

Stored and absorbed energy of fields in lossy chiral single-component metamaterialsI. Semchenko,¹ A. Balmakou,¹ S. Khakhomov,¹ and S. Tretyakov²¹*Department of General Physics, Francisk Skorina Gomel State University, 246019 Gomel, Belarus*²*Department of Radio Science and Engineering, Aalto University, P.O. Box 13000, FI-00076 Aalto, Finland*

(Received 29 April 2017; revised manuscript received 29 December 2017; published 24 January 2018)

Here we present theoretical results for estimation of electromagnetic field energy density and absorbed energy in dispersive lossy chiral single-component metamaterials which consist of an ensemble of identical helical resonators as inclusions. The shape of the helical resonator can vary over a wide range, from a straight wire to a flat split ring. An interaction of the inclusions with harmonic circularly polarized electromagnetic plane waves is studied. We focus on how the inclusion shape influences the mentioned metamaterial properties. The derived general solution for the problem is in good agreement with previous partial and alternative solutions obtained for split ring resonators, straight wires, and helices. The study reveals the optimal geometry of helical lossy resonators for their strongest selectivity of interaction with circularly polarized radiation.

DOI: [10.1103/PhysRevB.97.014432](https://doi.org/10.1103/PhysRevB.97.014432)**I. INTRODUCTION**

Knowledge of electromagnetic field energy stored and dissipated in various materials is important both from the physical point of view and for applications (for instance, these quantities define the efficiency and bandwidth of antennas, including nanoemitters). It is known that the electromagnetic field energy density in materials can be uniquely defined in terms of the effective material parameters only in the case of negligible losses [1]. For artificial materials based on metal or dielectric inclusions of various shapes, called metamaterials, absorption can be neglected when the operational frequency is far from the resonant frequencies of the inclusions and from the lattice resonance if the material is periodic. However, the most interesting phenomena take place in resonant regimes. If the material exhibits considerable losses near the frequency of interest, it is not possible to define the stored energy density in a general way, i.e., to express that in terms of the material permittivity and permeability functions [1]. Only if the internal structure of the medium is known and a specific dispersion model (like the Lorentz or Drude dispersion) is applicable is it possible to define and find the stored reactive energy in terms of the dispersion model parameters (the resonant frequency or the plasma frequency as well as the damping factor) even if losses are significant. Electromagnetic field energy density in such lossy materials has been addressed several times and a number of different methods have been developed [2–7]. In Ref. [2], for example, the energy of electromagnetic field in absorptive one-component dielectric nonchiral medium is determined. In Ref. [3], a general approach that allows one to determine the stored energy density in complex composite microwave materials has been presented (note that Ref. [7] gives an important correction to the method reported in Ref. [3]).

In this paper we study energy density and absorption in chiral composites formed by helical inclusions. Energy density in lossy chiral media has been studied in recent papers [8–10]. These papers determine the energy density for the case of linear polarization and complement our studies, which focus on the effects of polarization selectivity of interaction with circularly

polarized radiation in composites with helices of different shapes. Using microscopic and macroscopic models, paper [10] provides solutions for the average total energy density of the macroscopic quasimonochromatic electromagnetic field in regions of normal dispersion with negligible losses of a magnetoelectric medium and for the average energy density of the macroscopic quasimonochromatic electromagnetic field in a chiral medium with losses as a function of the refractive index and characteristic impedance of the medium. In comparison with the solutions obtained here, solutions from Ref. [10] cannot be used to study the pitch angle dependence and optimization of the electromagnetic properties of chiral media.

The approach proposed here is generally valid for metamaterials composed of an ensemble of identical particles of any helical shape including degenerate limiting cases of straight wires, split rings, and Ω -particles. To find most general expressions, we study lossy chiral metamaterials composed of an array of helices with variable shape (from the straight wire to the split ring resonator) at the main half-wavelength resonance: $L \approx \lambda/2$, where L is the full length of helix wire. Note that when the metamaterial is composed of any other kind of identical particles which cannot be derived from helices, the expressions should be obtained specifically for the particular particle shape, but the approach to derivations which we present here remains valid.

II. ALTERNATIVE APPROACHES**A. Lossless composites**

The constitutive relations for isotropic chiral materials or metamaterials have the form

$$\begin{aligned}\vec{D} &= \varepsilon_0 \varepsilon_r \vec{E} - j \sqrt{\varepsilon_0 \mu_0} \kappa_{em} \vec{H}, \\ \vec{B} &= \mu_0 \mu_r \vec{H} + j \sqrt{\varepsilon_0 \mu_0} \kappa_{me} \vec{E},\end{aligned}\quad (1)$$

where $\kappa_{em} = \kappa_{me}$ is the chirality parameter. For a chiral metamaterial in the absence of absorption, but at the presence of dispersion, the time-averaged energy density can be found

