Polariton Bose condensate in an open system: *Ab initio* approach

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In the framework of path-integral formalism and Keldysh technique for a nonequilibrium system we explore the kinetics of the polariton condensate in a quantum well embedded in an optical microcavity. We take into account pumping and leakage of excitons and photons. We make an *ab initio* derivation of the equations governing the dynamics of the condensates and reservoirs and show that the real open polariton system has a non-Markovian character at times comparable to the Rabi oscillation period.

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I. INTRODUCTION

Phase transitions in an open system are among the most exciting fields of modern condensed matter physics. The exciton-polariton Bose-Einstein condensation [1,2] in semiconductor microcavities possesses an essentially nonequilibrium character, which makes the polariton system open and coupled with external reservoirs.

The history of efforts to consider the polariton condensate as an open system is rather rich. The later 1990s are characterized by studies based on the Boltzmann equations [3–8]. Such an approach brought an opportunity to explore many key features of polariton condensation, such as the most significant mechanisms of the exciton and polariton relaxation and evolution of the occupation numbers towards the condensation in presence of particle transfer from a reservoir which is in turn pumped by a laser. One of the factors revealed in the course of the research was the problem of the bottleneck arising in consequence of the phase space narrowing in the region of the exciton-photon resonance. However, the Boltzmann equation technique is limited by mathematical cumbersomeness and inability to describe spontaneously occurring phase correlations and, thus, ignores the coherent character of the arising macroscopic state.

The work by Keeling and Berloff [9] was a remarkable attempt to suggest a model of the polariton condensate as an open dissipative system leaving the reservoir beyond consideration in order to avoid excessive difficulty. The pump and leakage were inserted directly into the equation for a polariton condensate in a phenomenological physically reasonable way.

Wouters, Carusotto, and Ciuti [10–12] described coupling of the polariton condensate with a reservoir of noncondensate excitons introducing a phenomenological function $R(n_R)$ into the Gross-Pitaevskii equation applied for the polariton condensate. Here, the argument n_R is the particle density in the reservoir, and an additional equation was involved to describe its time evolution.

Wouters and Savona [13] pointed out that the phase space of the excitons can be naturally divided into two parts: a low-energy polaritonic region where excitons resonantly interact with photons and resulting quasiparticles named lower and upper polaritons have an extremely small effective mass, and a high-energy excitonic reservoir with a high effective mass. So, we can introduce the boundary wave vector k_0, which separates the lower-energy region where the polariton condensate or quasicondensate arises and the rest of the phase space.

Baumberg and co-workers [14,15] overcame the problem of the bottleneck through the use of optical parametrical pumping (OPO). This experimental achievement motivated theoretical studies [16–19]. Simple models split the exciton reservoir into two parts: the first one accounts for resonant particle interchange with the condensate, the second one describes the rest of excitons [20].

Haug and co-workers [21] appropriated the first-principle consideration and made clear the structure of the phenomenological function $R(n_R)$ introduced by Wouters and Carusotto [11]. The authors described the experimentally observed spontaneous pattern formation [22] in the Markovian approximation presuming the description local in time.

In this paper, we develop an *ab initio* approach to the polariton condensate as an open system coupled with several external reservoirs (see also [23,24]). We start with the exciton-photon basis since the separate treatment of excitons and photons provides the natural way to introduce two different reservoirs for excitons and for photons.

We use path-integral formalism (the details of the technique applied to the polariton BEC are discussed in [25]) with the Schwinger-Keldysh time contour, as this technique constitutes the only approach to nonequilibrium problems which allows us to incorporate the Langevin equation, the Fokker-Planck equation, quantum kinetic equations, and Keldysh nonequilibrium Green functions. These all are powerful instruments of analysis [26,27].

We make an *ab initio* derivation of quantum kinetic equations for the coupled condensates of photons and excitons and for corresponding reservoirs. The kinetic equations for the condensates are obtained in the Langevin form. The corresponding Fokker-Planck equation is used to describe qualitatively the formation of the polariton condensate under

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simultaneous coherent and noncoherent pump. Here, we make an attempt to understand the phase transition dynamics in the polariton problem.

Using Keldysh nonequilibrium Green functions, we obtain a closed-form expression for the spectral function of the exciton condensate. The obtained relaxation time is found out to be of the same order of the period of the Rabi oscillations. This fact implies the non-Markovian character of the real polariton system at comparable times.

This fact should be considered in the light of arising technologies of femtosecond control of the polariton system state. One of the outstanding achievements here is the “full Poincare beams” experimental study [28] (see also [29–37]). As it was noted by the authors of this work, knowing the exciton reservoir character is of crucial importance for adequate interpretation of the obtained experimental results.

We derive an explicit solution of equations governing the dynamics of the polariton condensate coupled with the reservoir and discuss the effects arising due to the non-Markovian character of the open system.

II. ISOLATED EXCITON-PHOTON SYSTEM

We study a system of exciton polaritons in a semiconductor optical microcavity with an embedded quantum well. In the simple case when the quantum well possesses in-plane translational symmetry, the energy spectrum of excitons in the region of small in-plane momenta has the form

$$\varepsilon^{ex}_{\mathbf{q}} = \varepsilon^{ex}_0 + \frac{\hbar^2\mathbf{q}^2}{2m_{ex}}, \tag{1}$$

where ε is the dielectric constant of the medium, $\varepsilon^{ex}_0 = 2m_{ex}e^4/\varepsilon^2\hbar^2$ is the two-dimensional (2D) exciton binding energy, $m_{ex} = m_e + m_h$ is the 2D exciton mass, and m_e and m_h are the effective masses of an electron and a hole, respectively, $\mathbf{q}$ is the in-plane wave vector. We assume one-particle eigenstates describing noninteracting excitons to be found from the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2\nabla^2}{2m_{ex}} - \varepsilon^{ex}_{\mathbf{q}}\right)\chi_{\mathbf{q}}(\mathbf{x}) = 0. \tag{2}$$

In the case of the planar microcavity, the photons for small in-plane momenta have the following dispersion:

$$\varepsilon^{ph}_{\mathbf{q}} = \frac{\hbar c}{\sqrt{\varepsilon}}\sqrt{q_\perp^2 + q^2} \approx \varepsilon^{ph}_0 + \frac{\hbar^2\mathbf{q}^2}{2m_{ph}}. \tag{3}$$

Here, $\varepsilon^{ph}_0 = \pi\hbar cn/L\sqrt{\varepsilon}$ and $m_{ph} = \pi\hbar\sqrt{\varepsilon}/cL$ is the effective photon mass. We consider the lowest state $n = 1$. We will assume microcavities to possess in-plane translational symmetry and, so, the one-particle problem for photons reduces to

$$\left(-\frac{\hbar^2\nabla^2}{2m_{ph}} - \varepsilon^{ph}_{\mathbf{q}}\right)\psi_{\mathbf{q}}(\mathbf{x}) = 0. \tag{4}$$

In the functional approach, the grand-canonical function of the isolated exciton-photon system is the functional integral (see details, e.g., in [25])

$$Z_0 = \int D\chi\, D\bar{\chi}\, D\psi\, D\bar{\psi}\, e^{-\frac{i}{\hbar}S_0[\bar{\chi},\chi,\bar{\psi},\psi]}, \tag{5}$$

where the action S consists of four terms

$$S_0[\bar{\chi},\chi,\bar{\psi},\psi] = S_{ex}[\bar{\chi},\chi] + S_{ph}[\bar{\psi},\psi] + S_{\mathrm{Rabi}}[\bar{\chi},\chi,\bar{\psi},\psi] + S_{\mathrm{int}}[\bar{\chi},\chi], \tag{6}$$

describing excitons, photons, Rabi splitting, and interexciton interaction, respectively:

$$S_{ex}[\bar{\chi},\chi] = \int_C dt \int d\mathbf{x}\, \bar{\chi}(\mathbf{x},t) \times \left(i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2\nabla^2}{2m_{ex}} - \mu\right)\chi(\mathbf{x},t), \tag{7}$$

$$S_{ph}[\bar{\psi},\psi] = \int_C dt \int d\mathbf{x}\, \bar{\psi}(\mathbf{x},t) \times \left(i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2\nabla^2}{2m_{ph}} + E_0 - \mu\right)\psi(\mathbf{x},t), \tag{8}$$

$$S_{\mathrm{Rabi}}[\bar{\chi},\chi,\bar{\psi},\psi] = \int_C dt \int d\mathbf{x} \times \frac{\hbar\Omega}{2}[\bar{\psi}(\mathbf{x},t)\chi(\mathbf{x},t) + \psi(\mathbf{x},t)\bar{\chi}(\mathbf{x},t)], \tag{9}$$

$$S_{\mathrm{int}}[\bar{\chi},\chi] = -\frac{1}{2\hbar}\int_C dt \int d\mathbf{x}\, d\mathbf{x}'\, \bar{\chi}(\mathbf{x},t)\bar{\chi}(\mathbf{x}',t) \times V(\mathbf{x}-\mathbf{x}')\chi(\mathbf{x}',t)\chi(\mathbf{x},t). \tag{10}$$

Here, the integral $\int_c dt$ is taken along the Schwinger-Keldysh contour, $\hbar\Omega$ is the energy of Rabi splitting, μ is the chemical potential, in (8) $E_0 = \varepsilon^{ph}_0 - \varepsilon^{ex}_0$ is detuning between exciton and photon dispersion relations. $\chi(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ are the field operators written in the basis of coherent states and hence they constitute c-number functions connected with $\chi_{\mathbf{q}}(\mathbf{x})$ $\psi_{\mathbf{q}}(\mathbf{x})$ via relations

$$\chi(\mathbf{x},t) = \sum_{\mathbf{q}} \chi_{\mathbf{q}}(t)\, \chi_{\mathbf{q}}(\mathbf{x}), \quad \psi(\mathbf{x},t) = \sum_{\mathbf{q}} \psi_{\mathbf{q}}(t)\, \psi_{\mathbf{q}}(\mathbf{x}) \tag{11}$$

with the coefficients of expansion $\chi_{\mathbf{q}}(t)$ and $\psi_{\mathbf{q}}(t)$ dependent on time.

III. INTRODUCTION OF RESERVOIRS

Now, we insert our system into two thermal reservoirs separately for excitons and photons. We start with the exciton reservoir and introduce the momentum p_0 in order to separate particles with $p < p_0$ which we assume to belong to the exciton subsystem (for correspondent fields we keep notation χ) and particles with $p > p_0$ we assume to be contained in the exciton reservoir (their fields are denoted below as χ^R). We confine ourselves only to two main mechanisms of system-reservoir interplay. The first one is the direct exciton-exciton interaction and the second one is system-reservoir interaction via the semiconductor lattice.

The exciton-exciton interaction (10) resolves into the sum of terms containing different orders of χ and χ^R. In order to integrate out the exciton reservoir fields, we use the perturbation theory up to the second order in V_0. In the first order we keep all terms, i.e., proportional to $\bar{\chi}\bar{\chi}\chi\chi$, $\bar{\chi}^R\bar{\chi}^R\chi^R\chi^R$, and $\bar\chi\bar\chi^R\chi^R\chi$. These terms conserve the number of particles in the condensate and reservoir separately and do not lead to exciton condensate-reservoir exchange. So, we need in the second order of the perturbation theory to take into account this process. The very crucial point of our exploration is the statement that p_0 can be chosen pretty smaller than the momenta, which the most reservoir excitons have. Thus, the biggest contribution is made by the process when one exciton leaves or enters the condensate. Other processes of the interexciton scattering that lead to the condensate-reservoir exchange are substantially less probable. In the second order we leave only the term with the lowest order of χ:

$$\int_C dt\left(-\frac{V_0}{2F}\right)\sum_{\mathbf{p}_1,p_2,\hbar\mathbf{k}} \times\left[\bar\chi^R_{\mathbf{p}_2+\hbar\mathbf{k}}(t)\bar\chi^R_{\mathbf{p}_1-\hbar\mathbf{k}}(t)\chi_{\mathbf{p}_1}(t)\chi^R_{\mathbf{p}_2}(t)+\text{H.c.}\right]. \quad (12)$$

Here and elsewhere, we assume the exciton interaction to have the approximate local form $V(\mathbf{x}-\mathbf{x}')=V_0\,\delta(\mathbf{x}-\mathbf{x}')$, F is the area of quantization.

