Functional renormalization-group approach to the Pokrovsky-Talapov model via the modified massive Thirring fermions

P. A. Nosov,[1,2] Jun-ichiro Kishine,[3] A. S. Ovchinnikov,[4,5] and I. Proskurin[3,4]

[1]*Department of Physics, St. Petersburg State University, St. Petersburg 198504, Russia*

[2]*NRC Kurchatov Institute, Petersburg Nuclear Physics Institute, Gatchina 188300, Russia*

[3]*Division of Natural and Environmental Sciences, The Open University of Japan, Chiba 261-8586, Japan*

[4]*Institute of Natural Science, Ural Federal University, Ekaterinburg 620002, Russia*

[5]*Institute of Metal Physics, Ural Division, Russian Academy of Sciences, Ekaterinburg 620219, Russia*

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We consider a possibility of the topological Kosterlitz-Thouless (KT) transition in the two-dimensional Pokrovsky-Talapov model with a finite misfit parameter and discuss its relevance to the theory of critical behavior in thin films of monoaxial chiral helimagnets. For this purpose, the initial model is reformulated in terms of the two-dimensional relativistic model of massive Thirring fermions and the Wetterich's functional renormalization-group (RG) approach is employed. In the new formalism, the misfit parameter corresponds to an effective gauge field that can be included in the RG scheme on an equal footing with the other parameters of the theory. Our main result is that the presence of the misfit parameter, which may be attributed to the Dzyaloshinskii-Moriya interaction in the magnetic system, rules out the KT transition. To support this finding, we provide an additional intuitive explanation of the KT scenario breakdown by using the mapping onto the Coulomb gas model. In the framework of the model, the misfit parameter has a meaning of an effective in-plane electric field that prevents a formation of bound vortex-antivortex pairs.

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I. INTRODUCTION

In noncentrosymmetric magnetic materials, the intrinsic spin-orbit coupling may appear on a macroscopic level in the form of the asymmetric Dzyaloshinskii-Moriya interaction (DMI) that can stabilize topologically nontrivial magnetic phases. In some chiral magnets, such as, for example, MnSi [1–3], $Fe_{1-x}Co_xSi$ [4], FeGe [5,6], and Cu_2OSeO_3 [7] belonging to the cubic space group P_{2_13} and $CrNb_3S_6$ with the hexagonal space group P_{6_322} [8–11], the ground state in the form of a long-periodic helical magnetic modulation appears due to competition between the ferromagnetic exchange coupling and the DMI. By applying a magnetic field perpendicular to the helical axis in the compound $CrNb_3S_6$, the helix transforms into the soliton lattice state where a periodic array of helical twists within the same plane is formed. The evolution of the soliton lattice state in bulk crystals is well consistent with the effective *one-dimensional (1D) model* for *classical spins* addressed by Dzyaloshinskii in his seminal papers [12–14]. Recently, it has been demonstrated that thin films of $CrNb_3S_6$ exhibit essentially different intrinsic properties in comparison with bulk samples [15–17]. These experimental findings strongly motivate us to explore the nature of the soliton lattice phase in two dimensions (2D).

Being a particular case of the Frank and Van der Merwe (FVdM) theory [18], the 1D version of the Dzyaloshinskii's model cannot be applied to two-dimensional (2D) solids at nonzero temperatures because of the roughness transition [19]. However, the basic concept of FVdM theory, namely, domain walls formed by kinks, may be extended to 2D systems. The *classical 2D* sine-Gordon model with a misfit parameter giving a periodic modulation along a fixed direction is a good candidate to describe the situation at finite temperatures. This approach was developed by Pokrovsky and Talapov [20,21] being based on the assumption that the walls are roughly parallel and cross the whole sample from the top to the bottom neither crossing each other nor coming backwards. At $T = 0$, the results of the Pokrovsky-Talapov (PT) model are identical to the Dzyaloshinskii theory [22–24]. However, this model demonstrates a quite different behavior at finite temperatures predicting an existence of the commensurate-incommensurate (C-IC) transition [19]. The key features of the C-IC transition are conditioned by dynamics of the domain walls which meander entropically and interact with each other [25]. The model has been widely discussed in the context of the C-IC transition of adsorbates on a periodic potential [26] and was applied to a variety of commensurate-incommensurate systems, including, for instance, the ANNNI model [27], Josephson junctions in high-temperature superconductors [28], bilayer quantum-Hall junctions with an in-plane field [29], superconducting films [30], cold atoms [31], and fermionic atoms [32] in an optical lattice and graphene [33].

Phase transitions in the two-dimensional PT model have already been extensively discussed in the literature. In a low-temperature regime, where the walls destabilize the commensurate phase, the PT model can be exactly transformed into the 1D free spinless fermion problem and exactly solved [34,35]. Nonetheless, a detailed picture of phase transitions in this system at finite temperatures was proven to be quite nontrivial. The C-IC transition in the PT model was discussed in the context of the Wilsonian-type renormalization-group (RG) analysis, which was found to be dependent on how precisely RG procedure is set up [36,37]. It was established that the C-IC phase transition described by this formalism belongs to the Kosterlitz-Thouless universality class [38–40] with a peculiar feature: the RG flow changes when a lattice scaling parameter reaches a soliton separation. This corresponds to a succession of phase transitions above the critical temperature for the C phase and gives rise to the *floating modulated phase* (the "floating devil's staircase"), which is characterized by a power-law decay of correlation functions [41].

At higher temperatures, a transition into the *fluid phase* occurs when the domain wall pattern disappears, and the transition between the floating and the fluid phases is triggered by the vortices [42–45]. This transition can be captured as the vortex separation in the XY model at some critical temperature. Thus, the overall phase diagram, where the floating phase separates the commensurate phase from the fluid phase, is determined by the stability of two types of relevant topological excitations, domain walls and vortices. The former launch the C-IC transition, the latter trigger the transition between the floating and the fluid phases [41].

This hierarchical picture of phase transitions succeeded in the physics of the surface adsorbates [41]. However, in our opinion, application of this scheme to the temperature evolution of the soliton lattice order in thin films of the chiral helimagnets should be taken with care. In the adsorbates, one deals with an interface interaction between two subsystems, where the misfit parameter of the atomic arrangement is conditioned by an *extraneous* potential of the substrate. In contrast, in the chiral magnets, the *intrinsic* antisymmetric exchange acts along with the symmetric counterpart that can make the XY model inappropriate for the paramagnetic regime, since it ignores the effects of the DMI. Thus, the PT model with the misfit parameter being attributed to the antisymmetric exchange arises as a plausible candidate for description of the fluid phase in the 2D chiral helimagnets.

The primary aim of this paper is to find out whether the KT transition due to the vortices takes place in the situation when the DMI is fully taken into account. For this purpose, we map the PT model to the 2D massive Thirring (MT) fermion model [46,47]. Our analysis is closely related to the investigation of renormalizability of the 3D Thirring model by means of the functional renormalization group formulated in terms of the Wetterich equation [48]. The procedure gives flow equations for the mass of the two-dimensional Thirring model and the fictitious gauge field experienced by relativistic fermions, which may be matched with the magnetic field and the strength of the DM interaction, respectively, in the context of the chiral helimagnets. The RG transformations are complemented by a flow equation for the strength of the current-current coupling which can be compared with an in-plane anisotropy of exchange interactions.

In order to provide the readers with an intuitive picture, we supplement the rigorous functional RG analysis by a more physical approach based on a duality mapping between vortices and electrostatics that was actively employed in the theory of the KT transition [49]. We derive the partition function of point charges, corresponding to the given PT model of the chiral helimagnet, and demonstrate that the DM interaction brings forth an effective electric field directed perpendicularly to the chiral axis. A natural consequence of this electric field is a breakdown of KT transition that explains our rigorous results obtained within the Wetterich's RG scheme.

The paper is organized as follows. In Sec. II the PT model and the corresponding counterpart of the 2D Thirring model are formulated. Details of the functional RG calculation are outlined in Sec. III. In this section a picture of the RG flow is established by using the Thirring model and the nonperturbative RG in terms of the Wetterich equation. In Sec. IV the PT model is reformulated as the model of the two-dimensional Coulomb gas and the RG flows are perturbatively derived. Finally, a discussion of the obtained results and concluding remarks are given in Secs. V and VI, respectively.

