Power-law liquid in cuprate superconductors from fermionic unparticles

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Recent photoemission spectroscopy measurements (T. J. Reber *et al.*, arXiv:1509.01611) on cuprate superconductors have inferred that over a wide range of doping, the imaginary part of the electron self-energy scales as $\Sigma'' \sim (\omega^2 + \pi^2 T^2)^a$ with $a = 1$ in the overdoped Fermi-liquid state and $a < 0.5$ in the optimal to underdoped regime. We show that this non-Fermi-liquid scaling behavior can naturally be explained by the presence of a scale-invariant state of matter known as unparticles. We evaluate analytically the electron self-energy due to interactions with fermionic unparticles. We find that, in agreement with experiments, the imaginary part of the self-energy scales with respect to temperature and energy as $\Sigma'' \sim T^{2+2\alpha}$ and $\omega^{2+2\alpha}$, where α is the anomalous dimension of the unparticle propagator. In addition, the calculated occupancy and susceptibility of fermionic unparticles, unlike those of normal fermions, have significant spectral weights even at high energies. This unconventional behavior is attributed to the branch cut in the unparticle propagator which broadens the unparticle spectral function over a wide energy range and nontrivially alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) α. Our work presents evidence suggesting that unparticles might be important low-energy degrees of freedom in strongly coupled systems such as the cuprate superconductors.

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I. INTRODUCTION

Understanding the physics of cuprate superconductors involves identifying the low-energy degrees of freedom that can reproduce the bizarre features of the normal state which traditionally include T-linear resistivity, pseudogap, Fermi arcs, etc. Adding to the complexity are the recent angle-resolved photoemission spectroscopy (ARPES) measurements [1] of the cuprates that revealed that its well-known T-linear resistivity can be construed as a slice of a unified power-law scaling behavior. Over a wide range of doping levels, the measured scattering rates in the nonsuperconducting state scale with respect to temperature and frequency as $\Sigma'' \sim (\omega^2 + \pi^2 T^2)^a$, with only the scaling exponent a varying with doping. The power law smoothly varies from Fermi-liquid-like at overdoping, to one with $a \sim 0.5$ representing T-linear scattering rate at optimal doping, and to $a \lesssim 0.5$ at underdoping. Such a non-Fermi-liquid state of matter is dubbed a *power-law liquid*.

Theoretically, mechanisms yielding similar non-Fermi-liquid scalings have been extensively studied [2–11]. In a marginal Fermi liquid [2], a polarizability proportional to ω/T leads to T-linear resistivity, while a d-wave Pomeranchuk instability in two dimensions [3] yields self-energies with $\omega^{2/3}$ and $T^{2/3}$ dependence. In addition, similar behaviors can also be obtained by coupling quasiparticles with gauge bosons [4], Goldstone bosons [5], and critical bosons [6] near a quantum critical point [7]. Furthermore, strong coupling theories using the anti-de Sitter spacetime (AdS)/conformal field theory (CFT) correspondence [8] and Gutzwiller projection in hidden Fermi-liquid theory [9] also exhibit T-linear resistivity. In particular, the spectral functions calculated within the AdS/CFT formalism can also exhibit a range of power-law scaling when the scaling dimension of the boundary fermionic operator is tuned continuously [10,11].

Because of the recent unified scaling observations, it is natural to invoke a scale-invariant sector such as unparticles as the effective low-energy degrees of freedom in the cuprates. Proposed a decade ago as a scale-invariant sector within the standard model [12], unparticles can emerge in strong coupling theories as low-energy degrees of freedom. Exhibiting features similar to those of a fractional number of invisible massless particles [12], unparticles are an incoherent state of matter that lack any particlelike behavior. They can be construed as a product of states with a continuous distribution of masses [13–15] and can be constructed from theories in AdS [16].

While extensively studied in high-energy physics, unparticles remain relatively new in condensed matter physics. In the context of the cuprates, unparticles have been proposed to explain the absence of Luttinger's theorem in the pseudogap phase [17] using zeros in the Green function [18] and have also been found to yield unusual superconducting properties [17,19,20] and optical conductivity [21].

Unparticles can arise in the cuprates because any nontrivial infrared dynamics in a strongly correlated electron system is controlled by a critical fixed point. Consequently, scale invariance can be used to construct the form of the underlying propagator. This propagator which can acquire an anomalous dimension within the renormalization group approach is the unparticle propagator. Furthermore, in the context of AdS/CFT, one of us [22,23] showed that a massive scalar field in the bulk is generally dual to a nonlocal operator (i.e., a fractional Laplacian) on the boundary. The propagator of these operators is of a power-law form, just like the unparticle propagator. These results indicate that unparticles should generically exist in a strongly coupled system.

In the context of the Hubbard model near half-filling, dynamical spectral weight transfer [24] has long been observed to occur. The key implication is that the number of low-energy degrees of freedom exceeds the number of electrons the lower band can hold. Hence, the low-energy physics is not delineated by counting electrons alone. Such anomalous physics disappears in the overdoped regime. If the critical physics near optimal doping is due to the apparent non-Fermi-liquid behavior in the underdoped regime, then it is natural to suggest that the nonelectronlike degrees of freedom in the underdoped regime arise from a scale-invariant sector. Consequently, unparticle propagators are a natural starting point for describing such physics.

In this paper we show analytically that interactions between electrons and fermionic unparticles can reproduce the power-law liquid revealed in the cuprates by recent ARPES experiments [1]. This paper is a follow-up to our recent paper that focused on bosonic unparticles [25]. Here we find that, in agreement with the experiments, the electron self-energy due to interactions with fermionic unparticles exhibits power-law scaling with respect to both energy and temperature: $\Sigma'' \sim \omega^{2+2\alpha}$ and $T^{2+2\alpha}$, where α is the anomalous scaling of the unparticle propagator. In addition, we find that the occupancy number and susceptibility of fermionic unparticles, unlike those of normal fermions, have significant spectral weights even at high energies. These unconventional behaviors can be attributed to the branch cut in the unparticle propagator which broadens the unparticle spectral function over a wide energy range, and nontrivially alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) α.