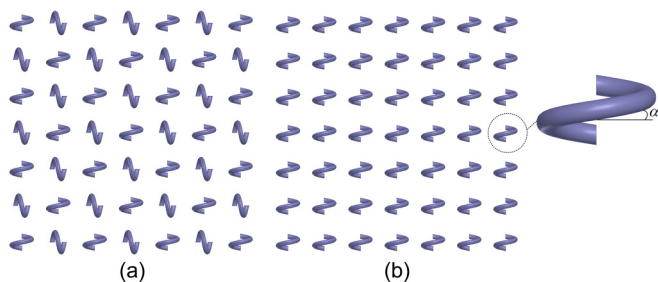


FIG. 1. An exemplary array (isotropic or uniaxial) composed of single-turn helices.

in terms of the material parameters¹:

$$\langle w \rangle = \frac{1}{4} \varepsilon_0 \frac{d}{d\omega} (\omega \varepsilon_r) \vec{E} \cdot \vec{E}^* + \frac{1}{4} \mu_0 \frac{d}{d\omega} (\omega \mu_r) \vec{H} \cdot \vec{H}^* - j \frac{1}{4} \sqrt{\varepsilon_0 \mu_0} \frac{d}{d\omega} (\omega \kappa_{em}) (\vec{E}^* \cdot \vec{H} - \vec{E} \cdot \vec{H}^*), \quad (2)$$

and these expressions are valid for any shape or shapes of composite constituents. For circularly polarized (CP) plane waves, we have derived the following formula:

$$\langle w \rangle = \frac{\varepsilon_0}{4} \frac{d(\omega \varepsilon_r)}{d\omega} (|\vec{E}_+|^2 + |\vec{E}_-|^2) + \frac{\varepsilon_0}{4} \frac{\varepsilon_r}{\mu_r} \frac{d(\omega \mu_r)}{d\omega} (|\vec{E}_+|^2 + |\vec{E}_-|^2) + \frac{\varepsilon_0}{2} \sqrt{\frac{\varepsilon_r}{\mu_r}} \frac{d(\omega \kappa)}{d\omega} (|\vec{E}_+|^2 - |\vec{E}_-|^2), \quad (3)$$

where $\kappa = \kappa_{em}$, and the right-hand (+) and left-hand (−) polarized waves are defined as follows: $\vec{E}_\pm = E_{0\pm} (\vec{x}_0 \mp j \vec{y}_0) \exp(j\omega t)$. For the field amplitudes of plane waves, the relation between the electric and magnetic field vectors reads $\vec{H}_\pm = \pm j \sqrt{\frac{\varepsilon_0 \varepsilon_r}{\mu_0 \mu_r}} \vec{E}_\pm$ [11, Chap. 2]. An interesting fact is that these relations between electric and magnetic field vectors do not contain the chirality parameter in explicit form. As a result, we arrive at a physically important conclusion that the Poynting vector for plane waves in a chiral medium also does not depend on the chirality parameter. However, expressions (2) and (3) lose validity if losses in the medium are not negligible.

Let us consider the following important case illustrated in Fig. 1 where the medium is composed of conductive helices. In particular, we are interested in the so-called “optimal” helices. Optimal helices, introduced and studied in Refs. [8–10, 12–18], interact only with waves of one of the two orthogonal circular polarizations. The geometrical parameters of such helices are known. For a single-turn helix, for example, the optimal pitch angle of $\alpha = 13.65^\circ$ is obtained at the condition of the fundamental half-wave resonance. Note that for metamaterials formed of optimal helices, the material parameters are related as

$$\varepsilon_r \approx \mu_r \approx 1 \pm \kappa. \quad (4)$$

This relation means that the metamaterial formed optimal helices exhibits identically strong dielectric, magnetic, and chiral properties. For the real and imaginary parts we have

$\varepsilon_r' \approx \mu_r' \approx 1 \pm \kappa'$ and $\varepsilon_r'' \approx \mu_r'' \approx \pm \kappa''$, respectively. For $\kappa' > 0$ (i.e., for the right-handed helix) it turns out that the CP mode with the amplitude E_{0-} is not absorbed. For $\kappa' < 0$ (i.e., for the left-handed helix) it leads to nonabsorption of the circular mode with the amplitude E_{0+} . The same conclusion holds for the stored energy.