II. MODEL

The Hamiltonian of the PT model in notations applicable to the 2D chiral helimagnet reads as

$$\frac{H}{T} = \int d^2r \left[\frac{1}{2}\frac{J_\perp}{T}(\partial_x\varphi)^2 + \frac{1}{2}\frac{J_{||}}{T}(\partial_y\varphi)^2 - \frac{D}{T}(\partial_y\varphi) - \frac{h}{T}\cos\varphi \right], \tag{1}$$

where $d^2r = dxdy$, and $J_\perp$ and $J_\parallel$ are the ferromagnetic exchange parameters. For $J_\perp \neq J_{||}$, the first two terms consider the in-plane anisotropy of the exchange interaction that reflects the situation in thin films of the monoaxial chiral helimagnet $Cr_{0.33}NbS_2$. The third term may be attributed to the DMI along the y axis, and is treated as a *misfit parameter* of the PT model, while the fourth describes the Zeeman energy in a transverse field $\mathbf{h} = h\hat{\mathbf{x}}$.

We note that the same form of the Hamiltonian may be used to consider various physical situations. For example, Bak suggested to use Eq. (1) for studying phase transitions in the 2D Ising ANNNI model [27]. In his analysis, the misfit parameter is given by the ratio of the competing exchange interactions, and φ parametrizes the order parameter fluctuations taken in the form of a spin density wave. Horovitz *et al.* [37] recognized Eq. (1) as being the Hamiltonian for the floating phase [37] of the 2D sine-Gordon model with a chemical potential coupled with the soliton density $\rho = (1/2\pi)\int dy\, \partial_y\varphi$. The chemical potential may be shifted away by the transformation $\varphi(x,y) \to \tilde{\varphi}(x,y) + 2\pi\rho y$ due to symmetry of the fluctuations $\tilde{\varphi}$ under spatial translations. This transformation changes the last term to $\cos(\tilde{\varphi} + 2\pi\rho x)$ and gives rise to the action for the fluctuations with the *periodic* boundary conditions. This transformation changes the last term to $\cos(\tilde{\varphi} + 2\pi\rho x)$ and gives rise to an action for the fluctuations with the *periodic* boundary conditions. The language of the 1D nonrelativistic fermions of the Tomonaga-Luttinger model provides a description of the low temperature phase below the C-IC phase transition, which requires a transformation of the original classical 2D model in Eq. (1) into its $(1+1)$-dimensional quantum counterpart followed by the fermion map [50,51].

In our study, we use the Hamiltonian (1) to examine and describe the disordered paramagnetic phase, when φ measures deviations from a *uniform* reference configuration. We emphasize that the uniform background excludes imposing of periodic boundary conditions either to identify a topological invariant term in the PT-model Hamiltonian or to make the phase shift and to rule out the misfit parameter as it was done in Ref. [52]. To deal with this situation, we use a map onto the 2D Thirring model of relativistic fermions. For this purpose, the changes $\varphi = \sqrt{T/J_\perp}\theta, \delta = D/T, \eta = h/T$, and $\beta = \sqrt{T/J_\perp}$ are adopted that yields the Euclidean action

$$\mathcal{S}_{\rm SG} = \int d^2r\bigg[\frac{1}{2}(\partial_x\theta)^2 + \frac{1}{2}\frac{J_{||}}{J_\perp}(\partial_y\theta)^2 - \beta\delta(\partial_y\theta) - \eta\cos(\beta\theta)\bigg], \tag{2}$$

which reproduces the isotropic 2D sine-Gordon model for $J_{||} = J_\perp$ and $\delta = 0$.

A transition between the classical 2D sine-Gordon and the 2D massive Thirring models is achieved by the rules

$$\frac{1}{8\pi}(\partial_\mu\theta)^2 \to \bar{\psi} i\sigma_\mu\partial_\mu\psi, \quad \frac{1}{2\pi i}\partial_y\theta \to \bar{\psi}\sigma_1\psi, \quad -\eta\cos\theta \to im\bar{\psi}\psi, \tag{3}$$

where m is the mass and σ_μ ($\mu = 1,2$) are the Pauli matrices. The Grassman valued fields are $\psi = (\psi_1,\psi_2)^T$ and $\bar{\psi} = \psi^*\sigma_1 = (\psi_2^*,\psi_1^*)$.

After the change $\beta^2\mathcal{S}_{\rm SG}/4\pi \to \mathcal{S}_{\rm Th}, \beta\theta \to \theta$, and the redefinitions $(x_1,x_2) \equiv (x,y), \Delta = \pi(J_{||}/J_\perp - 1), d = \beta^2\delta/2$, and $\tilde{m} = \beta^2 m/4\pi$ the Euclidean action of the 2D Thirring model is obtained:

$$\mathcal{S}_{\rm Th} = \int d^2r\bigg[\bar{\psi}(i\sigma_\mu D_\mu + i\tilde{m})\psi - \frac{\Delta}{2}(\bar{\psi}\sigma_1\psi)^2 - \frac{g}{2}(\bar{\psi}\sigma_\mu\psi)^2\bigg]. \tag{4}$$

The model described by this action may be called the modified massive Thirring model due the derivative $D_\mu = \partial_\mu - d\delta_{\mu,2}$ with the fictitious gauge field induced by the DM coupling. The current-current interaction of the strength g is also added, it involves the conserved current $j_\mu = \bar{\psi}\sigma_\mu\psi$.

Four-fermion terms may be simplified through the identity $(\bar{\psi}M\psi)^2 = \det M(\bar{\psi}\psi)^2$, where M is any 2×2 matrix [53] that converts (4) to the form

$$\mathcal{S}_{\rm Th} = \int d^2r\bigg[\bar{\psi}(i\sigma_\mu D_\mu + i\tilde{m})\psi + \bigg(\frac{\Delta}{2} + g\bigg)(\bar{\psi}\psi)^2\bigg]. \tag{5}$$

By using the Fourier transforms

$$\psi(\mathbf{x}) = \int \frac{d^2\mathbf{p}}{(2\pi)^2}e^{i\mathbf{px}}\psi_{\mathbf{p}} \equiv \int_p e^{i\mathbf{px}}\psi_{\mathbf{p}}, \tag{6}$$

and a similar expression for $\bar{\psi}(\mathbf{x})$ the action (5) takes the form in the momentum space

$$\mathcal{S}_{\rm Th} = \int \frac{d^2\mathbf{p}}{(2\pi)^2}\bar{\psi}_{\mathbf{p}}(-\not{p} - i\not{d} + i\tilde{m})\psi_{\mathbf{p}} + (2\pi)^2\bigg(\frac{\Delta}{2} + g\bigg) \times \int\prod_{i=1}^{4}\frac{d^2\mathbf{p}_i}{(2\pi)^2}(\bar{\psi}_{\mathbf{p}_1}\psi_{\mathbf{p}_2})(\bar{\psi}_{\mathbf{p}_3}\psi_{\mathbf{p}_4}) \times \delta(-\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} + \mathbf{p_4}), \tag{7}$$

where the slashed notations $\not{p} = p_\mu\sigma_\mu, \not{d} = d\sigma_1$ for the Dirac operators are used.

III. FUNCTIONAL RG

A nonperturbative analysis for the PT model at finite temperature is based on the effective average functional renormalization-group scheme by Wetterich [54,55]. This formalism, whose equations describe the scale dependence of the effective action, has proven highly powerful in studies of models from various fields of physics, ranging from condensed matter theory to high-energy physics. The nonperturbative RG methods have been used to analyze the KT transition on the basis of a microscopic action of the φ^4 model [56,57], a derivative expansion of the average action for the $O(2)$ linear σ model [58], and for the sine-Gordon model [59]. The distinctive feature of the nonperturbative RG techniques is that vortices are not explicitly introduced contrary to the traditional perturbative approaches that use a mapping to the Coulomb gas or sine-Gordon models [60].

Based on splitting of the corresponding Hamiltonian into the sine-Gordon part and the part depending only on a number of solitons present, the nonperturbative analysis was suggested for the PT model in Ref. [52]. A derivation of a functional RG transformation is validated by the assumption that the last part is unaffected by the RG transformation that is relevant for the floating phase regime when the periodic boundary conditions are allowed. The scaling equations, obtained this way, reproduce well known flow equations for the sine-Gordon model [40], which belongs to the universality class of the 2D XY spin model.