II. ELECTRON-FERMIONIC UNPARTICLE SCATTERING

A. Model

We consider a system of electrons in the presence of a background of fermionic unparticles. The action of the system in Matsubara-Fourier space is given by

$$S = T\sum_n\sum_p \psi_n^\dagger(p)G_0^{-1}(p,i\omega_n)\psi_n(p) + T\sum_n\sum_p \phi_n^\dagger(p)G_\alpha^{-1}(p,i\omega_n)\phi_n(p) + UT^3\sum_{m,n,l}\sum_{k,p,q}\psi_{m-l}^\dagger(k-q)\phi_{n+l}^\dagger(p+q)\phi_n(p)\psi_m(k), \tag{1}$$

where ψ is the nonrelativistic electron field, ϕ is the fermionic unparticle field, G_0 is the bare electron Green function

$$G_0(p,i\omega_n) = \frac{1}{i\omega_n - E_p}, \tag{2}$$

and G_α is the fermionic unparticle Green function

$$G_\alpha(k,i\omega_n) = \frac{1}{(i\omega_n - \epsilon_k + \mu)^{1-\alpha}}. \tag{3}$$

Here ϵ_k is the unparticle energy spectrum, $1-\alpha$ is the scaling exponent, and μ is the chemical potential. When $\alpha = 0$, the Green function reduces to that of a normal particle. In addition, U is the interaction between electrons and unparticles, and T is the temperature. The subscripts of the fields denote the dependence on the Matsubara frequency. In this model, the fermionic unparticles are assumed to exist up to a UV momentum cutoff Λ because they represent a low-energy description of some microscopic theory. For the unparticle Green function to be scale invariant, we set $\mu = 0$ when

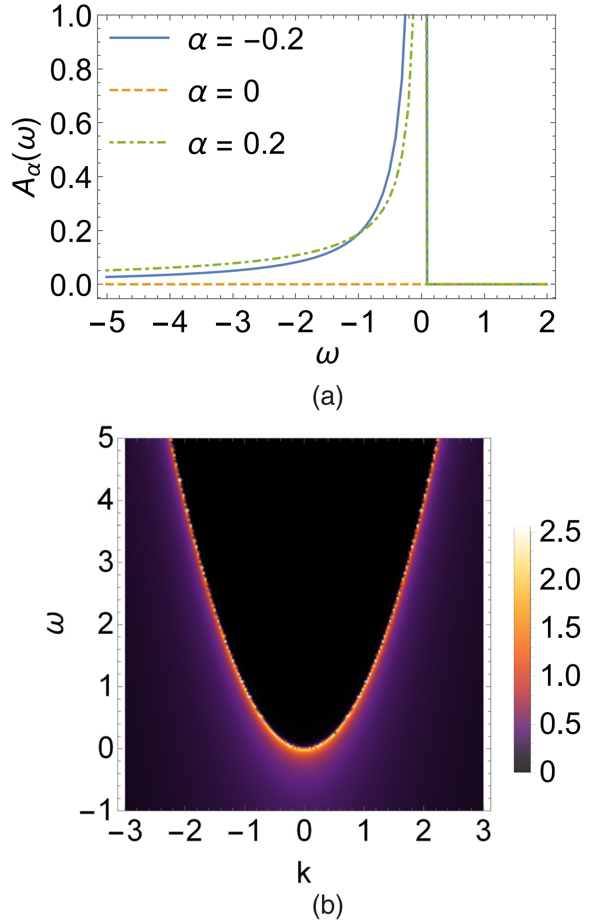

FIG. 1. (a) The spectral function $A_\alpha(\omega)$ of unparticles compared to that of particles. As α deviates from zero, the delta peak in the spectral function broadens due to the branch cut in the Green function. (b) The energy and momentum dependence of the unparticle spectral function for a quadratic energy spectrum $\epsilon_k \sim k^2$ with $\alpha = 0.5$. The broadening of the spectral function reflects the incoherent nature of unparticles.

$\alpha \neq 0$. While the literature in high-energy physics considers fermionic unparticles as relativistic four spinors within the standard model [26,27], here in the context of the cuprates, we consider them as nonrelativistic fermions. For simplicity, we also omit the normalization factor and the effects of spins.

In this paper we focus on unparticles with $-1 < \alpha < 1$. In this case, instead of a simple pole, the unparticle Green function has a branch cut, which we choose to be along the negative energy axis. That is, the branch cut of $z^{1-\alpha}$ is chosen to be along $-\infty < z < 0$ with the phase angle defined in the range $-\pi < \theta < \pi$. Figure 1 shows that, compared to particles, the spectral function of unparticles

$$A_\alpha(k,\omega) \equiv -\frac{1}{\pi}\text{Im}G_\alpha(k,\omega + i\eta) = \frac{1}{\pi}|\sin(\pi\alpha)|\frac{\theta(\epsilon_k - \omega)}{|\epsilon_k - \omega|^{1-\alpha}} \tag{4}$$

remains divergent at $\omega = \epsilon_k$, but has a broadened peak due to the presence of the branch cut, representing the incoherence of unparticles. Here $\theta(x)$ is the Heaviside step function. It is

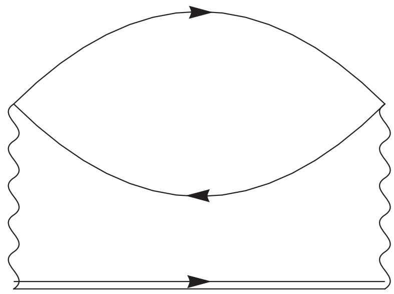

FIG. 2. The lowest-order Feynman diagram of the electron self-energy due to interactions between electrons and fermionic unparticles. The solid lines, double line, and wavy lines correspond to fermionic unparticles, electron, and the electron-unparticle interaction, respectively.

precisely the modeling of the broad incoherent background in the electron spectral function that unparticles are tailored to handle.

For the unparticle spectral function to satisfy the usual sum rule, a high energy cutoff is implicitly assumed when $\alpha > 0$. Similarly, the IR divergence when $\alpha < 0$ is regularized by an IR cutoff η, where a convenient choice for the spectral function is $A_\alpha(k,\omega) = \frac{1}{\pi}|\sin(\pi\alpha)|\frac{\theta(\epsilon_k-\omega)}{|\epsilon_k-\omega-i\eta|^{1-\alpha}}$. These cutoffs naturally arise from the fact that unparticles are effective degrees of freedom of some strongly interacting theories and so are scale invariant only within a certain energy range. These cutoffs also ensure the convergence of other observables, such as the susceptibility.