When we limit ourselves to the second order of the perturbation theory, our approach is consistent with the golden Fermi rule [see Eqs. (A6) and (B17)]. This fact justifies chosen accuracy of calculations.

The interaction via the semiconductor lattice requires one more subsystem to be taken into consideration: the gas of phonons, which will be regarded as equilibrium one. We add to the action the phonon part in the form

$$\begin{aligned}S_{\text{phon}}[\bar c,c,\bar\chi,\chi] &= S^0_{\text{phon}}[\bar c,c]+S_{\text{phon-ex}}[\bar c,c,\bar\chi,\chi]\\ &=\int_C dt\int_C dt'\sum_{\mathbf{k}}\bar c_{\mathbf{k}}(t)\hbar\omega_{\mathbf{k}}\\ &\quad\times\left[-\frac{1}{\omega_{\mathbf{k}}}\frac{\partial^2}{\partial t^2}-1\right]c_{\mathbf{k}}(t)\\ &\quad+\int_C dt\int_C dt'\sum_{\mathbf{kq}}\lambda_{\mathbf{k}}\\ &\quad\times\left[\chi^R_{\mathbf{q}}(t)\bar\chi_{\mathbf{k+q}}(t)c_{\mathbf{k}}(t)+\text{H.c.}\right]. \quad (13)\end{aligned}$$

Here, $c_{\mathbf{k}}$ are the phonon fields, $\omega_{\mathbf{k}}$ are the phonon frequencies. The first term is the own action of phonons and the second one accounts for the exciton-photon interaction. In this expression, we integrate out phononic fields and obtain the effective action with only excitonic degrees of freedom

$$S^{\text{eff}}_{\text{phon}}[\bar\chi,\chi]=-\frac{1}{\hbar}\int_C dt\int_C dt'\sum_{\mathbf{k}}\frac{\lambda^2_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}}\sum_{\mathbf{q}_1\mathbf{q}_2}\bar\chi^R_{\mathbf{q}_1-\mathbf{k}}(t)\times\chi_{\mathbf{q}_1}(t)D_{\mathbf{k}}(t,t')\chi^R_{\mathbf{q}_2}(t')\bar\chi_{\mathbf{q}_2+\mathbf{k}}(t'). \quad (14)$$

Here,

$$D_{\mathbf{k}}(t,t')=\hbar\int\frac{\omega^2_{\mathbf{k}}}{\omega^2-\omega^2_{\mathbf{k}}}e^{-i\omega(t-t')}\frac{d\omega}{2\pi} \quad (15)$$

is the zeroth-order Green function of phonons.

We introduce the photon reservoir with the help of the Caldeira-Legett model assuming the reservoir to be an ideal Bose gas which occupies the volume V_{phR} and has plane waves $\psi^R_k(\mathbf{x})=e^{i\mathbf{kx}}/\sqrt{V_{phR}}$ as eigenstates. In the basis of coherent states, the reservoir is described by a c-number function $\psi^R(\mathbf{x},t)=\sum_{\mathbf{k}}\psi^R_{\mathbf{k}}(t)\,\psi^R_{\mathbf{k}}(\mathbf{x})$.

The interaction between the photon subsystem and the reservoir adds to the action (6) two following terms:

$$\sum_{\mathbf{k}}\int_C dt\,\bar\psi^R_{\mathbf{k}}(t)\left(i\hbar\frac{\partial}{\partial t}-\varepsilon^{phR}_{\mathbf{k}}+\mu\right)\psi^R_{\mathbf{k}}(t) -\frac{1}{\sqrt{V_{phR}}}\sum_n\sum_{\mathbf{k}}\left[r_{n\mathbf{k}}\bar\psi_{\mathbf{q}}(t)\psi^R_{\mathbf{k}}(t)+\bar r_{n\mathbf{k}}\psi_{\mathbf{q}}(t)\bar\psi^R_{\mathbf{k}}(t)\right]. \quad (16)$$

The first term describes the reservoir degrees of freedom by itself, while the second one describes the interaction of the particle exchange type between the photon subsystem and the reservoir. It is convenient to extract explicitly from the double sum the term, which describes intensive coherent pumping in some prescribed modes with wave vectors $\mathbf{k}_1$ as coupling with classical fields $\Phi_{\mathbf{k}_1}$:

$$\frac{1}{\sqrt2}\sum_{k_1}\delta_{\mathbf{qk}_1}[\Phi_{\mathbf{k}_1}(t)\bar\psi_{\mathbf{q}}(t)+\bar\Phi_{\mathbf{k}_1}(t)\psi_{\mathbf{q}}(t)]. \quad (17)$$

The coefficient $1/\sqrt2$ is introduced for convenience.

IV. EQUATIONS FOR THE COUPLED EXCITON-PHOTON SYSTEM IN A RESERVOIR

Next, we integrate out the exciton reservoir in (14) and (12) using the perturbation theory up to the second order in $\lambda_{\mathbf{k}}$ and V_0, respectively. Analogously, we integrate out the photon reservoir in (16) and arrive to the effective action of the exciton-photon system

$$\begin{aligned}S^{\text{eff}}[\bar\chi,\chi,\bar\psi,\psi]=\sum_{\mathbf{q}}\int_C dt\int_C dt'\Big[&\bar\chi_{\mathbf{q}}(t)\Big\{\Big(i\hbar\frac{\partial}{\partial t}-\varepsilon^{ex}_{\mathbf{q}}+\mu\Big)\delta(t,t')-\hbar\Sigma^{ex}_{\mathbf{q}}(t,t')\Big\}\chi_{\mathbf{q}}(t')\\ &+\bar\psi_{\mathbf{q}}(t)\Big\{\Big(i\hbar\frac{\partial}{\partial t}-\varepsilon^{ph}_{\mathbf{q}}+\mu\Big)\delta(t,t')-\hbar\Sigma^{ph}_{\mathbf{q}}(t,t')\Big\}\psi_{\mathbf{q}}(t')\\ &+\sum_{k_1}\delta_{\mathbf{qk}_1}[\Phi_{\mathbf{k}_1}(t)\bar\chi_{\mathbf{q}}(t')+\bar\Phi_{\mathbf{k}_1}(t)\chi_{\mathbf{q}}(t')]\delta(t,t')+\frac{\hbar\Omega}{2}[\bar\psi_{\mathbf{q}}(t)\chi_{\mathbf{q}}(t')+\psi_{\mathbf{q}}(t)\bar\chi_{\mathbf{q}}(t')]\delta(t,t')\Big]. \quad (18)\end{aligned}$$

Here, the exciton self-energy $\hbar\Sigma^{ex}_{\mathbf{p}}(t,t')$ has the form

$$\hbar\Sigma^{ex}_{\mathbf{p}}(t,t')=\frac{i}{\hbar}\sum_{\mathbf{k}}\frac{\lambda^2_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}}D_{\mathbf{k}}(t,t')G^{exR}_{\mathbf{q-k}}(t,t')-\frac{V_0^2}{\hbar}\frac{1}{F^2}\sum_{\mathbf{q}_1,\mathbf{q}_2}G^{exR}_{\mathbf{q}_1}(t,t')G^{exR}_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(t',t)G^{exR}_{\mathbf{q}_2}(t,t') \quad (19)$$

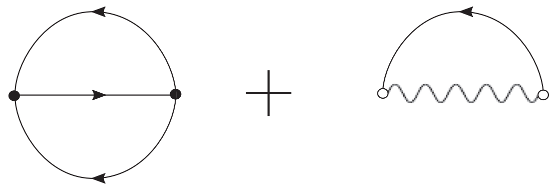

FIG. 1. Graphic representation of $\Sigma_{\mathbf{p}}^{ex}$. First diagram accounts for the direct exciton-exciton interaction, second diagram accounts for the exciton interaction via the semiconductor lattice. Straight lines denote the Green functions of reservoir excitons. Wave line denotes the phonon Green function. Solid circles denote the vertex of the exciton-exciton interaction, open circles denote the vertex of the exciton-phonon interaction.

and the photon self-energy $\hbar\Sigma_{\mathbf{q}}^{ph}(t,t')$ is

$$\hbar\Sigma_{\mathbf{q}}^{ph}(t,t') = \frac{1}{\hbar}\frac{1}{F}\sum_{\mathbf{k}} \bar{r}_{\mathbf{qk}} G_{\mathbf{k}}^{phR}(t,t') r_{\mathbf{qk}}. \quad (20)$$

Here, $G_{\mathbf{k}}^{exR}(t,t')$ and $G_{\mathbf{k}}^{phR}(t,t')$ are zeroth-order Green functions of the exciton and photon reservoirs, respectively, and will be discussed below. The diagrams used for the exciton self-energy $\hbar\Sigma_{\mathbf{p}}^{ex}$ calculation are depicted in Fig. 1.

In (19) the first term originates from the interaction with the reservoir via the lattice (14) and the second one originates from the direct exciton-exciton interaction (12).

Next, we project the parts "+" and "−" of the Schwinger-Keldysh contour on the axis of real time and perform the Keldysh rotation with the help of the substitutions $\chi(t_\pm) = 1/\sqrt{2}[\chi^{cl}(t) \pm \chi^{q}(t)]$, $\psi(t_\pm) = 1/\sqrt{2}(\psi^{cl}(t) \pm \psi^{q}(t))$ (see, e.g., [26,27] for details). As a result, we arrive to the following expression for the exciton part of the action:

$$\begin{aligned} S_{ex}[\bar{\psi}^{cl},\bar{\psi}^{q},\psi^{cl},\psi^{q}] &= \sum_{\mathbf{q}}\int_{-\infty}^{+\infty} dt \Big[\bar{\chi}_{\mathbf{q}}^{cl}(t)\Big(i\hbar\frac{\partial}{\partial t} - \varepsilon_{\mathbf{q}}^{ex} + \mu\Big)\chi_{\mathbf{q}}^{q}(t) \\ &+ \bar{\chi}_{\mathbf{q}}^{q}(t)\Big(i\hbar\frac{\partial}{\partial t} - \varepsilon_{\mathbf{q}}^{ex} + \mu\Big)\chi_{\mathbf{q}}^{cl}(t)\Big] \\ &- \hbar\sum_{\mathbf{q}}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt'\big[\bar{\chi}_{\mathbf{q}}^{cl}(t)\Sigma_{\mathbf{q}}^{ex(A)}(t,t')\chi_{\mathbf{q}}^{q}(t') \\ &+ \bar{\chi}_{\mathbf{q}}^{q}(t)\Sigma_{\mathbf{q}}^{ex(R)}(t,t')\chi_{\mathbf{q}}^{cl}(t') \\ &+ \bar{\chi}_{\mathbf{q}}^{q}(t)\Sigma_{\mathbf{q}}^{ex(K)}(t,t')\chi_{\mathbf{q}}^{q}(t')\big]. \end{aligned} \quad (21)$$

Here, $\Sigma_{\mathbf{q}}^{ex(A)}$, $\Sigma_{\mathbf{q}}^{ex(R)}$, and $\Sigma_{\mathbf{q}}^{ex(K)}$ are the advanced, retarded, and Keldysh exciton self-energies. The explicit calculation of $\Sigma_{\mathbf{q}}^{ex(R)}$ and $\Sigma_{\mathbf{q}}^{ex(K)}$ is presented in Appendix A (see also Sec. VI).

We can introduce a new complex field $\eta_{\mathbf{q}}^{ex}(t)$ and with the help of the Hubbard-Stratonovich transformation rewrite the last term in (21) as

$$\begin{aligned} &\frac{1}{\hbar}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt'\bar{\eta}_{\mathbf{q}}^{ex}(t)\Sigma_{\mathbf{q}}^{ex(K)-1}(t,t')\eta_{\mathbf{q}}^{ex}(t') \\ &- \int_{-\infty}^{+\infty} dt\big[\bar{\chi}_{\mathbf{q}}^{q}(t)\eta_{\mathbf{q}}^{ex}(t') + \chi_{\mathbf{q}}^{q}(t)\bar{\eta}_{\mathbf{q}}^{ex}(t')\big], \end{aligned} \quad (22)$$

where $\Sigma_{\mathbf{q}}^{ex(K)-1}$ is inverse of the matrix $\Sigma_{\mathbf{q}}^{ex(K)}$.

