Loss compensation in time-dependent elastic metamaterials

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Materials with properties that are modulated in time are known to display wave phenomena showing energy increasing with time, with the rate mediated by the modulation. Until now there has been no accounting for material dissipation, which clearly counteracts energy growth. This paper provides an exact expression for the amplitude of elastic or acoustic waves propagating in lossy materials with properties that are periodically modulated in time. It is found that these materials can support a special propagation regime in which waves travel at constant amplitude, with temporal modulation compensating for the normal energy dissipation. We derive a general condition under which amplification due to time-dependent properties offsets the material dissipation. This identity relates band-gap properties associated with the temporal modulation and the average of the viscosity coefficient, thereby providing a simple recipe for the design of loss-compensated mechanical metamaterials.

I. INTRODUCTION

Phononic crystals and metamaterials, which consist of periodic arrangements of scatterers and resonators in a solid or fluid matrix, have revolutionized the realm of acoustic and elastic wave propagation [1–3]. These structures have allowed the development of applications for the control and localization of mechanical energy that would be impossible to achieve with natural materials. Thus, gradient index lenses [4,5], cloaking shells [6–8], and hyperlenses [9], among other interesting devices, have been designed and experimentally tested.

However, most of the extraordinary applications of metamaterials are hindered by the strong dissipation they exhibit, especially near the resonant regime where the concentration of the fields in the scatterers is higher [10]. The performance of these structures could be considerably improved if combined with materials with gain. Additionally, other emerging applications related to $\mathcal{PT}$-symmetric systems [11,12], where gain and loss are combined, require the realization of materials with gain. Although gain has been introduced by means of electronic amplification in metamaterials [13] and in piezoelectric materials [14], this mechanism is difficult to implement for acoustic waves and at low frequencies, for which a more robust approach is required.

In this work we present a mechanism to provide gain and therefore to compensate dissipation in mechanical metamaterials based on materials with time-dependent properties. In these materials both the stiffness constant and the mass density are functions of time. The amplification properties of time-dependent media have been of interest for at least 60 years. The early studies focused on the parametric amplification in electrical transmission lines with time-varying inductance [15,16] or capacitance [17,18] and on wave propagation through dielectric media with time-varying properties [19,20]. More recent studies have considered both time-varying mechanical and time-varying electromagnetic materials [17,21–25]. Despite the wide interest in these materials, to the best of our knowledge the effects of dissipation on wave amplification in time-varying media have so far not been considered. A simplified model for the effect of a resistance element on amplification in transmission-line devices is to replace the system by a single-degree-of-freedom *RLC* circuit with time-varying capacitance [17]. This is a useful and instructive model which is repeated in this work in the context of elasticity, but it is important to note that it is not directly related to wave amplitudes. While time-dependent media can be thought of as difficult or nearly impossible to realize, it has to be taken into account that, essentially, they are tunable materials which can be quickly reconfigured. The domain of tunable and reconfigurable acoustic and elastic metamaterials is moving fast towards this direction, so that this concept could be doable in the framework of these structures.

We will derive the general properties of a time-dependent dissipative material, showing that the dissipation can be compensated by the amplification of the fields due to the time-dependent properties. It is found that the fields can either blow up or attenuate exponentially with time but that there is a special regime in which these effects compensate one another and the wave propagates at constant amplitude through the material. The demonstration is based on the analogy between periodically modulated materials in space and time, and it is valid for any periodic function of the constitutive parameters. A single-degree-of-freedom mechanical model is also considered and compared with the fully dynamic continuum model.

II. GAIN IN TIME-DEPENDENT MEDIA

The analogy between spatial and temporal modulation represented in Fig. 1(a) shows a classical layered material with
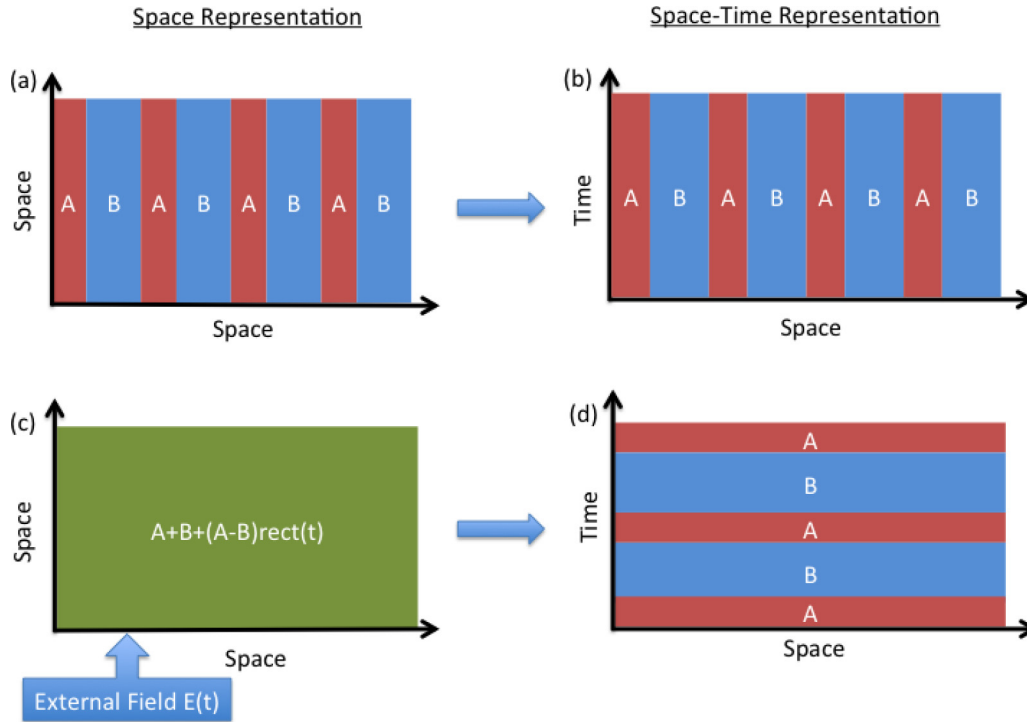
FIG. 1. Equivalences between materials with spatial and temporal modulation of their constitutive parameters. (a) A classical layered material in space and (b) its space-time representation. (c) A material whose properties are periodically changed in time by means of an external stimulus and (d) its space-time representation, which shows that it is a rotated version of (b).

