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Higher-order generalized hydrodynamics in one dimension: The noninteracting test

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We derive a dynamical equation that describes the exact time evolution in generic (inhomogeneous) noninteracting spin-chain models. Assuming quasistationarity, we develop a (generalized) hydrodynamic theory. The question at hand is whether some large-time corrections are captured by higher-order hydrodynamics. We consider in particular the dynamics after two chains, prepared in different conditions, are joined together. In these situations, a light cone, separating regions with macroscopically different properties, emerges from the junction. In free fermionic systems some observables close to the light cone follow a universal behavior, known as Tracy-Widom scaling. Universality means a weak dependence on the system's details, so this is the perfect setting where hydrodynamics could emerge. For the transverse-field Ising chain and the XX model, we show that hydrodynamics captures the scaling behavior close to the light cone. On the other hand, our numerical analysis suggests that hydrodynamics fails in more general models, whenever a condition is not satisfied.

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Over the past few years, we have experienced increased interest in the physics behind the nonequilibrium time evolution of inhomogeneous states. An example is the time evolution of two semi-infinite chains that are joined together after having been prepared in different equilibrium conditions [1,2]. This kind of setting allows one to investigate the transport properties of quantum many-body systems even if the system is isolated from the environment.

The first analytical results in this context were obtained in noninteracting models [3–24]. There, under the assumption of quasistationarity, a semiclassical picture applies where the information about the initial state is carried by free stable quasiparticles moving throughout the system. Similar results were obtained in the framework of conformal field theory and Luttinger liquid descriptions [25–37]. In the presence of interactions the situation was less clear [38–47], but, eventually, Refs. [48,49] showed that the continuity equations satisfied by the (quasi)local conserved quantities are sufficient to characterize the late-time behavior. The framework developed in Refs. [48,49] is now known as *generalized hydrodynamics* [48], where "generalized" is used to emphasize that integrable models have infinitely many (quasi)local charges [50]. We will generally omit "generalized" and refer to the system of equations derived in Refs. [48,49] as *first-order hydrodynamics*, 1^{st}GHD, to emphasize that it is a system of first-order partial differential equations.

Within 1^{st}GHD, it was possible to compute the profiles of local observables [48,49,51–57], to conjecture an expression for the time evolution of the entanglement entropy [58], and to efficiently calculate Drude weights [59–63]. There are, however, fundamental questions that cannot be addressed within 1^{st}GHD; diffusive transport [64–69] and large-time corrections [20–23] are two of them. The importance of these issues results in a considerable urge to fill these gaps [61], passing through refinements and reinterpretations of the theory [57,70–73].

In this Rapid Communication we carry out a preliminary analysis of whether higher-order hydrodynamics gives access to additional physical information. Since any refinement to the equations governing the dynamics must be able to pass the noninteracting test, we focus on generic noninteracting spin-chain models. We attack the problem in full generality, deriving first a dynamical equation that describes time evolution exactly. From that, we develop a "complete" hydrodynamic theory, GHD, based on the single assumption of quasistationarity. Within GHD, we compute some large-time corrections and compare them with exact numerical data. The result is puzzling: Higher-order hydrodynamics reproduces known asymptotic behaviors in the XX model and in the transverse-field Ising chain; the same hydrodynamic description, however, seems to fail in generic noninteracting models.

The system. We consider an infinite spin-$\frac{1}{2}$ chain described by a Hamiltonian of the form [74]

$$\boldsymbol{H}=\sum_{\ell\in\mathbb{Z}}\sum_{n\in\mathbb{N}_0}\sum_{\alpha,\beta\in x,y}J^{\alpha\beta}_{\ell,n}\boldsymbol{\sigma}^\alpha_\ell\boldsymbol{\Pi}^z_{\ell,n}\boldsymbol{\sigma}^\beta_{\ell+n}+\sum_{\ell\in\mathbb{Z}}J^z_\ell\boldsymbol{\sigma}^z_\ell, \tag{1}$$