To develop a nonperturbative RG formalism for the fluid phase, where no periodic boundary conditions are assumed, the 2D MT model turns out to be appropriate. One of the advantages of the approach is that nonlinear features of the sine-Gordon model appear in terms of a fermion interaction, what makes a sound basis for perturbative techniques [61,62] as long as the interaction is not very strong. Another important observation, the topological term of the PT model being linear in the scalar field, cannot be taken into account within the Wetterich formalism which operates only quadratic or higher order terms over fields. However, the mapping onto the Thirring fermions converts it to the quadratic form of Grassmann-valued fields, where a scaling behavior of the misfit parameter may be deduced.

The RG formulation by Wetterich is based on the effective average action Γ_k, which is a generalization of the effective action including only rapid modes, i.e., the fluctuations with $q^2 \geqslant k^2$, where k is an ultraviolet cutoff for slow modes [54]. This is achieved by adding a regulator (infrared cutoff) R_k to the full inverse propagator. The regulator decouples slow modes with momenta $q^2 \leqslant k^2$ by giving them a large mass, while high momentum modes are not affected.

The scale dependence of Γ_k is governed by the Wetterich equation

$$\partial_k\Gamma_k = -\frac{1}{2}{\rm Tr}\bigg[\frac{\partial_k R_k}{\Gamma^{(2)} + R_k}\bigg] = -\frac{1}{2}\tilde{\partial}_k\,{\rm Tr}\log(\Gamma^{(2)} + R_k), \tag{8}$$

with $\Gamma^{(2)}$ indicating the second functional derivative of Γ_k. The trace involves an integration over momenta as well as a summation over internal indices. The minus sign on the right-hand side of Eq. (8) is due to the Grassman nature of $\bar{\psi}$

and ψ [55]. The derivative $\tilde{\partial}_k$ acts only on the k dependence of R_k and not on $\Gamma^{(2)}$.

By definition, the average action equals the standard effective action for $k=0$, as the infrared cutoff is absent and all fluctuations are included. Similarly to (7) the effective action is defined as

$$\Gamma_k = \int \frac{d^2\mathbf{p}}{(2\pi)^2}\bar{\psi}_{\mathbf{p}}(-Z_k\not{p} - i\not{d}_k + i\tilde{m}_k)\psi_{\mathbf{p}} + (2\pi)^2 \times\left(\frac{\Delta_k}{2}+g_k\right)\int\prod_{i=1}^{4}\frac{d^2\mathbf{p}_i}{(2\pi)^2}(\bar{\psi}_{\mathbf{p}_1}\psi_{\mathbf{p}_2})(\bar{\psi}_{\mathbf{p}_3}\psi_{\mathbf{p}_4}) \times\delta(-\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3+\mathbf{p}_4), \tag{9}$$

where Z_k denotes scale dependent wave function renormalization for the fermionic fields. All parameters in the effective action are assumed to be scale dependent which is marked by the momentum-scale index k.

Using Eq. (8) fixed points associated with the four-fermion interactions can be simply examined. For this purpose, the inverse regularized propagator can be split into the field-independent $(\Gamma^{(2)}_{k,0}+R_k)$ and the field-dependent $(\Delta\Gamma^{(2)}_k)$ parts that yields

$$\begin{aligned}\frac{\partial_k R_k}{\Gamma^{(2)}+R_k} &= \tilde{\partial}_k\log\left(\Gamma^{(2)}_{k,0}+\Delta\Gamma^{(2)}_k+R_k\right)\\ &= \tilde{\partial}_k\log\left(\Gamma^{(2)}_{k,0}+R_k\right)+\tilde{\partial}_k\frac{\Delta\Gamma^{(2)}_k}{\Gamma^{(2)}_{k,0}+R_k}\\ &\quad -\frac{1}{2}\tilde{\partial}_k\left(\frac{\Delta\Gamma^{(2)}_k}{\Gamma^{(2)}_{k,0}+R_k}\right)^2+\cdots.\end{aligned} \tag{10}$$

For the relativistic fermions the regulator function may be chosen as [48,63]

$$R_k = -\frac{\delta_{p,q}}{(2\pi)^2}Z_k r\left(\frac{q^2}{k^2}\right)\begin{pmatrix}0 & \not{q}^T\\ \not{q} & 0\end{pmatrix}. \tag{11}$$

From the explicit calculations given in Appendix A the propagator is found to be

$$\begin{aligned}\left(\Gamma^{(2)}_{k,0}+R_k\right)^{-1} &= (2\pi)^2\delta_{p,q}\begin{pmatrix}0 & \frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2}\\ \frac{\not{\beta}_k^T+i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2} & 0\end{pmatrix}\\ &= (2\pi)^2\hat{G}_0\delta_{p,q},\end{aligned} \tag{12}$$

where $\not{\alpha}_k = -Z_k\not{q}[1+r(\frac{q^2}{k^2})] - i\not{d}_k \equiv a_k\not{q} - i\not{d}_k$ and $\not{\beta}_k = -Z_k\not{q}[1+r(\frac{q^2}{k^2})] + i\not{d}_k \equiv a_k\not{q} + i\not{d}_k$; $\alpha^2=\alpha_1^2+\alpha_2^2$ and $\beta^2=\beta_1^2+\beta_2^2$.

Similar derivation of the field-dependent part yields

$$\begin{aligned}\Delta\Gamma^{(2)}_k &= 2(2\pi)^2\left(\frac{\Delta_k}{2}+g_k\right)\\ &\quad\times\begin{pmatrix}-\bar{\psi}^T\bar{\psi} & \bar{\psi}^T\psi^T+\psi^T\bar{\psi}^T\\ \psi\bar{\psi}+\bar{\psi}\psi & -\psi\psi^T\end{pmatrix}\delta_{p,q}\\ &= 2(2\pi)^2\left(\frac{\Delta_k}{2}+g_k\right)\hat{G}_1\delta_{p,q},\end{aligned} \tag{13}$$

where the second functional derivative is evaluated for homogeneous (constant) background fields to account for the uniform background. In momentum space it means that $\Delta\Gamma^{(2)}_k$ is evaluated at $\psi_{\mathbf{p}}=(2\pi)^2\psi\delta(\mathbf{p})$ and the similar expression for $\bar{\psi}_{\mathbf{p}}$, where ψ $(\bar{\psi})$ is on the right-hand side, are constant [63].

We can then expand the flow equation in powers of the Grassman fields by combining Eqs. (8) and (10):

$$\begin{aligned}\partial_k\Gamma_k &= -\frac{1}{2}\mathrm{Tr}\left[\tilde{\partial}_k\log\left(\Gamma^{(2)}_{k,0}+R_k\right)\right]-\frac{1}{2}\mathrm{Tr}\left[\tilde{\partial}_k\frac{\Delta\Gamma^{(2)}_k}{\Gamma^{(2)}_{k,0}+R_k}\right]\\ &\quad+\frac{1}{4}\mathrm{Tr}\left[\tilde{\partial}_k\left(\frac{\Delta\Gamma^{(2)}_k}{\Gamma^{(2)}_{k,0}+R_k}\right)^2\right]+\cdots.\end{aligned} \tag{14}$$

The powers of $\frac{\Delta\Gamma^{(2)}_k}{\Gamma^{(2)}_{k,0}+R_k}$ can be calculated by simple matrix multiplication. The RG flow equations can be obtained straightforwardly by comparing the coefficients of the fermion interaction terms of the right-hand side of Eq. (14) with the couplings included in the anzatz (9). The calculations of the second (two-fermion beta function) and the third (four-fermion beta function) terms in Eq. (14) are relegated to Appendices B and C, respectively.

For practical computations of the flow equations the sharp cutoff regulator may be employed:

$$r\left(\frac{q^2}{k^2}\right) = \begin{cases}\infty, & q^2<k^2,\\ 0, & q^2>k^2,\end{cases} \tag{15}$$

which facilitates explicit evaluation of the threshold functions $\mathcal{L}_i$ $(i=1,2,3)$. Their detailed derivations are given in Appendix D.