alternating layers of materials A and B (a one-dimensional phononic crystal), and Fig. 1(b) shows its space-time representation, where we see that the properties (stiffness and mass density) remain constant in time but not in space. Now, let us assume that we have a medium whose material properties are sensitive to some external stimulus such as an electric or magnetic field, an applied stress, or even temperature. If this external stimulus $E(t)$ changes with time t, the properties of the material will be time dependent, as shown in Fig. 1(c), which represents a material in which the properties change from A to B periodically. Figure 1(d) shows the space-time representation of this material, where the properties are time dependent but constant along the space coordinate. The "space" representation of these two materials shows two completely different pictures [Figs. 1(a) and 1(c)]; however, the "space-time" representation shows a clear equivalence between these two problems.

The above equivalence between the spatial and temporal modulation of the materials is more evident from the equation for elastic waves. The one-dimensional wave equation for an inhomogeneous elastic material with mass density $\rho(x)$ and stiffness constant $C(x)$ is given by

$$\frac{\partial}{\partial x}\left(C(x)\frac{\partial u}{\partial x}\right) = \rho(x)\frac{\partial^2 u}{\partial t^2}, \qquad (1)$$

with $u = u(x,t)$ being the x component of the displacement vector. If the properties of the material change in time but not in space, the above wave equation is

$$\frac{\partial}{\partial t}\left(\rho(t)\frac{\partial u}{\partial t}\right) = C(t)\frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

from which it is clear that there is a direct relationship between the solutions of the equations for space and time modulations. Therefore, if we know the solution for the spatial modulation, we can obtain the solution for the temporal modulation by exchanging the roles of ρ and C.

Despite the formal analogy between spatial and temporal modulation of materials, there is a fundamental difference between these two situations concerning boundary and initial conditions. This difference manifests itself when we compare the transmission and reflection by a discontinuity in the material in space or time. The process is illustrated in Fig. 2: In the top panel we see the classical spatial transmission and reflection process in a layered material, while the bottom panel shows the analogous situation in time.

The top panel of Fig. 2 shows the process of reflection and transmission by a spatial discontinuity: a wave traveling through a given material arrives from the left and encounters the discontinuity (a layered material in this example), and a reflected wave is then excited, traveling backwards along the x direction; also, a transmitted wave appears at the other side of the slab, traveling forward in the x direction.

The bottom panel of Fig. 2 shows the equivalent situation in time: a wave is traveling through a given material in which, due to the application of a periodic temporal external stimulus, from $t = 0$ to $t = T_0$ its properties oscillate between two values labeled A and B and finally rest at its initial state. However, the "position" of the waves in the schematics is different since for $t < 0$ we have only one wave traveling forward along the x direction; obviously, the layered material in time cannot excite a wave traveling "backwards" in time. The reflected wave appears after the modulation period for $t > T_0$ and is, in fact,
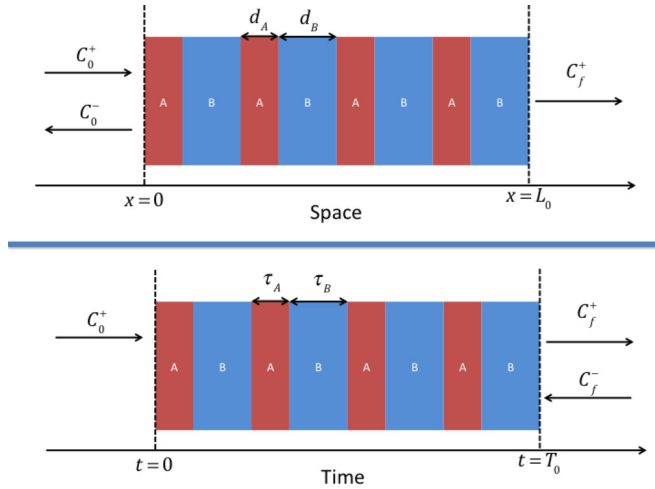
FIG. 2. Comparison of the excitation of transmitted and reflected waves by finite-layered materials in space (top panel) and time (bottom panel). The position of the reflected wave before (top panel) or after (bottom panel) the slab obeys causality, and it is responsible for the gain in energy for the temporal layered material (see text for further details).

a wave traveling backwards in x (it cannot travel backwards in time), so that the result of the modulation in time is the excitation of two waves traveling in opposite directions along the material, as before, but the different position of the reflected wave in the schematics will be the key to understanding the energy gain in the process.

The consequence of this distinction becomes clearer if we use layer theory, which relates the amplitudes of the incoming C_i^+ and outgoing C_i^- waves before ($i = 0$) and after ($i = f$) the layered structure by means of the scattering matrix M, so that for the spatial case we have

$$\begin{pmatrix} C_f^+ \\ C_f^- \end{pmatrix} = \begin{pmatrix} M_{11}^S & M_{12}^S \\ M_{21}^S & M_{22}^S \end{pmatrix} \begin{pmatrix} C_0^+ \\ C_0^- \end{pmatrix}. \tag{3}$$

For an incident wave coming from the left $C_f^- = 0$, and the reflection and transmission coefficients are defined as

$$r_S \equiv \frac{C_0^-}{C_0^+} = -\frac{M_{21}^S}{M_{22}^S}, \tag{4a}$$

$$t_S \equiv \frac{C_f^+}{C_0^+} = \frac{1}{M_{22}^S}. \tag{4b}$$

The determinant of the M matrix is unitary, and reciprocity shows that $M_{11} = M_{22}^*$ and $M_{21} = M_{12}^*$, relations that imply $|r_S|^2 + |t_S|^2 = 1$; that is, there is conservation of energy in the spatial case.

