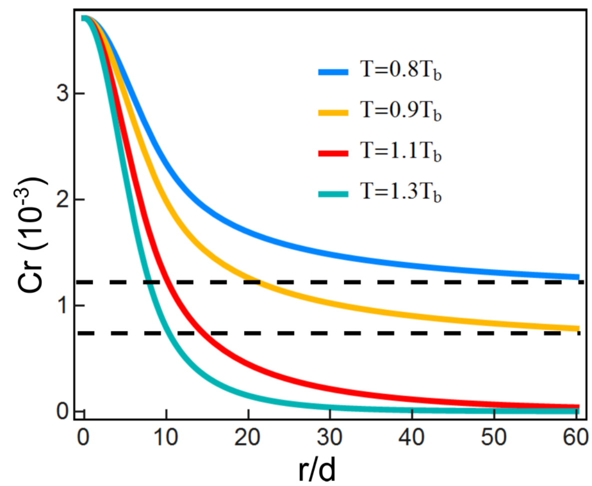


FIG. 3. Bispin entanglement as a function of the distance between two spins at various temperatures $T = 0.8T_b$ (the blue line), $0.9T_b$ (the orange line), $1.1T_b$ (the red line), and $1.3T_b$ (the cyan line). The dashed lines indicate the positions of the plateaus for two well-separated spins.

with the increase in temperature and results in a decreasing temperature dependence of concurrence as indicated in the first term of Eq. (12) whereas the excited magnon density contributes negatively. The two types of magnons compete with each other in determining the two-spin entanglement. The decreasing trend illustrated in Fig. 2 suggests that the condensed magnons dominate the contribution to concurrence. For $T > T_b$, only thermal magnons exist in the system. In this regime, $g_\nu(z) \approx z$, the solution of Eq. (10) gives $z^{-1} \approx T^{3/2}$ and $n_q = 1/(z^{-1}e^{bp^2} - 1) \approx T^{-1/2}$, then the magnon density $n_q$ will decrease as temperature increases, and the bispin entanglement decreases accordingly.

Different from the case of thermal magnons, the entanglement does not disappear when $T \to 0$ for condensed magnons due to the existence of a finite magnon density under the influence of external pumping. Furthermore, the temperature dependence of $-dC_r/dT$ is shown as dashed lines in Fig. 2. It takes on a maximum value at the condensation temperature, hence the second-order derivative $-dC_r^2/dT^2$ is discontinuous at $T_b$. This provides an alternative way to characterize phase transition between the normal phase and the condensed phase.

Interestingly, the phase transition between the normal phase and the condensed phase can be also interpreted in terms of off-diagonal long-range order (ODLRO). Figure 3 shows the two-spin entanglement/1-RDM ($C_{ij} = 2\langle a_i^\dagger a_j\rangle$) as a function of the distance between two spins. For $T < T_b$, the 1-RDM first decays and then saturates at a finite value as $r \to \infty$. This behavior testifies to the existence of ODRLO and further suggests that all the spins become long-range correlated in the condensed phase. This phenomenon is very similar to the Bose-Einstein atom condensate and superfluid [38]. For $T > T_b$, the 1-RDM approaches zero as $r \to \infty$, hence there is no ODLRO, and the spins are only correlated in a short range in the normal phase.

*Discussions and Conclusions.* First, our results provide a potential method to access the spin entanglement through the measurement of magnon density. Previously, the spatial entanglement of a normal (nonmagnetic) BEC was studied [41]; there it was proposed that the entanglement can be extracted by interacting with an external probe, which is inefficient. Here the magnetic degrees of freedom in the magnon BEC can be directly probed with the current technology. For example, one can measure the magnon density in a pumped magnetic system using Brillouin light scattering [11] or measure the spin-spin entanglement directly in ultracold systems in an optical lattice [42].

Second, our results should be still valid qualitatively in the higher temperature for magnetic systems with high Curie temperature and those materials with large spin-wave gaps where the excited magnons are still much smaller than the total number of spins. Then we can predict that the finite entanglement can survive at the higher temperature and this may be useful for quantum information science.