where $\boldsymbol{\sigma}^\alpha_\ell$ are Pauli matrices, $\boldsymbol{\Pi}^z_{\ell,n}=\prod_{j=\ell+1}^{\ell+n-1}\boldsymbol{\sigma}^z_j$ ($\boldsymbol{\Pi}^z_{\ell,0}$ is the identity $\mathbf{I}$), $\mathbb{Z}$ is the set of all the integers, and $\mathbb{N}_0$ is its non-negative subset. This class includes several paradigmatic models, as the transverse-field Ising chain [75] and the XY model [76]. Under the Jordan-Wigner transformation $\boldsymbol{a}_{2\ell-1}=\prod_{j<\ell}\boldsymbol{\sigma}^z_j\boldsymbol{\sigma}^x_\ell$, $\boldsymbol{a}_{2\ell}=\prod_{j<\ell}\boldsymbol{\sigma}^z_j\boldsymbol{\sigma}^y_\ell$, the Hamiltonian is mapped into a chain of noninteracting Majorana fermions ($\{\boldsymbol{a}_\ell,\boldsymbol{a}_n\}=2\delta_{\ell n}\mathbf{I}$, where $\{\cdot,\cdot\}$ is the anticommutator, and $\delta_{\ell n}$ is the Kronecker delta)

$$\boldsymbol{H}=\frac{1}{4}\sum_{\ell,n\in\mathbb{Z}}\boldsymbol{a}_\ell\mathcal{H}_{\ell n}\boldsymbol{a}_n. \tag{2}$$

Here, $\mathcal{H}$ is an infinite [77] purely imaginary antisymmetric matrix. Being quadratic, $\boldsymbol{H}$ is diagonal in a basis of Slater determinants. These are states $|\Gamma\rangle$ completely characterized (up to a phase) by the fermionic two-point functions, which can be organized in a purely imaginary antisymmetric matrix Γ, known as a "correlation matrix,"

$$\Gamma_{\ell n}=\delta_{\ell n}-\langle\Gamma|\boldsymbol{a}_\ell\boldsymbol{a}_n|\Gamma\rangle\,. \tag{3}$$

Thermal states are Slater determinants as well; the ground state, however, is not always a Slater determinant, as a symmetry could be spontaneously broken.

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Quadratic operators are closed under commutation, so a Slater determinant that time evolves under a Hamiltonian of the form (1) remains a Slater determinant. Specifically, the time-evolving state is as follows,

$$e^{-i\boldsymbol{H}t}\,|\Gamma\rangle = e^{-i\gamma_t}\,|e^{-i\mathcal{H}t}\Gamma e^{i\mathcal{H}t}\rangle\,, \tag{4}$$

where $e^{-i\gamma_t}$ is a phase.

The symbol. Let $\mathcal{M}$ be an infinite purely imaginary antisymmetric matrix with elements decaying sufficiently fast to zero the farther they are from the main diagonal. For a given positive integer κ, we define the (2κ)-by-(2κ) symbol $\hat{m}_x(e^{ip})$ of $\mathcal{M}$ in such a way that

$$\mathcal{M}_{2\kappa\ell_\kappa+i,2\kappa n_\kappa+j} = \int_{-\pi}^{\pi}\frac{dp}{2\pi}e^{i(\ell_\kappa-n_\kappa)p}\big[\hat{m}_{\frac{\ell_\kappa+n_\kappa}{2}}(e^{ip})\big]_{ij}, \tag{5}$$

where $i,j=1,\dots,2\kappa$, and $\ell_k,n_\kappa\in\mathbb{Z}$. This is a natural generalization of the standard definition of symbols of block-Toeplitz or block-circulant matrices [78], for which $\hat{m}_x(e^{ip})$ is independent of x. More generally, $\hat{m}_x(e^{ip})$ enters (5) only with $x\in\frac{1}{2}\mathbb{Z}$, where $\frac{1}{2}\mathbb{Z}$ is the set of all the integers and the half integers. In addition, if x is an integer (a half integer), the equation only fixes the π-periodic (antiperiodic) part of $\hat{m}_x(e^{ip})$ with respect to p. The undefined parts of the symbol are irrelevant and can be chosen arbitrarily. It is convenient to require the symbol to be Hermitian and to satisfy $[\hat{m}_x(e^{ip})]^t = -\hat{m}_x(e^{-ip})$, where t denotes transposition. We can then extend its definition in a smooth way so as to allow for real $x\in\mathbb{R}$.