The picture is different for the temporally layered material, where the reflected wave corresponds to the amplitude C_f^-. Layer theory is applied likewise; thus,

$$\begin{pmatrix} C_f^+ \\ C_f^- \end{pmatrix} = \begin{pmatrix} M_{11}^T & M_{12}^T \\ M_{21}^T & M_{22}^T \end{pmatrix} \begin{pmatrix} C_0^+ \\ C_0^- \end{pmatrix}. \tag{5}$$

and using $C_0^- = 0$, the reflection and transmission coefficients are given by

$$r_T \equiv \frac{C_f^-}{C_0^+} = M_{21}^T, \tag{6a}$$

$$t_T \equiv \frac{C_f^+}{C_0^+} = M_{11}^T. \tag{6b}$$

The transfer matrix M^T is obtained directly from M^S by changing $C \rightarrow \rho$, as discussed before, so that the unitarity and reciprocity relationships will be identical, and it can be easily shown that $|r_T|^2 + |t_T|^2 \geq 1$ (in fact, $|t_T|^2 \geq 1$); that is, there is increased wave energy. This gain in energy can be understood from the equivalence in the spatial case: since the values for the reflection and transmission coefficients have to be equal to or lower than 1, we have $|M_{22}| = |M_{11}| \geq 1$ for both matrices $\mathbf{M}^T$ and $\mathbf{M}^S$.

Interestingly, we see that the transmitted energy in the temporal case is the inverse of the transmitted energy in the spatial case. The roles of the mass density and the stiffness constant are interchanged between the two situations, which changes the elements of the matrix $\mathbf{M}$. The most important consequence is that, for a layered material of N periods, when waves propagate at the frequency of the band gap typical of periodic structures, the amplitude of the transmitted wave decreases exponentially with the number of layers, so that its equivalent temporal crystal will have an exponentially increasing gain of energy when the selected wave number lies in the band gap. As the number of periods becomes larger, the transmitted energy blows up, and the material becomes unstable, unless the modulation ceases. Therefore, the stability condition for an infinitely oscillatory medium is that the parameter oscillations are not strong enough to open a band gap in the dispersion curve.

The above effect can be quantified by means of layer theory, which shows that the M matrix of an N-layer material is given by [26]

$$\mathbf{M}_N^T = \mathbf{M}^T \frac{\sin N\Omega\tau}{\sin \Omega\tau} - \mathbf{I} \frac{\sin(N-1)\Omega\tau}{\sin \Omega\tau}, \tag{7}$$

where $\Omega = \Omega(k_0)$ defines the dispersion curve of the infinite periodic material for spatial wave number k_0. Clearly, within the band gap the element M_{11} has the form $e^{N\tau \operatorname{Im}(\Omega)}$, which grows exponentially with the number of periods N. Therefore, a periodically modulated material will be unstable if its (temporal) band structure presents a band gap, unless the modulation is of finite duration, in which case it will act simply as an amplifier.

III. LOSS COMPENSATION IN TIME-DEPENDENT MEDIA

When dissipation is introduced into the system, the space-time analogy is no longer valid, and the effect of gain is less evident. The main difference in the spatial case is that dissipation breaks the time-reversal symmetry, which means that the material is nonreciprocal in time and the transfer matrix M^S is no longer unitary. Although dissipation is a complex phenomenon with a strong dependence on frequency, the most common assumption in elasticity is to propose a complex stiffness constant directly proportional to the frequency,

$C \to C - i\omega\eta$, whose origin is the assumption that viscous forces are proportional to the velocity. This is equivalent to the following time-dependent constitutive equations:

$$\frac{\partial \sigma}{\partial x} = \frac{\partial}{\partial t}\left(\rho(t)\frac{\partial u}{\partial t}\right), \tag{8a}$$

$$\sigma = C(t)\frac{\partial u}{\partial x} + \eta(t)\frac{\partial^2 u}{\partial x \partial t}, \tag{8b}$$

where, like before, C, ρ, and the viscosity coefficient η are time dependent. With this model of dissipation, regardless of the temporal dependence of the constitutive parameters, it can be shown that the transfer matrix is given by [see Appendix A, Eq. (A15)]

$$\boldsymbol{M} = e^{-\Gamma k_0^2}\widehat{\boldsymbol{M}}, \tag{9}$$

where the Γ factor is given by

$$\Gamma = \frac{1}{2}\int_0^{T_0} \frac{\eta(t)}{\rho(t)}dt \tag{10}$$

and the matrix $\widehat{\mathbf{M}}$ satisfies unitarity and reciprocity.

The dissipation of the system is described by the exponential factor $e^{-\Gamma k_0^2}$; however, this dissipation can be compensated by the elements of the matrix $\widehat{\mathbf{M}}$, which is unitary, and therefore contributes to the gain of the system. For the specific case of a periodically modulated material, the reflection and transmission coefficients are given by

$$r_T = e^{-\Gamma k_0^2}\frac{\sin N\Omega\tau}{\sin \Omega\tau}\widehat{M}_{21}^T, \tag{11a}$$

$$t_T = e^{-\Gamma k_0^2}\left(\frac{\sin N\Omega\tau}{\sin \Omega\tau}\widehat{M}_{11}^T - \frac{\sin(N-1)\Omega\tau}{\sin \Omega\tau}\right), \tag{11b}$$

where now

$$\Gamma = \frac{N\tau}{2}\left\langle\frac{\eta(t)}{\rho(t)}\right\rangle, \tag{12}$$

with $\langle\cdot\rangle = 1/\tau\int_0^\tau dt$ being the average in the temporal unit cell τ. The above equations clearly establish the conditions for compensating the dissipation in the material. If the dispersion curve $\Omega = \Omega(k_0)$ is real, i.e., there is no band gap, all the contributions of the unitary matrix $\widehat{\mathbf{M}}$ are oscillatory in N, and as the number of periods (modulation time) increases, the amplitude of both the transmitted and reflected waves decreases because of the exponential factor $e^{-\Gamma k_0^2}$. If the modulation of the parameters is strong enough to open a band gap, the argument in the sinusoidal terms in Eq. (11a) becomes complex, and the sine becomes the hyperbolic sine, with an exponentially dominant term as N increases; therefore, both the transmission and reflection coefficients have terms of the form

$$e^{-\Gamma k_0^2}\sin N\Omega\tau \approx \exp\left(-\frac{N\tau}{2}\left\langle\frac{\eta(t)}{\rho(t)}\right\rangle k_0^2 + N\tau \operatorname{Im}(\Omega)\right). \tag{13}$$

