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Dynamics of Anderson localization in disordered wires

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We consider the dynamics of an electron in an infinite disordered metallic wire. We derive exact expressions for the probability of diffusive return to the starting point in a given time. The result is valid for wires with or without time-reversal symmetry and allows for the possibility of topologically protected conducting channels. In the absence of protected channels, Anderson localization leads to a nonzero limiting value of the return probability at long times, which is approached as a negative power of time with an exponent depending on the symmetry class. When topologically protected channels are present (in a wire of either unitary or symplectic symmetry), the probability of return decays to zero at long time as a power law whose exponent depends on the number of protected channels. Technically, we describe the electron dynamics by the one-dimensional supersymmetric nonlinear sigma model. We derive an exact identity that relates any local dynamical correlation function in a disordered wire of unitary, orthogonal, or symplectic symmetry to a certain expectation value in the random matrix ensemble of class AIII, CI, or DIII, respectively. The established exact mapping from a one- to a zero-dimensional sigma model is very general and can be used to compute any local observable in a disordered wire.

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Introduction. Quantum interference leads to localization of electrons in the presence of disorder. In one- (1D) and two-dimensional (2D) systems, even weak random potential localizes all eigenstates, while in three dimensions (3D) localization occurs when disorder is stronger than a certain threshold level [1–3]. In the past few years, the phenomenon of Anderson localization has witnessed a revival of activity due to discoveries made in several fields. On the experiment side, Anderson localization has been observed in a multitude of systems including cold atoms [4–6], light waves [7], ultrasound [8], as well as optically driven atomic systems [9]. On the theory side, dynamical phenomena such as thermalization and relaxation after a quantum quench in disordered systems have been the subject of growing interest [10–14]. This has been inspired, in part, by the discovery of many-body localization [15–19], which is an interacting analog of Anderson localization, and more recently by the proposal to diagnose quantum chaotic behavior by means of out-of-time-order correlations [20–24]. Furthermore, the discovery [25–31] and complete classification [32–36] of topological insulators has opened the door to a new arena where the interplay between disorder and topology leads to unusual localization-related effects. These include ultraslow (Sinai) diffusion at the critical phase between two topological insulator phases [37], as well as enhanced localization effects in systems where topologically protected and unprotected channels coexist [38,39].

Despite more than half a century since Anderson's original paper [40], there exist very few exact results [41] about electron dynamics in the Anderson-localized phase beyond the strictly 1D (single channel) case [42]. In particular, the absence of exact results for dynamical correlations in disordered wires (quasi-1D multichannel system) is rather surprising in light of the remarkable success of the field-theoretic approach to the problem in terms of the supersymmetric nonlinear sigma model (NLSM). The NLSM method has proven to be very efficient in describing static response [43–45] and has been successfully employed to obtain the conductance, its mesoscopic fluctuations [44,46,47], as well as the full distribution function of transmission eigenvalues [39,48–50] in disordered wires. In addition to being an effective model for localization problems in general, NLSM is a generic field theory arising in a number of other problems such as random banded matrices [45,51] and the dynamics of the quantum kicked rotor [52–56].

In this Rapid Communication, we provide an exact analytic expression for an arbitrary local dynamical correlation (LDC) function of a disordered metallic wire in one of the three Wigner-Dyson symmetry classes. This is done by showing that, rather surprisingly, any LDC of the supersymmetric 1D NLSM in the unitary, orthogonal, or symplectic class is given *exactly* by a corresponding correlation function of a zero-dimensional (0D) NLSM in one of the classes AIII, CI, and DIII, respectively. The latter can always be evaluated explicitly as a finite-dimensional integral.

Our result is quite general and can be used to compute any LDC such as correlations of the local density of states at different energies, out-of-time-order correlations of operators at nearby points, and diffusion probability of return. We will focus on the latter quantity since it is the simplest to compute and the most intuitive to understand [57–63].

Return probability quantifies the relaxation of a local density perturbation in time. It is readily observable in time-resolved measurements of the electron density profile, which has already been experimentally achieved in cold atom setups [4–6] and in disordered photonic crystals [64–66] where one of the spatial coordinates plays the role of time. Our theory allows for the existence of topologically protected channels, which models an edge of a 2D topological insulator. This, in particular, applies to the interface between two quantum Hall samples [38,67–70], graphene zigzag nanoribbons [71] and multiwall nanotubes, as well as doped Weyl semimetals in a strong magnetic field [39,72]. It can also be implemented with cold atoms by coupling the disordered quasi-1D wire [4–6] to the edge of a 2D Chern insulator [73,74].