First we will consider the case when the excitation frequency approaches the resonant frequency. Although the absorption of waves in this frequency range is negligible, the frequency dispersion of the metamaterial parameters is already manifested. In the absence of losses, that is, assuming that $\varepsilon_r = \varepsilon_r^*$, $\mu_r = \mu_r^*$, $\kappa_r = \kappa_r^*$, relation (3) holds approximately true in the frequency range under consideration. For the right-hand helices ($\kappa > 0$) the time-averaged energy density reads

$$\langle w \rangle \approx \frac{1}{2} \varepsilon_0 (|\vec{E}_+|^2 + |\vec{E}_-|^2) + \varepsilon_0 \frac{d}{d\omega} (\omega \kappa) |\vec{E}_+|^2, \quad (5)$$

where the first term on the right is the energy of the electromagnetic field in vacuum, and the second term is the energy of the helix in the field of only one circular mode as the second circular mode does not excite the helix. For the left-hand helices ($\kappa < 0$) the time-averaged energy density takes the following form:

$$\langle w \rangle \approx \frac{1}{2} \varepsilon_0 (|\vec{E}_+|^2 + |\vec{E}_-|^2) - \varepsilon_0 \frac{d}{d\omega} (\omega \kappa) |\vec{E}_-|^2. \quad (6)$$

Thus, the difference is in the sign of the second term on the right of (6). Of course, in this case the helix is activated by circular polarization of the opposite to the case (5) handedness.

B. Absorptive single-component media

To model absorptive single-component chiral media we adopt a classical model of free electrons oscillating harmonically with the frequency of the wave of excitation. Since the conduction electrons are bounded in a finite-length wire, these harmonic oscillations can be resonant. The conductor forming the metamaterial element has a finite length, so a standing wave of electric current appears in it. The amplitude of this wave increases significantly under the condition of resonance. Usually in practice, the elements of the metamaterial are made of a thin conducting wire or strip. Then the resonance condition is determined mainly by the length of this conductor. Resonances occur when an integer number of half-waves of electric current is laid on the length of the conductor. Therefore, the resonance frequencies for thin conductors can be approximately determined by the formula

$$\omega_{0n} = \frac{\pi c n}{L},$$

where n is an integer specifying the mode number of the oscillation, c is the speed of light in a vacuum, and L is the length of the conductor. Deviations from exact inverse proportionality are caused by two reasons: first, because the resonator is open and the wire has a finite diameter and, second, due to interactions of many identical inclusions in the metamaterial. All parameters of single particles are effective phenomenological parameters, which take into account the influence of particle interactions. In particular, the difference between the local fields acting on each particle and the averaged macroscopic field leads to a shift of the resonance frequency

and to compensation of radiation losses from individual helices [19].

To describe the resonance harmonic oscillations of conduction electrons in a standing wave of electric current arising in a metamaterial element, we must take into account not only the harmonic external field. We must also introduce into consideration a restoring force acting on the electrons in the direction opposite to their displacement along the wire. It is known from the theory of harmonic oscillations that the magnitude of this restoring force should be proportional to the displacement of the conduction electrons along the wire, i.e., $f_{\text{rest}} = -ks$. Therefore, such a force is called quasielastic. Here s is the electron displacement, k is the effective coefficient which can be expressed at the resonant frequency as $k = m\omega_0^2$. Let us repeat that the physical reason for the appearance of such a restoring force is the fact that the motion of conduction electrons is limited in a wire of finite dimensions.

As shown by numerical simulations [18], under the condition of half-wave resonance, the resonant frequency depends mainly on the length of the wire, if its thickness is small. In this case, the shape of the wire, that is, the shape of the metamaterial element, has little effect on the resonance frequency. For example, in paper [18] the change in the number of turns for optimal helices was investigated. In this case, the shape of the helix, that is, its pitch angle, also changed. It is shown that the relative change in the resonant frequency is about 3% for a constant wire length in the straightened state.

The used model of electrons oscillating along spiral trajectories is actually a classical model of natural chiral material; see, for example, the classical book [20]. The oscillator model of small and resonant wire (and other) metal scatterers is a classical one, used before by many authors. This model has been successfully validated by experiments; as an example of experimental validation specifically for helices, we can refer to paper [21]. In paper [21], the antenna model is used to calculate the polarizabilities of the helix and the electromagnetic field scattered by the helix. However, that paper did not calculate the energy of the electromagnetic field in the structure formed by the helices.

The basic approximation used in our paper is that only one resonant frequency of electron oscillations is considered. At the same time for metamaterials such an approximation is justified, since they consist of resonant “meta-atoms.” It should be recalled that natural chiral substances have usually molecular absorption spectra, characterized by a certain absorption band. Unlike molecules of natural substances, meta-atoms have discrete absorption spectra, in which the resonant frequencies differ significantly from each other. In the presence of several resonant frequencies of a meta-atom, it is necessary to add the field energy calculated for each resonance separately. The method used by us makes it possible to trace directly how the field energy in a metamaterial is composed of the potential and kinetic energies of oscillating conduction electrons. In known particular cases the results coincide with those obtained using other approaches to calculating the field energy in the medium, for example in the framework of the equivalent-circuit model. Known special cases, considered earlier in papers [3–7], concern media formed by straight wires and split rings, that is, artificial structures possessing dielectric and magnetic properties. In this paper, we consider a metamaterial consisting

of metal helices, that is, a more general case of an artificial structure that simultaneously possesses dielectric, magnetic, and chiral properties. We answer the question of how the energy of the electromagnetic field changes if the medium is characterized not only by the dielectric permittivity and magnetic permeability, but also by the chirality parameter. Since in the chiral metamaterial the eigenmodes of the electromagnetic field have circular polarization, it can be assumed that the electromagnetic energy will be different for the right and left circularly polarized waves.