The reader can picture the symbol as the Fourier transform, in the direction of the antidiagonals, of a smooth function that matches the matrix block elements at the vertices of a square lattice. In the present context, the meaning of the symbol becomes more transparent when $\hat{m}_x(e^{ip})$ has a weak dependence on x. For example, if the Hamiltonian is invariant under a shift by κ sites, its symbol can be chosen to be independent of x, $\hat{h}_x(e^{ip})=\hat{h}(e^{ip})$; then, it turns out that the excitation energies $\varepsilon_n(p)$ are the eigenvalues of $\hat{h}(e^{ip})$, in the sense that $\hat{h}(e^{ip})=\sum_{n=1}^{\kappa}\varepsilon_n(p)\mathrm{P}_n(p)-\varepsilon_n(-p)\mathrm{P}_n^t(-p)$, where $\mathrm{P}_n(p)$ and $\mathrm{P}_n^t(-p)$ are projectors orthogonal to one another. Analogous relations can be found considering the symbol $\hat{\Gamma}_x(e^{ip})$ of the correlation matrix of a locally quasistationary state [51], which is a state that, in a sufficiently small space-time interval, resembles a steady state. Specifically, the eigenvalues of $\hat{\Gamma}_x(e^{ip})$ can be interpreted [79] (up to additive constants and multiplicative factors) as the densities of the excitations over the ground state; they are also known as "root densities."

The equations governing the dynamics. We find that the degrees of freedom in the definition of the symbol can be used to recast time evolution (4) in the form of a Moyal dynamical equation [80]

$$i\partial_t\hat{\Gamma}_{x,t}(e^{ip}) = \hat{h}_x(e^{ip})\star\hat{\Gamma}_{x,t}(e^{ip})-\hat{\Gamma}_{x,t}(e^{ip})\star\hat{h}_x(e^{ip}). \tag{6}$$

Here, $\star$ denotes the Moyal star product [81], defined as $\hat{f}_x(e^{ip})\star\hat{g}_x(e^{ip}) = e^{i\frac{\partial_q\partial_x-\partial_p\partial_y}{2}}\hat{f}_x(e^{ip})\hat{g}_y(e^{iq})|_{\substack{q=p\\y=x}}$. This is a fundamental equation for noninteracting systems in infinite spin-$\frac{1}{2}$ chains.

If the Hamiltonian is invariant under a shift by κ sites, (6) can be solved; its solution reads

$$\hat{\Gamma}_{x,t}(e^{ip}) = \iint_{-\infty}^{\infty}\frac{dydq}{2\pi}e^{iq(x-y)} \times e^{-it\hat{h}(e^{i(p+\frac{q}{2})})}\hat{\Gamma}_{y,0}(e^{ip})e^{it\hat{h}(e^{i(p-\frac{q}{2})})}. \tag{7}$$

This can be interpreted as an exact Wigner description [82] of the dynamics in noninteracting spin-chain models. The solution (7) applies to any Slater determinant time evolving under any homogeneous noninteracting Hamiltonian.

We point out that connections between matrix multiplication and Moyal star product have been already established (see, e.g., Ref. [83]). Since the formalism in terms of symbols is not used much in the present context, (6) and (7) are not widely known; nevertheless, their structure can be recognized in equations emerging within semiclassical approximations, as in Refs. [84,85].

Hydrodynamics. Equation (6) is a useful tool, but it cannot be easily generalized to interacting models. In addition, even when (7) applies, the explicit calculation of the integrals, but also their numerical evaluation, can be difficult. Notwithstanding, one is often interested in particular aspects of the dynamics that are not expected to depend on all the system's details. For these two reasons, we pivot to a description that, *a priori*, is only an approximation; we develop a hydrodynamic theory.