We can undertake all operations performed above with the photons; this part is omitted for the sake of brevity. We note only that the coherent pumping term acquires the form

$$\sum_{k_1}\int_{-\infty}^{+\infty} dt\, \delta_{\mathbf{qk}_1}\big[\Phi_{\mathbf{k}_1}(t)\bar{\psi}_{\mathbf{q}}^{q}(t) + \bar{\Phi}_{\mathbf{k}_1}(t)\psi_{\mathbf{q}}^{q}(t)\big] \quad (23)$$

and the Rabi cross action (9) after all transformations takes the form

$$\begin{aligned} \sum_{\mathbf{q}}\int_{-\infty}^{+\infty} dt\frac{\hbar\Omega}{2}\big[&\bar{\psi}_{\mathbf{q}}^{q}(t)\chi_{\mathbf{q}}^{cl}(t) + \bar{\psi}_{\mathbf{q}}^{cl}(t)\chi_{\mathbf{q}}^{q}(t) \\ &+ \psi_{\mathbf{q}}^{q}(t)\bar{\chi}_{\mathbf{q}}^{cl}(t) + \psi_{\mathbf{q}}^{cl}(t)\bar{\chi}_{\mathbf{q}}^{q}(t)\big]. \end{aligned} \quad (24)$$

Taking in (21) and (24) the terms linear in $\bar{\chi}_{\mathbf{q}}^{q}(t)$ and setting their sum to zero, we obtain the equation for the exciton subsystem

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\chi_{\mathbf{q}}(t) &= \Big(\frac{\hbar^2 q^2}{2m_{ex}} - \mu\Big)\chi_{\mathbf{q}}(t) \\ &+ V_0 N_{ex}\chi_{\mathbf{q}}(t) + 2V_0 N_{exR}(t)\chi_{\mathbf{q}}(t) + \frac{\hbar\Omega}{2}\psi_{\mathbf{q}}(t) \\ &+ \hbar\int_{-\infty}^{+\infty} dt'\,\Sigma_{\mathbf{q}}^{ex(R)}(t,t')\chi_{\mathbf{q}}(t') + \eta_{\mathbf{q}}^{ex}(t), \end{aligned} \quad (25)$$

where $N_{ex}(t) = \sum_{\mathbf{p},|\mathbf{p}|<p_0}\bar{\chi}_{\mathbf{p}}(t)\chi_{\mathbf{p}}(t)$ is the condensate exciton density and $N_{exR}(t) = \sum_{\mathbf{p},|\mathbf{p}|>p_0}\bar{\chi}_{\mathbf{p}}(t)\chi_{\mathbf{p}}(t)$ is the reservoir exciton density.

Analogously, we get the equation for the photon subsystem

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{q}}(t) &= \Big(\frac{\hbar^2 q^2}{2m_{ph}} - \mu\Big)\psi_{\mathbf{q}}(t) + \sum_{k_1}\delta_{\mathbf{qk}_1}\Phi_{\mathbf{k}_1}(t) \\ &+ \frac{\hbar\Omega}{2}\chi_{\mathbf{q}}(t) + \hbar\int_{-\infty}^{+\infty} dt'\,\Sigma_{\mathbf{q}}^{ph(R)}(t,t')\psi_{\mathbf{q}}(t') \\ &+ \eta_{\mathbf{q}}^{ph}(t). \end{aligned} \quad (26)$$

The fields $\eta_{\mathbf{q}}^{ex}(t)$ and $\eta_{\mathbf{q}}^{ph}(t)$ play the role of a Langevin noise with time correlations

$$\big\langle\eta_{\mathbf{q}}^{ex}(t)\bar{\eta}_{\mathbf{q}}^{ex}(t')\big\rangle = i\hbar^2\Sigma_{\mathbf{q}}^{ex(K)}(t,t'), \quad (27)$$

$$\big\langle\eta_{\mathbf{q}}^{ph}(t)\bar{\eta}_{\mathbf{q}}^{ph}(t')\big\rangle = i\hbar^2\Sigma_{\mathbf{q}}^{ph(K)}(t,t'). \quad (28)$$

If we consider the condensate as a unique macroscopically occupied state, i.e., set $p_0 = 0$, we obtain

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\chi_0(t) &= -\mu\chi_0(t) + V_0|\chi_0(t)|^2\chi_0(t) \\ &+ 2V_0 N_{exR}(t)\chi_0(t) + \frac{\hbar\Omega}{2}\psi_0(t) + \eta_0^{ex}(t) \\ &+ N_{exR}^2(t)\,\hbar\int_{-\infty}^{+\infty} dt'\,\tilde{\Sigma}_0^{ex(R)}(t,t')\chi_0(t'), \end{aligned} \quad (29)$$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_0(t) &= \Phi_0(t) - \mu\psi_0(t) + \frac{\hbar\Omega}{2}\chi_0(t) + \eta_0^{ph}(t) \\ &+ \hbar\int_{-\infty}^{+\infty} dt'\,\Sigma_0^{ph(R)}(t,t')\psi_0(t'). \end{aligned} \quad (30)$$

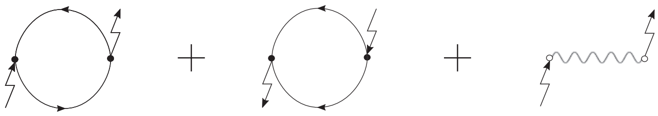

FIG. 2. Graphic representation of $\Sigma_{\mathbf{p}}^{exR}$. First and second diagrams account for the direct exciton-exciton interaction, and third diagram accounts for the exciton interaction via the semiconductor lattice. Zigzag arrows denote the exciton condensate. See other notations in the capture of Fig. 1.

Here, we introduce the normalized retarded exciton self-energy $\tilde{\Sigma}_0^{ex(R)}(t,t')$ such that $\Sigma_0^{ex(R)}(t,t') = N_{exR}^2(t)\tilde{\Sigma}_0^{ex(R)}(t,t')$ in order to explicitly include the reservoir exciton density $N_{exR}(t)$ into the equations.

The equation for the exciton reservoir density can be derived in a similar way. We use diagrams presented in Fig. 2 for the reservoir exciton self-energy $\hbar\Sigma_{\mathbf{p}}^{exR}$ calculation. The resulting equation is

$$\frac{\partial}{\partial t}N_{exR}(t) = F(t) - 2N_{exR}^2(t)\mathrm{Im}\times\left[\bar{\chi}_0(t)\int_{-\infty}^{+\infty}dt'\tilde{\Sigma}_0^{ex(R)}(t,t')\chi_0(t')\right] - 2\gamma_{exR}N_{exR}(t). \tag{31}$$

Here, $F(t)$ is the incoherent pump into the exciton reservoir, γ_{exR} is the reservoir exciton decay rate.

V. EQUATIONS IN THE LOWER-UPPER POLARITON BASIS

It is convenient to introduce exciton and photon reservoirs using the exciton-photon representation, nevertheless, in many cases it is preferable to deal with lower and upper polaritons. The change of the exciton-polariton basis to the lower-upper polariton one is performed with the help of Hopfield's tranformation

$$P_0^L = \kappa_1^L\chi_0 + \kappa_2^L\psi_0, \tag{32}$$

$$P_0^U = \kappa_1^U\chi_0 + \kappa_1^U\psi_0, \tag{33}$$

where $\kappa_1^L, \kappa_2^L, \kappa_1^U, \kappa_2^U$ are Hopfield's coefficients for $\mathbf{q} = 0$:

$$\kappa_1^{L,U} = \frac{\hbar\Omega/2}{\sqrt{\left(\varepsilon_0^{ph} - \varepsilon_0^{(L,U)}\right)^2 + \hbar^2\Omega^2/4}},\quad \kappa_2^{L,U} = \frac{\varepsilon_0^{(L,U)} - \varepsilon_0^{ph}}{\sqrt{\left(\varepsilon_0^{ph} - \varepsilon_0^{(L,U)}\right)^2 + \hbar^2\Omega^2/4}}. \tag{34}$$

Here, $\varepsilon_0^{(L,U)}$ are the lower and upper polariton dispersions for $\mathbf{q} = 0$:

$$\varepsilon_0^{(L,U)} = \frac{1}{2}\left(E_0 + \varepsilon_0^{ex} + \varepsilon_0^{ph}\right) \pm\frac{1}{2}\sqrt{\left(E_0 + \varepsilon_0^{ex} - \varepsilon_0^{ph}\right)^2 + \hbar^2\Omega^2}. \tag{35}$$

After the basis transformation we obtain for the lower polariton

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}P_0^L(t) &= \left(\varepsilon_0^L - \mu\right)P_0^L(t) + \kappa_2^L\Phi(t) \\ &\quad + \kappa_1^L V_0\left|\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right|^2 \\ &\quad \times\left[\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right] \\ &\quad + 2\kappa_1^L V_0 N_{exR}(t)\left[\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right] \\ &\quad + \hbar\int_{-\infty}^{+\infty}dt'\Sigma_0^{L(R)}(t,t')P_0^L(t') \\ &\quad + \hbar\int_{-\infty}^{+\infty}dt'\Sigma_0^{LU(R)}(t,t')P_0^U(t') + \eta_0^L(t) \end{aligned} \tag{36}$$

and for the upper polariton

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}P_0^U(t) &= \left(\varepsilon_0^U - \mu\right)P_0^U(t) + \kappa_2^U\Phi(t) \\ &\quad + \kappa_1^U V_0\left|\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right|^2 \\ &\quad \times\left[\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right] \\ &\quad + 2\kappa_1^U V_0 N_{exR}(t)\left[\kappa_1^L P_0^L(t) + \kappa_1^U P_0^U(t)\right] \\ &\quad + \hbar\int_{-\infty}^{+\infty}dt'\Sigma_0^{U(R)}(t,t')P_0^U(t') \\ &\quad + \hbar\int_{-\infty}^{+\infty}dt'\Sigma_0^{LU(R)}(t,t')P_0^L(t') + \eta_0^U(t), \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Sigma_0^{L(R)} &= \left(\kappa_1^L\right)^2\Sigma_0^{ex(R)} + \left(\kappa_2^L\right)^2\Sigma_0^{ph(R)}, \\ \Sigma_0^{U(R)} &= \left(\kappa_1^U\right)^2\Sigma_0^{ex(R)} + \left(\kappa_2^U\right)^2\Sigma_0^{ph(R)}, \\ \Sigma_0^{LU(R)} &= \kappa_1^L\kappa_1^U\Sigma_0^{ex(R)} + \kappa_2^L\kappa_2^U\Sigma_0^{ph(R)}, \end{aligned} \tag{38}$$

and

$$\eta_0^L = \kappa_1^L\eta_0^{ex} + \kappa_2^L\eta_0^{ph},\quad \eta_0^U = \kappa_1^U\eta_0^{ex} + \kappa_2^U\eta_0^{ph}. \tag{39}$$

If the energy of the excited state is close to the ground one and, thus, upper polaritons do not exist, we can use only one equation, which takes the form

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}P_0^L(t) &= \left(\varepsilon_0^L - \mu\right)P_0^L(t) + \kappa_2^L\Phi(t) \\ &\quad + \left(\kappa_1^L\right)^3 V_0\left|P_0^L(t)\right|^2 P_0^L(t) \\ &\quad + 2\left(\kappa_1^L\right)^2 V_0 N_{exR}(t)P_0^L(t) \\ &\quad + \eta_0^L(t) + \hbar\int_{-\infty}^{+\infty}dt'\Sigma_0^{L(R)}(t,t')P_0^L(t'). \end{aligned} \tag{40}$$

In this case, we have for the exciton reservoir density the equation

$$\begin{aligned} \frac{\partial}{\partial t}N_{exR}(t) &= F(t) - 2\left(\kappa_1^L\right)^2 N_{exR}^2(t) \\ &\quad \times\mathrm{Im}\left[\bar{P}_0^L(t)\int_{-\infty}^{+\infty}dt'\tilde{\Sigma}_0^{ex(R)}(t,t')P_0^L(t')\right] \\ &\quad - 2\gamma_{exR}N_{exR}(t). \end{aligned} \tag{41}$$

Thus, in Secs. IV and V we derived the equations which govern the dynamics of the polariton condensate as an open system. Equations (29), (30), (36), and (37) are the generalized equations of the Gross-Pitaevskii type and contain new terms such as Langevin noises and time nonlocal terms, which arise due to coupling of the polariton condensate with the external reservoirs. The role of the Langevin noises is most crucial at the stage of the condensate formation, which will be considered in Sec. IX, when the density of the emerging condensate is quite low. As it follows from the analysis of the condensate spectral function appropriated in the next section, the time nonlocal terms play a substantial role, when the energy of the condensate is comparable to the Rabi one. In time domain it corresponds to the case when the condensate evolutes considerably at the times of the order of the period of the Rabi oscillations. This case takes place in the applications of the polariton condensate to quantum information processing purposes (see also the discussion at the end of Sec. X).