Since both the decaying and growing factors are proportional to $N\tau$, this exponential term will be compensated and set constant if the condition

$$\Delta(k_0) \equiv \frac{1}{2}\left\langle\frac{\eta(t)}{\rho(t)}\right\rangle k_0^2 - \operatorname{Im}(\Omega) = 0 \tag{14}$$
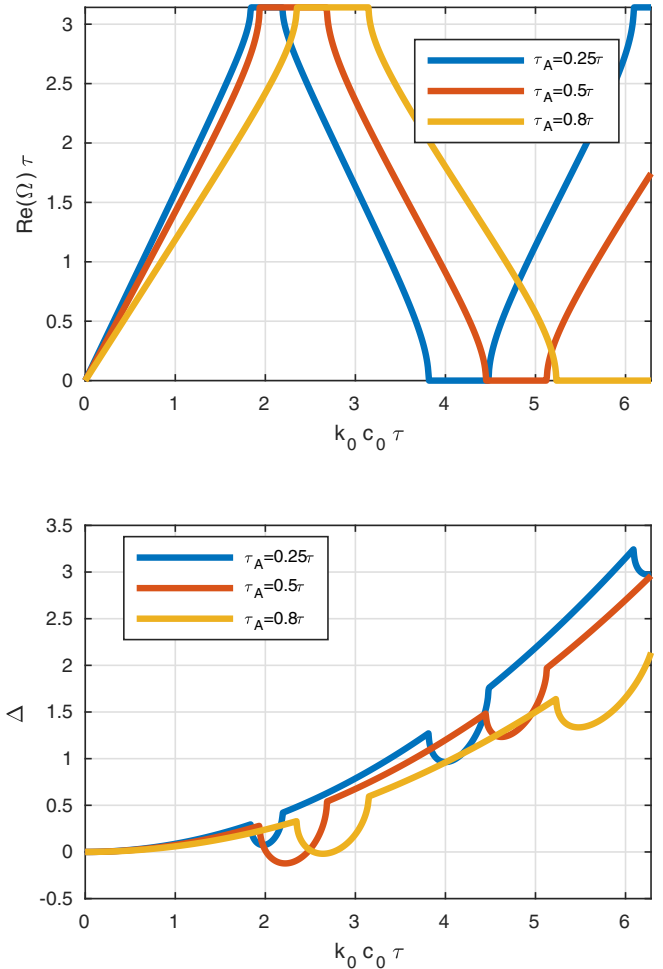

FIG. 3. Top: Real part of the dispersion curve $\Omega = \Omega(k_0)$ for a two-layer time-dependent medium for different values of τ_A. Bottom: The parameter Δ of Eq. (14) as a function of k_0 for three different values of τ_A. The regions of Δ that are positive, zero, and negative correspond to the situations in which the field is dissipated, stabilized, and amplified, respectively.

is satisfied, and the transmitted energy will be stable with the modulation time T_0 since all the exponential terms (the decaying and the growing ones) have disappeared from the expressions. The energy will therefore propagate along the material without dissipation or amplification. The quantity $\Delta(k^0)$ is therefore the parameter determining the stability of the material. If there is a frequency region where this quantity is negative, the material will be unstable since the energy will blow up exponentially with the number of periods N. This parametric amplification can also be used to gain energy in a controllable way, using the fact that the dissipation Δ can be a small quantity. It is interesting to compare the stability condition with the analogous criterion for a single-degree-of-freedom damped oscillator with time-varying parameters (see Appendix B).

The above results are now illustrated via some numerical examples. Further details regarding the calculations and the expressions employed can be found in the Appendix A.

Figure 3 shows the dispersion curve $\Omega = \Omega(k_0)$ for a two-layer periodic material of time period $\tau = \tau_A + \tau_B$ with elastic
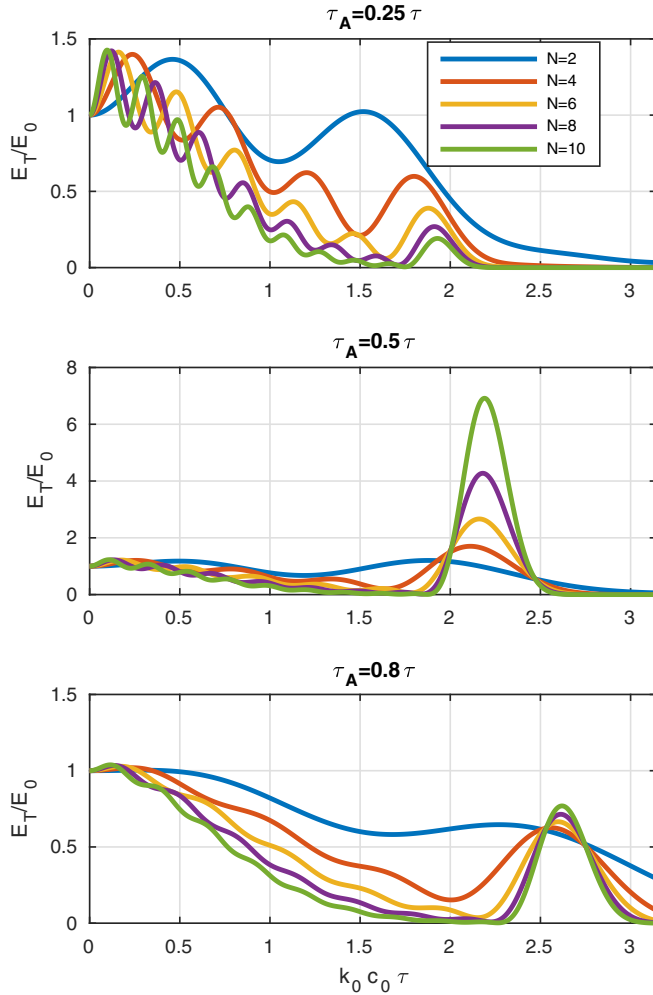
FIG. 4. Total energy gain after a periodic modulation of the material for different values of the number of periods N. Simulations are shown for $\tau_A = 0.25\tau$ (top), $\tau_A = 0.5\tau$ (middle), and $\tau_A = 0.8\tau$ (bottom), displaying dissipation, gain, and stabilization, respectively.