As a result, the RG equations take the form

$$\partial_t\bar{m} = \bar{m}+\frac{\bar{m}\bar{\lambda}}{4\pi\bar{d}^2}\frac{(1-4\bar{d}^2+4\bar{m}^2)}{\sqrt{\left[2\left(\frac{1-4\bar{d}^2+4\bar{m}^2}{4\bar{d}}\right)^2+1\right]^2-1}}, \tag{16}$$

$$\partial_t\bar{d} = \bar{d}-\frac{\bar{\lambda}}{4\pi\bar{d}}+\frac{\bar{\lambda}}{4\pi\bar{d}}\frac{(1-4\bar{d}^2+4\bar{m}^2)\left(1+\frac{1}{8\bar{d}^2}[1-4\bar{d}^2+4\bar{m}^2]\right)}{\sqrt{\left[2\left(\frac{1-4\bar{d}^2+4\bar{m}^2}{4\bar{d}}\right)^2+1\right]^2-1}}, \tag{17}$$

$$\partial_t\bar{\lambda} = -\frac{2\bar{\lambda}^2}{\pi}\left(1-\frac{\bar{m}^2}{\bar{d}^2}\right)\frac{1}{\sqrt{\left[2\left(\frac{1-4\bar{d}^2+4\bar{m}^2}{4\bar{d}}\right)^2+1\right]^2-1}}, \tag{18}$$

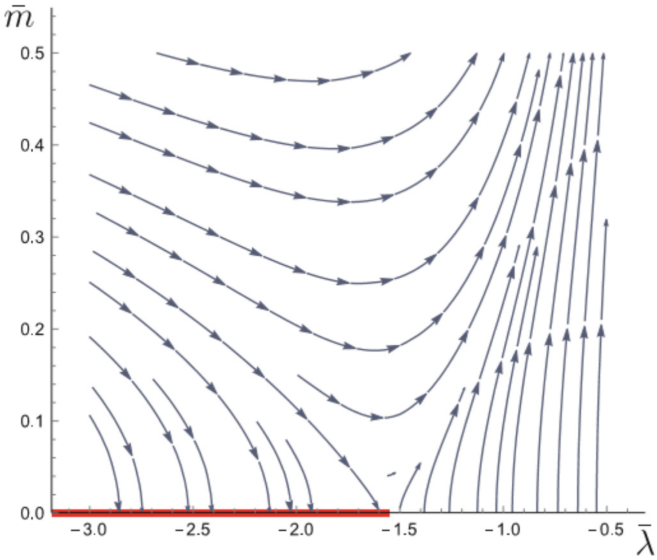

FIG. 1. Flow diagrams for the RG [Eqs. (20) and (21)]. The fixed points are shown by the red line.

where the "RG time" $|t| = \ln(\Lambda/k)$ and the dimensionless quantities $\bar{\lambda} = \lambda_k/2$, $\bar{m} = (2k)^{-1}\tilde{m}_k$, $\bar{d} = (2k)^{-1}d_k$ are expressed in units of the running scale k. The additional factor 2 on the right-hand side is input to eliminate a multiple regulator dependence (see also a discussion in Ref. [48]) and to reach a consistence between the SG and MT theories.

Before discussing the results of model (5) we first focus on the more trivial case of the massive Thirring (or, equivalently, the sine-Gordon) model. The SG and the MT models are equivalent provided the coupling constants β and g of the two models are related through the relation [46,47]

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}. \tag{19}$$

As for the sine-Gordon model, the system undergoes a continuous KT phase transition at $\beta^2 = 8\pi$. Given the equivalence between the SG and MT models, the transition point for the massive Thirrring model is $g = -\pi/2$ [64]. For $\beta^2 < 8\pi$ ($g > -\pi/2$) the coupling constant g flows to strong coupling regime that indicates an opening of a gap in the spectrum, and relevant degrees of freedom are massive fermionic solitons. For $\beta^2 > 8\pi$ (or $g < -\pi/2$) the weak-coupling regime arises, where the coupling g flows to zero, and relevant degrees of freedom are massless bosons.

If $\bar{d} = 0$ Eqs. (16)–(18) are restricted to

$$\partial_t \bar{m} = \bar{m}\left[1 + \frac{2\bar{\lambda}}{\pi(1 + 4\bar{m}^2)}\right], \tag{20}$$

$$\partial_t \bar{\lambda} = \frac{16}{\pi}\frac{\bar{\lambda}^2\bar{m}^2}{(1 + 4\bar{m}^2)^2}. \tag{21}$$

These flow equations reproduce the well-known scaling equations of the KT type. The RG trajectories remain in the plane $(\bar{m}, \bar{\lambda})$, the corresponding flow diagram is shown in Fig. 1, where there exists a line of fixed points with $\bar{m} = 0$ and finite $\bar{\lambda}^* < \bar{\lambda}_{KT} = -\pi/2$.

The corresponding parametric flow at finite $\bar{d}$ value is shown in Fig. 2. The flow is seen to initially closely follow

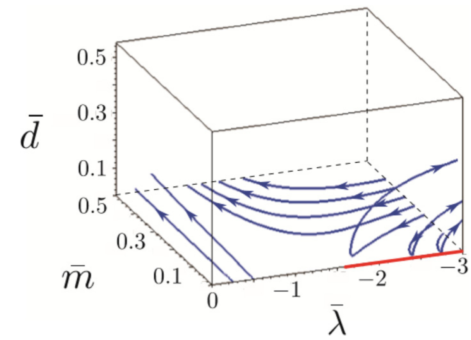

FIG. 2. The RG trajectories of the 2D Thirring-like PT model. The fixed points of the case $\bar{d} = 0$ are shown by the red line.

the KT flow at $\bar{d} = 0$, approaching the fixed line at $\bar{m} = 0$, but ultimately departing from it in a flow toward the high-temperature phase. We may conclude that in the presence of the linear gradient term, or DM interaction, no KT transition may exist in the fluid phase.

IV. COULOMB GAS MODEL

Many phenomena, which are difficult to interpret in the fermion language, have simple semiclassical explanations via the boson description, and the current problem is not an exception. A remarkable feature found in early studies [38–40,42] is that the defect-mediated transition of the 2D XY model and its analogs can be mapped to the insulator-conductor transition of a two-dimensional Coulomb gas. To elucidate the origin of the flow of the Thirring model we formulate the 2D Coulomb gas model by using discrete vector calculus on a square lattice for the Hamiltonian (5), where, for simplicity, the representation is restricted by the isotropic case $J_{||} = J_{\perp}$. For definiteness, all sums run over the sites of a square lattice although the transformation described below are easily generalized.

The partition function defined on such a lattice is of the form

$$\mathcal{Z} = \int \mathcal{D}\varphi \, \exp\left[\beta\tilde{J}\sum_{\langle ij\rangle}\cos(\varphi_i - \varphi_j - \alpha_{ij}) + \beta h \sum_i \cos\varphi_i\right], \tag{22}$$

where henceforth $\beta = (k_B T)^{-1}$ is the inverse temperature, $\tilde{J} = S^2\sqrt{J^2 + D^2}$ is the effective exchange parameter, $h = Sh_x$, and the bond angle is given by $\alpha_{ij} = \alpha = \tan^{-1}(D/J)$ for the ij link along the z axis and zero otherwise. The first sum runs over all nearest neighbor sites within the (xz) plane.