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TABLE I. Sigma-model manifolds for Wigner-Dyson classes $Q \in \mathcal{G}(2n)/\mathcal{G}(n) \times \mathcal{G}(n)$. The parameter γ accounts for the size of the matrix and normalizes the supertraces. The effective 0D sigma model defined on the group manifold $\mathcal{G}(2n)$ is used in the integral representation (10).

Class	γ	$\mathcal{G}(n)$	Noncompact	Compact	Topology
Unitary	1	AIII	$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$	$\mathrm{U}(n)$	$\mathbb{Z}$
Orthogonal	2	CI	$\mathrm{SO}(2n,\mathbb{C})/\mathrm{SO}(2n)$	$\mathrm{Sp}(2n)$	0
Symplectic	2	DIII	$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(2n)$	$\mathrm{O}(2n)$	$\mathbb{Z}_2$

Formalism. We consider a model of an infinite quasi-1D metallic wire with $N \gg 1$ conducting channels with or without time-reversal symmetry (TRS) $\mathcal{T}$. The system belongs to one of three Wigner-Dyson symmetry classes: unitary (no TRS), orthogonal ($\mathcal{T}^2 = 1$), or symplectic ($\mathcal{T}^2 = -1$). In the absence of TRS, the numbers of left- and right-moving channels generally differ by an integer m that represents a topological invariant and corresponds to the number of chiral topologically protected channels. The presence of TRS enforces the number of left- and right-moving channels to be the same. In this case, it is possible to have a single helical topologically protected channel if $\mathcal{T}^2 = -1$ (symplectic class) and the total number of channels N is odd.

Any LDC of a disordered system can be expressed as the disorder-averaged product of Green's functions. Dynamical correlations involve Green's functions at two different energies, whereas local correlations involve Green's functions between spatially close points within the localization length $\xi = Nl$, where l is the mean-free path. The main quantity we will consider in this work is the return probability $W(t)$, which is the probability that a diffusing electron returns to the starting point after time t. It can be expressed in terms of the disorder average of two Green's functions as

$$W(t) = \int \frac{d\omega\, e^{-i\omega t}}{4\pi^2 \nu} \langle G^R_{\epsilon+\omega}(x,x') G^A_\epsilon(x',x)\rangle\Big|_{x'\to x}, \qquad (1)$$

with ν being the density of states. The limit $x' \to x$ implies that $l \ll |x' - x| \ll \xi$; the first inequality excludes any nonuniversal ballistic effects.

Disorder averaging of a product of Green's functions can be performed following the standard procedure [43–45] that starts by writing this product as a Gaussian integral over supervector field ψ. Averaging over disorder leads to a quartic term in ψ that is decoupled with the help of a supermatrix field Q. The effective field theory in terms of Q is obtained by means of a saddle-point approximation followed by a gradient expansion.

The resulting action at an imaginary frequency $\omega = i\Omega$ has the form of a nonlinear sigma model [38,39,43–45]

$$S = -\frac{\pi\nu}{4\gamma} \int dx\ \mathrm{str}[D(\partial_x Q)^2 - 2\Omega\Lambda Q] + S_{\mathrm{top}},$$
$$S_{\mathrm{top}} = \frac{m}{2} \int dx\ \mathrm{str}(T^{-1}\Lambda\partial_x T), \qquad Q = T^{-1}\Lambda T. \qquad (2)$$

Here D is the diffusion constant and γ is given in Table I. The topological term S_{top} involves an integer number m denoting the difference between the number of left- and right-moving channels in a unitary wire, or the total number of channels in a symplectic wire. The matrices T and Q operate in the direct product of retarded-advanced, Bose-Fermi, and (if TRS is present) time-reversal spaces in addition to the space of n replicas. The latter is required to compute an average of $2n$ Green's functions [75]. The matrix Λ is $\mathrm{diag}\{1, -1\}_{\mathrm{RA}}$.