properties $\{\rho_A, C_A, \eta_A\} = \{1, 1, 0.1\}$ during the time τ_A and $\{\rho_B, C_B, \eta_B\} = \{1, 3, 0.2\}$ in the remaining part of the period $\tau_B = \tau - \tau_A$. Figure 3 shows the behavior for $\tau_A = 0.25\tau, 0.5\tau$ and 0.8τ. The bottom panel shows the parameter Δ in Eq. (14) as a function of wave number k_0. The curves deviate from parabolic shape only within the band gaps where $\text{Im}(\Omega) \neq 0$.

Observe that for $\tau_A = 0.5\tau$ there is a region for which $\Delta < 0$, which means that the field will be amplified as a function of the number of periods N of the temporal modulation, while for $\tau_A = 0.8\tau$ there is a region with $\Delta = 0$, so that the field will be stabilized there, despite the fact that the material is dissipative.

Figure 4 shows the gain in energy $E_T/E_0 = |r_T|^2 + |t_T|^2$ after the modulation of the material's properties for the system in Fig. 3. Results are shown for different values of the number of periods N and for different values of τ_A. Clearly, for $\tau_A = 0.25\tau$ there is a progressive dissipation of energy as a function of N (top panel), while for $\tau_A = 0.5\tau$ the energy increases as a function of N within the band gap (middle panel). Finally, for $\tau_A = 0.8\tau$ there is a situation of stabilization since within the band gap the energy tends to be stable as a function of N. It must be pointed out that these three situations depend only on the modulation period τ_A/τ, which is straightforward to change in practice since it will be the duration for which the external stimulus is in one state or the other, so that the situation of gain-dissipation-stabilization can be externally controlled in these materials.

Figure 5 shows the time evolution of a Gaussian pulse in a time-dependent material under the condition of gain, i.e., $\tau_A = 0.5\tau$. The pulse is chosen to have central frequency at the peak of gain shown in the middle panel of Fig. 4. The left, middle, and right panels of Fig. 5 show the initial pulse, the response after the temporal modulation has begun, and the excitation of the reflected and transmitted wave packets when the modulation ceases, respectively. The full time evolution for this configuration is presented in the supplementary movie [27], which clearly illustrates how the wave packet is strongly localized in space during the amplification process and subsequently propagates after the modulation has stopped. The example shown here corresponds to $N = 25$, but the supplementary movie [27] illustrates the response for $N = 75$, in which the amplification is more evident.

IV. SUMMARY

In summary, we have presented a general theory for time-dependent media showing that in the absence of dissipation these materials display gain when the modulation of the parameters is of finite duration. For continuous and periodic temporal modulation of the material's properties, energy blows up exponentially in band gaps, indicating the possibility of material instability. By extending the theory to consider real-
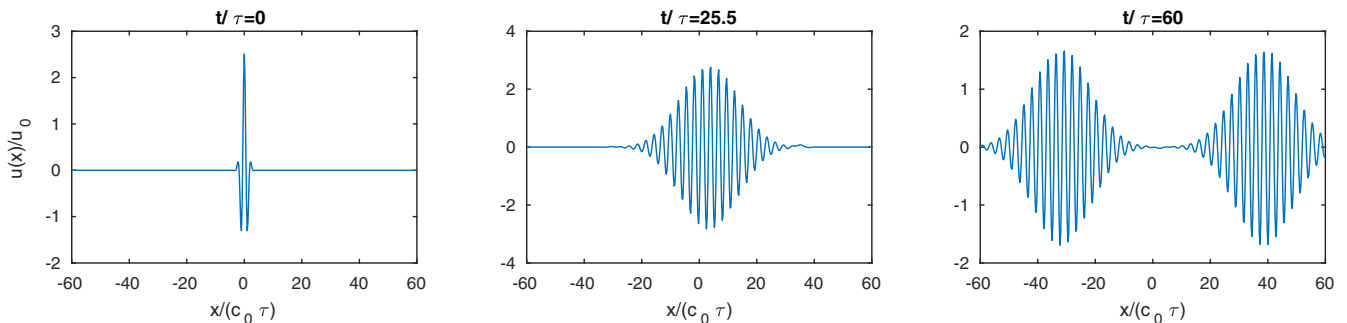

FIG. 5. Time evolution of a short pulse in a time-dependent material. It is clear how after the modulation two pulses propagating along opposite directions have been created and amplified.

istic dissipative materials we have shown that the energy can decrease or increase exponentially, depending on a balance equation which relates the band gap growth to the average value of dissipative parameters. A general equation describes the condition under which the gain due to the band gap and the losses due to dissipation compensate each other, so that the material, despite being dissipative, maintains constant energy. These results are valid for any type of time modulation and can provide the basis for the design of loss-compensated metamaterials and devices.

APPENDIX A: TIME-DEPENDENT MEDIUM

1. Basic equations with space-time parameters

The field variables are displacement $u(x,t)$, velocity $v = \partial_t u$, and strain $\varepsilon = \partial_x u$. The equilibrium equation and the stress constitutive relation are

$$\partial_x \sigma = \partial_t(\rho v), \tag{A1a}$$

$$\sigma = C\varepsilon + \eta \partial_x v, \tag{A1b}$$

where the material parameters are density $\rho(x,t)$, stiffness $C(x,t)$, and viscosity $\eta(x,t)$. Equations (A1) together give the pointwise energy balance

$$\partial_t\left(\tfrac{1}{2}\rho v^2 + \tfrac{1}{2}C\varepsilon^2\right) + \partial_x(-v\sigma)$$
$$= \tfrac{1}{2}\varepsilon^2\partial_t C - \tfrac{1}{2}v^2\partial_t\rho - \eta(\partial_x v)^2. \tag{A2}$$

The right-hand terms in (A2) are clearly producers of energy; the first two could be either sources or drains, while the final term is the expected viscous loss.