A duality mapping between vortices and electrostatics is derived in detail in Appendix D. The resulting partition function for point charge particles reads as

$$\mathcal{Z}_{\text{eff}} = \sum_{\{q(\mathbf{r})\}} \exp\left[\pi K_0 \sum_{\mathbf{r}\neq\mathbf{r}'} q(\mathbf{r}) \log|\mathbf{r} - \mathbf{r}'| q(\mathbf{r}') + 2\pi \sum_{\mathbf{r}} (\mathbf{E}_x \cdot \mathbf{r}) q(\mathbf{r})\right] y_0^{\sum_{\mathbf{r}} q^2(\mathbf{r})}, \tag{23}$$

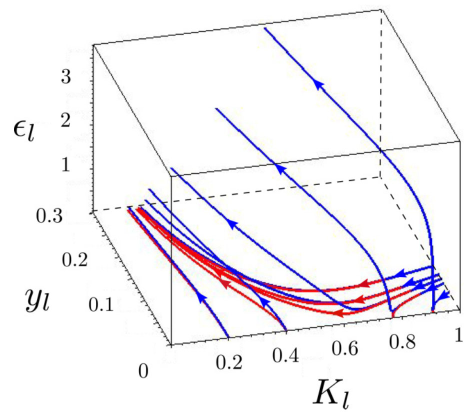

FIG. 3. The RG flows of the 2D Coulomb gas counterpart of the PT model (blue) and for the XY model (red), $\epsilon_l = 0$.

where the bare values for the vortex coupling $K_0 = \beta \tilde{J}$ and for the vortex pair fugacity $y_0 = \exp(-\beta\pi^2\tilde{J}/2)$. The first term in the exponential of Eq. (23) describes the charge-charge interactions, the second includes the sum of the self-energies associated with each elementary charge $q(\mathbf{r})$, which arises due to the effective uniform x-direction field $\mathbf{E}_x = \alpha\beta\tilde{J}\mathbf{e}_x = \epsilon\,\mathbf{e}_x$. The question whether a topological order realizes is thereby mapped, as in the conventional Kosterlitz-Thouless transition, to the problem of screening in the Coulomb gas, albeit now with modified terms due to the DM interaction.

The scaling equations can then be obtained from (23) in the limit of small fugacity. The procedure parallels what was detailed in Refs. [65,66]. At a general minimum scale $a = e^l$ the renormalized vortex coupling K_l, the vortex pair fugacity y_l, and the topological electric field ϵ_l obey the scaling equations

$$\frac{dK_l}{dl} = -4\pi^3 K_l^2 y_l^2, \tag{24}$$

$$\frac{dy_l}{dl} = (2 - \pi K_l + \pi\epsilon_l)y_l, \tag{25}$$

$$\frac{d\epsilon_l}{dl} = \epsilon_l + 4\pi^3 y_l^2 K_l \epsilon_l. \tag{26}$$

An advantage of the nonperturbative RG approach of the previous section should be noted. In contrast to the RG of the 2D Coulomb gas model, a smallness of the magnetic field is not required.

Figure 3 shows numerical solutions of RG flows. For $\epsilon = 0$ it reproduces the well-known $K_l(y_l)$ scale dependencies of the KT theory that tend asymptotically to zero (infinity) for $T > T_{\mathrm{KT}}$, $k_{\mathrm{B}}T_{\mathrm{KT}} = \pi J/2$, and to constant (zero) for $T < T_{\mathrm{KT}}$. The results for nonzero electric fields are also presented in this figure. Apparently, at all temperatures K_l tends asymptotically to zero and y_l tends to infinity, i.e., vortex pairs are unbound by the electric field. It means that the breaking apart of dipoles by the topological field begins to exceed the vortex attractive interaction. In addition, the plot clearly demonstrates that both models, the Thirring model (5) with a fictitious gauge field and the Coulomb gas in an electric field, are in the same universality class.

This concludes the RG analysis which shows that the DM interaction is relevant, it creates an effective electric field perpendicular to the direction of the DM vector on a lattice and eliminates the KT transition.

V. DISCUSSIONS

To understand the significance of our results and compare them with pertinent calculations on simple magnetic models which exhibit periodically modulated structures, we provide a view of the critical behavior of the 2D chiral helimagnet from a slightly broader perspective. The first relevant modeling system is the 2D anisotropic Ising model with competing nearest and next-nearest neighboring interactions (ANNNI model) [67]. It has been observed that the ANNNI model displays a variety of interesting physical features related to an emergence of the modulated phase. While the 3D version of the model exhibits an infinity of phases where periodicity is commensurate with the lattice, what is known as the *devil's staircase*, the 2D version behaves in a completely different way. At nonzero temperature, the devil's staircase is replaced by the floating modulated phase, which is followed by the paramagnetic phase at higher temperatures. The transition from the floating phase to the paramagnetic one is of the KT type [27] and it is a consequence of vortex separation at a critical temperature higher than that of the C-IC transition. The latter is estimated through the analogy between ANNNI and the 2D sine-Gordon models: the C-IC takes place when the domain wall free energy balances the chemical potential determined by the misfit parameter.

The transition sequence is clearly in line with the general scheme suggested by Nelson and Halperin in the theory of melting in two dimensions [45] and would seem to be relevant for the 2D chiral helimagnet. However, despite an apparent similarity of the spiral spin structures in the chiral helimagnet and the ANNNI model, they have a profound difference at which level the chiral symmetry is broken [68]. In the ANNNI model the chiral symmetry is not violated at the level of a Hamiltonian, but an emerging helimagnetic order breaks it spontaneously. On the other hand, in the chiral helimagnet, the Hamiltonian breaks explicitly the chiral symmetry due to the presence of the DMI. Therefore, the helicoidal structure of the ANNNI model does not have any macroscopical protectorate, which justifies *ad hoc* an application of the XY model for the high-temperature disordered phase. In contrast, our approach implies that the disordered phase in the 2D chiral helimagnet should take into account the remnant spin correlations caused by the DMI.

Our conclusion that in the 2D helimagnets vortices do not contribute to melting of the floating phase is amenable to direct verification by Lorentz microscopy. In this respect, we mention that peculiarities of the floating phase near the C-IC phase transition has been also discussed for the 2D sine-Gordon model within the RG scheme [37]. This approach treats the system as a grand-canonical ensemble with a chemical potential coupled to the soliton density, which is independent from any microscopic mechanism of an appearance of the modulated phase, and can be applied with an equal success to both the ANNNI and the chiral helimagnet. A linear growth of the soliton density with a temperature and a similar dependence for the critical index η may be considered as hallmarks of the floating phase in these systems. The first has been observed in the Monte Carlo studies of the 2D ANNNI model as a reduction of the average width of the stripe domains with a growth of temperature [69].

In the low temperature regime the fermion model can be formulated, where the domain walls or kinks, which are nothing but solitons with short-range repulsion between them, are represented by Tomonaga-Luttinger fermions moving in one spatial direction, while the other axis represents time. In this formalism, a finding of the free energy of the initial 2D sine-Gordon system amounts to determining of the ground state energy of the fermion Hamiltonian. The soliton density of the sine-Gordon model becomes equivalent to the density of Tomonaga-Luttinger fermions. The misfit parameter is mapped to a chemical potential of the fermions, whereas the strength of the pinning term generates a gap needed for creation of soliton excitations (additional kinks). At the commensurate phase, the chemical potential is within the gap dividing the empty upper conducting band from the the completely filled lower valence band of the Tomonaga-Luttinger quasiparticles. At zero temperature, the system is a Mott insulator. Heating the system and passing through the critical temperature of the C-IC transition puts the chemical potential above the gap, when the states of the conduction band are getting occupied [70], thus indicating the onset of the floating phase.

VI. SUMMARY

A possibility of the topological KT transition in the 2D sine-Gordon model with the misfit parameter, when the phase describes fluctuations around an uniform reference configuration with no kinks, is investigated by using the functional renormalization-group (RG) approach by Wetterich. Our main result is that the misfit parameter, which can be identified as the Dzyaloshinsky-Moriya interaction in the chiral helimagnet, makes such a transition prohibited. In order to argue this conclusion the initial boson model has been reformulated in terms of the 2D theory of relativistic fermions using an analogy between the 2D sine-Gordon and the massive Thirring models. In the new formalism the misfit parameter corresponds to an effective gauge field that enables us to include it in the renormalization-group procedure on an equal footing with the other parameters of the theory. With the new fermionic action at hand, we apply the Wetterich equation to obtain flow equations and demonstrate that these RG equations reproduce a KT type of behavior for the zero misfit. However, any small nonzero value of the quantity rules out a possibility of the topological transition. To confirm these findings, a description of the problem in terms of the effective 2D Coulomb gas model is developed. Within this approach, the breakdown of the KT scenario becomes transparent. The misfit parameter results in the appearance of an effective in-plane electric field that prevents a formation of bound vortex-antivortex dipoles. The discussion is presented on how these results are embedded in a general hierarchy of phase transitions in this 2D system and their relation with a melting of the soliton lattice in thin films of chiral helimagnets.