The energy of one representative electron is

$$W_e = \Pi_e + K_e = \frac{ks^2}{2} + \frac{mv^2}{2},$$

where s is the electron displacement along the helix, v is the electron velocity along the helix, m is the electron mass, and k is the effective coefficient describing the quasielastic force on the electron in the direction opposite to its displacement. The final result from this model coincides with earlier results [2,3], in particular, for a straight wire oscillator and split ring resonator. As a particular case, first let us consider the electrons oscillating along a straight wire.

The equation of motion of an electron in a one-component medium having one resonance frequency is $m\ddot{x} = -eE_x - \kappa x - \gamma\dot{x}$, where the electric field $E_x = E_{0x} \exp(j\omega t)$ and displacement $x = x_0 \exp(j\omega t)$ are time harmonic. The solution for the equation of motion takes the form of the following complex function:

$$x = -\frac{e}{m} \frac{E_x}{\omega_0^2 - \omega^2 + j\omega\Gamma}, \tag{7}$$

where $\Gamma = \gamma/m$. The electric field which excites one particular inclusion, called the local field, is different from the averaged macroscopic field E_x . This difference is taken into account by the use of effective, macroscopic value of the resonance frequency ω_0 , which depends not only on the parameters of a single inclusion but also on the inclusion concentration. Also, the loss factor Γ here measures only the dissipation loss in the particles, since the radiation loss is compensated due to particle interactions [19].

We introduce the potential and kinetic energies in the following well-known way:

$$\langle \Pi \rangle_t = \frac{1}{2} k \langle x'^2 \rangle_t, \quad \langle K \rangle_t = \frac{1}{2} m \langle v'^2 \rangle_t. \tag{8}$$

Here x' and v' are the real parts of the electron’s displacement and its speed, which in the general case are defined as complex functions. The angle brackets stand for time averaging. Taking into consideration the obtained solution for the equation of motion, we can find the sum of the potential and kinetic energies (8):

$$\langle \Pi \rangle_t + \langle K \rangle_t = \frac{1}{4} \frac{e^2}{m} E_{0x} E_{0x}^* \psi(\omega), \tag{9}$$

which is a summation taken for only one oscillating charged particle (e.g., electron) and where

$$\psi(\omega) = \frac{\omega_0^2 + \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

is a parameter introduced for convenience. Therefore, to find the volumetric energy density, we need to take into account the concentration of conducting electrons $N = N_e V_c N_c$, where N_e is the concentration of electrons in metal, and $V_c = LS$ is the volume of one conducting element forming the metamaterial (in this case it is a straight wire). Finally, N_c is the concentration of the conductive inclusions. In addition, we add the energy of the field itself and obtain

$$\langle w_{el} \rangle_t = \frac{1}{4} \varepsilon_0 \left(1 + N \frac{e^2}{m \varepsilon_0} \psi(\omega) \right) E_x E_x^*. \quad (10)$$

In this expression, the energy of the conduction electrons and the energy of the field are averaged over the oscillation period.

We see that at this stage of the simulation a well-known expression appears for the angular plasma frequency

$$\omega_p^2 = \frac{N e^2}{m \varepsilon_0}.$$

In particular, for a metamaterial composed of straight wires, the relative permittivity takes the form

$$\varepsilon_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}. \quad (11)$$

Using this relation for the angular plasma frequency the volumetric energy density can be written as

$$\langle w_{el} \rangle_t = \frac{1}{4} \varepsilon_0 [1 + \omega_p^2 \psi(\omega)] E_x E_x^*. \quad (12)$$

Formula (12) was obtained earlier [2–4] by different approaches.

For the energy of electric field absorbed per unit volume and time, the following relation is known:

$$\langle Q_{el} \rangle_t = -\frac{1}{2} \omega \varepsilon_0 \varepsilon_r'' |E|^2,$$

where

$$\varepsilon_r'' = -N \frac{e^2}{m \varepsilon_0} \frac{\omega \Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} < 0$$

is the imaginary part of the relative permittivity. We note that the energy of the field decreases as the dissipative forces make a negative work on electrons decelerating them. As is usually done, we can consider the work of forces that slow electrons dA_{dis} , performed over a period of time dt in a physically small volume dV . In this case the following relation is true:

$$\frac{dA_{\text{dis}}}{dt dV} = -\gamma (v')^2 N = -Q_{el}, \quad (13)$$

where $v' = \text{Re}(\dot{x})$ is the real part of the electron velocity, the retarding force of a single electron is $-\gamma v'$, and the corresponding power is $-\gamma (v')^2$. Going to a unit volume, the power needs to be multiplied by the concentration of electrons N . The formula (13) confirms that if the absorbed energy is positive ($Q_{el} > 0$) then the work of dissipative forces dA_{dis} is negative.