Since the IR/UV cutoffs are free parameters in our model, they can in principle be made sufficiently small/large for experimental agreement. However, what ultimately sets the cutoffs cannot be an extrinsic scale. In the cuprates, phonons with energies around 10 meV [28] are intrinsically present. Since our model does not take into account these phonons, they naturally set a low-energy scale η at which the power-law scaling breaks down. This scale is well below the energy of 0.1 eV up to which the ARPES experiments are measured [1] and therefore is low enough to explain the experiments.

B. Electron self-energy

For a constant interaction U between electrons and fermionic unparticles, Fig. 2 illustrates the lowest-order contribution to the electron self-energy $\Sigma(k,i\omega_n)$ within a perturbative approach. This can be written as

$$\Sigma(k,i\omega_n) = -U^2\sum_q T\sum_{i\omega_m} G_0(k-q,i\omega_n-i\omega_m) \times \chi_\alpha(q,i\omega_m), \tag{5}$$

where

$$\chi_\alpha(q,i\omega_m) = \sum_p T\sum_{i\omega_n} G_\alpha(p,i\omega_n)G_\alpha(p-q,i\omega_n-i\omega_m) \tag{6}$$

is the unparticle susceptibility, and $G_0(p,i\omega_m)$ is the electron Green function. While unparticle-particle interactions in the standard model are constrained by experiments to be weak [12], the coupling strength U here in the cuprates can be significant.

Appendix A details our analytic evaluation of the Matsubara sums in Eqs. (5) and (6) using standard contour integration techniques. After analytic continuation $i\omega_n \to \omega + i\eta$, we write the imaginary part of the electron self-energy in the standard form

$$\Sigma''(k,\omega) = -U^2\sum_p \chi''_\alpha(k-p,\omega-E_p) \times [n_B(\omega-E_p)+n_F(-E_p)], \tag{7}$$

where

$$\chi''_\alpha(q,\omega) = \pi\int_{-\infty}^{\infty} dz[n_F(z)-n_F(z-\omega)] \times \sum_p A_\alpha(p,z)A_\alpha(p-q,z-\omega). \tag{8}$$

Here $n_{F/B}(z) = (e^{z/T}\pm 1)^{-1}$ is the Fermi (Bose) distribution, and E_k is the electron energy spectrum.

To understand how the electron self-energy depends on the anomalous dimension α, it is insightful to consider the scattering phase space. This phase space is governed by the function $\tilde{S}''_\alpha$ given by

$$\tilde{S}''_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,\omega) = [n_B(\omega-\epsilon_3)+n_F(-\epsilon_3)][\bar{\kappa}_\alpha(\epsilon_2,\epsilon_1-\omega+\epsilon_3) - \bar{\kappa}_\alpha(\epsilon_1,\epsilon_2+\omega-\epsilon_3)], \tag{9}$$

$$\bar{\kappa}_\alpha(\epsilon,\epsilon') = \pi\int_{-\infty}^{\infty} dz\, n_F(z)A_\alpha(z-\epsilon)A_\alpha(z-\epsilon'), \tag{10}$$

such that

$$\Sigma''(k,\omega) = -U^2\sum_{pq}\tilde{S}''_\alpha(\epsilon_p,\epsilon_{p-q},E_{k-q},\omega). \tag{11}$$

This function $\tilde{S}''_\alpha$ describes the amount of scattering at different energies.

To elucidate the analytic structure of $\tilde{S}''_\alpha$, we note that, for $\alpha > 0$ in the $T \to 0$ limit, the integral $\bar{\kappa}$ evaluates to the closed-form expression

$$\bar{\kappa}(\epsilon,\epsilon') = \frac{1}{1-2\alpha}\frac{1}{\pi}\sin^2(\pi\alpha)\left[\frac{2}{\xi(\epsilon,\epsilon')}\right]^{1-2\alpha} {}_2F_1\left[1-\alpha,\frac{1}{2}-\alpha;\frac{3}{2}-\alpha;\left|\frac{\epsilon-\epsilon'}{\xi(\epsilon,\epsilon')}\right|^2\right], \tag{12}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and $\xi(\epsilon,\epsilon') = \max(|\epsilon-\epsilon'|,\epsilon+\epsilon')$. As in the Fermi-liquid case, $\alpha = 0$,

$$\tilde{S}''_{\text{FL}}(\epsilon_1,\epsilon_2,\epsilon_3,\omega) = \pi\delta(\epsilon_1-\epsilon_2-\omega+\epsilon_3)[\theta(-\epsilon_1)\theta(-\epsilon_3)\theta(\epsilon_2) + \theta(\epsilon_1)\theta(\epsilon_3)\theta(-\epsilon_2)], \tag{13}$$

we find that the analogous expression for unparticles $\tilde{S}''_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,\omega)$ diverges when $\epsilon_1-\epsilon_2+\epsilon_3-\omega = 0$ and $\epsilon_1\epsilon_2 < 0$. However, given that the unparticle chemical potential $\mu = 0$, this divergence does not occur because ϵ_1,ϵ_2 are nonnegative. In addition, unlike the Fermi-liquid result, $\tilde{S}''_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,\omega)$ can be nonzero for other values of energies $\epsilon_1,\epsilon_2,\epsilon_3,\omega$. These features are illustrated in Fig. 3. These nonzero values provide additional contributions to the electron