The matrix T is an element of a Lie supergroup $\mathcal{G}(2n)$ given in Table I for the three classes [76]. The matrix Q, parametrized as $T^{-1}\Lambda T$, is invariant under left multiplication $T \mapsto KT$ by any matrix K that commutes with Λ. As a result, Q belongs to the coset space $\mathcal{G}(2n)/\mathcal{G}(n) \times \mathcal{G}(n)$ [77]. We restrict T and K to have unit superdeterminant $\mathrm{sdet}\, T = \mathrm{sdet}\, K = 1$, which is necessary for the proper definition of S_{top} in Eq. (2) [38].

The topological term S_{top} is not invariant under gauge transformations $T \mapsto KT$ but rather changes by an integral of a total derivative, much like the action of a charged particle in an external magnetic field [38]. In the three symmetry classes, the value of S_{top} is either identically zero (orthogonal), 0 or $i\pi$ (symplectic), or an arbitrary imaginary number (unitary). Hence the value of m is immaterial in an orthogonal wire. In symplectic wires, only the parity of m is relevant, distinguishing the cases of even and odd number of channels. In the unitary class, m corresponds to the imbalance between left- and right-moving channels.

Evolution operator and correlation functions. Any LDC is expressed in the sigma-model language as the expectation value of a function of Q at a single point

$$\langle F(Q)\rangle = \int \mathcal{D}Q\, F[Q(x=0)] e^{-S[Q]}, \qquad (3)$$

with the action $S[Q]$ given by Eq. (2). Equation (3) represents a path integral over the field configurations of the matrix Q with the operator $F(Q)$ inserted at $x = 0$ (the point at which the observable is computed). In particular, the return probability $W(t)$, defined in Eq. (1), can be written as

$$W(t) = -\nu \int \frac{d\omega\, e^{-i\omega t}}{16\gamma^2}\, \mathrm{str}\langle kP_+QkP_-Q\rangle, \qquad P_\pm = \frac{1 \pm \Lambda}{2}. \qquad (4)$$

Here $k = \mathrm{diag}\{1, -1\}_{\mathrm{BF}}$ is the grading matrix.

Calculation of the expectation value (3) is facilitated by defining the evolution operator

$$\psi_m(T) = \int_{x=0,\,T(0)=T}^{x=\infty,\,T(\infty)=1} \mathcal{D}Q\, e^{-S[Q]}. \qquad (5)$$

which is a path integral on the half-infinite wire in terms of $Q = T^{-1}\Lambda T$. The boundary conditions at $x = \infty$ are applied to fix the gauge. We write the evolution operator as a function of T rather than Q to emphasize its gauge dependence. Under a gauge transformation $T \mapsto KT$, it transforms as

$$\psi_m(KT) = (\mathrm{sdet}\, K_R)^m \psi_m(T) = (\mathrm{sdet}\, K_A)^{-m} \psi_m(T) \qquad (6)$$

in full analogy to a wave function in magnetic field.

In Eq. (6), $K_{R/A}$ are the two (retarded and advanced) blocks of the matrix K, each from the supergroup $\mathcal{G}(n)$. The restriction $\mathrm{sdet}\, K = 1$ ensures the equivalence of the two expressions in Eq. (6). The product $\psi_m(T)\psi_{-m}(T)$ is gauge invariant and hence depends on Q only. This allows us to write the expectation value $\langle F(Q)\rangle$ as an ordinary rather than path integral:

$$\langle F(Q)\rangle = \int dQ\, \psi_{-m}(T) F(Q) \psi_m(T). \tag{7}$$

The function $\psi_m(T)$ can be identified with the zero mode of the transfer matrix Hamiltonian corresponding to the action (2) with the coordinate x playing the role of a fictitious imaginary time. Under evolution in x, all nonzero modes exponentially decay, hence only the zero mode survives in a half-infinite wire. The transfer matrix Hamiltonian contains a kinetic term, represented by the Laplace-Beltrami operator on the sigma-model manifold, and a potential term str ΛQ [78].

