Second quantization of Leinaas-Myrheim anyons in one dimension and their relation to the Lieb-Liniger model

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In one spatial dimension, anyons in the original description of Leinaas and Myrheim are formally equivalent to locally interacting bosons described by the Lieb-Liniger model. This allows an interesting reinterpretation of interacting bosons in the context of anyons. We elaborate on this parallel, particularly including the many-body bound states from the attractive Lieb-Liniger model. In the anyonic context these bound states are created solely by quantum-statistical attraction and coined the quantum-statistical condensate, which is shown to be more robust than the Bose-Einstein condensate. We introduce the second quantization formalism for the present anyons and construct the generalized Jordan-Wigner transformation that connects them to the bosons of the Lieb-Liniger model.

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I. INTRODUCTION

In modern mesoscopic systems, electronic excitations are effectively confined to a lower-dimensional world. An unexpected consequence of such a reduced spatial dimension is the occurrence of particles that neither obey Fermi nor Bose statistics. These are known as anyons [1–6]. Especially in two dimensions, anyons have been theoretically extensively studied [7–9] and indicated to exist in several experimental systems [10–15]. The spatial exchange of two-dimensional anyons and the accompanied, fixed unitary transformation of the anyonic wave function could pave the way to topological quantum computing [16]. Exchangeability is also apparent in special one-dimensional systems, e.g, ringlike ones or T structures [17,18]. Sparked by this idea, the interest in lower-than-two-dimensional anyons has recently increased, especially in conjunction with the possible detection of Majorana bound states in quantum wires [19–23]. Those are expected to be non-Abelian anyons, potentially applicable in topological quantum computing [18,24] as well.

In this paper, we describe a first-principles theory of one-dimensional anyons whose exchange is prohibited by the geometry of the system, i.e., anyons on a line and in a box. To this end, we employ the concepts introduced in the fundamental work of Leinaas and Myrheim [1] extended to many-particle systems. Their approach follows one fundamental idea: to set up the proper classical theory of indistinguishable particles and subsequently quantize it. In two dimensions, this results in "standard" Chern-Simons anyons, that can be interpreted as bosons with an attached flux acquiring an Aharonov-Bohm phase when physically exchanged [6,9]. This renders the work of Leinaas and Myrheim [1] one of the standard references in the field. Their theory is less frequently applied to one spatial dimension. There, a manifold of different theories exists [5,19,25–40], which also describe locally interacting anyons [41–44]; see Appendix A for a brief summary. In particular, Leinaas-Myrheim anyons have to be contrasted to the emergent excitations of the Calogero-Sutherland model, the Haldane-Shastry chain, and the fractional quasiparticles in Tomonaga-Luttinger liquids [19,35–39] that are as well called anyons [37]. The defining property of these kinds of anyons is that the wave function acquires a fixed phase κ when the coordinates of two anyons get permuted, in complete analogy to the statistical angle in two spatial dimensions. While this behavior is seen as natural, it conflicts with the impenetrability of anyons, an essential ingredient to deriving the two-dimensional theory of anyons in the framework of Leinaas and Myrheim. The question arises how anyonic wave functions can acquire a phase upon the exchange of coordinates if the particles themselves cannot be exchanged. Interestingly, corresponding theories are still described by the approach of Leinaas and Myrheim with a proper continuation procedure [37]. To strengthen the plausibility of Leinaas' and Myrheim's approach, we furthermore want to stress its fundamental depth. First, it does not require an underlying theory of additional, constituting particles. Only the indistinguishability of the considered particles, the validity of canonical quantization for flat spaces, and the Hermiticity of the Hamiltonian is assumed. Second, Leinaas-Myrheim anyons appear naturally when two-dimensional Chern-Simons anyons are confined to one dimension by a potential. In the process of the dimensional crossover, the complete statistical angle gets gradually absorbed and encoded into the scattering behavior of the anyons [32], which eliminates the need for an additional statistical phase in one spatial dimension. As a concrete physical example, we can imagine a fractional quantum Hall insulator [11] where anyonic bulk excitations are confined to one spatial dimension by an electric potential.