C. Lossy chiral media: Helix pitch angle dependence

For the helical model considering a more complicated way of electrons' movement, we take the following equation of

motion:

$$m\ddot{s} = -ks - \gamma\dot{s} - eE_s, \quad (14)$$

where $s = s_0 \exp(j\omega t)$ is the displacement of the charged particle along the helical path, s_0 is the displacement amplitude, $E_s = E_x \sin \alpha$ is the component of field which is tangential to the surface of the helix, the axis of the helix is directed along the x axis, α is the helix pitch angle which is found through

$$\sin \alpha = \pm \frac{1}{\sqrt{1 + q^2 r^2}}.$$

Here the “+” sign corresponds to the right-hand helix while the “−” sign corresponds to the left-hand helix, where $|q| = 2\pi/h$, h is the helix pitch, r is the helix radius, $q > 0$ for a right-handed helix, and $q < 0$ for a left-handed helix. Equation (14) is the standard equation for forced oscillations of a particle in the presence of losses. The left-hand side of the equation contains the product of mass by acceleration of the electron, and on the right-hand side the sum of all the forces acting on the electron is written. The first term on the right side of the equation describes the returning force acting opposite to the displacement of the electron. Without this force, resonant oscillations of conduction electrons would be impossible. The second term on the right-hand side of Eq. (14) describes the retarding force, which leads to losses of the mechanical energy of the oscillating electron. The third term on the right-hand side of the equation describes the electric force acting on the electron from the electromagnetic wave. The acceleration of the electron and all these forces are directed along the tangent to the trajectory of the electron, that is, along the helical conductor axis.

The fundamental resonance of a finite-length piece of thin conducting wire takes place when the length of the conductor is approximately equal to half the wavelength of the electromagnetic field. If the wire curvature radius is very large compared with the wire diameter (which is the considered case) and the wire does not cross or touch itself, changes in the conductor wire shape do not lead to any significant shift in the resonance frequency. Whatever the shape of the thin conductor, the electric current in it is directed along the wire axis. Consequently, the coefficient in Eq. (14) for a representative electron depends primarily on the total length of the thin conductor, and not on the shape of the conductor. The coefficient k has the form $k = m\omega_0^2 = m(\frac{\pi c}{L})^2$, where L is the length of the conductor for any of the considered shapes. For example, if the conductor has the form of a single-turn helix, we can write L in an explicit form, $L = \sqrt{(2\pi r)^2 + h^2}$, where r is the helix radius and h is the helix pitch. Passing to a straight wire, we have $r = 0$ and $h = L$.

Let us take into consideration external magnetic field $B_x = B_{0x} \exp(j\omega t)$ oscillating along the helix axis and permeating the loops of the helix. According to Faraday's law of electromagnetic induction, we can write

$$-E2\pi r = \pi r^2 \frac{\partial B_x}{\partial t}, \quad (15)$$

where E is the component of vortex electrical field orthogonal to the helix axis. The last relation results in

$$E = -\frac{1}{2} r j \omega B_x. \quad (16)$$

Now we can calculate the component of electric field tangential to the helix:

$$E_s = E \cos \alpha = -j\omega \frac{r}{2} B_x \cos \alpha,$$

where the cosine of helix pitch angle is

$$\cos \alpha = \frac{qr}{\sqrt{1+q^2r^2}}.$$

The electron's displacement along the helix, stipulated by both electric and magnetic fields, takes the following form:

$$s = -\frac{e}{m} \frac{E_x \sin \alpha - j\frac{\omega r}{2} B_x \cos \alpha}{\omega_0^2 - \omega^2 + j\omega\Gamma}. \quad (17)$$

Generalizing the methods which we used above for metamaterials composed of straight wires, we calculate the time-averaged energy of oscillating electrons localized in a helix:

$$\begin{aligned} \langle \Pi \rangle_t + \langle K \rangle_t = \frac{e^2}{4m} \psi(\omega) \left\{ E_{0x} E_{0x}^* \sin^2 \alpha + \frac{\omega^2 r^2}{4} B_{0x} B_{0x}^* \cos^2 \alpha \right. \\ \left. + j\frac{\omega r}{2} \sin \alpha \cos \alpha (E_{0x} B_{0x}^* - E_{0x}^* B_{0x}) \right\}. \end{aligned} \quad (18)$$

For relation (18), we can consider two special limiting cases:

(a) Assuming $\alpha = \frac{\pi}{2}$ (straight wires), we transit to formula (9).