2. Space-harmonic solution

We focus on time-dependent material properties: $\rho = \rho(t)$, $C = C(t)$, and $\eta = \eta(t)$. Assume that the variables have separate space and time dependence,

$$u(x,t) = \mathrm{Re}\, u(t)e^{ik_0 x}, \tag{A3}$$

where $u(t)$ is a complex-valued quantity and k_0 is real valued and positive. Similar expressions follow for the other variables, and from here on we consider $v(t) = \partial_t u(t)$, $\varepsilon(t) = ik_0 u(t)$, and $\sigma(t) = ik_0[Cu(t) + \eta v(t)]$ to be complex quantities with the space-harmonic factor $e^{ik_0 x}$ omitted but understood, analogous to how we consider time-harmonic motion.

Equations (A1) become

$$\partial_t \mathbf{U} = -ik_0\begin{bmatrix} 0 & \rho^{-1} \\ C & -ik_0\eta\rho^{-1} \end{bmatrix}\mathbf{U}, \tag{A4}$$

where

$$\mathbf{U}(t) \equiv \begin{pmatrix} \varepsilon \\ -\pi \end{pmatrix} \text{ and } \pi(t) = \rho(t)v(t) \tag{A5}$$

is the momentum. The propagator, or transfer matrix, for $\mathbf{U}$ is not unitary. The connection with unitarity and reciprocity can be made by first defining the speed, impedance, and nondimensional viscosity,

$$c = \sqrt{C/\rho}, \ z = \rho c, \ \zeta = \frac{k_0\eta}{2z}. \tag{A6}$$

Consider wave solutions of Eqs. (A1) for constant material properties ρ, C, and η of the form $(u,\sigma) = (u_0,\sigma_0)\exp\{ik_0(x - \lambda_0 ct)\}$. The nondimensional frequency λ_0 satisfies

$$\lambda_0^2 + 2i\zeta\lambda_0 - 1 = 0, \tag{A7}$$

which implies that propagating waves in $t > 0$ occur only for $\zeta < 1$; otherwise, the wave is exponentially decaying with time. Hereafter, is it assumed that the damping factor ζ is less than the critical value of unity.

Equation (A4) can now be rewritten

$$\partial_t \mathbf{V}(t) = \mathbf{Q}(t)\mathbf{V}(t), \tag{A8}$$

where

$$\mathbf{V}(t) = e^{\int_0^t k_0 c\zeta\, dt}\, \mathbf{U}(t),$$
$$\mathbf{Q} = -ik_0 c\sqrt{1-\zeta^2}\mathbf{A}, \ \mathbf{A} = \frac{1}{\sqrt{1-\zeta^2}}\begin{bmatrix} i\zeta & z^{-1} \\ z & -i\zeta \end{bmatrix}. \tag{A9}$$

The property $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix, leads to the usual unitary properties for the propagator and other related results (see below). The restriction $\zeta < 1$ implies that the effects of dissipation are described entirely through the exponential term in Eq. (A9).

We next consider the propagator and transfer matrices using well-known methods [28]. The following results are for arbitrary time-dependent $\mathbf{Q}(t)$ of the form defined in (A9).

3. Propagator and transfer matrices

Let $\mathbf{P}(t)$ be the 2×2 matrix solution of the differential initial value problem

$$\partial_t \mathbf{P}(t) = \mathbf{Q}(t)\mathbf{P}(t), \ \mathbf{P}(0) = \mathbf{I}. \tag{A10}$$

The propagator matrix $\mathbf{P}$ relates the state vector $\mathbf{V}(t)$ at one time with that at another, $\mathbf{V}(t) = \mathbf{P}(t)\mathbf{V}(0)$, and it has the usual properties of an undamped propagator, such as a determinant of 1. The actual "damped" propagator for $\mathbf{U}(t)$ follows from (A9) as $e^{-k_0^2\Gamma(t)}\mathbf{P}$ since

$$\mathbf{U}(t) = e^{-k_0^2\Gamma(t)}\mathbf{P}(t)\mathbf{U}(0) \tag{A11}$$

and

$$\Gamma(t) = \frac{1}{2}\int_0^t \frac{\eta(t)}{\rho(t)}dt. \tag{A12}$$

For future reference we define

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{Z}(t) = \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}. \tag{A13}$$

We assume a finite layer of thickness (time) T with uniform properties outside. Then

$$\mathbf{W}(T) = \mathbf{M}\mathbf{W}(0), \quad \mathbf{W}(t) \equiv \begin{pmatrix} C^+(t) \\ C^-(t) \end{pmatrix}. \quad \text{(A14)}$$

A forward- (backward-) traveling wave satisfies $-\pi = z\varepsilon$ ($-\pi = -z\varepsilon$). The forward and backward components $\mathbf{W}$ are connected to the state vector $\mathbf{U}$ via the impedance matrix by $\mathbf{U} = \mathbf{Z}\mathbf{W}$, so that $\mathbf{M} = \mathbf{Z}^{-1}(T)\mathbf{P}(t)\mathbf{Z}(0)$. The periodicity assumption implies $\mathbf{Z}(T) = \mathbf{Z}(0)$, so that

$$\mathbf{M} = e^{-\Gamma k_0^2}\widehat{\mathbf{M}}, \quad \widehat{\mathbf{M}} = \mathbf{Z}^{-1}(0)\mathbf{P}(T)\mathbf{Z}(0), \quad \text{(A15)}$$

where the damping constant is

$$\Gamma \equiv \Gamma(T). \quad \text{(A16)}$$

Note that the term $e^{-\Gamma k_0^2}$ comes from the definition (A14).