As a twist of history, Leinaas' and Myrheim's theory of one-dimensional anyons (1977) turns out to be formally equivalent to the Lieb-Liniger model, describing locally interacting bosons in one spatial dimension (1963) [45]. With "formally equivalent" we mean that the equations appearing in both theories are identical. However, the calculation of physical observables is different. This leads to both systems being described by the same equations but exhibiting different phenomenologies (see Sec. II below). Lieb and Liniger derived the solutions of their model by first rewriting it employing boundary conditions, not knowing that these equations would several years later be employed by Leinaas and Myrheim to describe one-dimensional anyons. While the Lieb-Liniger model has substantially advanced since 1963, the results have, to the best of our knowledge, not been carried over to the theory of Leinaas-Myrheim anyons. In this work, we close this gap. This includes the calculation of observables [19,20,46–48] for confined anyons: the energy spectrum, momentum density, and finite-size density oscillations. Our results are applicable to quasiparticle excitations in quasi-one-dimensional systems, such as interacting cold-atom/ion chains and edge liquids of topological insulators, that potentially carry anyonic excitations [19,20,40,49,50]. Additionally, the interpretation of interacting bosons as anyons introduces a fresh perspective into the established field of the Lieb-Liniger model. For instance, the results for confined anyons also describe Lieb-Liniger bosons in a box, where the Dirichlet boundary conditions result in modified Bethe ansatz equations [51,52]. Motivated by the anyonic interpretation, we develop the second quantization formalism for Leinaas-Myrheim anyons and, hence, an alternative second quantization for the Lieb-Liniger model. While presenting the formalism, we take particular care of including complex momenta, which, as usual, describe spatially bound states. In the exact many-body solutions, they build up a stable quantum phase, which we call the quantum-statistical condensate. At first sight, the existence of bound states is surprising in the anyonic interpretation, but immediately becomes clear when interpreted as attractively interacting bosons in the Lieb-Liniger model. Lieb and Liniger have first disregarded this regime as unphysical and unstable [45]. More recent work has albeit revealed its soundness [53–58]. Furthermore, it has been pointed out that, within the attractive regime, additional gaslike phases may exist [59,60].

The structure of the paper is as follows. In Sec. II, we concisely review both relevant models, i.e., the Lieb-Liniger model and the model of Leinaas and Myrheim for one-dimensional anyons and state their formal equivalence. In Sec. III, we construct the anyonic wave functions, which are the basis of the second quantization formalism that we derive in Sec. IV. We also provide the generalized Jordan-Wigner transformation from Lieb-Liniger bosons to Leinaas-Myrheim anyons. The Bethe ansatz equations for systems of finite size are discussed in Sec. V, which are consequently, in Sec. VI, applied to derive some properties of anyons in a box. The case of a negative statistical parameter is covered in Sec. VII, where we introduce the quantum-statistical condensate and the interpretation of the clusters as individual anyons themselves. We conclude our work in Sec. VIII.

II. MODEL

We start by reviewing the models of Lieb-Liniger and Leinaas and Myrheim and highlight how their formal equivalence still results in different phenomenology. The Lieb-Liniger model [45] describes a number of n locally interacting bosons in one dimension. In real space, the system is represented by its totally symmetric wave function Ψ, which maps n real numbers to a complex one, and is governed by the Hamiltonian

$$\mathcal{H}_{\mathrm{LL}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\partial_{x_j}^2 + 2c\sum_{i\neq j}\delta(x_i - x_j). \tag{1}$$