(b) Assuming $\alpha = 0$ (split rings) and then multiplying by the electrons' concentration N and adding the energy of the magnetic field itself (in vacuum), we get

$$\begin{aligned} \langle w_m \rangle_t = \frac{1}{4} \mu_0 |H|^2 + (\langle \Pi \rangle_t + \langle K \rangle_t) N \\ = \frac{1}{4} \mu_0 |H|^2 + \frac{1}{4} N \frac{e^2}{m} \psi(\omega) \frac{\omega^2 r^2}{4} B_{0x} B_{0x}^*. \end{aligned} \quad (19)$$

For the metamaterial with artificial magnetic properties due to currents in split-ring resonators, the relative permeability is equal to

$$\mu_r = 1 + \frac{A_1 \omega^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}, \quad (20)$$

where

$$A_1 = \frac{1}{4} \mu_0 \frac{N e^2}{m} r^2.$$

Now, the time-averaged energy of magnetic field is

$$\langle w_m \rangle_t = \frac{1}{4} \mu_0 [1 + A_1 \omega^2 \psi(\omega)] |H|^2. \quad (21)$$

Formula (21) was reported earlier in Refs. [3,4] for structures with artificial magnetic properties. We recall that in this formula, as above, the factor 1/4 appears after averaging the energy density over time.

Next, let us return to the more complicated case, when electrons oscillate in the helix so that the metamaterial exhibits simultaneous dielectric, magnetic, and chiral properties. In this

case, we use the general relation for the energy of a single electron (9) multiplied by the concentration of the conductive electrons and add the energies of both electric and magnetic fields:

$$\langle w \rangle_t = \frac{1}{4} \varepsilon_0 E_{0x} E_{0x}^* + \frac{1}{4} \mu_0 H_{0x} H_{0x}^* + (\langle \Pi \rangle_t + \langle K \rangle_t) N. \quad (22)$$

We use the material equations (1), which are true for the first-order spatial dispersion effects, i.e., for the cases where the second-order spatial derivatives of the fields can be neglected in considering particle response (usually it means that chirality is weak). The material equations (1) contain the chirality parameter $\kappa = \kappa_{em}$ only in the first power, therefore in the following equations we should keep the chirality parameter also only in the first power. The conservation of the chirality parameter at higher powers is not expedient in the calculations, since their accuracy is limited by the material equations originally used. Taking this into account, we use the approximate relation $\vec{B} = \mu_0 \vec{H}$ in the transformation of formulas (18) and (22) and get

$$\begin{aligned} \langle w \rangle_t = \frac{\varepsilon_0}{4} E_{0x} E_{0x}^* \left(1 + \frac{1}{A \varepsilon_0} \psi(\omega) \right) + \frac{\mu_0}{4} H_{0x} H_{0x}^* \\ \times \left(1 + \frac{\mu_0 M^2}{A} \psi(\omega) \right) + \frac{1}{4} \frac{\mu_0 M}{A} \\ \times j (E_{0x} H_{0x}^* - E_{0x}^* H_{0x}) \psi(\omega). \end{aligned} \quad (23)$$

This equation contains in the third term on the right the specific torsion of the spiral trajectory of the electron q in the first power, which enters into the coefficient M . Consequently, the chiral properties of the metamaterial are taken into account in this expression in the first-order approximation, as it should be done. The advantage of formula (23) is that it gives the ultimate value for the volumetric energy density at $\omega = \omega_0$, i.e., at the resonance frequency where the absorption is strong. Here we use the following shorthand notations:

$$\begin{aligned} A = m \frac{r^2 q^2 + 1}{N e^2}, \\ M = \frac{r^2 q \omega}{2}. \end{aligned}$$

For metamaterials composed of helices, the relative permittivity, permeability, and chirality factors are

$$\begin{aligned} \varepsilon_r = 1 + \frac{1}{A \varepsilon_0} \frac{\omega_0^2 - \omega^2 - j\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}, \\ \mu_r = 1 + \mu_0 \frac{M^2}{A} \frac{\omega_0^2 - \omega^2 - j\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}, \\ \kappa = \frac{M}{A} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\omega_0^2 - \omega^2 - j\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}. \end{aligned}$$

For plane waves, $\langle w \rangle_t$ takes the form

$$\begin{aligned} \langle w \rangle_t &= \frac{1}{4} \varepsilon_0 (|E_{0+}|^2 + |E_{0-}|^2) \left(1 + \frac{1}{A \varepsilon_0} \psi(\omega) \right) \\ &+ \frac{1}{4} \varepsilon_0 \left| \frac{\varepsilon_r}{\mu_r} \right| (|E_{0+}|^2 + |E_{0-}|^2) \left(1 + \frac{\mu_0 M^2}{A} \psi(\omega) \right) \\ &+ \sqrt{\varepsilon_0 \mu_0} \frac{M}{4A} \left(\sqrt{\frac{\varepsilon_r^*}{\mu_r^*}} + \sqrt{\frac{\varepsilon_r}{\mu_r}} \right) (|E_{0+}|^2 - |E_{0-}|^2) \psi(\omega). \end{aligned} \quad (24)$$

For the optimal helices at the resonance frequency, $\frac{M}{c} = \pm 1$ or $\frac{r^2 q \omega}{c} = \pm 2$, $\varepsilon_r = \mu_r$, “+” corresponds to right-hand helices, while “-” corresponds to left-hand ones.