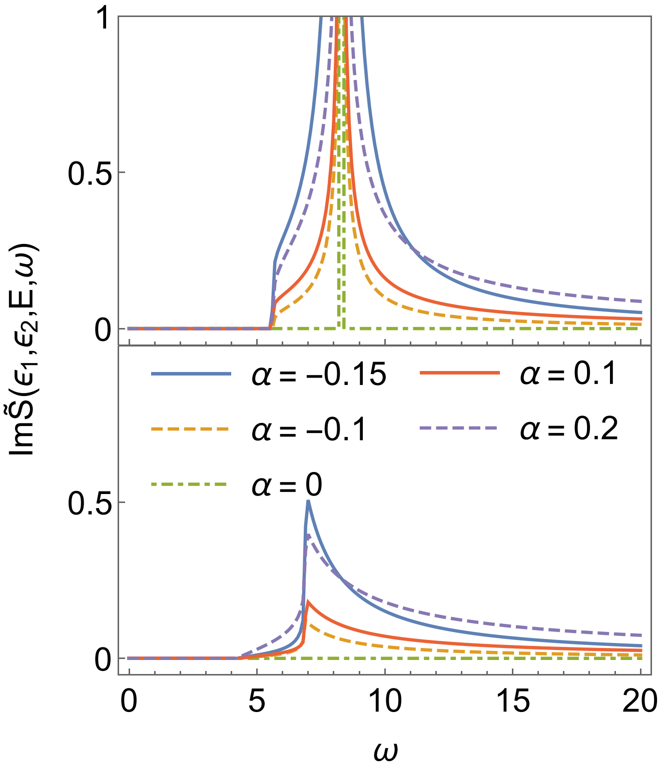

FIG. 3. Top: Plot of the self-energy Matsubara sum $\tilde{S}''_\alpha(\epsilon_1,\epsilon_2,E,\omega)$ at $T = 0.01$ for $\epsilon_1 = 2.7$, $\epsilon_2 = -1.4$, and $E = 4.2$. Bottom: Same plot but with $\epsilon_2 = 1.4$. Compared to the Fermi-liquid result, the unparticle one has additional nontrivial contributions.

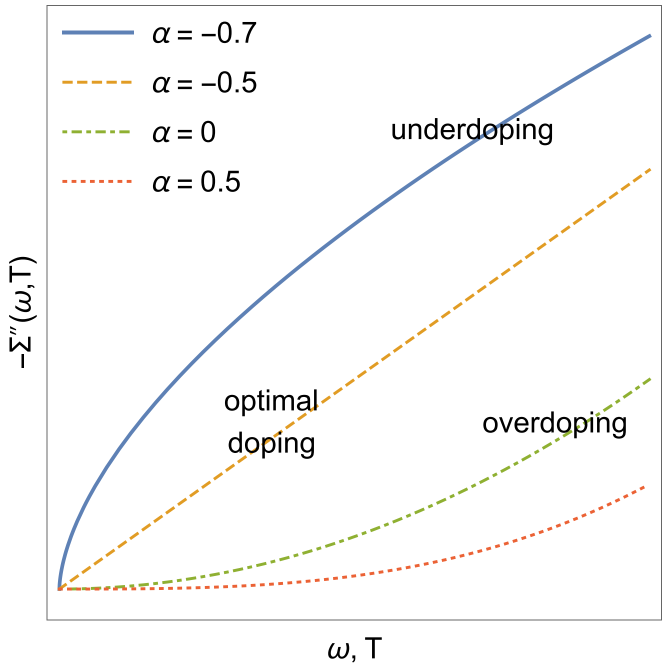

FIG. 4. Schematic of the energy and temperature dependence of the electron self-energy, showing deviations from Fermi-liquid theory. In the cuprates, unparticles with $\alpha \lesssim 0$, $\alpha \approx -0.5$, and $\alpha < -0.5$ correspond to overdoping, optimal doping, and underdoping, respectively.

self-energy, and can be attributed to the broadening of the unparticle spectral function illustrated in Fig. 1.

Next, to determine the scaling form of the electron self-energy in the $T \to 0$ limit, we note that the unparticle spectral function scales as

$$\begin{aligned} A_\alpha(\lambda\omega) &= -\frac{1}{\pi}\lim_{\eta\to 0}\mathrm{Im}G_\alpha(\lambda\omega + i\eta) \\ &= -\frac{1}{\pi}\lambda^{-1+\alpha}\lim_{\eta\to 0}\mathrm{Im}G_\alpha(\omega + i\eta) \\ &= \lambda^{-1+\alpha}A_\alpha(\omega). \end{aligned} \tag{14}$$

Consequently, we have

$$\bar{\kappa}_\alpha(\lambda\epsilon,\lambda\epsilon') = \lambda^{-1+2\alpha}\bar{\kappa}_\alpha(\epsilon,\epsilon'), \tag{15}$$

$$\tilde{S}''_\alpha(\lambda\epsilon_1\lambda\epsilon_2,\lambda\epsilon_3,\lambda\omega) = \lambda^{-1+2\alpha}\tilde{S}''_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,\omega). \tag{16}$$

Then, approximating the density of states to be constant near the Fermi level, we find that the imaginary part of the electron self-energy in the $T \to 0$ limit becomes

$$\begin{aligned} \Sigma''(k,\omega) &= -U^2 \sum_{p_1p_2p_3}\delta_{p_1+p_3,p_2+k}\tilde{S}''_\alpha(\epsilon_{p_1},\epsilon_{p_2},E_{p_3},\omega) \\ &\sim -U^2\int d\epsilon_1 d\epsilon_2 dE\ \tilde{S}''_\alpha(\epsilon_1,\epsilon_2,E,\omega), \end{aligned} \tag{17}$$

which scales with respect to energy ω as

$$\Sigma''(k,\lambda\omega) = \lambda^{2+2\alpha}\Sigma''(k,\omega). \tag{18}$$

Therefore, the electron self-energy due to electron-unparticle interactions behaves as $\Sigma'' \sim \omega^{2+2\alpha}$ at low temperatures, deviating from the Fermi-liquid behavior of $\Sigma''_{\mathrm{FL}} \sim \omega^2$. In the $\omega \to 0$ limit, a similar argument shows that $\Sigma'' \sim T^{2+2\alpha}$ at low energies. Summarized in Fig. 4, these scaling behaviors of the electron self-energy are our main result; they hold for $-1 < \alpha < 1$, and do not depend on the specific form of the electron energy spectrum E_k. For $\alpha \lesssim 0$, this non-Fermi-liquid state of matter quantitatively corresponds to the power-law liquid revealed in the cuprates by the recent ARPES measurements [1].