Here, m denotes the mass of the particles and c is the real-valued interaction strength that has the dimension of momentum. The δ functions can be directly implemented into the wave function by demanding boundary conditions, which, because of the symmetry of the wave function, turn out to be the so-called Robin boundary conditions

$$(\partial_{x_{j+1}} - \partial_{x_j})\,\Psi(\boldsymbol{x})|_{x_j\to x_{j+1}} = c\;\Psi(\boldsymbol{x})|_{x_j\to x_{j+1}}, \tag{2}$$

for each j between 1 and $n-1$, and we restrict ourselves to the region $\mathcal{R} = \{\boldsymbol{x} \mid x_1 < x_2 < \cdots < x_n\}$ of the parameter space [45]. In exchange for the boundary conditions, the Hamiltonian on $\mathcal{R}$ becomes the one of free particles, i.e.,

$$\mathcal{H}_{\mathrm{LL}}|_{\mathcal{R}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\partial_{x_j}^2. \tag{3}$$

Let us now recapitulate and slightly extend the theory of Leinaas and Myrheim [1] for indistinguishable quantum particles. First, consider n classical particles on a line. The spatial configurations of a system of distinguishable particles would be described by tuples of positions $\boldsymbol{x} = (x_1,\ldots,x_n)$. Because the particles are indistinguishable, however, using tuples is ambiguous: For $n = 2$, (x_1,x_2) and (x_2,x_1) label the same configuration. Instead, we employ the sets $\{x_1,\ldots,x_n\}$ of n distinct positions. The family of all these sets is called configuration space $\mathcal{R}$ and inherits various properties by local equivalence to $\mathbb{R}^n$. Here, the notational correspondence of the configuration space to the parameter region of the Lieb-Liniger model is on purpose, since the real-space variables $x_1 < \cdots < x_n$ parametrize $\mathcal{R}$. To obtain the quantum-mechanical theory, space and momentum variables get promoted to operators acting on the wave functions $\Psi : \mathcal{R} \to \mathbb{C}$. We consider, for concreteness, the particles to obey the free Hamiltonian of Eq. (3) as well. However, electromagnetic potentials and particle interactions can be added without changing the general formalism. Finally, we demand the Hamiltonian $\mathcal{H}$ to be Hermitian. Interestingly, Hermiticity is granted if and only if Ψ fulfills the Robin boundary conditions of Eq. (2) [32]. In this context, the interaction strength c is called the statistical parameter $\eta \equiv c$.

In conclusion, both theories use the same differential equation and boundary conditions, which constitutes a formal equivalence between them. The phenomenology of both models, however, can differ significantly. The reason for this is the differing calculation of physical observables. For the model of Leinaas and Myrheim, given an operator A, its expectation value is calculated by an integral over the configuration space $\mathcal{R}$ only, according to

$$\langle A\rangle_{\mathrm{LM}} = \int_{x_{n-1}}^{\infty} dx_n \cdots \int_{x_2}^{x_3} dx_2 \int_{-\infty}^{x_2} dx_1 \Psi^*(\boldsymbol{x})A\Psi(\boldsymbol{x}), \tag{4}$$

while for the Lieb-Liniger model, the integration region is the full real space, such that

$$\langle A\rangle_{\mathrm{LL}} = \int_{-\infty}^{\infty} d^n\boldsymbol{x}\,\Psi^*(\boldsymbol{x})A\Psi(\boldsymbol{x}). \tag{5}$$

In the latter equation, $\Psi(\boldsymbol{x})$ has been symmetrically continued by $\Psi(\boldsymbol{x}) := \Psi(\pi(\boldsymbol{x}))$, where π is a permutation such that $\pi(\boldsymbol{x})$ is in $\mathcal{R}$. The fact that Eqs. (4) and (5) can differ is known in the context of impenetrable bosons, i.e., the Tonks-Girardeau gas, and free fermions. While the former is the limit of the Lieb-Liniger theory at infinite repulsion $c \to \infty$, the latter is the limit of the Leinaas-Myrheim theory for $\eta \to \infty$. However, impenetrable one-dimensional bosons differ from free fermions, for instance, by their momentum distribution [61].

The twofold interpretation of Eq. (2) in combination with Eq. (3) provides simple explanations of seemingly complicated facts. For instance, if the bosons attract each other, i.e., $c < 0$, they form clusters [58]. This intuitive property of the Lieb-Liniger model seems surprising for Leinaas-Myrheim anyons, where the attraction would be mediated by the statistics only [62].