To that aim, we add the hypothesis of *quasistationarity*: We assume that $|\Gamma\rangle\sim|\Gamma^{\mathrm{h}}\rangle$ is a locally quasistationary state at every time (from now on, the superscripts h and h_j stand for "within GHD" and "within jth-order GHD"). If $\boldsymbol{H}$ is invariant under a shift by κ sites, this is equivalent to asking for the symbol of the correlation matrix to locally commute with the symbol of the Hamiltonian, i.e., $[\hat{\Gamma}^{\mathrm{h}}_{x,t}(e^{ip}),\hat{h}(e^{ip})]=0$. This condition can be enforced on (6) by extracting the diagonal part of (6) in a basis that diagonalizes $\hat{h}(e^{ip})$; we then find

$$i\partial_t\hat{\Gamma}^{\mathrm{h}}_{x,t}(e^{ip}) = \iint_{-\infty}^{\infty}\frac{dqdy}{2\pi}e^{iq(x-y)} \times\big\langle\!\big\langle\hat{h}\big(e^{i(p+\frac{q}{2})}\big)-\hat{h}\big(e^{i(p-\frac{q}{2})}\big)\big\rangle\!\big\rangle(e^{ip})\hat{\Gamma}^{\mathrm{h}}_{y,t}(e^{ip}), \tag{8}$$

where $\langle\langle\hat{a}(e^{ik})\rangle\rangle(e^{ip})$ denotes the diagonal part of $\hat{a}(e^{ik})$. This is a *complete (generalized) hydrodynamic equation* that includes all the contributions from arbitrarily high-order spatial derivatives. In fact, (8) can be put in a more familiar form if $\kappa+1$ is not smaller than the range [86] of the Hamiltonian. In that case (8) reads [87]

$$\partial_t\rho^{\mathrm{h}}_{n;x,t}(p)+v_n(p)\big[\rho^{\mathrm{h}}_{n;x+\frac{1}{2},t}(p)-\rho^{\mathrm{h}}_{n;x-\frac{1}{2},t}(p)\big]=0, \tag{9}$$

where $n=1,\dots,\kappa$, $\rho^{\mathrm{h}}_{n;x}(p)$ are the root densities, and $v_n(p)=\varepsilon_n'(p)$ are the velocities of the excitations. The first-order hydrodynamic equation is recovered in the limit of weak inhomogeneity, which allows one to expand the last two terms of (9) about x, ignoring the contributions from spatial derivatives higher than the first. That is the equation that Refs. [48,49] generalized to interacting integrable models. Incidentally, if we apply it to (9) the same prescription that lifted its first-order approximation to a theory for interacting integrable models, we obtain

$$\partial_t \rho^{\mathrm{h}}_{n;x,t} + v^{\mathrm{h}}_{n;x+\frac{1}{2}} \rho^{\mathrm{h}}_{n;x+\frac{1}{2},t} - v^{\mathrm{h}}_{n;x-\frac{1}{2}} \rho^{\mathrm{h}}_{n;x-\frac{1}{2},t} = 0. \tag{10}$$

Here, the velocity depends on x (and, in turn, it is affected by the assumption of quasistationarity) because it is dressed by the interaction [88].

At this stage, (10) is nothing but a speculation; we must first understand what physical information is contained in (9) in the very noninteracting case. Indeed, (9) is based on the assumption of quasistationarity, which is known to be exact only in particular limits when (9) reduces to its first-order approximation [80]. The main goal of this Rapid Communication is to assess whether the hypothesis of quasistationarity, which leads to (8) and (9), has a validity that goes beyond the regime identified in Refs. [48,49].

Dynamics close to the light cone. Let us imagine to prepare two semi-infinite chains in different stationary conditions, for example, at different temperatures. Let us then join the chains together so as to form a single infinite chain. The state is let to evolve under the merged Hamiltonian, which we assume to be homogeneous. Qualitatively, a light cone, which separates regions with macroscopically different properties, emerges from the junction. 1$^{\mathrm{st}}$GHD turns out to capture the limit of large time at any position. On the other hand, large-time corrections are generally beyond its capabilities.