VI. SPECTRAL FUNCTION OF THE EXCITON CONDENSATE

The spectral function $A_0^{ex}(\omega)$ of the exciton condensate embedded into the exciton reservoir is connected with the function $\Sigma_0^{ex(R)}$ by the relation

$$A_0^{ex}(\omega) = \frac{-2\,\mathrm{Im}\,\Sigma_0^{ex(R)}(\omega)}{\left(\omega - \mathrm{Re}\,\Sigma_0^{ex(R)}(\omega)\right)^2 + \left(\mathrm{Im}\,\Sigma_0^{ex(R)}(\omega)\right)^2}. \quad (42)$$

The function $\Sigma_0^{ex(R)}$ can be calculated starting with the expression (19) (see Fig. 1 as well). We begin with the second term, accounting the direct exciton-exciton interaction (see [21]). The calculation presented in detail in Appendix A leads to the following result:

$$\tilde{\Sigma}_0^{ex\text{-}ex(R)}(t,t') = i\,\frac{V_0^2}{\hbar^2}\,\frac{e^{-(\gamma_{ex}+i\Omega/2)(t-t')}}{1+\left[\frac{k_BT}{\hbar}(t-t')\right]^2}\theta(t-t'). \quad (43)$$

Here, T is the exciton reservoir temperature, γ_{ex} describes the collision broadening. It is convenient to introduce the function $S^{ex}(\omega)$ connected with the Fourier transform $\tilde{\Sigma}_{ee\,0}^{ex(R)}(\omega)$ of the function (43) via the relation $S^{ex}(\omega) = \hbar\,\mathrm{Im}\,\tilde{\Sigma}_{ee\,0}^{ex\text{-}ex(R)}(\omega)/(\pi V_0^2)$. The function $S^{ex}(\omega)$ can be obtained in closed form into limiting cases: for $\gamma_{ex} \to 0$

$$S^{ex}(\omega) = \frac{1}{2k_BT}e^{-\frac{|\hbar\omega-\hbar\Omega/2|}{k_BT}}, \quad (44)$$

and for $T \to 0$

$$S^{ex}(\omega) = \frac{1}{\pi\hbar}\frac{\gamma_{ex}}{(\omega-\Omega/2)^2+\gamma_{ex}^2}. \quad (45)$$

The function $S^{ex}(\omega)$ is normalized so that it obeys the sum rule $\hbar\int S^{ex}(\omega)d\omega = 1$. It is shown on Fig. 3 for CdTe microcavities. The exciton reservoir temperature T in (43) is taken to be 20 K, which is higher than a bath temperature in experiments [20–22], in order to take into account the nonequilibrium nature of the exciton reservoir. The curve *a* for $\gamma_{ex} = 0$ is given by (44). The decay rate $\hbar\gamma_{ex}$ ranges in papers [20–22] from 10^{-2} eV up to 10 eV, thus curves *a*, *b*, *c*, and *d* demonstrate the growing influence of γ_{ex} upon $S^{ex}(\omega)$.

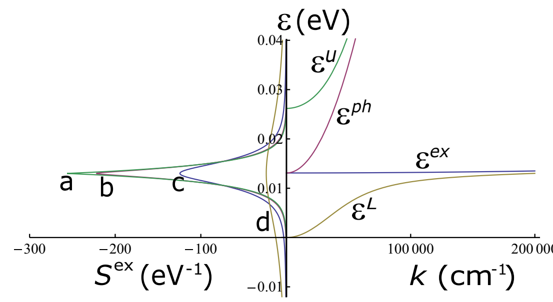

FIG. 3. We consider CdTe microcavities and use $V_0 = 1.8\times10^{-3}$ meV μm^2, $\hbar\Omega = 26$ meV, $T = 20$ K, (a) $\hbar\gamma_{ex} = 0$ meV, (b) $\hbar\gamma_{ex} = 0.1$ meV, (c) $\hbar\gamma_{ex} = 1$ meV, (d) $\hbar\gamma_{ex} = 10$ meV (see the text).

We see that for most values of γ_{ex} the exciton reservoir has the distinct non-Markovian character and only for extremely high γ_{ex} (curve *d*) it can be regarded as Markovian one. In case of the Markovian exciton reservoir $\tilde{\Sigma}_{ee\,0}^{ex(R)}(t,t') \sim \delta(t-t')$ and we have the local in time behavior in (40) and (41).

To estimate the effect of the exciton reservoir on the condensate, we now compare the condensate interexciton interaction energy $(\kappa_1^L)^3 V_0|P_0^L(t)|^2$ and the energy of the reservoir-condensate interaction $N_{exR}^2(t)\,\hbar\int_{-\infty}^{+\infty}dt'\tilde{\Sigma}_{ee\,0}^{ex(R)}(t,t')P_0^L(t')/P_0^L(t)$ in Eq. (40). Using (43) and values presented for CdTe microcavities in the caption of Fig. 3, we find that second energy exceeds the first one beginning with $N_{exR} \sim 10^{10}$ cm^{-2}. However, such direct comparison is suitable in case of *a Markovian reservoir only*. General situation is more complicated because of the different time-integral behavior at different energies of the condensate. Thus, when the energy of the condensate is close to the ground-state energy, rapidly changing $\tilde{\Sigma}_{ee\,0}^{ex(R)}(t,t')$ makes the integral value negligibly small.

The calculation of the first term in (19) (see [3,21]), which accounts for the condensate-reservoir exciton interaction via the semiconductor lattice, is presented in Appendix B. The result is

$$\hbar\,\mathrm{Im}\,\Sigma_0^{ex\text{-}a(R)}(\omega) = \frac{D^2k_BT}{4\hbar v_s^3\rho_0}N_x\,f_a\!\left(\frac{\hbar\omega-\hbar\Omega/2}{k_BT}\right), \quad (46)$$

where

$$f_a(x) = \theta(x)[2e^{-x}-1-x] + \theta(-x)[1-x]. \quad (47)$$

Here, $D = D_e + D_h$, where D_e and D_h are electron and hole deformation potentials, v_s is the velocity of sound, ρ_0 is the material density, $\theta(x)$ is the Heaviside function. For GaAs microcavities, where $D_e = -7$ eV, $D_h = 2.7$ eV, $\rho_0 = 5.4$ g cm^{-3}, and $v_s = 5.2\times10^5$ cm s^{-1}, we obtain $\frac{D^2k_BT}{4\hbar v_s^3\rho_0} = 3\times10^{-5}$ meV μm^2.

The exciton-phonon interaction and direct exciton-exciton interaction have the same order at

$$N_{x0} = \frac{1}{2\pi\hbar}\frac{D^2}{V_0^2}\frac{(k_BT)^2}{v_s^3\rho_0}. \quad (48)$$

For GaAs microcavities, where we take $V_0 = 6\times10^{-3}$ meV μm^2, the estimation gives $N_{x0} \sim 10^7$ cm^{-2}.

Here, we treat the phonons in the total system (a quantum well embedded into a cavity) as 3D phonons. If we consider

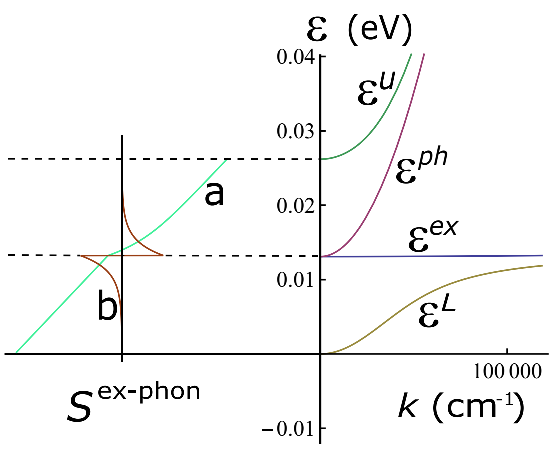

FIG. 4. Functions (a) $f(x)$ from (47) and (b) $f_a^{2D}(0,x)$ from (50). $T = 20$ K, $\hbar\Omega = 26$ meV.

acoustic phonons as 2D quasiparticles with the quantized momentum $k_z = \frac{\pi}{L_z}$, where L_z is the quantum well thickness, we obtain

$$\hbar\,\mathrm{Im}\Sigma_0^{ex\text{-}a2D(R)}(\omega) = \frac{\pi^2\hbar D^2}{2L_z^2 k_B T\rho_0 v_s} N_x \sqrt{1+\frac{2m}{\hbar k_z^2}|\omega-\Omega/2|} \times e^{-\frac{\hbar|\omega-\Omega/2|}{k_B T}}\mathrm{sgn}(\omega-\Omega/2) = \frac{\pi^2\hbar D^2}{2L_z^2 k_B T\rho_0 v_s} N_x \times f_a^{2D}\left(\frac{2mk_B T L_z^2}{\pi^2\hbar},\frac{\hbar\omega-\hbar\Omega/2}{k_B T}\right), \quad (49)$$

where

$$f_a^{2D}(a,x) = \sqrt{1+a|x|}e^{-|x|}\mathrm{sgn}(x). \quad (50)$$

The functions $f_a(x)$ from (47) and $f_a^{2D}(0,x)$ from (50) are depicted in Fig. 4. For GaAs microcavities $\frac{\pi^2\hbar D^2}{2L_z^2 k_B T\rho_0 v_s} = 3\times 10^{-7}$ meV μm^2. This magnitude is negligibly small in comparison with the experimental results, whereas in contrast the estimate made for (46) is close to them. This result leads to the statement that acoustic phonons in an optical microcavity must be treated as 3D particles. As it can be seen in Fig. 4, the functions $f_a(x)$ and $f_a^{2D}(a,x)$ are qualitatively different, so, our result contradicts the widespread opinion that the spectrum density of the exciton-phonon channel is localized in the region of the exciton reservoir energies [21].

The calculation for optical phonons with energy $\hbar\omega_0$ gives the following expression:

$$\hbar\,\mathrm{Im}\Sigma_0^{ex\text{-LO}(R)}(\omega) = \frac{e^2\hbar\omega_0}{\sqrt{2m_{ex}}(k_B T)^{3/2}}\left(\frac{1}{\varepsilon_\infty}-\frac{1}{\varepsilon_0}\right) \times N_x\, f_{\mathrm{LO}}\left(\frac{\hbar(\omega-\Omega/2-\omega_0)}{k_B T}\right), \quad (51)$$

where

$$f_{\mathrm{LO}}(x) = -\frac{e^{-x}}{\sqrt{x}}\theta(x). \quad (52)$$

Here, e is the electron charge, ε_∞ and ε_0 are high-frequency and static dielectric constants. For GaAs microcavities, where $\hbar\omega_0 = 35.3$ meV, $\varepsilon_0 = 12.9$, $\varepsilon_\infty = 10.9$, the estimation gives $\frac{e^2\hbar\omega_0}{\sqrt{2m_{ex}}(k_B T)^{3/2}}(\frac{1}{\varepsilon_\infty}-\frac{1}{\varepsilon_0}) = 5\times 10^{-3}$ meV μm^2, however, the argument of the function $f_{\mathrm{LO}}(x)$ obeys the inequality $x<0$ for all acceptable energies of the condensate, thus optical phonons do not play any role.