C. Susceptibility

The scaling behavior of the electron self-energy can be traced back to the unparticle susceptibility χ_α given by Eq. (8). Figure 5 illustrates the unparticle susceptibility in the $q \to 0$ and $T \to 0$ limit for a quadratic energy spectrum $\epsilon_k \sim k^2$ in two dimensions. We note three features distinctive from the analogous free electron susceptibility. First, the unparticle susceptibility is nonzero despite the chemical potential being restricted to be zero on account of scale invariance. This is unlike normal particles for which a zero chemical potential necessarily implies that there is zero filling and hence zero susceptibility. Second, the unparticle susceptibility does not have a cutoff at high energies. Third, from

$$\begin{aligned} \chi''_\alpha(q=0,\omega) \propto &\int dz[\theta(-z) - \theta(\omega - z)] \\ &\times \int d\epsilon A_\alpha(z-\epsilon)A_\alpha(z-\omega-\epsilon), \end{aligned} \tag{19}$$

we see that the susceptibility scales as $\chi''(0,\omega) \sim \omega^{2\alpha}$. Such a scaling form ensures that when $\alpha < 0$ ($\alpha > 0$), the susceptibility is enhanced (suppressed) at low energies, as shown

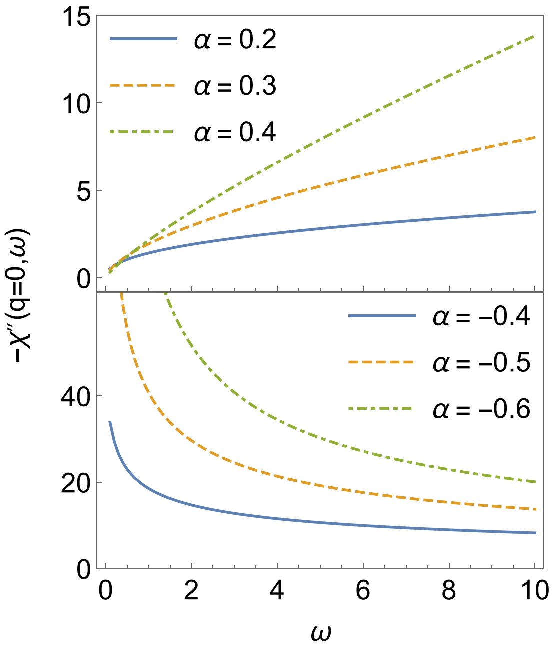

FIG. 5. The energy dependence of the unparticle susceptibility χ''_α for a quadratic energy spectrum $\epsilon_k \sim p^2$ in the $T \to 0$ and $q \to 0$ limit, for various values of α. The scaling behavior $\chi''_\alpha \sim \omega^{2\alpha}$ is associated with the scaling of the electron self-energy depicted in Fig. 4. Note that these plots are only qualitatively accurate due to issues with numerical stability.

in Fig. 5. Such an enhancement (suppression) is crucial for the increased (decreased) scattering rate, as quantified by the electron self-energy in the previous subsection. These features completely violate the usual susceptibility sum rule and can be attributed to the broadening of the unparticle spectral function. As $|\alpha|$ decreases, the features become less pronounced, as expected.

Similar non-Fermi-liquid behavior induced by the enhancement of low energy susceptibility also occurs, for example, in systems where large portions of the Fermi surface are nested with a single nesting wave vector [29,30], and in multiband models with orbital fluctuations [31]. Additionally, the self-energy of a Fermi liquid in the presence of weak impurities has an imaginary part of the form $\Sigma'' \sim (E - E_f)^{d/2}$, where d is the spatial dimension [32]. Non-Fermi-liquid behavior in this case can also be understood as an enhancement in the low energy spectrum of the susceptibility [32].

When $\alpha < 0$, the scaling behavior $\chi''_\alpha(q = 0,\omega) \sim \omega^{2\alpha}$ may seem to suggest a divergence as $\omega \to 0$. However, when the IR divergence of the spectral function is regularized as described above, the susceptibility in fact remains finite and continuous. What happens is that the scaling behavior is true only for ω larger than some energy scale dependent on the IR cutoff.

D. Occupancy

In Fermi-liquid theory, the quadratic scaling of the electron self-energy follows from a phase-space argument involving the occupancy of electrons. Therefore, it can be illuminating to explore how this argument is modified in the case of unparticles

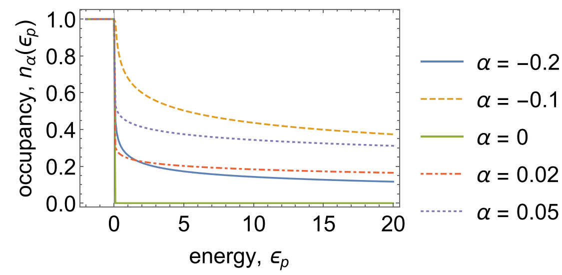

FIG. 6. The energy ϵ_p dependence of the unparticle occupancy at $T = 0$. The significant occupancy at large ϵ_p differs from the Fermi distribution.

by computing the occupancy for unparticles,

$$n_\alpha(\epsilon_p) = \int_{-\infty}^{\infty} dz\, n_F(z) A_\alpha(z - \epsilon_p) e^{z0^+}. \tag{20}$$

Figure 6 shows that in the $T \to 0$ limit, unlike the Fermi distribution for particles, the occupancy of unparticles is significant even when ϵ_p is large. This counterintuitive result can be understood by noting that the occupancy number measures the filling of states at momentum p, instead of at energy ϵ_p. This distinction is important because, unlike the particle case, the unparticle spectral function is broadened over a wide energy range. Consequently, even unparticles with a large ϵ_p possess a significant amount of low energy states that are filled at low temperatures. For $\alpha < 0$, these states enlarge the scattering phase space in the electron self-energy by enhancing the low energy susceptibility bubble, resulting in the non-Fermi-liquid behavior described in the preceding section. In addition, the occupancy is notably nonsymmetric, reflecting the particle-hole asymmetry of the unparticle Green function. This enhancement of phase space undoubtedly reflects the enhanced scattering rate that ultimately grows linearly with temperature as opposed to the standard T^2 in the Fermi-liquid case.