There is indication that the coincidentally looking formal equivalence between locally interacting bosons and Leinaas-Myrheim anyons is in fact not coincidental. To explain this, we refer to the connection between bosons and anyons in two spatial dimensions. There, anyons are equivalent to bosons that acquire an Aharonov-Bohm flux when circling around each other. Hence, also in two dimensions, anyons are equivalent to particularly interacting bosons. The reason for this is that the configuration space for two-dimensional anyons has holes, the points where the position of a pair of particles would coincide. The holes themselves are irrelevant concerning scattering since particles can move around them by infinitesimally altering their path. However, the holes potentially induce a holonomy in the wave function, the Aharonov-Bohm phase. In one spatial dimension, the holes no longer induce a holonomy because particles cannot be exchanged. However, the holes themselves become relevant as, by normal propagation, particles can scatter off each other at some time. Then, the holes induce the boundary condition of Eq. (2) and thereby again serve as the origin of the bosonic interaction that connects bosons and anyons.

In the remaining course of the paper, we present and interpret the solutions to the equivalent models from the more rarely employed anyonic point of view.

III. CONSTRUCTION OF THE WAVE FUNCTIONS

We next construct all wave functions that fulfill Eq. (2), thereby combining the solutions of the attractive and the repulsive Lieb-Liniger model [45,58], with the final aim to provide the second quantization formalism for Leinaas-Myrheim anyons. To this end, we employ the ansatz [45] $\Psi(\boldsymbol{x}) = \int_{\boldsymbol{k}\in\mathbb{C}^n} d^n k\, \alpha(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}}$. Complex momenta $\boldsymbol{k}$ are explicitly included. These are needed to describe anyonic bound states that form for a negative statistical parameter. In momentum space, the boundary conditions translate to

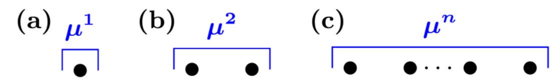
$$\alpha(\boldsymbol{k}) = e^{-i\phi_\eta(k_{j+1}-k_j)}\alpha\big(\sigma_j \boldsymbol{k}\big) \quad \text{if } k_{j+1} - k_j \neq i\eta, \tag{6}$$

$$\alpha(\boldsymbol{k}) = 0 \quad \text{if } k_{j+1} - k_j = i\eta. \tag{7}$$

Here, σ_j denotes the elementary permutation which permutes the jth and $(j+1)$th element of a tuple and

$$\phi_\eta(k_{j+1} - k_j) = 2\arctan\big[\eta/(k_{j+1} - k_j)\big] \tag{8}$$

is the statistical phase. By iteration, these conditions connect coefficients of relatively permuted momenta $\alpha(\boldsymbol{k}) = e^{i\phi_\eta^P(\boldsymbol{k})}\alpha(P\boldsymbol{k})$. Here, $P = \sigma_{j_1}\cdots\sigma_{j_r}$ is a general permutation written with an r as small as possible and $\phi_\eta^P(\boldsymbol{k}) = \sum_{i=1}^r \phi_\eta[(\sigma_{j_1}\cdots\sigma_{j_i}\boldsymbol{k})_{j_i} - (\sigma_{j_1}\cdots\sigma_{j_i}\boldsymbol{k})_{j_i+1}]$. The basis functions are therefore of the form $\Psi_{\boldsymbol{k}}(\boldsymbol{x}) \propto \sum_{P\in S_n} e^{i\phi_\eta^P(\boldsymbol{k})} e^{i(P\boldsymbol{k})\boldsymbol{x}}$. On physical grounds, divergent elements of this set need to be excluded. This is done by only permitting special values of $\boldsymbol{k}$. These are build up by tuples $\boldsymbol{\mu}$ of complex momenta where the difference between adjacent momenta $\mu_{j+1} - \mu_j$ is $-i\eta$. These tuples are called strings in the context of the Lieb-Liniger model [58]. Within the anyonic context, we call them clusters. Examples of clusters are sketched in Fig. 1.