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APPENDIX A: THE INVERSE PROPAGATOR

The matrix of second derivatives of Γ_k with respect to the fermion fields deduced from Eq. (9):

$$\Gamma_k^{(2)}(p,q) = \begin{pmatrix} \overrightarrow{\partial}_{\psi^T_{-p}} \Gamma_k \overleftarrow{\partial}_{\psi_q} & \overrightarrow{\partial}_{\psi^T_{-p}} \Gamma_k \overleftarrow{\partial}_{\bar{\psi}^T_{-q}} \\ \overrightarrow{\partial}_{\bar{\psi}_p} \Gamma_k \overleftarrow{\partial}_{\psi_q} & \overrightarrow{\partial}_{\bar{\psi}_p} \Gamma_k \overleftarrow{\partial}_{\bar{\psi}^T_{-q}} \end{pmatrix} \tag{A1}$$

results in the field-independent part

$$\Gamma_{k,0}^{(2)}(p,q) = \frac{\delta_{p,q}}{(2\pi)^2}\begin{pmatrix} 0 & -Z_k \not{p}^T + i\not{a}^T - i\tilde{m} \\ -Z_k\not{p} - i\not{a} + i\tilde{m} & 0 \end{pmatrix}. \tag{A2}$$

Then the inverted form of the regularized propagator reads

$$\left(\Gamma_{k,0}^{(2)} + R_k\right)(p,q) = \frac{\delta_{p,q}}{(2\pi)^2}\begin{pmatrix} 0 & -Z_k \not{p}^T\left[1 + r\left(\frac{q^2}{k^2}\right)\right] + i\not{a}^T - i\tilde{m} \\ -Z_k\not{p}\left[1 + r\left(\frac{q^2}{k^2}\right)\right] - i\not{a} + i\tilde{m} & 0 \end{pmatrix}, \tag{A3}$$

the inverse of the matrix yields the result (12). To find the form of the field-dependent part (13) the property $\bar{\psi}\psi = -\psi^T\bar{\psi}^T$ appears to be useful.

APPENDIX B: TWO-FERMION BETA FUNCTION

Let us now derive the RG flow equations for the couplings that were involved in the part of the action which is quadratic in the fermionic fields ψ and $\bar{\psi}$. From the series (14) it is clear that only the term

$$-\frac{1}{2}\mathrm{Tr}\left[\frac{\Delta\Gamma_k^{(2)}}{\Gamma_{k,0}^{(2)}+R_k}\right] = -\left(\frac{\Delta_k}{2}+g_k\right)\Omega\int\frac{d^2q}{(2\pi)^2}\mathrm{Tr}[\hat{G}_1\hat{G}_0], \tag{B1}$$

where Ω is the volume of the system, contributes to the RG flow of the needed couplings, and $\hat{G}_1$ is defined by Eq. (13).

An elementary calculation gives

$$\begin{aligned}\mathrm{Tr}[\hat{G}_1\hat{G}_0] &= -\frac{\alpha_{k\mu}}{\alpha_k^2+\tilde{m}_k^2}(\bar{\psi}\sigma_\mu\psi)+\frac{\beta_{k\mu}}{\beta_k^2+\tilde{m}_k^2}(\bar{\psi}\sigma_\mu\psi)-i\tilde{m}_k\left(\frac{1}{\alpha_k^2+\tilde{m}_k^2}+\frac{1}{\beta_k^2+\tilde{m}_k^2}+\right)\bar{\psi}\psi\\ &= -2i\tilde{m}_k\frac{a_k^2q^2-d_k^2+\tilde{m}_k^2}{\left(a_k^2q^2-d_k^2+\tilde{m}_k^2\right)^2+4a_k^2d_k^2q_1^2}(\bar{\psi}\psi)+2id_k\frac{a_k^2q^2-d_k^2+\tilde{m}_k^2-2a_k^2q_1^2}{\left(a_k^2q^2-d_k^2+\tilde{m}_k^2\right)^2+4a_k^2d_k^2q_1^2}(\bar{\psi}\sigma_1\psi)\\ &\quad -4id_k\frac{a_k^2q_1q_2}{\left(a_k^2q^2-d_k^2+\tilde{m}_k^2\right)^2+4a_k^2d_k^2q_1^2}(\bar{\psi}\sigma_2\psi).\end{aligned} \tag{B2}$$

The third term drops out of Eq. (B1) after integration over the momentum $\mathbf{q}$ that brings forth

$$-\frac{1}{2}\mathrm{Tr}\left[\tilde{\partial}_k\frac{\Delta\Gamma_k^{(2)}}{\Gamma_{k,0}^{(2)}+R_k}\right] = 2i\Omega\tilde{m}_k\left(\frac{\Delta_k}{2}+g_k\right)\tilde{\partial}_k\mathcal{L}_1(\bar{\psi}\psi)-2i\Omega d_k\left(\frac{\Delta_k}{2}+g_k\right)\tilde{\partial}_k\mathcal{L}_2(\bar{\psi}\sigma_1\psi), \tag{B3}$$

where the threshold functions are

$$\mathcal{L}_1 = \int\frac{d^2q}{(2\pi)^2}\frac{a_k^2q^2-d_k^2+\tilde{m}_k^2}{\left(a_k^2q^2-d_k^2+\tilde{m}_k^2\right)^2+4a_k^2d_k^2q_1^2}, \tag{B4}$$

$$\mathcal{L}_2 = \int\frac{d^2q}{(2\pi)^2}\frac{a_k^2q^2-d_k^2+\tilde{m}_k^2-2a_k^2q_1^2}{\left(a_k^2q^2-d_k^2+\tilde{m}_k^2\right)^2+4a_k^2d_k^2q_1^2}. \tag{B5}$$

The ansatz for the kinetic term in the effective action (9) gives

$$\partial_k\Gamma_k = -i\Omega(\partial_kd_k)(\bar{\psi}\sigma_1\psi)+i\Omega(\partial_k\tilde{m}_k)\bar{\psi}\psi. \tag{B6}$$

In our approximation, the RG running of Z is trivial, i.e., $\partial_kZ=0$. Thus, the associated anomalous dimension $\eta=-k\partial_k\ln Z$ is zero. Therefore, in what follows, we set the wave-function renormalization as $Z=1$.

Comparing coefficients of the quadratic contributions [Eqs. (B3) and (B6)] to the exact flow equations yields

$$\partial_km_k = 2\left(\frac{\Delta_k}{2}+g_k\right)m_k\tilde{\partial}_k\mathcal{L}_1, \tag{B7}$$

$$\partial_kd_k = 2\left(\frac{\Delta_k}{2}+g_k\right)d_k\tilde{\partial}_k\mathcal{L}_2. \tag{B8}$$

APPENDIX C: FOUR-FERMION BETA FUNCTION

Formula for the four-fermion beta function reads

$$\frac{1}{4}\mathrm{Tr}\left[\tilde{\partial}_k\left(\frac{\Delta\Gamma_k^{(2)}}{\Gamma_{k,0}^{(2)}+R_k}\right)^2\right] = \left(\frac{\Delta_k}{2}+g_k\right)^2\Omega\int\frac{d^2q}{(2\pi)^2}\mathrm{Tr}[\hat{G}_1\hat{G}_0\hat{G}_1\hat{G}_0]. \tag{C1}$$

By evaluating

$$\mathrm{Tr}[\hat{G}_1\hat{G}_0\hat{G}_1\hat{G}_0] = \left\{-\det\left(\frac{\not{\alpha}_k-i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2}\right)-\det\left(\frac{\not{\beta}_k-i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right)+2\mathcal{X}\left(\frac{\not{\alpha}_k-i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2},\frac{\not{\beta}_k+i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right)\right\}(\bar{\psi}\psi)^2, \tag{C2}$$

where the function with the matrix arguments is defined by

$$\mathcal{X}(\hat{M},\hat{N}) = \tfrac{1}{2}(M_{11}N_{22}-M_{12}N_{21}-M_{21}N_{12}+M_{22}N_{11}), \tag{C3}$$

the four-fermion terms are straightforwardly appear.