In conclusion, we have investigated the two-spin entanglement around the critical temperature of a magnon BEC. The two-spin entanglement could be approximated as the one-particle reduced density matrix of the system. Under condensation temperature, the entanglement even exists when the two spins are separated by an infinitely long distance. This suggests the existence of ODLRO in the magnon condensate where the long-range entanglement of spins is determined by the significant occupation of the ground state. As temperature increases above the condensation temperature, the ODLRO vanishes. Around the condensation point, the second-order derivative of entanglement with temperature takes on a minimum at the condensation temperature, which provides a new indicator of the phase transition from a normal magnon gas to a condensed phase.

*Note added.* Recently, we became aware of a paper [43] on the study of magnon-magnon entanglement in the magnon condensates.

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## APPENDIX: DERIVATION OF THE BISPIN ENTANGLEMENT

The bispin entanglement is quantified by the concurrence,

$$C_{ij} = \frac{1}{2}\max\Big(0, \left|\left\langle\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\right\rangle\right| - \sqrt{\left(1+\left\langle\sigma_i^z\sigma_j^z\right\rangle\right)^2 - \left\langle\sigma_i^z+\sigma_j^z\right\rangle^2}\Big). \tag{A1}$$

Using the Holstein-Primakoff transformation, $S_i^+ = \sqrt{2S - a_i^\dagger a_i}a_i \approx \sqrt{2S}a_i$, $S_i^- = a_i^\dagger\sqrt{2S - a_i^\dagger a_i} \approx \sqrt{2S}a_i^\dagger$, $S_i^z = S - a_i^\dagger a_i$ where long-wave excitation is assumed such that higher-order terms of $a_i$ and $a_i^\dagger$ are disregarded. $a_i, a_i^\dagger$ are the magnon creation and annihilation operators as claimed in the main text, and we have

$$\begin{aligned}\left\langle\sigma_i^x\sigma_j^x\right\rangle &= 4\left\langle S_i^x S_j^y\right\rangle \\ &= \langle(S_i^\dagger + S_i^-)(S_j^\dagger + S_j^-)\rangle \\ &= \langle(a_i + a_i^\dagger)(a_j + a_j^\dagger)\rangle.\end{aligned} \tag{A2}$$

In a similar way, we have

$$\left\langle\sigma_i^y\sigma_j^y\right\rangle = -\langle(a_i - a_i^\dagger)(a_j - a_j^\dagger)\rangle. \tag{A3}$$

Therefore,

$$\left\langle\sigma_i^x\sigma_j^x\right\rangle + \left\langle\sigma_i^y\sigma_j^y\right\rangle = 2\langle a_i a_j^\dagger + a_i^\dagger a_j\rangle. \tag{A4}$$

Moreover,

$$\begin{aligned}&\left(1+\left\langle\sigma_i^z\sigma_j^z\right\rangle\right)^2 - \left\langle\sigma_i^z+\sigma_j^z\right\rangle^2 \\ &= (1+4S^2)^2 - 4(2S)^2 + O(a_i^\dagger a_i a_j^\dagger a_j, a_i^\dagger a_i a_j^\dagger a_j a_i^\dagger a_i) \\ &= O(a_i^\dagger a_i a_j^\dagger a_j, a_i^\dagger a_i a_j^\dagger a_j a_i^\dagger a_i) \\ &\approx 0.\end{aligned} \tag{A5}$$

These higher-order terms are not considered in our model since they are sufficiently small under the long-wave excitation limit as claimed in the main text. Hence, the bispin entanglement is simplified as $C_{ij} \approx \frac{1}{2}(|\langle\sigma_i^x\sigma_j^x\rangle + \langle\sigma_i^y\sigma_j^y\rangle|) = \langle a_i a_j^\dagger + a_i^\dagger a_j\rangle = 2\langle a_i a_j^\dagger\rangle$ where the symmetry $C_{ij} = C_{ji}$ is used to guarantee the validity of the last equal sign. This form is the same as the one-particle density matrix [37].


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