We wonder whether some corrections can still be computed within GHD. Since off-diagonal contributions are neglected, it is not reasonable to expect hydrodynamics to capture generic corrections. It could be effective, however, for corrections exhibiting universal properties, such as the ones studied in Ref. [20]. There, the time evolution of a domain-wall state under a free fermionic Hamiltonian (XX model) was considered. The authors were able to establish a connection between the probability distributions of particular observables close to the light cone and the distribution functions of the largest eigenvalues of the Gaussian unitary random matrix ensemble [89]. In particular, the two-point functions of the fermions lying in a region, around the edge, scaling as $t^{\frac{1}{3}}$, have corrections that decay as $t^{-\frac{1}{3}}$, and they can be written in terms of the so-called Airy kernel,

$$K_1(u,v) = [\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)]/(u-v). \tag{11}$$

An analogous behavior was observed years before [5], studying the transverse-field Ising chain, and, recently, Refs. [22,23] pointed out that, also in that case, the large-time corrections are characterized by the Airy kernel.

The presumptive universality of these corrections makes them a perfect candidate to test GHD. As a first step, we argue that, if the scaling behavior close to the light cone can be described by complete hydrodynamics, then it can also be described by hydrodynamics at the third order. The latter is obtained by expanding at the third order, either the integrand in (8) about p, as if q were close to zero, or, equivalently, the last two terms of (9) about x. For the sake of simplicity, we assume that the Hamiltonian is one-site shift invariant. For $\kappa = 1$, third-order hydrodynamics reads

$$\partial_t \rho^{\mathrm{h}_3}_{x,t}(p) + v(p)\partial_x \rho^{\mathrm{h}_3}_{x,t}(p) + [w(p)/24]\partial_x^3 \rho^{\mathrm{h}_3}_{x,t}(p) = 0, \tag{12}$$

where $w(p)$ will be reported in (15) [if the couplings are nearest neighbor, $w(p) = v(p)$]. The solution to (12) is

$$\rho^{\mathrm{h}_3}_{x,t}(p) = \int_{-\infty}^{\infty} dy\, \mathrm{Ai}(y) \rho^{\mathrm{h}_3}_{x - v(p)t - \frac{y}{2}\sqrt[3]{w(p)t},0}(p), \tag{13}$$

where $\mathrm{Ai}(y)$ is the Airy function. We focus on the situation where $v(p)$ has a unique global maximum at $p = \bar{p}$ and $v''(\bar{p})$ is nonzero. The maximal velocity $v(\bar{p})$ determines the speed at which the light cone propagates to the right. A sketch of a proof of the equivalence between (13) and complete hydrodynamics is reported in the Supplemental Material [90].

By Wick's theorem, the expectation value of any observable can be written in terms of the correlation matrix Γ (3). In the hypothesis of quasistationarity, we apply (8), and Γ is replaced by the following correlation matrix,

$$\Gamma^{\mathrm{h}}_{2\ell+i,2m+j}(t) = 2\int_{-\pi}^{\pi} dp \left(\rho^{\mathrm{h}}_{\frac{\ell+m}{2},t}(p) - \frac{1}{4\pi} \right) \{\cos[(\ell-m)p] \times A_{i,j}(p) + i\sin[(\ell-m)p] B_{i,j}(p)\}. \tag{14}$$

Here, $A(p) = \frac{\tilde{\sigma}(p)+\tilde{\sigma}(-p)}{2}$ and $B(p) = \mathrm{I} + \frac{\tilde{\sigma}(p)-\tilde{\sigma}(-p)}{2}$, with $\tilde{\sigma}(p) = \mathrm{sgn}\{\hat{h}(e^{ip}) - \frac{\mathrm{I}}{2}\mathrm{tr}[\hat{h}(e^{ip})]\}$; the dispersion relation is $\varepsilon(p) = \mathrm{tr}[\frac{\mathrm{I}+\tilde{\sigma}(p)}{2}\hat{h}(e^{ip})]$, and the velocity is $v(p) = \varepsilon'(p)$; the function $w(p)$ appearing in (12) can be written as