III. DISCUSSION AND CONCLUSIONS

While our model likely exhibits conventional Fermi-liquid behavior at energy scales below the IR cutoff, what happens in that regime does not detract from the main point of our paper. That is, the electron self-energy above the IR cutoff exhibits the power-law scaling observed experimentally. In fact, the ARPES measurements [1] also possess an inherent cutoff due to a limited energy resolution of 4 meV [33]. Consequently, the behavior of the cuprates at lower energies remains unclear.

As discussed in Ref. [1], a sublinear scaling of the electron self-energy can be interpreted as having a vanishing quasiparticle residue Z in Fermi-liquid theory. This signifies that interactions with fermionic unparticles with $\alpha < -0.5$ cause electrons to behave completely incoherently, which is unsurprising given the nature of unparticles. Nevertheless, since $\Sigma''(\omega = 0, T = 0) = 0$, the Fermi surface remains sharp [34].

We can similarly calculate the self-energy of unparticles due to self-interactions, that is, when the electron line in Fig. 2 is replaced by another unparticle line. Naively, we expect the self-energy to scale as $\Sigma'' \sim \omega^{2+3\alpha}$ and $T^{2+3\alpha}$. This result, as well as the susceptibility and occupancy calculated above, can in principle be observed experimentally. However, any meaningful comparison with experimental observations would require further knowledge about the form of couplings between unparticles and external fields.

While the unparticle approach may resemble Anderson's proposal of a 2D Luttinger liquid [35], the two models differ for two main reasons. First, Anderson's model acquires anomalous properties because the current-carrying degrees of freedom themselves are scale-invariant objects. In the present unparticle picture, the current-carrying degrees of freedom are still electrons; their unusual properties arise from scattering off scale-invariant unparticles. Second, in Anderson's model, the extension of Luttinger liquid to two dimensions lacks a rigorous basis: bosonization by transforming fermions to particle-hole excitations requires the particle-hole pairs to be long lived, which is guaranteed only in 1D. On the other hand, as is well known in conformal field theory, the unparticle idea is completely general regardless of spatial dimension; the construction of unparticles depends only on symmetry considerations. In fact, as shown previously by one of us [17], scale-invariant matter constructed using gauge/gravity duality has its anomalous dimension a function of spatial dimension and a mass of the bulk scalar field.

While dimensional considerations may suggest the self-energy's power-law scaling, they are insufficient as a proof due to technical complications. Since the unparticle propagator has branch cuts instead of poles, the convergence of the self-energy contour integral requires a careful choice of contour. In fact, the convergence of certain components of the integral depends on α, which is not obvious from the formal expression of the self-energy. Moreover, an explicit evaluation elucidates the way the scattering phase space is altered by an enhancement of the imaginary part of the low-energy susceptibility. Obtaining this simple physical interpretation would have been impossible from trivial power counting.

For the perturbative approach we have adopted to be meaningful, two conditions need to be satisfied. First, the contributions at each order of perturbation are finite. Second, the perturbation series converge. For the first condition, Fig. 3(b) and Eq. (12) show that the Matsubara sum $\tilde{S}_\alpha$ in the self-energy converges for all energies when $\alpha < \frac{1}{2}$ at least in the limit $T \to 0$. Consequently, the self-energy is finite, as required. For the second condition to be satisfied, one can consider a model in which both fermionic unparticles and electrons satisfy an SU(N) gauge group. For an unparticle-electron interaction given by U/N, the effective electron-electron interaction (the fermionic unparticle pair bubble) is of the order U^2/N. One can then follow the same analysis of Ref. [36] which outlines the details of the $1/N$ expansion. Nevertheless, for simplicity, we just assume that the electron interaction with the unparticle sector is small.

Our recent paper [25] studied the effects of bosonic unparticles on the electron self-energy. While similar scaling behaviors were obtained, there are a few subtle differences. First, while a unitarity bound constrains the scaling dimension of bosonic unparticles, we do not know of any such constraint for fermionic unparticles. This freedom allows for a more qualitative agreement with experiments. Second, unlike the results in the bosonic case, there is no dependence on the dimensionality in the scaling of Σ''. This state of affairs is obtained because we approximate the density of states of both electrons and fermionic unparticles to be constant near the Fermi level. Third, our susceptibility plots in Fig. 5 differ from that in Ref. [25], because a nonzero chemical potential was previously adopted.

In conclusion, we showed analytically that interactions between electrons and fermionic unparticles—a scale-invariant state of matter—can produce the power-law liquid revealed in the cuprates by recent ARPES experiments [1]. In particular, we found that, at low temperatures and energies, the electron self-energy due to interactions with fermionic unparticles exhibits power-law scaling with respect to energy and temperature: $\Sigma'' \sim \omega^{2+2\alpha}$ and $T^{2+2\alpha}$, where α is the anomalous scaling of the unparticle propagator. This non-Fermi-liquid behavior can be attributed to the broadening of the unparticle spectral function over a wide energy range, which drastically alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) α. Although unparticles have zero chemical potential as required by scale invariance, they nevertheless can contribute to the electron self-energy due to the same broadening. Our results present evidence suggesting that unparticles might be important low-energy degrees of freedom in the cuprates, and should inspire the interpretation of other experimental data using unparticles.

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APPENDIX: ANALYTIC EVALUATION OF MATSUBARA SUMS

1. Susceptibility

The unparticle susceptibility defined by Eq. (6) involves the fermionic Matsubara sum

$$S_\alpha(\epsilon_1,\epsilon_2,i\omega_n) = T\sum_{i\omega_m} G_\alpha(i\omega_m - \epsilon_1)G_\alpha(i\omega_m - i\omega_n - \epsilon_2),$$

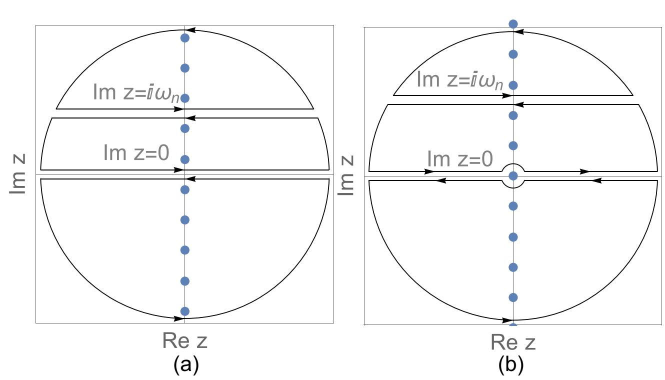

FIG. 7. The contours used to evaluate the Matsubara sums in (a) the unparticle susceptibility and (b) the electron self-energy.

where $i\omega_n$ is a bosonic Matsubara frequency. Using Cauchy's residue theorem, we rewrite the Matsubara sum as

$$S_\alpha(\epsilon_1,\epsilon_2,i\omega_n) = -\frac{1}{2\pi i}\oint_C dz\, n_F(z)G_\alpha(z-\epsilon_1)G_\alpha(z-i\omega_n-\epsilon_2),$$

where $n_F(z) = (e^{z/T}+1)^{-1}$ is the Fermi distribution. Since the integrand is analytic except along $\mathrm{Im}z = 0$ and $\mathrm{Im}z = i\omega_n$, we use the contour C illustrated in Fig. 7(a).