For metamaterials based on only right-handed optimal helices, the following relation holds:

$$\langle w \rangle_t \approx \frac{1}{2} \varepsilon_0 (|E_{0+}|^2 + |E_{0-}|^2) + \frac{1}{A} |E_{0+}|^2 \psi(\omega). \quad (25)$$

The first term in (25) is the energy of two CP modes in vacuum combining the energies of electric and magnetic fields; the second term is the energy of interaction of one mode with the helices, as the orthogonal mode is not interacting. For metamaterials based on only left-handed optimal helices, the following relation is obtained:

$$\langle w \rangle_t \approx \frac{1}{2} \varepsilon_0 (|E_{0+}|^2 + |E_{0-}|^2) + \frac{1}{A} |E_{0-}|^2 \psi(\omega). \quad (26)$$

The relation (26) shows that, in this case, the metamaterial interacts only with the orthogonal mode of circular polarization.

Calculating losses of energy of electromagnetic fields in a metamaterial, we find that the energy absorbed per unit volume and time is equal to $\langle Q \rangle_t = \langle \gamma (\dot{v})^2 N \rangle_t$ [see the explanations after formula (13)].

For helices the axes of which are oriented along the x axis, the absorbed energy reads

$$\begin{aligned} \langle Q \rangle_t &= -\frac{\omega}{2} \{ \varepsilon_0 \varepsilon_r'' |E_{0x}|^2 + \mu_0 \mu_r'' |H_{0x}|^2 \\ &+ j \sqrt{\varepsilon_0 \mu_0} (E_{0x} H_{0x}^* - E_{0x}^* H_{0x}) \kappa'' \}. \end{aligned} \quad (27)$$

If the structure is isotropic, that is, the helices are oriented along x and y axes in equal concentrations, we need to take into account components of the fields E_{0y} , H_{0y} . In this case, formula (27) can be easily generalized. For better clearness of calculations, as before, we consider only helices oriented along the x axis. If we subtract the energy of the field in vacuum, the energy stored in the helices on the frequency ω_0 reads

$$\langle w \rangle_t(\omega_0) - \frac{1}{4} \varepsilon_0 |E_{0x}|^2 - \frac{1}{4} \mu_0 |H_{0x}|^2 = \langle w_{\text{stor}} \rangle_t(\omega_0), \quad (28)$$

$$\begin{aligned} \langle w_{\text{stor}} \rangle_t(\omega_0) &= \frac{1}{2} \frac{1}{\Gamma^2} \frac{1}{A} \{ |E_{0x}|^2 + \mu_0^2 M^2 |H_{0x}|^2 \\ &+ \mu_0 M j (E_{0x} H_{0x}^* - E_{0x}^* H_{0x}) \}, \end{aligned} \quad (29)$$

$$\frac{\langle w_{\text{stor}} \rangle_t(\omega_0)}{\langle Q \rangle_t(\omega_0) T} = \frac{\omega_0}{2\pi \Gamma}, \quad T = \frac{2\pi}{\omega_0}. \quad (30)$$

These formulas show that when the damping factor Γ is increasing, the stored energy (29) reduces more rapidly and substantially rebating the absorbed energy for the period of

field variation. For small but nonzero Γ the stored energy considerably increases the absorbed energy per time period T . For plane waves, the absorbed energy can be presented as follows:

$$\begin{aligned} \langle Q \rangle_t &= -\frac{\omega}{2} \varepsilon_0 \left\{ \left(\varepsilon_r'' + \mu_r'' \left| \frac{\varepsilon_r}{\mu_r} \right| \right) (|E_{0+}|^2 + |E_{0-}|^2) \right. \\ &\left. + \kappa'' \left(\sqrt{\frac{\varepsilon_r^*}{\mu_r^*}} + \sqrt{\frac{\varepsilon_r}{\mu_r}} \right) (|E_{0+}|^2 - |E_{0-}|^2) \right\}, \end{aligned} \quad (31)$$

where $E_{0\pm}$ are the amplitudes of CP waves, double prime denotes the imaginary part, and the asterisk denotes the complex conjugate.

Formulas (24) and (31) obtained for helically structured metamaterials coincide with already known results for linearly polarized waves in metamaterials formed by straight wires or split ring resonators. This is confirmed by expressions (12) and (21) obtained via various techniques [2–4]. However, such metamaterials do not possess chiral properties, in contrast to metamaterials consisting of helices. In the latter case, the metamaterial exhibits selective properties that are substantially different for the right-handed and left-handed CP waves. Therefore, the energy of the field, stored and absorbed in the chiral metamaterial, must be calculated specifically for CP waves. In the framework of other known approaches, such an energy calculation was not previously performed.