$$w(p) = -v''(p) - 3\partial_p\{\varepsilon_{\mathrm{e}}^2(p)\det[\partial_p\tilde{\sigma}(p)]\}/[2\varepsilon_{\mathrm{e}}(p)], \tag{15}$$

where $\varepsilon_{\mathrm{e}}(p) = [\varepsilon(p) + \varepsilon(-p)]/2$. We parametrize the root density at the initial time as $\rho_{x,0}(p) = \rho^+(p) + [\rho^-(p) - \rho^+(p)]\theta_H(-x)$, where θ_H is the Heaviside step function, and ρ^- and ρ^+ are the root densities describing the states of the two original semi-infinite chains [92]. We consider observables that lie close to the light cone. These observables can be fully described by the reduced density matrix of a spin block consisting of the sites $S = \{r, r+1, \ldots, r+|S|-1\}$ lying around the edge. The reduced density matrix is a Slater determinant with the following block correlation matrix,

$$\left[\Gamma^{\mathrm{h}(S)}_{\ell,m}\right]_{i,j}(t) = \Gamma^{\mathrm{h}}_{2(\ell+r)+i,2(m+r)+j}(t), \tag{16}$$

where $\ell,m = 0,\ldots,|S|-1$ and $i,j = 1,2$. Following Ref. [20], we consider the scaling limit where the time is large and both $r - v(\bar{p})t$ and the subsystem's length $|S|$ are proportional to $t^{\frac{1}{3}}$. Assuming the various functions to be sufficiently smooth around $p = \bar{p}$, we find

$$\Gamma^{\mathrm{h}(S)}_{\ell,m}(t) \approx \Gamma^{\mathrm{h}(S)}_{\ell,m}(0) + 4\pi\left(\frac{2|\alpha|}{-\bar{v}''t}\right)^{\frac{1}{3}} [\bar{\rho}^- - \bar{\rho}^+] K_\alpha(x_\ell, x_m) \times \{\cos[(\ell-m)\bar{p}]\bar{A} + i\sin[(\ell-m)\bar{p}]\bar{B}\}, \tag{17}$$

where $\bar{f}$ stands for $f(\bar{p})$; $\alpha = \mathrm{sgn}(\bar{w})(\frac{|\bar{w}|}{-\bar{v}''})^{\frac{1}{2}}$, and we introduced the rescaled variables

$$x_j = 2^{\frac{1}{3}} \frac{j + r - \bar{v}t}{\sqrt[3]{-\bar{v}''\alpha^2 t}}; \tag{18}$$

the kernel $K_\alpha(u,v)$ is defined as follows,

$$K_\alpha(u,v) = 2^{\frac{2}{3}} \int_0^\infty dy\, \mathrm{Ai}\left[\mathrm{sgn}(\alpha) 2^{\frac{2}{3}}\left(y + \frac{u+v}{2}\right)\right] \times \sin[\alpha(u-v)\sqrt{y}]/[\alpha\pi(u-v)]. \tag{19}$$