The integrals along the paths at large radius vanish when $\alpha < 1/2$. For $\alpha \geqslant 1/2$, a convergence factor e^{z0^+} can be included so that the same integrals vanish. Consequently, the nonvanishing contributions to the contour integral are those along the branch cuts:

$$\begin{aligned}
I_{b1}(\epsilon_1,\epsilon_2,i\omega_n) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\, n_F(z)[G_\alpha(z^+-\epsilon_1) - G_\alpha(z^- -\epsilon_1)]G_\alpha(z-i\omega_n-\epsilon_2)\\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\, n_F(z)2i\mathrm{Im}[G_\alpha(z^+-\epsilon_1)]G_\alpha(z-i\omega_n-\epsilon_2)\\
&= \int_{-\infty}^{\infty} dz n_F(z)A_\alpha(z-\epsilon_1)G_\alpha(z-i\omega_n-\epsilon_2),\\
I_{b2}(\epsilon_1,\epsilon_2,i\omega_n) &= -\frac{1}{2\pi i}\int_{-\infty+i\omega_n}^{\infty+i\omega_n} dz\, n_F(z)G_\alpha(z-\epsilon_1)[G_\alpha(z^+-i\omega_n-\epsilon_2) - G_\alpha(z^- -i\omega_n-\epsilon_2)]\\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\, n_F(z+i\omega_n)G_\alpha(z+i\omega_n-\epsilon_1)[G_\alpha(z^+-\epsilon_2) - G_\alpha(z^- -\epsilon_2)]\\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\, n_F(z)G_\alpha(z+i\omega_n-\epsilon_1)2i\mathrm{Im}G_\alpha(z^+-\epsilon_2)\\
&= \int_{-\infty}^{\infty} dz\, n_F(z)G_\alpha(z+i\omega_n-\epsilon_1)A_\alpha(z-\epsilon_2).
\end{aligned}$$

Here $z^\pm = z \pm i\eta$, with $\eta = 0^+$ being a positive real infinitesimal. After analytic continuation $i\omega_n \to \omega + i\eta$, the imaginary part of the Matsubara sum becomes

$$\begin{aligned}
\mathrm{Im}S_\alpha(\epsilon_1,\epsilon_2,\omega+i\eta) &= \int_{-\infty}^{\infty} dz\, n_F(z)[A_\alpha(z-\epsilon_1)\mathrm{Im}G_\alpha(z-\omega-i\eta-\epsilon_2) + \mathrm{Im}G_\alpha(z+\omega+i\eta-\epsilon_1)A_\alpha(z-\epsilon_2)]\\
&= \int_{-\infty}^{\infty} dz\, n_F(z)[A_\alpha(z-\epsilon_1)\pi A_\alpha(z-\omega-\epsilon_2) - \pi A_\alpha(z+\omega-\epsilon_1)A_\alpha(z-\epsilon_2)]\\
&= \pi\int_{-\infty}^{\infty} dz\, n_F(z)A_\alpha(z-\epsilon_1)A_\alpha(z-\omega-\epsilon_2) - (\epsilon_1 \leftrightarrow \epsilon_2, \omega \to -\omega)\\
&\equiv \bar{\kappa}_\alpha(\epsilon_1,\epsilon_2+\omega) - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1-\omega),
\end{aligned}$$

where we have defined

$$\bar{\kappa}_\alpha(\epsilon,\epsilon') = \pi\int_{-\infty}^{\infty} dz\, n_F(z)A_\alpha(z-\epsilon)A_\alpha(z-\epsilon').$$

For $\alpha > 0$, we can evaluate this exactly in the $T \to 0$ limit using the unparticle spectral function in Eq. (4):

$$\begin{aligned}
\bar{\kappa}_\alpha(\epsilon,\epsilon') &= \frac{1}{\pi}\sin^2(\pi\alpha)\int_{-\infty}^{\min(\epsilon,0,\epsilon')} dz\frac{1}{(\epsilon-z)^{1-\alpha}}\frac{1}{(\epsilon'-z)^{1-\alpha}}\\
&= \frac{1}{\pi}\sin^2(\pi\alpha)2^{1-2\alpha}\int_{\xi(\epsilon,\epsilon')}^{\infty}\frac{dz}{[z^2-(\epsilon-\epsilon')^2]^{1-\alpha}}\\
&= \frac{1}{\pi}\sin^2(\pi\alpha)\frac{1}{2}\left[\frac{2}{\xi(\epsilon,\epsilon')}\right]^{1-2\alpha}\int_0^1 dt\frac{t^{-\frac{1}{2}-\alpha}}{\left[1-t\left|\frac{\epsilon-\epsilon'}{\xi(\epsilon,\epsilon')}\right|^2\right]^{1-\alpha}}\\
&= \frac{1}{\pi}\sin^2(\pi\alpha)\frac{1}{1-2\alpha}\left[\frac{2}{\xi(\epsilon,\epsilon')}\right]^{1-2\alpha}{}_2F_1\left[1-\alpha,\frac{1}{2}-\alpha;\frac{3}{2}-\alpha;\left|\frac{\epsilon-\epsilon'}{\xi(\epsilon,\epsilon')}\right|^2\right],
\end{aligned}$$

where $\xi(\epsilon,\epsilon') = \max(|\epsilon-\epsilon'|,\epsilon+\epsilon')$, and ${}_2F_1(a,b;c,z)$ is the hypergeometric function.