Performed estimates show that we can neglect $\Sigma_0^{ex\text{-}a(R)}$ in comparison with $\Sigma_0^{ex\text{-}ex(R)}$ for the reservoir exciton densities $N_x \gg N_{x0}$, so hereafter we shall take into account only $\Sigma_0^{ex\text{-}ex(R)}$.

VII. GREEN FUNCTIONS OF THE PHOTON CONDENSATE

The shape of the photon condensate self-energy $\Sigma_0^{ph(R)}(\omega)$ depends on technological features of the optical microcavity fabrication, so in order to escape a discussion on this topic we prefer for simplicity to choose a model one-peak form

$$\mathrm{Im}\Sigma_0^{ph(R)}(\omega) = -A\frac{\gamma_{ph}}{(\omega-\Omega/2)^2+\gamma_{ph}^2}, \quad (53)$$

where γ_{ph} is the decay rate of reservoir phonons. In time domain the retarded self-energy of the photon condensate reads as

$$\Sigma_0^{ph(R)}(t-t') = -iAe^{-(\gamma_{ph}+i\Omega/2)(t-t')}\theta(t-t') \quad (54)$$

and the Keldysh component is given by the expression

$$\Sigma_0^{ph(K)}(t-t') = -iAe^{-i\frac{\Omega}{2}(t-t')-\gamma_{ph}|t-t'|}. \quad (55)$$

VIII. MARKOVIAN LIMIT

Equations describing the dynamics of a Markovian system offer the property of time locality. Equation (40) becomes local in time in the regime $\gamma_{ex},\gamma_{ph} \gg 1$, when the time integral can be substituted according to the correspondence

$$\hbar\int_{-\infty}^{+\infty} dt'\,\Sigma_0^{L(R)}(t,t')P_0^L(t') \to i\hbar R(t)P_0^L(t),$$

where

$$R(t) = \frac{(\kappa_1^L)^2 V_0^2}{\hbar\gamma_{ex}}N_{exR}^2(t) - \frac{(\kappa_2^L)^2 A}{\hbar\gamma_{ph}}. \quad (56)$$

We come to the known system of equations (see, e.g., [11,21])

$$i\hbar\frac{\partial}{\partial t}P_0^L(t) = \left(\varepsilon_0^L-\mu\right)P_0^L(t)+\kappa_2^L\Phi(t) + \tilde{V}_0\left|P_0^L(t)\right|^2P_0^L(t) + 2\left(\kappa_1^L\right)^2V_0N_{exR}(t)P_0^L(t) + i\hbar R(t)P_0^L(t)+\eta_0^L(t), \quad (57)$$

$$\frac{\partial}{\partial t}N_{exR}(t) = F(t)+\frac{(\kappa_1^L)^2V_0^2}{\hbar\gamma_{ex}}N_{exR}^2(t)\left|P_0^L(t)\right|^2 - 2\gamma_{exR}N_{exR}(t). \quad (58)$$

Equation (57) has the Langevin form including the Gaussian noise $\eta_0^L(t)$ with time correlations

$$\left\langle\eta_0^L(t)\bar{\eta}_0^L(t')\right\rangle = i\hbar^2\Sigma_0^{L(K)}(t,t'), \quad (59)$$

where

$$\Sigma_0^{L(K)}(t,t') = \left(\kappa_1^L\right)^2 \Sigma_0^{ex(K)}(t,t') + \left(\kappa_2^L\right)^2 \Sigma_0^{ph(K)}(t,t') = -\frac{2i}{\hbar^2}\left(\frac{\left(\kappa_1^L\right)^2 V_0^2}{\gamma_{ex}} N_{exR}^2(t) + \frac{\left(\kappa_2^L\right)^2 A}{\gamma_{ph}}\right)\delta(t-t'). \tag{60}$$

Here, we used Eqs. (A14) and (55). This formula expresses the well-known fact of statistical theory: if $Y = X_1 + X_2$ where X_1 and X_2 are stochastic variables, then their dispersions are connected by the equality $\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2$.

Note that in the Markovian limit the equilibrium condition for the stationary regime of the polariton condensate takes the simple form $R(t) = 0$.

IX. CONDENSATE FORMATION IN PRESENCE OF THE COHERENT PUMPING

Here, we qualitatively study the lower polariton condensate formation under the simultaneous coherent and incoherent pumping with the help of the Fokker-Planck equation. We start with Eq. (57) and neglect the nonlinear term under assumption that the condensate is just at the outset of the formation and, so, its density is sufficiently low. In addition, the low condensate density can not affect the considerably larger density of the reservoir excitons, and so, we can exclude the reservoir degree of freedom from our consideration treating $R(t)$ as an assigned function of time. We come to the equation of the Langevin type

$$i\hbar\frac{\partial}{\partial t}P_0^L(t) = \kappa_2^L\Phi(t) + i\hbar R(t)P_0^L(t) + \eta_0^L(t). \tag{61}$$

It is useful to write $P_0^L = \sqrt{n_0}e^{i\phi}$, $\eta_0^L = |\eta_0|e^{i\beta}$, denote $\kappa_2^L\Phi/\hbar = \Phi_0 e^{i\alpha}$ and rewrite the equation in new variables n_0 and ϕ. We come to the result

$$\begin{aligned}\frac{\partial n_0}{\partial t} &= 2\Phi_0\sqrt{n_0}\sin(\alpha-\phi) + 2Rn_0 + \frac{2|\eta_0|}{\hbar}\sqrt{n_0}\sin(\beta-\phi),\\ \frac{\partial\phi}{\partial t} &= -\Phi_0\frac{\cos(\alpha-\phi)}{\sqrt{n_0}} - \frac{|\eta_0|}{\hbar}\frac{\cos(\beta-\phi)}{\sqrt{n_0}}.\end{aligned} \tag{62}$$

Stochastic equations of the Langevin type are a popular instrument of digital explorations, whereas for the analytical study the deterministic Fokker-Planck equation is more convenient. The path integral constitutes a natural way in passing from a Langevin equation to a corresponding Fokker-Planck equation (see details in [27]), which in our case has the form

$$i\hbar\frac{\partial}{\partial t}P = -\frac{\partial}{\partial P_0^L}\left(\kappa_2^L\Phi + i\hbar RP_0^L\right)P + \frac{\partial}{\partial \bar{P}_0^L}\left(\kappa_2^L\bar{\Phi} - i\hbar R\bar{P}_0^L\right)P - \frac{\partial^2}{\partial P_0^L\partial\bar{P}_0^L}\left(\hbar\Sigma_0^{L(K)}P\right). \tag{63}$$

Making the change of variables $(P_0^L,\bar{P}_0^L)\to(r,\phi)$, where $P_0^L = re^{i\phi}$, and going to the properly normalized distribution $W = rP$ we obtain

$$\frac{\partial}{\partial t}W = -R\left(1 + r\frac{\partial}{\partial r}\right)W + \Phi_0\left[\frac{1}{r}\cos(\alpha-\phi)\frac{\partial}{\partial\phi} - \sin(\alpha-\phi)\left(-\frac{1}{r}+\frac{\partial}{\partial r}\right)\right]W + \frac{i}{4}\left[\frac{\partial}{\partial r}\left(-\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\left(\Sigma_0^{L(K)}W\right). \tag{64}$$

In the absence of the coherent pumping Φ_0 and incoherent pumping R, the evolution of the system resolves into diffusion consisting in the last line of Eq. (64). It is worth to note the fast decrease of the phase diffusion with the increasing condensate density and the presence of the centrifugal drift potential $(i/4)r^{-1}$ (see [26]). At sufficiently high pumping, the diffusion terms can be omitted and the system evolutes along the characteristics, which obey the equations

$$\begin{aligned}\frac{\partial}{\partial t}r(t) &= \Phi_0(t)\sin[\alpha-\phi(t)] + R(t)r(t),\\ \frac{\partial}{\partial t}\phi(t) &= -\Phi_0(t)\frac{\cos[\alpha-\phi(t)]}{r(t)}.\end{aligned} \tag{65}$$

The equations coincide with deterministic parts of Eq. (62) after the substitution $r = \sqrt{n_0}$. We see from the second equation of (65) that the phase of the condensate changes up to the magnitude $\phi_0 = \alpha - \pi/2$ at a time of the order of $\hbar\sqrt{n_0}/(\kappa_2^L\Phi)$. An analysis shows that the phase $\tilde{\phi}_0 = \alpha + \pi/2$ is a point of unstable equilibrium. Therefore, this is a mechanism for the coherent pumping imposing a certain phase upon the condensate.

X. CONDENSATE DYNAMICS GOVERNED BY THE COHERENT PUMPING: NON-MARKOVIAN EFFECTS

In this section, we derive an explicit solution of the set of equations (36) and (37) and discuss the dynamics of the polariton system coupled with the reservoir. First, we make some simplifications. As we are interested in the influence of the condensate-reservoir coupling on the dynamics of the condensate, it is instructive to consider only low densities of the polariton condensate, for which the nonlinear terms can be omitted, meanwhile, the densities are assumed to be high enough to neglect the Langevin terms, which are substantial only in the limit of extremely low densities (see Sec. IX). We limit ourselves only to the case of the zeroth detuning $E_0 = 0$, for which $\varepsilon_0^L = -\hbar\Omega/2$, $\varepsilon_0^U = \hbar\Omega/2$, $\kappa_1^L = -\kappa_2^L = \kappa_1^U = \kappa_2^U = 1/\sqrt{2}$. In this regime $\mu = -\hbar\Omega/2$ as shown in [25]. We start with the homogeneous integrodifferential equations

$$i\hbar\frac{\partial}{\partial t}P_0^L(t) = \hbar\int_0^t dt'\,\Sigma_0^{L(R)}(t-t')P_0^L(t') + \hbar\int_0^t dt'\,\Sigma_0^{LU(R)}(t-t')P_0^U(t'), \tag{66}$$

$$i\hbar\frac{\partial}{\partial t}P_0^U(t) = \hbar\Omega P_0^U(t) + \hbar\int_0^t dt'\,\Sigma_0^{U(R)}(t-t')P_0^U(t') + \hbar\int_0^t dt'\,\Sigma_0^{LU(R)}(t-t')P_0^L(t') \tag{67}$$

and discuss the Cauchy problem with initial conditions $P_0^L(0)$ and $P_0^U(0)$.

The system can be analyzed in the Laplace domain. We perform the Laplace transformation

$$f(s) = \int_0^\infty e^{-st} f(t)dt \tag{68}$$

of the system (66) and (67), obtain Laplace transforms $P_0^L(s)$ and $P_0^U(s)$ from the system of two algebraic equations, and make the inverse Laplace transformation via the formula

$$f(t) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} e^{st} f(s)ds. \tag{69}$$

The result is

$$P_0^L(t) = P_0^L(0)\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^{st}\,\frac{s+i\Omega-\sigma^U(s)}{\Delta(s)} + P_0^U(0)\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^{st}\,\frac{\sigma^{LU}(s)}{\Delta(s)}, \tag{70}$$

$$P_0^U(t) = P_0^L(0)\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^{st}\,\frac{\sigma^{LU}(s)}{\Delta(s)} + P_0^U(0)\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^{st}\,\frac{s-\sigma^L(s)}{\Delta(s)}, \tag{71}$$

where

$$\Delta(s) = [s-\sigma^L(s)][s+i\Omega-\sigma^U(s)] - [\sigma^{LU}(s)]^2. \tag{72}$$

Here, $\sigma^L(s)$, $\sigma^U(s)$, and $\sigma^{LU}(s)$ are Laplace transforms of the functions $\Sigma_0^{L(R)}(t)$, $\Sigma_0^{U(R)}(t)$, and $\Sigma_0^{LU(R)}(t)$ correspondingly.