The main result of this Rapid Communication is an explicit integral representation of $\psi_m(T)$ that we construct as

$$\psi_m(T) = \int dK (\operatorname{sdet} K_R)^m \phi(KT). \tag{8}$$

This integral runs over $K \in \mathcal{G}(n) \times \mathcal{G}(n)$ constrained by sdet $K = 1$. For any function $\phi(T)$, the above integral represents an average over the gauge group K with the weight $(\operatorname{sdet} K_R)^m$ that ensures the correct transformation properties (6). Choosing the function ϕ to be

$$\phi(T) = \exp\left[-\frac{\kappa}{2\gamma} \operatorname{str} P_\pm (T + T^{-1})\right], \quad \kappa = 4\pi\nu\sqrt{D\Omega}, \tag{9}$$

we observe that the integral (8) is annihilated by the transfer-matrix Hamiltonian, which is shown explicitly in the Supplemental Material [78], and hence indeed provides an explicit expression for the zero mode.

Several comments are in place here about the expression for the evolution operator [Eqs. (8) and (9)]. First, the integral (8) can be equivalently written with the factor $(\operatorname{sdet} K_A)^{-m}$, while the function $\phi(T)$ contains any of the two projection operators $P_\pm$ defined in Eq. (4). It turns out that the result of integration is independent of the choice of $P_\pm$. In both cases, integration over K in Eq. (8) reduces to integration over K_R or K_A within the group $\mathcal{G}(n)$, since the integrand depends only on one of the two blocks of K. Second, the very existence of the zero mode relies crucially on supersymmetry. Both compact and noncompact replica sigma models do not possess a zero mode and the function defined in Eqs. (8) and (9) *does not* vanish under the action of the transfer-matrix Hamiltonian. However, the result of such an action *does* vanish in the replica limit $n \to 0$. This means that the simple integral representation for the evolution operator is an exclusive feature of symmetric superspaces not shared by their compact or noncompact nonsupersymmetric counterparts. Third, the expression (8) already captures the correct topological properties of the three classes. The determinant factor is always 1 in the orthogonal class and thus drops out for any m, while it equals ± 1 in the symplectic class making it sensitive only to the parity of m. In the unitary class, the determinant represents a phase factor and hence distinguishes all integer values of m.

An arbitrary LDC can now be expressed using Eqs. (7), (8), and (9). The integral for $\langle F(Q)\rangle$ contains the functions ψ_m and ψ_{-m}. We choose the form with K_R integral in (8) for one of them and with the K_A integral for the other. This amounts to using two different projectors $P_\pm$ for the two functions. The integrals over Q, K_R, and K_A can be combined into a single integral over $T \in \mathcal{G}(2n)$ leading to the remarkably simple expression

$$\langle F(Q)\rangle = \int_{\mathcal{G}(2n)} dT (\operatorname{sdet} T)^m F(T^{-1}\Lambda T) \times \exp\left[-\frac{\kappa}{2\gamma} \operatorname{str}(T + T^{-1})\right], \tag{10}$$

where the assumption sdet $T = 1$ has been dropped.

Integrals of the type (10) were previously studied in the context of Gaussian ensembles of random chiral matrices [79]. Equation (10) relates any local correlation function of a 1D sigma model at frequency Ω to the correlation function of a 0D sigma model at frequency $\kappa \sim \sqrt{\Omega}$ *in a different symmetry class*. The unitary, orthogonal, and symplectic classes map to classes AIII, CI, and DIII, respectively (see Table I).

Return probability. We now demonstrate the power of Eq. (10) and compute the return probability, Eq. (4). We employ the minimal $n = 1$ model and use a specific parametrization of $T \in \mathcal{G}(2)$ whose details are given in the Supplemental Material [78]. The result takes the simplest form in terms of the inverse dimensionless time z:

$$W(t) = \frac{F(z)}{8\pi\nu D}, \qquad z = \frac{1}{\tau} = \frac{8\pi^2\nu^2 D}{t}. \tag{11}$$

The function $F(z)$ is given by

$$F_m^{\mathrm{U}}(z) = \frac{2e^{-z}}{3}[(2z + m + 2)I_m(z) + zI_{m+1}(z)], \tag{12a}$$

$$F^{\mathrm{O}}(z) = 1 + \frac{e^{-z}}{3}[(3z+5)I_0(z) + (3z+4)I_1(z)], \tag{12b}$$

$$F_{\mathrm{e/o}}^{\mathrm{Sp}}(z) = F^{\mathrm{O}}(z) - 2 \pm \frac{e^{-z/2}}{3}(z+2). \tag{12c}$$

Here $I_m(z)$ denotes the modified Bessel function. These simple expressions capture the complete crossover between classical diffusion at short times $\tau \ll 1$ and localization at long times $\tau \gg 1$.