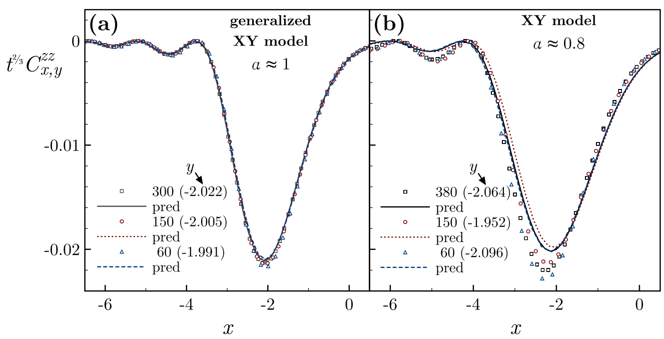

FIG. 1. The connected two-point function of s^z as a function of the rescaled positions x,y (18). The symbols are exact numerical data (in a chain with 1801 spins) at three different times, (a) $t = 60,150,300$ and (b) $t = 60,150,380$; after that, a thermal state with inverse temperature $\beta = 1$ is put in contact with the infinite-temperature state. The lines are the predictions (20). (a) is for a generalized XY model with $J^{yy}_{\ell,0} = -1$, $J^z_\ell = -2$, $J^{xx}_{\ell,1} \approx -0.0796$, and $J^{yy}_{\ell,1} \approx 0.3294$ [see (1)], for which $\alpha \approx 1$. (b) is for an XY model with $J^{xx}_{\ell,0} = 0.15$, $J^{yy}_{\ell,0} = -1.15$, and $J^z_\ell = -2$, for which $\alpha \approx 0.8$. (a) unveils an excellent agreement between data and predictions. On the other hand, in (b) the data do not seem to approach the predictions as the time is increased.

Using a representation of the product of two Airy functions derived in Ref. [93], it is simple to show that, if $\alpha > 0$, $K_\alpha(u,v)$ can be expressed in terms of the Airy kernel (11) as $K_\alpha(u,v) = K_1(\frac{1+\alpha}{2}u + \frac{1-\alpha}{2}v, \frac{1+\alpha}{2}v + \frac{1-\alpha}{2}u)$. In the transverse-field Ising chain and in the XX model, the parameter α is equal to unity, and we recover (11).

It is worth noting that (17) describes the asymptotic behavior only if (i) the difference in the root densities at the initial time is nonzero at $p = \bar{p}$, and (ii) the expectation value of the observable does not accidentally zero the term. The former case is closely related to the situation studied in Ref. [53] for a similar protocol in the XXZ model; the latter case is discussed in Ref. [22] considering the critical Ising model.

A comparison with Ref. [23] shows that, in the transverse-field Ising chain, GHD gives the correct asymptotic behavior. The same conclusion can be drawn for the XX model. For more general systems, we have analyzed the behavior close to the light cone numerically. Figure 1 shows the edge profile of the connected two-point function of $s^z = \frac{1}{2}\sigma^z$, i.e., $C^{zz}_{x_\ell,x_m} = \langle s^z_{\ell+1} s^z_{m+1}\rangle - \langle s^z_{\ell+1}\rangle\langle s^z_{m+1}\rangle$, after putting in contact a thermal state at inverse temperature β with the infinite temperature state. In that case we have ($x \neq y$)

$$C^{zz}_{x,y} \xrightarrow{\text{GHD}} -\left[2\pi \cos\bar{\theta} \tanh\left(\frac{\beta\bar{\varepsilon}}{2}\right) K_\alpha(x,y) \sqrt[3]{\frac{2\alpha}{\bar{v}'' t}}\right]^2. \quad (20)$$

In Fig. 1(a) we report data for an "ornate" generalized XY model with nonzero coupling constants also between next-nearest-neighbor spins. The agreement between (20) and the numerical data is excellent. This example is representative of all the models with $\alpha = 1$, for which GHD seems to capture the behavior close to the light cone. In Fig. 1(b) we report data for an XY model for which $\alpha \approx 0.8$. The predictions are only in fair agreement with the numerical data, and we do not see a substantial reduction of the discrepancy when the time is increased. In fact, it seems that the data always approach (17) if α is replaced by 1. This indicates that GHD fails whenever $\alpha \neq 1$.

Summary and discussion. We have derived a Moyal dynamical equation that describes the exact time evolution in noninteracting spin-chain models. Assuming quasistationarity, we developed a higher-order generalized hydrodynamic theory. We identified the neighborhood of the light cone (emerging from the junction of two steady states) as a region where higher-order hydrodynamics could improve on the first-order theory. In the XX model and in the transverse-field Ising chain, our expectations are met. In more general systems, we report a discrepancy every time a particular condition on the velocity of the quasiparticle excitations is not satisfied. Our key finding is that higher-order generalized hydrodynamics does not generally describe the system better than the first-order theory. This null result undermines the validity of the quasistationarity assumption beyond the regime identified in Refs. [48,49]. It does not rule out, however, the possibility of a second-order hydrodynamic theory which is different from the first-order one only in the presence of interactions.

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