The following identities are readily derived:

$$\mathbf{Q}^\dagger = -\mathbf{J}\mathbf{Q}\mathbf{J}, \quad \text{(A17a)}$$

$$\mathbf{P}^{-1}(t) = \mathbf{P}(-t) = \mathbf{J}\mathbf{P}^\dagger(t)\mathbf{J}, \quad \text{(A17b)}$$

$$\det \mathbf{P}(t) = \det \widehat{\mathbf{M}} = 1, \quad \text{(A17c)}$$

where the dagger ($\dagger$) indicates the Hermitian transpose (transpose plus conjugate). These identities imply properties for the elements of $\mathbf{P}$ and $\mathbf{M}$ (and ones for $\widehat{\mathbf{M}}$ similar to those for $\mathbf{M}$):

$$P_{11}^* = P_{11},\ P_{22}^* = P_{22},\ P_{12}^* = -P_{12},\ P_{21}^* = -P_{21},$$

$$M_{11}^* = M_{22},\ M_{12}^* = M_{21},$$

$$\mathbf{P}(t) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \text{(A18)}$$

where the asterisk ($*$) indicates the complex conjugate.

According to the definition (A14), the reflection and transmission coefficients for the time-varying medium can be found using $C^-(0) = 0$ as

$$r_T = \frac{C^-(T)}{C^+(0)} = M_{21}, \quad \text{(A19a)}$$

$$t_T = \frac{C^+(T)}{C^+(0)} = M_{11}. \quad \text{(A19b)}$$

These coefficients satisfy

$$|t_T|^2 - |r_T|^2 = e^{-2\Gamma k_0^2}, \quad \text{(A20)}$$

implying magnification of the forward-traveling wave ($|t_T| \geqslant 1$) in the absence of damping ($\zeta = 0$).

4. Piecewise constant regions

For instance, if $\mathbf{Q}$ is constant as a function of t, then the solution of (A10) is

$$\mathbf{P}(t) = e^{\mathbf{Q}t} = \cos\phi\,\mathbf{I} - i\sin\phi\,\mathbf{A}, \quad \phi = k_0 c\sqrt{1-\zeta^2}\,T. \quad \text{(A21)}$$

For a layer $\{c, z, \zeta\}$ sandwiched in time by the uniform medium with impedance z_0,

$$\mathbf{M} = e^{-\Gamma k_0^2}\left(\cos\phi\,\mathbf{I} - \frac{i\sin\phi}{2\sqrt{1-\zeta^2}}\right.$$

$$\left. \times \begin{bmatrix} \frac{z}{z_0} + \frac{z_0}{z} + i2\zeta & \frac{z}{z_0} - \frac{z_0}{z} + i2\zeta \\ \frac{z_0}{z} - \frac{z}{z_0} + i2\zeta & -\left(\frac{z}{z_0} + \frac{z_0}{z}\right) \end{bmatrix}\right). \quad \text{(A22)}$$

The reflection and transmission amplitudes follow from Eqs. (A19) and (A22) as

$$|r_T|^2 = e^{-2\Gamma k_0^2}\frac{\sin^2\phi}{1-\zeta^2}\left[\frac{1}{4}\left(\frac{z}{z_0} - \frac{z_0}{z}\right)^2 + \zeta^2\right], \quad \text{(A23a)}$$

$$|t_T|^2 = e^{-2\Gamma k_0^2}\left\{\frac{\sin^2\phi}{1-\zeta^2}\left[\frac{1}{4}\left(\frac{z}{z_0} - \frac{z_0}{z}\right)^2 + \zeta^2\right] + 1\right\}. \quad \text{(A23b)}$$

5. Bloch-Floquet

If the time modulation is T periodic, then there exist Bloch-Floquet solutions of the form $u(t + T) = u(t)e^{-i\omega T}$, with similar expressions for the other variables. The frequency ω satisfies the eigenvalue condition

$$\det\left[e^{-k_0^2\Gamma}\mathbf{P}(T) - e^{-i\omega T}\mathbf{I}\right] = 0. \quad \text{(A24)}$$

Therefore,

$$\omega = \pm\Omega - i\frac{k_0^2\Gamma}{T}, \quad \text{(A25)}$$

where Ω is the undamped Bloch-Floquet frequency,

$$\cos\Omega T = \tfrac{1}{2}\,\mathrm{tr}\,\mathbf{P}(T). \quad \text{(A26)}$$

Note that the frequency ω is complex valued when damping is present. In the absence of damping ($\eta = 0$) the frequency is real valued in the pass bands and complex in stop bands.

6. Example: Two-layer system

The materials are denoted 1 and 2, with duration t_1 and t_2, $t_1 + t_2 = T$; then (A26) is

$$\cos\Omega T = \cos k_0 c_1 t_1 \cos k_0 c_2 t_2$$

$$- \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right)\sin k_0 c_1 t_1 \sin k_0 c_2 t_2. \quad \text{(A27)}$$

This can be used to look at the response within band gaps.

For instance, suppose the speeds and temporal thicknesses are the same in both layers ($c_1 = c_2 = c$, $t_1 = t_2 = T/2$). The first band gap, defined by $\cos\Omega T < -1$, is

$$k_0 \in \frac{1}{cT}(\pi - \theta, \pi + \theta),$$

$$\cos\frac{\theta}{2} = \frac{2}{\sqrt{\frac{z_1}{z_2}} + \sqrt{\frac{z_2}{z_1}}}, \quad 0 < \theta < \pi. \quad \text{(A28)}$$

The frequency Ω has its largest imaginary value in the middle of the band gap, $k_0 cT = \pi$, where

$$\Omega = \frac{\pi}{T} + \frac{i}{T}\left|\ln\frac{z_1}{z_2}\right|. \quad \text{(A29)}$$

Thus, the loss from damping compensates the gain of the temporal modulation if the two are matched according to $\Delta(k_0) = 0$, which in this case becomes

$$\frac{1}{2}\left\langle\frac{\eta(t)}{\rho(t)}\right\rangle k_0^2 = \frac{1}{T}\left|\ln\frac{z_1}{z_2}\right|. \quad \text{(A30)}$$

7. Energy evolution

Averaging Eq. (A2) over a wavelength $2\pi/k_0$ yields a purely time dependent energy result,

$$
\partial_t \left(\frac{1}{2} C |\varepsilon|^2 + \frac{1}{2} \rho^{-1} |\pi|^2 \right)
$$
$$
= \frac{1}{2} |\varepsilon|^2 \partial_t C + \frac{1}{2} |\pi|^2 \partial_t \rho^{-1} - \eta \frac{k_0^2}{\rho^2} |\pi|^2. \quad (A31)
$$

This form of the energy balance is instructive since it involves the quantities ε and π that vary smoothly in time according to Eq. (A4). It is also interesting to note that ε and π are the dual variables to v and σ, which define the state vector in the spatially modulated material. The two sets of variables appear on equal footing in the Willis equations [29].