For better understanding this general solution behavior it is useful to make the particular choice of the functions $\Sigma_0^{(R)}$. Let us take $\Sigma_0^{ex(R)}(t-t') = ia_{ex}e^{-(\gamma_{ex}+i\Omega/2)(t-t')}$ and $\Sigma_0^{ph(R)}(t-t') = -ia_{ph}e^{-(\gamma_{ph}+i\Omega/2)(t-t')}$, where $a_{ex},a_{ph} = \text{const}(t) > 0$. From (38) we obtain $\Sigma_0^{L(R)} = \Sigma_0^{U(R)} = (\Sigma_0^{ex(R)}+\Sigma_0^{ph(R)})/2$, $\Sigma_0^{LU(R)} = (\Sigma_0^{ex(R)}-\Sigma_0^{ph(R)})/2$. The Laplace transforms are

$$\sigma^L(s) = \sigma^U(s) = \frac{1}{2}\left(\frac{a_{ex}}{s+i\Omega/2+\gamma_{ex}} - \frac{a_{ph}}{s+i\Omega/2+\gamma_{ph}}\right),$$
$$\sigma^{LU}(s) = \frac{1}{2}\left(\frac{a_{ex}}{s+i\Omega/2+\gamma_{ex}} + \frac{a_{ph}}{s+i\Omega/2+\gamma_{ph}}\right). \tag{73}$$

The next step is to close the contour of integration in (70) and (71) and calculate integrals using residues. The problem is that the substitution of these expressions into (72) converts Δ into a polynomial of the sixth order in s. Thus, aiming to obtain a closed-form expression, we restrict ourselves to a simpler problem, when only the lower polariton condensate exists. The problem reduces to the equation

$$i\hbar\frac{\partial}{\partial t}P_0^L(t) = \hbar\int_0^t dt'\,\Sigma_0^{L(R)}(t-t')P_0^L(t'), \tag{74}$$

which has the following solution:

$$P_0^L(t) = P_0^L(0)\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^{st}\,\frac{s+i\Omega-\sigma^U(s)}{\tilde{\Delta}(s)}, \tag{75}$$

where $\Delta^L(s) = s - \sigma^L(s)$ is the polynomial of third order in s, hence, the poles of the integrand s_1, s_2, and s_3 can be obtained in the explicit form. The solution becomes

$$P_0^L(t) = P_0^L(0)\sum_{i=1\ldots3} e^{s_i t}(s_i+i\Omega/2+\gamma_{ex}) \times (s_i+i\Omega/2+\gamma_{ph})\prod_{j=1\ldots3,\, j\neq i}\frac{1}{s_i-s_j}. \tag{76}$$

We see that the solution is represented by the set of exponents with complex factors depending on time and constant prefactors.

The expression for s_1 shows that the coupling to the exciton reservoir increases the energy of the lower polariton condensate, whereas in contrast the coupling to the photon reservoir decreases the energy. The real part of s_1 is responsible for the condensate density evolution. The equilibrium condition for the condensate stability under the incoherent exponential factors calculated to first order in a_{ex} and a_{ph},

$$s_1 = -\frac{i}{\Omega}(a_{ex}-a_{ph}),\quad s_2 = -i\frac{\Omega}{2}\left(1+\frac{2a_{ph}}{\Omega^2}\right) - \gamma_{ph},$$
$$s_3 = -i\frac{\Omega}{2}\left(1-\frac{2a_{ex}}{\Omega^2}\right) - \gamma_{ex}. \tag{77}$$

Pumping is the equality $\text{Re}\, s_1 = 0$, which resolves into a cumbersome explicit expression, in contrast to the simple result $R(t) = 0$ [see Eq. (56)] in the Markovian case. Remaining factors express the reaction of the open system at the frequencies close to the reservoir ones with corresponding decay rates.

We can formally solve the same simplified problem for the upper polariton condensate in the absence of the lower one and analogously obtain three exponential factors: one close to $-i\Omega$ and two close to $-i\Omega/2$. These factors are roots of the cubic equation $\Delta^U(s) = s + i\Omega - \sigma^U(s)$. So, we now turn back to the original problem (70) and (71) and come to reach the conclusion that the evolution of the full polariton system is described by a set of six exponential functions. Among the factors of these functions, one is close to 0, one is close to $-i\Omega$, and remaining four are close to $-i\Omega/2$.

Our choice of the function $\Sigma_0^{ex(R)}$ in the form $ia_{ex}e^{-(\gamma_{ex}+i\Omega/2)t}$ corresponds to the case of the zeroth temperature [see (19)]. If the temperature of the exciton reservoir is different from zero, the Laplace transform of $\Sigma_0^{ex(R)}$ does not have a simple explicit form like in (73) and the problem becomes complicated. We can draw a conclusion that at nonzero temperatures of the exciton reservoir, the reaction of the system does not resolve into the set of the exponential functions with characteristic energies in their factors, but has more complicated character.

Note that the obtained homogeneous solution can be used for the calculation of the system response on the coherent pumping $\Phi(t)$. We can write the solution (70) and (71) in a short form

$$\begin{bmatrix} P_{0\,hom}^L(t) \\ P_{0\,hom}^U(t)\end{bmatrix} = \begin{bmatrix} m_{LL}(t) & m_{LU}(t) \\ m_{LU}(t) & m_{UU}(t)\end{bmatrix}\begin{bmatrix} P_0^L(0) \\ P_0^U(0)\end{bmatrix}. \tag{78}$$

If at time $t=0$ the polariton condensate has components $P_0^L(0)$, $P_0^U(0)$ and the coherent pumping $\Phi(t)$ turns on, then the further dynamics is governed by the sum of the homogeneous solution (78) and the convolution of components of the matrix $\mathbf{m}(t)$ from (78) and the function $\Phi(t)$:

$$P^L_{0\,\mathrm{con}}(t) = -i\frac{\kappa_2^L}{\hbar}\int_0^t dt' m_{LL}(t-t')\Phi(t') - i\frac{\kappa_2^U}{\hbar}\int_0^t dt' m_{LU}(t-t')\Phi(t'),$$
$$P^U_{0\,\mathrm{con}}(t) = -i\frac{\kappa_2^L}{\hbar}\int_0^t dt' m_{LU}(t-t')\Phi(t') - i\frac{\kappa_2^U}{\hbar}\int_0^t dt' m_{UU}(t-t')\Phi(t'). \quad (79)$$

The obtained solution can be involved in an analysis of experimental results of the exploration [28], in which authors excited time-resolved Rabi oscillations of a polariton condensate by a wide pulse of a Gaussian shape. Furthermore, the developed formalism provides an alternative to the treatment of the exciton-polariton qubit appropriated by the authors of [32] in the frame of the Markovian approximation. Qubits based on polariton condensate manipulations open a new dimension for quantum information processing such as quantum calculations, quantum cloning, and quantum memory.

XI. CONCLUSION

With the use of path-integral formalism involving the integration along the Schwinger-Keldysh time contour and Keldysh technique, we formulate the universal approach which describes the nonequilibrium system of excitons and photons, their pumping and leakage, and the coupled exciton-photon condensate arising.

We employ the exciton-photon basis and treat a system of coupled excitons and photons. Such an approach provides a natural way to introduce thermal reservoirs. We make the microscopical derivation of equations governing the dynamics of the polariton condensate. The obtained equations include nonlocal in time terms and the Langevin noise. The correlation function of the Langevin noise is shown to be not Gaussian. This fact in conjunction with the time nonlocality of the obtained equations points at the non-Markovian character of the polariton condensate regarded as an open system embedded into the thermal reservoir. The non-Markovian character is substantial at times of the order of the Rabi oscillation period.

We obtain the effective Fokker-Plank equation for the open polariton system and use it to study the process of the polariton condensate formation under the coherent and incoherent pumping, and make clearer the mechanism for the coherent pumping imposing a certain phase upon the arising condensate.

We perform the *ab initio* derivation of the exciton condensate spectral function considering the direct interexciton interaction and exciton interaction via the semiconductor lattice as the channels of the condensate-reservoir particle exchange. The obtained closed-form expression helps to make simplifying but realistic assumptions about nonlocal terms in the equations mentioned above. These equations are used for the analysis of the system response to the external disturbance. We find their explicit solution, which shows that at zeroth temperature the system response represents the set of six oscillations. Among them, two oscillations have frequencies close to those of the isolated lower and upper polariton condensates and others are the damped oscillations arising in consequence of the polariton condensate coupling to the external thermal reservoirs. The equilibrium condition of the condensate stability under the incoherent pumping is obtained. The non-Markovian behavior of the polariton condensate at nonzero temperatures is a subject of future exploration.

APPENDIX A: DIRECT INTRAEXCITON CONDENSATE-RESERVOIR INTERACTION

Starting with the second term of (19) (see also [21]) we obtain

$$\hbar\Sigma^{ex(R)}_{\mathbf{q}}(t_1-t_2) = \hbar\Sigma^{ex(>)}_{\mathbf{q}}(t_1-t_2) - \hbar\Sigma^{ex(<)}_{\mathbf{q}}(t_1-t_2) = -\frac{V_0^2}{\hbar F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}$$
$$\times\big[G^{exR(>)}_{\mathbf{q}_1}(t_1-t_2)G^{exR(<)}_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(t_2-t_1)G^{exR(>)}_{\mathbf{q}_2}(t_1-t_2) - G^{exR(<)}_{\mathbf{q}_1}(t_1-t_2)G^{exR(>)}_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(t_2-t_1)G^{exR(<)}_{\mathbf{q}_2}(t_1-t_2)\big]$$
$$= -\frac{V_0^2}{2\hbar F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}\int\frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3}e^{-i(\omega_1-\omega_2+\omega_3)(t_1-t_2)}\big[G^{exR(>)}_{\mathbf{q}_1}(\omega_1)G^{exR(<)}_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(\omega_2)G^{exR(>)}_{\mathbf{q}_2}(\omega_3)$$
$$- G^{exR(<)}_{\mathbf{q}_1}(\omega_1)G^{exR(>)}_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(\omega_2)G^{exR(<)}_{\mathbf{q}_2}(\omega_3)\big]. \quad (A1)$$

For excitons we use the Green functions

$$G^{exR(<)}_{\mathbf{q}}(\omega) = -iA_{\mathbf{q}}(\omega)f_{\mathbf{q}}, \quad G^{exR(>)}_{\mathbf{q}}(\omega) = -iA_{\mathbf{q}}(\omega)(1+f_{\mathbf{q}}), \quad (A2)$$

where

$$A_{\mathbf{q}}(\omega) = \frac{2\gamma_{ex}}{(\omega-e_{\mathbf{q}}/\hbar)^2+\gamma_{ex}^2} \approx 2\pi\delta(\omega-e_{\mathbf{q}}/\hbar), \quad (A3)$$

$f_{\mathbf{q}}$ are the occupation numbers of the reservoir excitons, $e_{\mathbf{q}} = \frac{\hbar\Omega}{2} + \frac{\hbar^2\mathbf{q}^2}{2m_{ex}}$ are the energies of the corresponding states, γ_{ex} describes the collision broadening. We will use the Boltzmann distribution $f_{\mathbf{q}} = Pe^{-\frac{e_{\mathbf{q}}}{k_BT}}$, $P = \frac{2\pi\hbar^2}{m_{ex}k_BT}N_{exR}$, regarding that $f_{\mathbf{q}} \ll 1$.