By employing the ansatz (9) for the effective action after some elementary algebra we get for the constant fields

$$\Gamma_k = \Omega\left(\frac{\Delta_k}{2} + g_k\right)(\bar{\psi}\psi)^2. \tag{C4}$$

The flow for the coupling constant $\lambda_k = (\Delta_k/2 + g_k)$ is obtained by comparing both sides of Eq. (14) and by using Eqs. (C1) and (C4):

$$\partial_k\lambda_k = \lambda_k^2\tilde{\partial}_k\mathcal{L}_3. \tag{C5}$$

The flow involves the threshold function

$$\mathcal{L}_3 = \int\frac{d^2q}{(2\pi)^2}\left\{-\det\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2}\right) - \det\left(\frac{\not{\beta}_k - i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right) + 2\mathcal{X}\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2},\frac{\not{\beta}_k + i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right)\right\}. \tag{C6}$$

Equations (B7), (B8), and (C5) represent the main result of the two sections.

APPENDIX D: THRESHOLD FUNCTIONS

The flow equations include single integrals due to the one-loop structure of the Wetterich equation, the threshold functions, which contain details of the regularization scheme. The definitions of the threshold functions are given by Eqs. (B4), (B5), and (C6). In the flow equations $\tilde{\partial}_k$ is defined to act on the regulator's k dependence. The sharp cut-off regulator (15) has the remarkable feature that all threshold integrals can be done explicitly.

Indeed, in the polar coordinates

$$\tilde{\partial}_k\mathcal{L}_1 = \tilde{\partial}_k\int_k^\Lambda\frac{dq}{(2\pi)^2}\int_0^{2\pi}d\phi\frac{q\left(q^2-d_k^2+\tilde{m}_k^2\right)}{\left(q^2-d_k^2+\tilde{m}_k^2\right)^2+4d_k^2q^2\cos^2\phi}, \tag{D1}$$

where Λ is the ultraviolet cutoff, and we take into account that $a=-1$ for the regulator (15).

The needed dependence on k appears only in the lower limit of the integration over q. Therefore, one obtain

$$k\tilde{\partial}_k\mathcal{L}_1 = -\frac{1}{8\pi^2}\frac{\left(k^2-d_k^2+\tilde{m}_k^2\right)}{d^2}\int_0^{2\pi}\frac{d\phi}{2\left(\frac{k^2-d_k^2+\tilde{m}_k^2}{2kd_k}\right)^2+1+\cos 2\phi}. \tag{D2}$$

Once the simple integration is performed, we get

$$k\tilde{\partial}_k\mathcal{L}_1 = -\frac{\left(k^2-d_k^2+\tilde{m}_k^2\right)}{4\pi d_k^2}\frac{1}{\sqrt{\left[2\left(\frac{k^2-d_k^2+\tilde{m}_k^2}{2kd_k}\right)^2+1\right]^2-1}}. \tag{D3}$$

Similarly, the scale derivative of $\mathcal{L}_2$ is given by

$$k\tilde{\partial}_k\mathcal{L}_2 = \left[1+\frac{1}{2d^2}(k^2-d^2+\tilde{m}^2)\right]k\tilde{\partial}_k\mathcal{L}_1 + \frac{k^2}{4\pi d^2}. \tag{D4}$$

To find the RG running of $\mathcal{L}_3$ we first note that

$$\det\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2}\right) = -\frac{1}{\alpha_k^2+\tilde{m}_k^2},\quad \det\left(\frac{\not{\beta}_k + i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right) = -\frac{1}{\beta_k^2+\tilde{m}_k^2}, \tag{D5}$$

and

$$\mathcal{X}\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2},\frac{\not{\beta}_k + i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right) = \frac{\tilde{m}_k^2-\alpha_{1k}\beta_{1k}-\alpha_{2k}\beta_{2k}}{\left(\alpha_k^2+\tilde{m}_k^2\right)\left(\beta_k^2+\tilde{m}_k^2\right)}. \tag{D6}$$

Therefore,

$$-\det\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2}\right) - \det\left(\frac{\not{\beta}_k - i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right) + 2\mathcal{X}\left(\frac{\not{\alpha}_k - i\tilde{m}_k}{\alpha_k^2+\tilde{m}_k^2},\frac{\not{\beta}_k + i\tilde{m}_k}{\beta_k^2+\tilde{m}_k^2}\right)$$
$$= \frac{4\tilde{m}_k^2+(\alpha_{1k}-\beta_{1k})^2+(\alpha_{2k}-\beta_{2k})^2}{\left(\alpha_k^2+\tilde{m}_k^2\right)\left(\beta_k^2+\tilde{m}_k^2\right)}. \tag{D7}$$

For the sharp cutoff (15), for which $\alpha_{1k}-\beta_{1k}=-2id_k$ and $\alpha_{2k}-\beta_{2k}=0$, the last expression reads

$$\frac{4\left(\tilde{m}_k^2-d_k^2\right)}{\left(q^2-d_k^2+\tilde{m}_k^2\right)^2+4q_1^2d_k^2}. \tag{D8}$$

Through the insertion of the result into Eq. (C6) one obtain

$$\mathcal{L}_3 = \frac{1}{\pi^2}\int_k^\Lambda dq\, q\int_0^{2\pi} d\phi\, \frac{\left(\tilde{m}_k^2 - d_k^2\right)}{\left(q^2 - d_k^2 + \tilde{m}_k^2\right)^2 + 4q^2 d_k^2\cos^2\phi}. \tag{D9}$$

From the definition it follows that

$$k\tilde{\partial}_k\mathcal{L}_3 = \frac{4\left(\tilde{m}_k^2 - d_k^2\right)}{\left(k^2 - d_k^2 + \tilde{m}_k^2\right)^2} k\,\tilde{\partial}_k\mathcal{L}_1. \tag{D10}$$

APPENDIX E: THE ELECTROSTATIC MODEL

To introduce the duality mapping we replace [60] in the partition function (22):

$$e^{\beta\tilde{J}\cos\Phi_{ij}} \to e^{\beta\tilde{J}}\sum_{m=-\infty}^{m=+\infty}\exp\left[-\frac{\beta\tilde{J}}{2}(\Phi_{ij} - 2\pi m)^2\right], \tag{E1}$$

where $\Phi_{ij} = \varphi_i - \varphi_j - \alpha_{ij}$, and use the Poisson sum formula which states

$$\sum_{m=-\infty}^{+\infty} f(m) = \sum_{l=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\phi\, f(\phi)e^{2\pi i l\phi}. \tag{E2}$$

This yields

$$e^{\beta\tilde{J}\cos\Phi_{ij}} \to \frac{1}{\sqrt{2\pi\beta\tilde{J}}}\sum_{l_{ij}=-\infty}^{+\infty} e^{\beta\tilde{J}}\exp\left[-\frac{l_{ij}^2}{2\beta\tilde{J}} + il_{ij}\Phi_{ij}\right]. \tag{E3}$$

After substituting the result into (22) and omitting nonessential factors one find

$$\mathcal{Z} = \int\mathcal{D}\varphi\sum_{\{l_{ij}\}}\exp\left(-\sum_{\langle ij\rangle}\left[\frac{l_{ij}^2}{2\beta\tilde{J}} - il_{ij}\Phi_{ij}\right] + \beta h\sum_i\cos\varphi_i\right). \tag{E4}$$

We now define a vector field $l_\mu(\mathbf{r})$ ($\mu = 1,2$) that is directed from the starting point $\mathbf{r}$, which is the left-hand side or lower side of the link between the sites i and j, to the other side of the link. The vector field takes the value l_{ij} on the link. The partition function is then just the sum over all possible values $l_\mu(\mathbf{r})$ of the form

$$\mathcal{Z} = \sum_{\{l_\mu(\mathbf{r})\}}\int\mathcal{D}\varphi\,\exp\left(-\sum_{\mathbf{r},\mu}\left[\frac{l_\mu^2(\mathbf{r})}{2\beta\tilde{J}} - il_\mu(\mathbf{r})\{\varphi(\mathbf{r}) - \varphi(\mathbf{r}+\mathbf{a}_\mu)\} + il_\mu(\mathbf{r})\alpha_\mu\right] + \beta h\sum_{\mathbf{r}}\cos\varphi(\mathbf{r})\right), \tag{E5}$$

where α_μ coincides with α_{ij} on the ij link, and $\mathbf{a}_\mu$ is the lattice unit.