The return probability $F(\tau)$ in the absence of any topological channels is plotted in Fig. 1. At short times $\tau \ll 1$, all the curves approach the result for classical diffusion $F = \sqrt{2/\pi\tau}$. The leading correction to the classical result is given by ± 1 for the orthogonal/symplectic class indicating weak localization/antilocalization. In the unitary class, localization correction $(5/4)\sqrt{\tau/2\pi}$ appears only in the second order.

At long times $\tau \gg 1$, all curves approach a nonzero saturating value indicating localization. This value is $4/3$ for the unitary and symplectic classes and $8/3$ for the orthogonal class. This is consistent with the fact that the localization length in the latter case is twice shorter [47,80]. The function $F(\tau)$ approaches its saturation value as a power law $\sim 1/\tau^3$, $1/\tau^2$, and $1/\tau^5$ in the unitary, orthogonal, and symplectic classes, respectively.

Return probabilities in symplectic wires with even and odd number of channels are compared in Fig. 2. Short-time asymptotics of $F(\tau)$ is insensitive to the parity to all orders. This shows that the effects of $\mathbb{Z}_2$ topology are invisible on the perturbative weak localization level [38]. At long times, the curve for odd number of channels decays to zero as

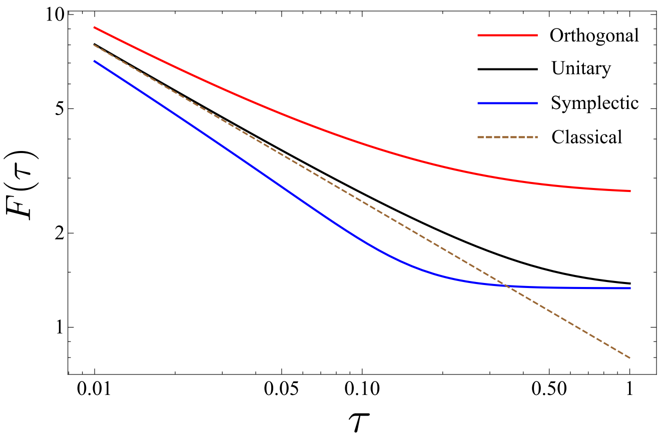

FIG. 1. Return probability $F(\tau)$ as a function of dimensionless time τ (logarithmic scale) for the unitary (black), orthogonal (red), and symplectic (blue) classes together with the result for classical diffusion (dashed).

$\sim 1/\tau^2$ indicating delocalization due to the presence of a single topologically protected channel.

Return probability in a unitary wire is shown in Fig. 3 for different values of the channel imbalance m. For $m \neq 0$, the curves decay to zero as $\sim 1/\tau^m$ indicating delocalization. The decay power increases with m since delocalization is enhanced with increasing the number of topologically protected channels. It is instructive to compare this result to the classical picture of diffusion accompanied with a unidirectional drift due to protected chiral channels [38,72]. In the classical limit, the return probability is given by $F(\tau) = \sqrt{2/\pi\tau}\, e^{-m^2\tau/2}$ and decays exponentially at long times. This corresponds to a Gaussian wave packet that spreads as $\sqrt{2Dt}$ and drifts with a constant velocity $m/2\pi\nu$. Localization corrections turn this exponential decay of return probability into a power law indicating that the drifting wave packet leaves a "fat tail" behind.

Discussion and conclusion. The main result of this Rapid Communication is the identity (10) that relates an arbitrary local correlation function of the 1D NLSM at finite frequency to the correlation function of a 0D NLSM in a different symmetry class. The latter can be evaluated explicitly as a finite-dimensional integral. The result applies to supersymmetric models with an arbitrary number of replicas, is valid for disordered metallic wires in the presence or absence of time-reversal symmetry, and allows for an arbitrary topological index m. It remains to be seen whether the result can be generalized further to superconducting and chiral symmetry classes. The exact identity between correlation functions of the 1D and 0D NLSM raises an intriguing possibility that similar relations could also hold in higher dimensions.