Equation (A31) indicates that the energy is not necessarily smoothly varying since instantaneous jumps in C and/or ρ lead to instantaneous nonzero changes in the total energy. In order to understand this further, for the moment we ignore damping and consider a uniform medium with properties $\{C_0, \rho_0\}$ that instantaneously switches to $\{C_1, \rho_1\}$ at time $t = 0$. Equation (A31) with $\eta = 0$ implies the energy equation

$$
E = E_0 + \frac{1}{2} \left[|\varepsilon_0|^2 (C_1 - C_0) + |\pi_0|^2 (\rho_1^{-1} - \rho_0^{-1}) \right] H(t), \quad (A32)
$$

where $E = \frac{1}{2} C |\varepsilon|^2 + \frac{1}{2} \rho^{-1} |\pi|^2$ is the total energy; E_0, ε_0, and π_0 are the values just before $t = 0$; and $H(t)$ is the Heaviside step function. According to Eq. (A32) the instantaneous change could increase or decrease the total energy. For instance, if the wave before the switch is propagating in one direction, $\pi_0 = \pm z_0 \varepsilon_0$, then the energy increases if $\frac{z_1}{z_0} + \frac{z_0}{z_1} > 2\frac{c_0}{c_1}$; otherwise, it decreases, where $C_1 = c_1 z_1$, $\rho_1^{-1} = c_1/z_1$.

Now suppose that the medium properties revert at time $t = T > 0$ to those before the switch at $t = 0$; then the subsequent energy for $t > T$ is

$$
E_T = E_0 + \frac{1}{2}(|\varepsilon_0|^2 - |\varepsilon_1|^2)(C_1 - C_0)
$$
$$
+ \frac{1}{2}(|\pi_0|^2 - |\pi_1|^2)(\rho_1^{-1} - \rho_0^{-1}), \quad (A33)
$$

where the values ε_1 and π_1 at $t = T$ are related to those at $t = 0$ by the propagator

$$
\begin{pmatrix} \varepsilon_1 \\ -\pi_1 \end{pmatrix} = \begin{bmatrix} \cos(k_0 c_1 T) & -i z_1^{-1} \sin(k_0 c_1 T) \\ -i z_1 \sin(k_0 c_1 T) & \cos(k_0 c_1 T) \end{bmatrix} \begin{pmatrix} \varepsilon_0 \\ -\pi_0 \end{pmatrix}. \quad (A34)
$$

Hence,

$$
E_T = E_0 + \frac{C_0}{2} \left(\frac{z_1^2}{z_0^2} - 1 \right) \left(|\varepsilon_0|^2 - \frac{|\pi_0|^2}{z_1^2} \right) \sin^2(k_0 c_1 T), \quad (A35)
$$

implying that the energy can increase or decrease relative to E_0 depending on the wave dynamics at $t = 0$. However, assuming the dynamic state just before the first switch at $t = 0$ is a wave traveling in only one direction, i.e., $\pi_0 = \pm z_0 \varepsilon_0$, then the energy after reversion is

$$
E_T = E_0 + \frac{E_0}{2} \left(\frac{z_1}{z_0} - \frac{z_0}{z_1} \right)^2 \sin^2(k_0 c_1 T). \quad (A36)
$$

This is always greater than or equal to the initial energy.

In summary, for wave incidence in one direction on the temporal slab of width T, (i) the evolved energy is nondecreasing, (ii) it remains constant for any temporal slab width $T > 0$ if and only if the impedance remains constant, and (ii) for a given impedance mismatch, the energy increase is maximum for T such that $\sin^2(k_0 c_1 T) = 1$.

APPENDIX B: A SINGLE-DEGREE-OF-FREEDOM MODEL

Consider a mass-spring-damper system with time-varying stiffness

$$
m\ddot{u}(t) + c\dot{u}(t) + (K + \Delta K \cos 2\omega_n t)u(t) = 0, \quad (B1)
$$

where $\omega_n = \sqrt{K/m}$ is the undamped natural frequency. The displacement solution has the form $u(t) = v(t)e^{(\mu - \zeta)\omega_n t}$, where $v(t)$ is periodic and $\zeta = \frac{c}{2\omega_n m}$ is the damping factor. For small values of $\frac{\Delta K}{K}$ the parameter μ follows from [30]. In particular,

$$
\text{Re } \mu \approx \frac{\Delta K}{8K}. \quad (B2)
$$

Note that this differs by a factor of 2 from the analogous result in [17]. Hence, the solution will grow exponentially if

$$
\frac{\Delta K}{8K} > \zeta. \quad (B3)
$$

In order to compare this with the full-wave result of Eq. (14) we note that the latter implies exponential growth if $\text{Im}(\Omega) > \frac{1}{2} \langle \frac{\eta(t)}{\rho(t)} \rangle k_0^2$. Assuming time-independent wave speed $c = \sqrt{C/\rho}$ and nondimensional viscosity ζ of Eq. (A6), the growth condition can be expressed

$$
\frac{\text{Im}(\Omega)}{\Omega_0} > \zeta, \quad (B4)
$$

where $\Omega_0 \equiv ck_0$. In both the simple model (B3) and the full-wave solution (B4), the condition for exponential growth involves the nondimensional damping factor ζ. In the single-degree-of-freedom model it competes with the relative change in stiffness. However, in the full-wave case, the damping competes with an indirect quantity which depends upon the Bloch-Floquet response.

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