	(a)	(b)	(c)
Re $\{\boldsymbol{k}\}$	$(K_{\boldsymbol{\mu}^1})$	$(K_{\boldsymbol{\mu}^2}, K_{\boldsymbol{\mu}^2})$	$(K_{\boldsymbol{\mu}^n}, K_{\boldsymbol{\mu}^n}, \ldots, K_{\boldsymbol{\mu}^n})$
Im $\{\boldsymbol{k}\}$	(0)	$-\frac{\eta}{2}(-1,1)$	$-\eta\left(\frac{1-n}{2}, \frac{3-n}{2}, \ldots, \frac{n-1}{2}\right)$

FIG. 1. Examples of composite anyons described by clusters $\boldsymbol{\mu}$, called strings in the context of the Lieb-Liniger model. These are the fundamental building blocks of the anyonic wave functions and can be conceived as individual particles. For clusters of more than one anyon, $\eta < 0$ is implicit. (a) Single anyon; (b) two-anyon bound state; (c) maximally bound cluster of n anyons: the quantum-statistical condensate.

Physically, clusters with more than one element represent composite anyons whose constituents move collectively, separated by a characteristic length scale of $1/\eta$. They can be conceived as individual particles themselves. For positive η, clusters only consist of single particles, and hence describe free (unbound) anyons. In order to uniquely label the basis functions, we introduce the cluster ordering O. This is done in direct analogy to the ordering that needs to be introduced to label fermionic basis states in standard quantum many-body theory [63]. To apply O to a tuple $\mathcal{D}$ of clusters, first take the union of the clusters' momenta. Then sort all momenta by their real parts (smaller values first). If there are momenta with equal real parts, sort them by their imaginary parts (again, smaller values first).

In conclusion, the basis functions are given by momenta that describe composite and free anyons in momentum space. Given an ordered tuple $O(\mathcal{D})$ of clusters, the corresponding basis function obtains the form

$$\Psi_{(\boldsymbol{k}=O(\mathcal{D}))}(\boldsymbol{x}) = N_{\boldsymbol{k}} \sum_{P\in S_n} e^{i\phi_\eta^P(\boldsymbol{k})} e^{i(P\boldsymbol{k})\boldsymbol{x}}, \tag{9}$$

where $N_{\boldsymbol{k}}$ is the normalization [64] and $e^{i\phi_\eta^P(\boldsymbol{k})}$ plays the role of a generalized Slater determinant.

IV. SECOND QUANTIZATION

An advantage of the anyonic interpretation of Eqs. (2) and (3) is that a second quantization of the solution is reasonably motivated. In contrast, this endeavor seems to be discouraged in the bosonic picture of the Lieb-Liniger model, where the bosons are already given in their second quantized form. The formalism can facilitate the calculation of various properties, similar to the original second quantization of bosons and fermions. Details on the formalism are presented in Appendix B. Given the basis wave functions of Eq. (9), second quantization amounts to defining creation operators to construct all basis states from a vacuum state [65]. For a cluster $\boldsymbol{\mu}$, we define its creation operator by

$$a^\dagger_{\boldsymbol{\mu}}\Psi_{O(\mathcal{D})} = \sqrt{M(\boldsymbol{\mu})+1}\, e^{i\Phi^{\boldsymbol{\mu}}_\eta(\mathcal{D})}\Psi_{O(\{\boldsymbol{\mu}\}\cup\mathcal{D})} \tag{10}$$