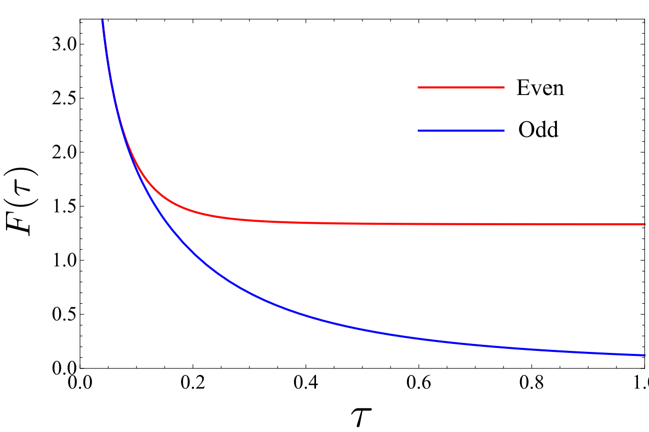

FIG. 2. Return probability $F(\tau)$ in a symplectic wire with an even (red) and odd (blue) total number of channels.

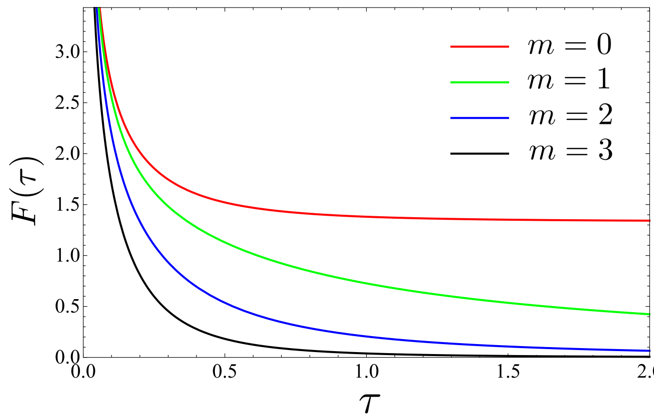

FIG. 3. Return probability $F(\tau)$ in a unitary wire for several different values of the channel imbalance m.

The identity (10) was applied to study diffusion probability of return, which is the simplest local dynamical observable. We obtained exact analytic expressions (12) that cover the complete crossover from the short-time semiclassical (weak localization) regime to the long-time strong localization regime. The return probability has a nonzero value at long times indicating complete localization in wires without topologically protected channels (Fig. 1). This saturation value is approached as a power law in time with an exponent that depends on the symmetry class. In the presence of protected channels, the return probability decays to zero as a power law in time (Figs. 2 and 3) with an exponent that depends on the topological index m. This power-law decay arises due to quantum interference effects and is in sharp contrast with the exponential decay predicted by the classical model of diffusion and drift.

The general result (10) can be used to compute various physical observables in disordered systems *exactly*. In addition to the diffusion probability of return considered here, these observables include out-of-time-order correlations [81], correlations of the local density of states at different energies [41] (which can be probed in optical response experiments), zero-bias anomaly in disordered wires in the presence of short-range interactions [82], strong Anderson localization peak in cold atom quantum quenches [83–85], as well as the proximity effect at the interface between a superconductor and a disordered wire [86,87].

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[75] In the minimal model (one replica), which is sufficient to compute the average of two Green's functions [Eq. (1)], the matrices T and Q have size 8×8 (4×4) in the presence (absence) of TRS.

[76] Strictly speaking, T should belong to a proper Lie supergroup $\tilde{\mathcal{G}}(2n)$ related to $\mathcal{G}(2n)$ in Table I by analytic continuation. The group $\tilde{\mathcal{G}}(2n)$ is given by $\mathrm{U}(n,n|2n)$, $\mathrm{OSp}(2n,2n|4n)$ and $\mathrm{SpO}(2n,2n|4n)$ for the unitary, orthogonal, and symplectic classes, respectively.

[77] Here, $\mathcal{G}(n)$ in the denominator should be analytically continued to a proper *compact* Lie supergroup given by $\mathrm{U}(n|n)$, $\mathrm{OSp}(2n|2n)$, and $\mathrm{SpO}(2n|2n)$ for the unitary, orthogonal, and symplectic classes, respectively.

[78] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.96.201105 for the explicit expression for the transfer matrix Hamiltonian, the proof that the zero mode provided by Eqs. (8) and (9) is the annihilated by this Hamiltonian, and an explicit description of the parametrization of the sigma model manifold as well as the construction of the zero mode in the minimal model for the unitary, orthogonal and symplectic classes.

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