The Fourier transform of $\hbar\Sigma_{\mathbf{q}}^{ex(R)}(t_1-t_2)$ is

$$\Sigma_{\mathbf{q}}^{ex(R)}(\omega)=\int dt\, e^{i\omega t}\Sigma_{\mathbf{q}}^{ex(R)}(t)=-\frac{V_0^2}{\hbar^2F^2}(-i)^3\sum_{\mathbf{q}_1\mathbf{q}_2}[(1+f_{\mathbf{q}_1})f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(1+f_{\mathbf{q}_2})-f_{\mathbf{q}_1}(1+f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}})f_{\mathbf{q}_2}]$$
$$\times\int_0^\infty dt\, e^{i[\omega-(e_{\mathbf{q}_1}-e_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}+e_{\mathbf{q}_2})/\hbar]t-\gamma_{ex}t}$$
$$=\frac{V_0^2}{\hbar^2F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}\frac{1}{\omega-(e_{\mathbf{q}_1}-e_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}+e_{\mathbf{q}_2})/\hbar+i\gamma_{ex}}[(1+f_{\mathbf{q}_1})f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(1+f_{\mathbf{q}_2})-f_{\mathbf{q}_1}(1+f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}})f_{\mathbf{q}_2}].\quad\text{(A4)}$$

In the limiting case $\gamma_{ex}\to 0$, the imaginary part of $\Sigma_{\mathbf{q}}^{ex(R)}(\omega)$ has the form

$$\mathrm{Im}\,\Sigma_{\mathbf{q}}^{ex(R)}(\omega)=-\frac{\pi V_0^2}{\hbar F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}\delta(\hbar\omega-e_{\mathbf{q}_1}+e_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}-e_{\mathbf{q}_2})[(1+f_{\mathbf{q}_1})f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(1+f_{\mathbf{q}_2})-f_{\mathbf{q}_1}(1+f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}})f_{\mathbf{q}_2}].\quad\text{(A5)}$$

The corresponding quantum kinetic equation reads as [26]

$$\frac{\partial n_{\mathbf{q}}}{\partial t}=\frac{2\pi}{\hbar}V_0^2\frac{1}{F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}\delta(\varepsilon_{\mathbf{q}}-e_{\mathbf{q}_1}+e_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}-e_{\mathbf{q}_2})$$
$$\times[(1+n_{\mathbf{q}})f_{\mathbf{q}_1}(1+f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}})f_{\mathbf{q}_2}$$
$$-n_{\mathbf{q}}(1+f_{\mathbf{q}_1})f_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}}(1+f_{\mathbf{q}_2})].\quad\text{(A6)}$$

Here, $n_{\mathbf{q}}$ are the occupation numbers of the exciton condensate, $\varepsilon_{\mathbf{q}}=\frac{\hbar^2\mathbf{q}^2}{2m_{ex}}$ are the energies of the corresponding states. In the approximation $f_{\mathbf{q}_1+\mathbf{q}_2}=0$ (see [21]), the expression (A4) for $\mathbf{q}=0$ can be rewritten as

$$\Sigma_0^{ex(R)}(t)=i\frac{V_0^2}{\hbar^2F^2}\sum_{\mathbf{q}_1\mathbf{q}_2}e^{-i\frac{e_{\mathbf{q}_1}-e_{\mathbf{q}_1+\mathbf{q}_2}+e_{\mathbf{q}_2}}{\hbar}t-\gamma_{ex}t}f_{\mathbf{q}_1}f_{\mathbf{q}_2}$$
$$=i\frac{V_0^2}{\hbar^2}\int_0^\infty\frac{2\pi q_1dq_1}{(2\pi)^2}\int_0^\infty\frac{q_2dq_2}{(2\pi)^2}\int_0^{2\pi}d\phi\, f_{\mathbf{q}_1}f_{\mathbf{q}_2}$$
$$\times\exp\left[i\left(\frac{\hbar^2q_1q_2\cos\phi}{\hbar\, m}-\frac{\Omega}{2}\right)t-\gamma_{ex}t\right].\quad\text{(A7)}$$

Next, we use the formula

$$\int_0^{2\pi}d\phi\, e^{ia\cos\phi}=2\pi J_0(a)\quad\text{(A8)}$$

and obtain

$$\Sigma_0^{ex(R)}(t)=i\frac{V_0^2}{\hbar^2}\frac{1}{(2\pi)^2}e^{-(i\Omega/2+\gamma_{ex})t}$$
$$\times\int_0^\infty q_1dq_1\int_0^\infty q_2dq_2J_0\left(\frac{\hbar q_1q_2t}{m}\right)f_{\mathbf{q}_1}f_{\mathbf{q}_2}$$
$$=i\frac{V_0^2}{\hbar^2}\frac{P^2}{(2\pi)^2}e^{-(i\Omega/2+\gamma_{ex})t}\int_0^\infty q_1dq_1$$
$$\times\int_0^\infty q_2dq_2J_0\left(\frac{\hbar q_1q_2t}{m}\right)e^{-\frac{\beta\hbar^2}{2m}(q_1^2+q_2^2)}.\quad\text{(A9)}$$

Next, we integrate over q_2 using the formula

$$\int_0^\infty q\,dq\,J_o(aq)e^{-bq^2}=\frac{e^{-a^2/4b}}{2b}\quad\text{(A10)}$$

and obtain the integral

$$\int_0^\infty dq_1q_1\frac{e^{-a^2q_1^2/4b}}{2b}e^{-bq_1^2}=\frac{1}{a^2+4b^2},\quad\text{(A11)}$$

where $a=\hbar t/m$ and $b=\hbar^2\beta/2m$. Thus, the result is

$$\Sigma_0^{ex(R)}(t)=i\frac{V_0^2N_x^2}{\hbar^2}\frac{e^{-(\gamma_{ex}+i\Omega/2)t}}{1+\left(\frac{k_BT}{\hbar}t\right)^2}\theta(t-t').\quad\text{(A12)}$$

To find the Keldysh component of the self-energy we can start with the relation

$$\hbar\Sigma_{\mathbf{q}}^{ex(K)}(t_1-t_2)=\hbar\Sigma_{\mathbf{q}}^{ex(>)}(t_1-t_2)+\hbar\Sigma_{\mathbf{q}}^{ex(<)}(t_1-t_2)\quad\text{(A13)}$$

and obtain analogously

$$\Sigma_0^{ex(K)}(t)=-i\frac{V_0^2N_x^2}{\hbar^2}\frac{e^{-i\frac{\Omega}{2}t-\gamma_{ex}|t|}}{1+\left(\frac{k_BT}{\hbar}t\right)^2}.\quad\text{(A14)}$$

APPENDIX B: INTRAEXCITON CONDENSATE-RESERVOIR INTERACTION VIA THE LATTICE

We start from the classical action [38]

$$S_{ph}^{cl}=\int dt\,L_{ph}^{cl}=\frac{1}{2}\int dt\,d\mathbf{x}\left(\rho_0\frac{\partial d_i}{\partial t}\frac{\partial d_i}{\partial t}-B\frac{\partial d_i}{\partial x_j}\frac{\partial d_i}{\partial x_j}\right).\quad\text{(B1)}$$

Here, $\mathbf{d}$ is the displacement vector, ρ_0 is the density of ions, and B is the adiabatic bulk modulus $B=-V(\partial P/\partial V)_S$. The summation over repeating indexes is assumed.

Next, we introduce boson annihilation and creation operators $\hat{c}_{\mathbf{k}}$ and $\hat{c}_{\mathbf{k}}^+$ such that $[\hat{c}_{\mathbf{k}},\hat{c}_{\mathbf{k}'}^+]=\delta_{\mathbf{k},\mathbf{k}'}$, $\hat{c}_{\mathbf{k}}^+=\hat{c}_{-\mathbf{k}}$ and construct the phonon field operator

$$\hat{\phi}(\mathbf{x},t)=\sum_{\mathbf{k}}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2F}}[\hat{c}_{\mathbf{k}}(t)e^{i\mathbf{kx}}+\hat{c}_{\mathbf{k}}^+(t)e^{-i\mathbf{kx}}],\quad\text{(B2)}$$

where $\omega_{\mathbf{k}} = v_s k$, $v_s = \sqrt{B/\rho_0}$ the velocity of sound. We rewrite the displacement vector $\mathbf{d}$ as the operator

$$\hat{\mathbf{d}}(\mathbf{x},t) = -\frac{i}{\sqrt{\rho_0}} \sum_{\mathbf{k}} \sqrt{\frac{\hbar}{2F\omega_{\mathbf{k}}}} \frac{\mathbf{k}}{k} [\hat{c}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{x}} - \hat{c}^{+}_{\mathbf{k}}(t) e^{-i\mathbf{k}\mathbf{x}}]. \tag{B3}$$

Substituting $\hat{\mathbf{d}}(\mathbf{x},t)$ into (B1) and integrating over $\mathbf{x}$ we have

$$\frac{\rho_0}{2}\int dt\, d\mathbf{x} \frac{\partial d_i}{\partial t}\frac{\partial d_i}{\partial t} = \int dt \sum_{\mathbf{k}} \frac{\hbar}{2\omega_{\mathbf{k}}}\left[\frac{\partial \hat{c}_{\mathbf{k}}}{\partial t}\frac{\partial \hat{c}^{+}_{\mathbf{k}}}{\partial t} + \frac{\partial \hat{c}^{+}_{\mathbf{k}}}{\partial t}\frac{\partial \hat{c}_{\mathbf{k}}}{\partial t}\right] \tag{B4}$$

and

$$-\frac{B}{2}\int dt\, d\mathbf{x} \frac{\partial d_i}{\partial x_j}\frac{\partial d_i}{\partial x_j} = -\int dt \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} [\hat{c}_{\mathbf{k}}\hat{c}^{+}_{\mathbf{k}} + \hat{c}^{+}_{\mathbf{k}}\hat{c}_{\mathbf{k}}]. \tag{B5}$$

Thus, in the basis of the coherent states the phonon action $S^{(0)}_{ph}$ reads as

$$S^{(0)}_{ph} = \int dt \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\left[\frac{1}{\omega^2_{\mathbf{k}}}\frac{\partial \bar{c}_{\mathbf{k}}}{\partial t}\frac{\partial \bar{c}_{\mathbf{k}}}{\partial t} - \bar{c}_{\mathbf{k}}\bar{c}_{\mathbf{k}}\right] = \hbar \sum_{\mathbf{k}} \int dt\, dt' \bar{c}_{\mathbf{k}}(t') \tilde{D}^{-1}_{\mathbf{k}}(t-t') c_{\mathbf{k}}(t), \tag{B6}$$

where

$$\tilde{D}^{-1}_{\mathbf{k}}(t-t') = \omega_{\mathbf{k}}\left[-\frac{1}{\omega^2_{\mathbf{k}}}\frac{\partial^2}{\partial t^2} - 1\right]\delta(t-t'). \tag{B7}$$

The corresponding propagator $\tilde{D}_{\mathbf{k}}(t-t')$ obeys the equation

$$\omega_{\mathbf{k}}\left(-\frac{1}{\omega^2_{\mathbf{k}}}\frac{\partial^2}{\partial t^2} - 1\right)\tilde{D}_{\mathbf{k}}(t-t') = \delta(t-t') \tag{B8}$$

and, so, equals $\tilde{D}_{\mathbf{k}}(t-t') = (1/\hbar\omega_{\mathbf{k}}) D_{\mathbf{k}}(t-t')$ where

$$D_{\mathbf{k}}(t-t') = \hbar \int \frac{\omega^2_{\mathbf{k}}}{\omega^2 - \omega^2_{\mathbf{k}}} e^{-i\omega(t-t')} \frac{d\omega}{2\pi} \tag{B9}$$

is the phonon propagator.

Now, we introduce the bare phonon Green functions $D^{(>)}(\mathbf{x},t) = -i\langle\tilde{\phi}(\mathbf{x},t)\tilde{\phi}(0,0)\rangle_0$ and $D^{(<)}(\mathbf{x},t) = -i\langle\tilde{\phi}(0,0)\tilde{\phi}(\mathbf{x},t)\rangle_0$ where for $\hat{\phi}(\mathbf{x},t)$ we use the expression (B2) with $\hat{c}_{\mathbf{k}}(t) = \hat{c}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}$ and the subscript "0" denotes averaging over the equilibrium distribution. Substituting (B2) we obtain

$$D^{(>)}(\mathbf{x},t) = -i\sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2F}[\langle\hat{c}_{\mathbf{k}}\hat{c}^{+}_{\mathbf{k}}\rangle_0 e^{i\mathbf{k}\mathbf{x} - i\omega_{\mathbf{k}} t} + \langle\hat{c}^{+}_{\mathbf{k}}\hat{c}_{\mathbf{k}}\rangle_0 e^{-i\mathbf{k}\mathbf{x} + i\omega_{\mathbf{k}} t}] \tag{B10}$$

and

$$D^{(<)}(\mathbf{x},t) = -i\sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2F}[\langle\hat{c}_{\mathbf{k}}\hat{c}^{+}_{\mathbf{k}}\rangle_0 e^{-i\mathbf{k}\mathbf{x} + i\omega_{\mathbf{k}} t} + \langle\hat{c}^{+}_{\mathbf{k}}\hat{c}_{\mathbf{k}}\rangle_0 e^{i\mathbf{k}\mathbf{x} - i\omega_{\mathbf{k}} t}]. \tag{B11}$$

The Fourier transforms have the form

$$D^{(>)}(\mathbf{k},\omega) = \int dt\, d\mathbf{x}\, D^{(0)>}(\mathbf{x},t) e^{-i(\mathbf{k}\mathbf{x} - \omega t)} = -2\pi i \frac{\hbar\omega_{\mathbf{k}}}{2}[(1+N_{\mathbf{k}})\delta(\omega - \omega_{\mathbf{k}}) + N_{\mathbf{k}}\delta(\omega + \omega_{\mathbf{k}})] \tag{B12}$$

and

$$D^{(<)}(\mathbf{k},\omega) = -2\pi i \frac{\hbar\omega_{\mathbf{k}}}{2}[(1+N_{\mathbf{k}})\delta(\omega + \omega_{\mathbf{k}}) + N_{\mathbf{k}}\delta(\omega - \omega_{\mathbf{k}})]. \tag{B13}$$

The Hamiltonian of the exciton-phonon interaction can be written as

$$\hat{H}^{ex\text{-}ph} = \sum_{\mathbf{k}} \lambda_{\mathbf{k}}[\hat{\chi}^{+}_{\mathbf{p}+\mathbf{k}}\hat{\chi}_{\mathbf{k}}\hat{c}_{\mathbf{k}} + \hat{\chi}_{\mathbf{p}+\mathbf{k}}\hat{\chi}^{+}_{\mathbf{k}}\hat{c}^{+}_{\mathbf{k}}], \tag{B14}$$

where $\lambda_{\mathbf{k}}$ is the strength of the exciton-phonon interaction and will be discussed later.