To evaluate the sum we shall make use of

$$\sum_{\mathbf{r},\mu} l_\mu(\mathbf{r})\{\varphi(\mathbf{r}) - \varphi(\mathbf{r}+\mathbf{a}_\mu)\} = \sum_{\mathbf{r},\mu}\{l_\mu(\mathbf{r}) - l_\mu(\mathbf{r}-\mathbf{a}_\mu)\}\varphi(\mathbf{r}) \tag{E6}$$

to transform the partition function into

$$\mathcal{Z} = \sum_{\{l_\mu(\mathbf{r})\}}\int\mathcal{D}\varphi\,\exp\left(-\sum_{\mathbf{r},\mu}\left[\frac{l_\mu^2(\mathbf{r})}{2\beta\tilde{J}} - i\{l_\mu(\mathbf{r}) - l_\mu(\mathbf{r}-\mathbf{a}_\mu)\}\varphi(\mathbf{r}) + il_\mu(\mathbf{r})\alpha_\mu\right] + \beta h\sum_{\mathbf{r}}\cos\varphi(\mathbf{r})\right). \tag{E7}$$

We wish to perform integration over $\varphi(\mathbf{r})$ from 0 to 2π. The goal is easily accomplished with the aid of the Jacobi-Anger expansion

$$e^{z\cos\varphi} = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z)\cos(k\varphi), \tag{E8}$$

where $I_k(z)$ is the modified Bessel function of the first kind.

The φ integrals can be then done immediately which reduces the partition function to a sum over the bond variables $l_\mu(\mathbf{r})$ with a set of δ functions restricting these variables at every site:

$$\mathcal{Z} \propto \sum_{\{l_\mu(\mathbf{r})\}}\exp\left[-\sum_{\mathbf{r}\mu}\left(\frac{l_\mu^2(\mathbf{r})}{2\beta\tilde{J}} + il_\mu(\mathbf{r})\alpha_\mu\right)\right]\prod_{\mathbf{r}}\left\{\sum_{\kappa(\mathbf{r})=-\infty}^{\infty} I_{\kappa(\mathbf{r})}(\beta h)\,\delta\left[\sum_\mu[l_\mu(\mathbf{r}) - l_\mu(\mathbf{r}-\mathbf{a}_\mu)] - \kappa(\mathbf{r})\right]\right\}. \tag{E9}$$

A presence of the magnetic field violates the "zero divergence" condition

$$\sum_{\mu}[l_{\mu}(\mathbf{r}) - l_{\mu}(\mathbf{r}-\mathbf{a}_{\mu})] = 0, \tag{E10}$$

giving the effective integer-valued charges $\kappa(\mathbf{r})$ confined to lattice sites.

To gain an insight into the nature of the constraints imposed by the δ functions it is worthy to note that $l_{\mu}(\mathbf{r})$ can be splitted into the longitudinal and transverse parts [71]

$$l_{\mu}(\mathbf{r}) = m(\mathbf{r}) - m(\mathbf{r}+\mathbf{a}_{\mu}) + \sigma(\mathbf{r}) - \sigma(\mathbf{r}+\mathbf{a}_{\mu}) + \varepsilon_{\mu\nu}[n(\mathbf{r}) - n(\mathbf{r}-\mathbf{a}_{\nu})], \tag{E11}$$

where $\varepsilon_{\mu\nu}$ is the standard antisymmetric tensor, the $m(\mathbf{r})$ and $n(\mathbf{r})$ are integers, and $|\sigma(\mathbf{r})| < 1$.

The transverse vector field $\mathbf{n}(\mathbf{r}) = n(\mathbf{r})\mathbf{e}_3$, where $\mathbf{e}_3$ is perpendicular to the plane of the system, realizes the discrete version of the equation $\mathbf{l}(\mathbf{r}) = \mathrm{rot}\,\mathbf{n}(\mathbf{r})$,

$$l_1(\mathbf{r}) = n(\mathbf{r}) - n(\mathbf{r}-\mathbf{a}_2), \quad l_2(\mathbf{r}) = -n(\mathbf{r}) + n(\mathbf{r}-\mathbf{a}_1), \tag{E12}$$

that ensure that the zero divergence condition (E10) is properly satisfied.

The δ-function condition in Eq. (E9) can, in turn, be satisfied if the longitudinal part obeys the discrete Poisson equation

$$-\sum_{\mu}[m(\mathbf{r}+\mathbf{a}_{\mu}) + m(\mathbf{r}-\mathbf{a}_{\mu}) - 2m(\mathbf{r}) + \sigma(\mathbf{r}+\mathbf{a}_{\mu}) + \sigma(\mathbf{r}-\mathbf{a}_{\mu}) - 2\sigma(\mathbf{r})] = \kappa(\mathbf{r}). \tag{E13}$$

Here the $m(\mathbf{r})$ are required to be integer valued and $\sigma(\mathbf{r})$ are adjusted to keep Eq. (E13).

Given that we are primarily focusing on a role of the DM interaction, we restrict ourselves to the case of vanishing magnetic fields $\beta h \to 0$, when $I_{\kappa(\mathbf{r})}(\beta h)$ can be replaced by the delta symbol $\delta_{\kappa,0}$ and the trivial solution $m(\mathbf{r}) + \sigma(\mathbf{r}) = 0$ can be taken for Eq. (E13).

By substituting (E12) in Eq. (E9) and taking account of $\alpha_{\mu} = \alpha\delta_{\mu,1}$ we find

$$\mathcal{Z} = \sum_{\{n(\mathbf{r})\}} \exp\left[-\sum_{\mathbf{r}\mu}\frac{1}{2\beta\tilde{J}}[n(\mathbf{r}) - n(\mathbf{r}-\mathbf{a}_{\mu})]^2 - i\alpha\sum_{\mathbf{r}}[n(\mathbf{r}) - n(\mathbf{r}-\mathbf{a}_2)]\right]. \tag{E14}$$

Rewriting the sum running over integers $n(\mathbf{r})$ through the Poisson formula (E2) one obtains

$$\mathcal{Z} = \int\mathcal{D}\phi\sum_{\{q(\mathbf{r})\}}\exp\left[-\frac{1}{2\beta\tilde{J}}\sum_{\mathbf{r},\mu}(\hat{\Delta}_{\mu}\phi)^2 - i\alpha\sum_{\mathbf{r}}\hat{\Delta}_2\phi + 2\pi i\sum_{\mathbf{r}}q(\mathbf{r})\phi(\mathbf{r})\right], \tag{E15}$$

where the lattice difference is defined as $\hat{\Delta}_{\mu}\phi(\mathbf{r}) = \phi(\mathbf{r}) - \phi(\mathbf{r}-\mathbf{a}_{\mu})$.

Making use of the parallel translation in the functional space [72], $\phi(\mathbf{r}) \to \phi(\mathbf{r}) - i\alpha\beta\tilde{J}x_2$, and carrying out Gaussian integration over $\phi(\mathbf{r})$ we are led to

$$\mathcal{Z}_{\text{eff}} \propto \sum_{\{q(\mathbf{r})\}}\exp\left\{-2\pi^2\beta\tilde{J}\sum_{\mathbf{r},\mathbf{r}'}q(\mathbf{r})G(\mathbf{r}-\mathbf{r}')q(\mathbf{r}') + 2\pi\alpha\beta\tilde{J}\sum_{\mathbf{r}}q(\mathbf{r})x_2\right\}. \tag{E16}$$

The lattice Green function takes the form [40]

$$G(\mathbf{r}-\mathbf{r}') = \int\frac{d^2k}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{4 - 2\cos k_x - 2\cos k_y} \approx -\frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right) - \frac{1}{4} + G(0), \tag{E17}$$

where the last term does not contain divergent terms. Interpreting $q(\mathbf{r})$ as an electric charge at the position $\mathbf{r}$ and the logarithmic potential as the Coulomb potential in two dimensions, the term with $G(0)$ disappears if the charge neutrality condition $\sum_{\mathbf{r}} q(\mathbf{r}) = 0$ is imposed. The remaining part of $G(\mathbf{r}-\mathbf{r}')$ leads to the result (23).

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