2. Self-energy

The electron self-energy defined by Eq. (5) involves the bosonic Matsubara sum

$$\tilde{S}_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,i\omega_n) = T\sum_{i\omega_{m'}} G_0(i\omega_n - i\omega_{m'} - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,i\omega_{m'})$$
$$= \frac{1}{2\pi i}\oint_{C'} dz' n_B(z')G_0(i\omega_n - z' - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,z') + TG_0(i\omega_n - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,0),$$

where $i\omega_n$ is a fermionic Matsubara frequency, and $n_B(z) = (e^{z/T} - 1)^{-1}$ is the Bose distribution. Since the integrand has a branch cut along $\mathrm{Im}z' = 0$ and a pole on a line $\mathrm{Im}z' = i\omega_n$, we adopt the contour C' shown in Fig. 7(b). If $\alpha < 1$, the integrals along the paths at large radius vanish.

The integrals along the small circle of radius r around the origin require special consideration. Since $S_\alpha(\epsilon_1,\epsilon_2,z')$ is analytic in the upper (lower) half-plane, we see that $S_\alpha(\epsilon_1,\epsilon_2,z') \to S_\alpha(\epsilon_1,\epsilon_2,0)$ in the limit $z' \to 0$ for z' in the same domain. Hence, as the radius $r \to 0$, the integral along the small circle reduces to

$$\frac{1}{2\pi i}\oint_{|z'|=r} dz' n_B(z')G_0(i\omega_n - z' - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,z') \sim \left[\frac{1}{2\pi i}\oint_{|z'|=r} dz' n_B(z')\right]G_0(i\omega_n - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,0)$$
$$= -TG_0(i\omega_n - \epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,0),$$

which exactly cancels the $i\omega_{m'} = 0$ term in $\tilde{S}_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,i\omega_n)$. This cancellation can be physically motivated. First, notice that the imaginary part of the term contains the factor $\delta(\omega - \epsilon_3)$ after analytic continuation. Then, since $\omega = \epsilon_3$ corresponds to no energy transfer between unparticles and electrons, such a term understandably should not contribute to the electron self-energy.

Then, the nonvanishing contributions to $\tilde{S}_\alpha$ are simply the integrals along the lines $\mathrm{Im}z' = 0$ and $\mathrm{Im}z' = i\omega_n$:

$$\tilde{I}_{b1}(\epsilon_1,\epsilon_2,\epsilon_3,i\omega_n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz' n_B(z')G_0(i\omega_n - z',\epsilon_3)[S_\alpha(\epsilon_1,\epsilon_2,z'^+) - S_\alpha(\epsilon_1,\epsilon_2,z'^-)]$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz' n_B(z')G_0(i\omega_n - z',\epsilon_3)2i\,\mathrm{Im}S_\alpha(\epsilon_1,\epsilon_2,z'^+)$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dz' n_B(z')G_0(i\omega_n - z',\epsilon_3)[\bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z') - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z')],$$

$$\tilde{I}_{b2}(\epsilon_1,\epsilon_2,\epsilon_3,i\omega_n) = \frac{1}{2\pi i}\int_{-\infty+i\omega_n}^{\infty+i\omega_n} dz' n_B(z')[G_0(i\omega_n - z'^+,\epsilon_3) - G_0(i\omega_n - z'^-,\epsilon_3)]S_\alpha(\epsilon_1,\epsilon_2,z')$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz' n_B(z' + i\omega_n)[G_0(-z'^+,\epsilon_3) - G_0(-z'^-,\epsilon_3)]S_\alpha(\epsilon_1,\epsilon_2,z' + i\omega_n)$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz'[-n_F(z')]2i\,\mathrm{Im}[G_0(-z'^+,\epsilon_3)]S_\alpha(\epsilon_1,\epsilon_2,z' + i\omega_n)$$
$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz' n_F(z')2i\pi A_0(-z',\epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,z' + i\omega_n)$$
$$= -\int_{-\infty}^{\infty} dz' n_F(z')A_0(-z',\epsilon_3)S_\alpha(\epsilon_1,\epsilon_2,z' + i\omega_n).$$

Here $A_0 = -\frac{1}{\pi}\mathrm{Im}G_0$ is the spectral function of G_0, and we take principal values of the integrals in $\tilde{I}_{b1}$ due to the cancellation mentioned above. Then, analytic continuation $i\omega_n \to \omega + i\eta$ gives

$$\mathrm{Im}\tilde{I}_{b1}(\epsilon_1,\epsilon_2,\epsilon_3,\omega + i\eta) = -\int_{-\infty}^{\infty} dz' n_B(z')A_0(\omega - z',\epsilon_3)[\bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z') - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z')],$$

$$\mathrm{Im}\tilde{I}_{b2}(\epsilon_1,\epsilon_2,\epsilon_3,\omega + i\eta) = -\int_{-\infty}^{\infty} dz' n_F(z')A_0(-z',\epsilon_3)[\bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z' + \omega) - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z' - \omega)]$$
$$= -\int_{-\infty}^{\infty} dz' n_F(z' - \omega)A_0(\omega - z',\epsilon_3)[\bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z') - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z')],$$

$$\mathrm{Im}\tilde{S}(\epsilon_1,\epsilon_2,\epsilon_3,\omega + i\eta) = -\int_{-\infty}^{\infty} dz'[n_B(z') + n_F(z' - \omega)]A_0(\omega - z',\epsilon_3)[\bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z') - \bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z')]$$
$$= \int_{-\infty}^{\infty} dz' A_0(\omega - z',\epsilon_3)[n_B(z') + n_F(z' - \omega)][\bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - z') - \bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + z')].$$

Finally, using $A_0(\omega) = \delta(\omega)$ gives

$$\mathrm{Im}\tilde{S}_\alpha(\epsilon_1,\epsilon_2,\epsilon_3,\omega + i\eta) = [n_B(\omega - \epsilon_3) + n_F(-\epsilon_3)][\bar{\kappa}_\alpha(\epsilon_2,\epsilon_1 - \omega + \epsilon_3) - \bar{\kappa}_\alpha(\epsilon_1,\epsilon_2 + \omega - \epsilon_3)].$$

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