Starting with the first term of (19), we obtain

$$\begin{aligned}\hbar\Sigma^{ex\text{-}a(R)}_{\mathbf{q}}(t_1-t_2) &= \frac{i}{\hbar F}\sum_{\mathbf{k}} \frac{\lambda^2_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}}\left[D^{(>)}_{\mathbf{k}}(t_1-t_2) G^{exR(>)}_{\mathbf{q}-\mathbf{k}}(t_1-t_2) - D^{(<)}_{\mathbf{k}}(t_1-t_2) G^{exR(<)}_{\mathbf{q}-\mathbf{k}}(t_1-t_2)\right] \\ &= -\frac{i}{2\hbar}\sum_{\mathbf{k}} \lambda^2_{\mathbf{k}}\{(1+f_{\mathbf{q}-\mathbf{k}})[(1+N_{\mathbf{k}})e^{-i(e_{\mathbf{q}-\mathbf{k}}/\hbar + \omega_{\mathbf{k}})(t_1-t_2)} + N_{\mathbf{k}} e^{-i(e_{\mathbf{q}-\mathbf{k}}/\hbar - \omega_{-\mathbf{k}})(t_1-t_2)}] \\ &\quad - f_{\mathbf{q}-\mathbf{k}}[(1+N_{\mathbf{k}})e^{-i(e_{\mathbf{q}-\mathbf{k}}/\hbar - \omega_{-\mathbf{k}})(t_1-t_2)} + N_{\mathbf{k}} e^{-i(e_{\mathbf{q}-\mathbf{k}}/\hbar + \omega_{\mathbf{k}})(t_1-t_2)}]\}.\end{aligned} \tag{B15}$$

In the limiting case $\gamma_{ex} \to 0$ the imaginary part of $\Sigma^{ex\text{-}a(R)}_{\mathbf{q}}(\omega)$ reads as

$$\begin{aligned}\mathrm{Im}\Sigma^{ex\text{-}a(R)}_{\mathbf{q}}(\omega) = -\frac{\pi}{2\hbar F}\sum_{\mathbf{k}} \lambda^2_{\mathbf{k}}\{&(1+f_{\mathbf{q}-\mathbf{k}})(1+N_{\mathbf{k}})\delta(\hbar\omega - e_{\mathbf{q}-\mathbf{k}} - \hbar\omega_{\mathbf{k}}) + (1+f_{\mathbf{q}+\mathbf{k}})N_{\mathbf{k}}\delta(\hbar\omega - e_{\mathbf{q}+\mathbf{k}} + \hbar\omega_{\mathbf{k}}) \\ &- f_{\mathbf{q}+\mathbf{k}}(1+N_{\mathbf{k}})\delta(\hbar\omega - e_{\mathbf{q}+\mathbf{k}} + \hbar\omega_{\mathbf{k}}) - f_{\mathbf{q}-\mathbf{k}}N_{\mathbf{k}}\delta(\hbar\omega - e_{\mathbf{q}-\mathbf{k}} - \hbar\omega_{\mathbf{k}})\}.\end{aligned} \tag{B16}$$

The corresponding quantum kinetic equation can be written as [26]

$$\begin{aligned}\frac{\partial n_{\mathbf{q}}}{\partial t} = \frac{2\pi}{\hbar}\frac{1}{F}\sum_{\mathbf{k}} \lambda^2_{\mathbf{k}}\{&(1+n_{\mathbf{q}})f_{\mathbf{q}+\mathbf{k}}(1+N_{\mathbf{k}})\delta(\varepsilon_{\mathbf{q}} - e_{\mathbf{q}+\mathbf{k}} + \hbar\omega_{\mathbf{k}}) + (1+n_{\mathbf{q}})f_{\mathbf{q}-\mathbf{k}}N_{\mathbf{k}}\delta(\varepsilon_{\mathbf{q}} - e_{\mathbf{q}-\mathbf{k}} - \hbar\omega_{\mathbf{k}}) \\ &- n_{\mathbf{q}}(1+f_{\mathbf{q}-\mathbf{k}})(1+N_{\mathbf{k}})\delta(\varepsilon_{\mathbf{q}} - e_{\mathbf{q}-\mathbf{k}} - \hbar\omega_{\mathbf{k}}) - n_{\mathbf{q}}(1+f_{\mathbf{q}+\mathbf{k}})N_{\mathbf{k}}\delta(\varepsilon_{\mathbf{q}} - e_{\mathbf{q}+\mathbf{k}} + \hbar\omega_{\mathbf{k}})\}.\end{aligned} \tag{B17}$$

For acoustic phonons, the strength of the exciton-phonon interaction is

$$\lambda_{\mathbf{k}} = A^{e}_{\mathbf{k}} I^{\parallel}_{e}(k_{\parallel}) I^{\perp}_{e}(k_z) + A^{h}_{\mathbf{k}} I^{\parallel}_{h}(k_{\parallel}) I^{\perp}_{h}(k_z), \tag{B18}$$

where

$$A^{e,h}_{\mathbf{k}} = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\rho_0 L_z F}}\frac{D_{e,h}}{v_s}, \quad k = \sqrt{k_z^2 + k_{\parallel}^2}, \tag{B19}$$

$$I^{\perp}_{e,h}(k_z) \approx 1, \quad I^{\parallel}_{e,h}(k_{\parallel}) = \left[1 + \left(\frac{m_{e,h} a_0^{2\text{D}} k_{\parallel}}{2(m_e + m_h)}\right)^2\right]^{-3/2} = \left[1 + \left(\frac{k_{\parallel}}{k^{e,h}_{\parallel}}\right)^2\right]^{-3/2}, \quad k^{e,h}_{\parallel} = \frac{2(m_e + m_h)}{m_{e,h} a_0^{2\text{D}}}. \tag{B20}$$

Here, D_e and D_h are electron and hole deformation potentials, m_e and m_h are electron and hole masses, $a_0^{2\text{D}}$ is the Bohr radius of a 2D exciton. If the momentum $k_{\parallel}$ obeys the condition $k_{\parallel} \ll k^{e}_{\parallel}, k^{h}_{\parallel}$, we can regard $\lambda_{\mathbf{k}} \sim \sqrt{k}$.

For GaAs microcavities $D_e = -7$ eV, $D_h = 2.7$ eV, $m_e = 6.1 \times 10^{-29}$ g, $m_h = 4.1 \times 10^{-28}$ g, $a_0^{2\text{D}} = 5.9 \times 10^{-7}$ cm. Thus, the momentum $k^{e}_{\parallel} = 2.6 \times 10^{7}$ cm^{-1} is greater than the momenta of the reservoir excitons.

We take $\mathbf{q} = 0$, $N_{\mathbf{k}} = 0$, $\lambda^2_{\mathbf{k}} = \frac{\alpha}{F L_z} k^2$, $\alpha = \frac{\hbar D^2}{2\rho_0 v_s}$, $D = D_e + D_h$, denote $k_{\parallel} \to k$, and obtain

$$\begin{aligned}
\hbar\,\text{Im}\Sigma_0^{ex\text{-}a(R)}(\omega) &= -\frac{\pi}{2\hbar}\sum_{\mathbf{k}} \lambda^2_{\mathbf{k}}\big[(1 + f_{-\mathbf{k}})\delta(\hbar\omega - e_{-\mathbf{k}} - \hbar\omega_{\mathbf{k}}) - f_{\mathbf{k}}\delta(\hbar\omega - e_{\mathbf{k}} + \hbar\omega_{\mathbf{k}})\big] \\
&= \frac{\pi}{2\hbar} P\alpha \int_0^{\infty} \frac{k\,dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sqrt{k_z^2 + k^2}\, e^{-\frac{\hbar^2 k^2}{2mk_B T}} \\
&\quad\times \left[\delta\left(\hbar\omega - \frac{\hbar^2 k^2}{2m} + \hbar v_s \sqrt{k_z^2 + k^2}\right) - \delta\left(\hbar\omega - \frac{\hbar^2 k^2}{2m} - \hbar v_s \sqrt{k_z^2 + k^2}\right)\right] \\
&= \frac{D^2 k_B T}{4\hbar v_s^3 \rho_0} N_x\, f_a\left(\frac{\hbar\omega - \hbar\Omega/2}{k_B T}\right),
\end{aligned} \tag{B21}$$

where

$$f_a(x) = \theta(x)[2e^{-x} - 1 - x] + \theta(-x)[1 - x]. \tag{B22}$$

For optical phonons with energy $\hbar\omega_0$ the procedure of the calculation is very similar. We consider here only the emission of phonons and write

$$\hbar\,\text{Im}\Sigma_0^{ex\text{-LO}(R)}(\omega) = -\frac{\pi}{2\hbar}\sum_{\mathbf{k}} \lambda^2_{\mathbf{k}}(1 + f_{-\mathbf{k}})\delta(\hbar\omega - e_{-\mathbf{k}} - \hbar\omega_0), \tag{B23}$$

where

$$\lambda_{\mathbf{k}} = \big(I^{\parallel}_{e}(k_{\parallel}) + I^{\parallel}_{h}(k_{\parallel})\big)\frac{e}{k}\sqrt{\frac{\hbar\omega_0}{F L_1}\left(\frac{1}{\varepsilon_{\infty}} - \frac{1}{\varepsilon_0}\right)}. \tag{B24}$$

Here, L_1 is the length of quantization in z direction, ε_{∞} and ε_0 are high-frequency and static dielectric constants. Thus, we have

$$\lambda^2_{\mathbf{k}} \approx 2\frac{e^2}{k^2 + k_z^2}\frac{\hbar\omega_0}{F L_1}\left(\frac{1}{\varepsilon_{\infty}} - \frac{1}{\varepsilon_0}\right), \tag{B25}$$

where we denote $k_{\parallel} \to k$. After the integration we obtain

$$\begin{aligned}
\hbar\,\text{Im}\Sigma_0^{ex\text{-LO}(R)}(\omega) &= \frac{e^2\hbar\omega_0}{4\pi\hbar^2}\left(\frac{1}{\varepsilon_{\infty}} - \frac{1}{\varepsilon_0}\right)\sqrt{\frac{2m}{k_B T}} P\ f_{\text{LO}}\left(\frac{\hbar(\omega - \Omega/2 - \omega_0)}{k_B T}\right) \\
&= \frac{e^2\hbar\omega_0}{\sqrt{2m}(k_B T)^{3/2}}\left(\frac{1}{\varepsilon_{\infty}} - \frac{1}{\varepsilon_0}\right) N_x f_{\text{LO}}\left(\frac{\hbar(\omega - \Omega/2 - \omega_0)}{k_B T}\right),
\end{aligned} \tag{B26}$$

where

$$f_{\text{LO}}(x) = -\frac{e^{-x}}{\sqrt{x}}\theta(x). \tag{B27}$$

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