III. NUMERICAL EXAMPLES

For a quantitative analysis of obtained theoretical results, we have plotted a typical dependence of stored and dissipated energies $\langle w \rangle_t^\pm$ and $\langle Q \rangle_t^\pm$ as functions of the frequency and the helix pitch angle. Here, signs \pm stand for the right-handed and left-handed CP waves, respectively. Note that we call a CP wave “right-handed” when its electric field vector rotates clockwise if the observer looks in the wave propagation direction. The numerical results are presented in Fig. 2 for the following parameter values: $N = 2 \times 10^{17} \text{ m}^{-3}$, $\Gamma = 0.03\omega_0$.

From the analysis of Fig. 2, it is clear that there is a special geometry of single-turn right-handed helices having the pitch angle of about 13.7° at which the interaction with the left-handed mode is minimal. We call this type of helice “optimal.” However, there is one more extreme pitch angle near 48° at which the interaction with the right-handed mode has its maximum. It is also interesting to estimate the pitch angle for the maximum energy difference between two modes. The energy spread is maximal for 45° . Note that Figs. 2(c) and 2(d) are in logarithmic scale. At this pitch angle, however, the energy of the left-handed mode is substantially greater than for the optimal helix. A similar trend is observed for the absorbed energy in Fig. 2(d). It is interesting to compare these results with the conclusions of paper [22], where the notion of “objects of maximum chirality” was introduced. In that paper, “maximum chirality” corresponds to the optimal helices, introduced and studied in Refs. [12–18], that is, to helices which interact only with waves of one of the two orthogonal polarizations. However, the present results show that extreme chirality can be possibly defined based on other criteria: maximally strong interaction with one of the CP modes or maximum difference of reactive energies of an object in the field of right- or left-hand polarized waves. Clearly, the most

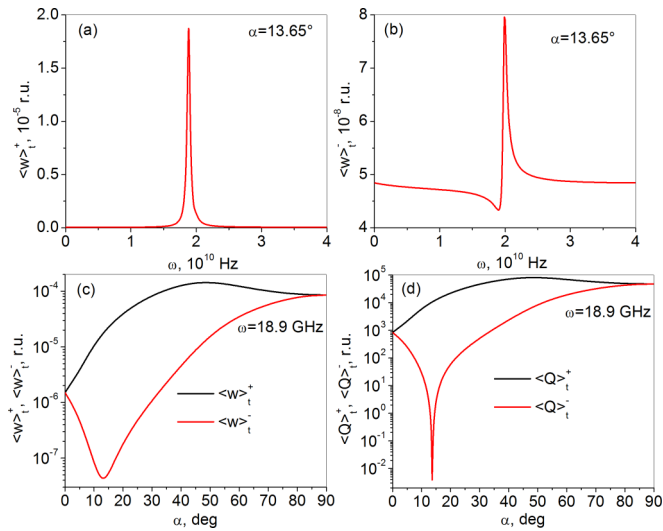


FIG. 2. (a) Energy density for optimal helices ($\alpha = 13.65^\circ$) in the field of a right-handed CP wave (24) in relative units (r.u.) vs frequency, (b) energy density for optimal helices ($\alpha = 13.65^\circ$) in the field of a left-handed CP wave (24) vs frequency, (c) energy density for arbitrary (in general, nonoptimal) helices (24) vs helix pitch angle for the right-handed (+) and left-handed (-) modes at the resonant frequency ($\omega = 18.9$ GHz), (d) absorbed energy for the same helices (31) vs helix pitch angle for the right-handed (+) and left-handed (-) modes at the resonant frequency ($\omega = 18.9$ GHz).

appropriate criterion is defined by the thought application of the studied chiral object.

IV. CONCLUSION

Electromagnetic energy density in a dispersive chiral structure which is made of helices has been analytically determined taking into account strong dispersion and losses. The results have been obtained using several approaches: the general approach, the helical model, and the approach based on the model of a single-component medium. The stored energy density and the absorbed energy in metamaterials composed of helices have been determined depending on the frequency and the helix pitch angle. We have found the geometrical parameters of the helix for maximal selectivity of interaction of helices with right- and left-handed CP waves and discussed criteria for definition of “maximally chiral” objects. The results can be used in optimizing chiral shapes for specific applications from microwave to optics. Although this study is focused on chiral effects, the developed methods can be used to study more general bi-anisotropic particles such as Ω or pseudo-chiral particles [23].

ACKNOWLEDGMENT

The work was supported by Belarusian Republican Foundation for Fundamental Research (BRFFR) Project No. F15SO-047.

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