and linear continuation to all states. Here, $M(\boldsymbol{\mu})$ is the number of clusters $\boldsymbol{\mu}$ in $\mathcal{D}$. The phase $\Phi^{\boldsymbol{\mu}}_\eta(\mathcal{D}) = \sum_{\tilde{\boldsymbol{\mu}}<\boldsymbol{\mu}}\varphi^{\tilde{\boldsymbol{\mu}},\boldsymbol{\mu}}_\eta$ is composed of the cluster-cluster exchange phases $\varphi^{\tilde{\boldsymbol{\mu}},\boldsymbol{\mu}}_\eta = \sum_{i=1}^{N(\boldsymbol{\mu})}\sum_{i=j}^{N(\tilde{\boldsymbol{\mu}})}\phi_\eta(\tilde{\mu}_j-\mu_i)$. Here, $\tilde{\boldsymbol{\mu}}<\boldsymbol{\mu}$ if $\tilde{\boldsymbol{\mu}}$ is ordered to the left of $\boldsymbol{\mu}$ by cluster ordering and $N(\boldsymbol{\mu})$ denotes the number of anyons in $\boldsymbol{\mu}$. Employing Eq. (10), the algebra of the cluster creation operators is

$$a^\dagger_{\boldsymbol{\mu}_1}a^\dagger_{\boldsymbol{\mu}_2} = e^{i\varphi^{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}_\eta}a^\dagger_{\boldsymbol{\mu}_2}a^\dagger_{\boldsymbol{\mu}_1}. \tag{11}$$

To be concrete, we consider the case of unbound anyons, described by clusters with exactly one element. Here,

$$\begin{aligned} a^\dagger_p a^\dagger_q &= e^{i\phi_\eta(p-q)}a^\dagger_q a^\dagger_p,\\ a_p a^\dagger_q &= e^{-i\phi_\eta(p-q)}a^\dagger_q a_p + \delta(p-q), \end{aligned} \tag{12}$$

where the annihilation operator a_p is the Hermitian conjugate of $a^\dagger_p$, which is the shorthand notation for $a^\dagger_{(p)}$.

It is striking that the one-dimensional anyonic algebra depends on the relative momentum instead of providing a fixed statistical phase as familiar from two-dimensional anyons [1] and the different types of one-dimensional anyons mentioned in the Introduction [19,35–39].

Employing the anyonic second quantization, the Hamiltonian of Eq. (3) becomes

$$\mathcal{H} = \sum_{\boldsymbol{\mu}}\epsilon_{\boldsymbol{\mu}}a^\dagger_{\boldsymbol{\mu}}a_{\boldsymbol{\mu}}, \tag{13}$$

and describes free anyonic clusters. In Eq. (13), the sum runs over all possible clusters $\boldsymbol{\mu}$, which have the energy

$$\epsilon_{\boldsymbol{\mu}} = \frac{\hbar^2}{2m}\left(n_{\boldsymbol{\mu}}K^2_{\boldsymbol{\mu}} - \frac{1}{12}\eta^2(n_{\boldsymbol{\mu}}-1)n_{\boldsymbol{\mu}}(n_{\boldsymbol{\mu}}+1)\right). \tag{14}$$

The latter equation is derived in Appendix C. Here, K_μ is the real-valued center-of-mass momentum (see Fig. 1) and $n_{\boldsymbol{\mu}}$ is the number of bare anyons forming the cluster. One can see that the contribution of clusters to the energy separates, which further motivates their interpretation as individual particles themselves. Furthermore, each cluster contributes by its kinetic energy and its internal binding energy. This is reflected in the first and second summand of Eq. (14), respectively.

Given the momentum-space operator algebra of Eq. (12), we can address the algebra of the real-space operators $\Psi^\dagger(x) = \int_{-\infty}^{\infty}dp\frac{e^{ipx}}{\sqrt{2\pi}}a^\dagger_p$. We obtain

$$\begin{aligned} \{\Psi(x),\Psi^\dagger(y)\} &= \delta(x-y)\\ &\quad + \int_0^\infty dz\frac{2e^{-\frac{z}{|\eta|}}}{|\eta|}\Psi^\dagger(y-z)\Psi(x-z),\\ \{\Psi^\dagger(x),\Psi^\dagger(y)\} &= \int_0^\infty dz\frac{2e^{-\frac{z}{|\eta|}}}{|\eta|}\Psi^\dagger(y-z)\Psi^\dagger(x+z), \end{aligned} \tag{15}$$

where $\{\ldots,\ldots\}$ denotes the anticommutator. Here, $\lim_{\eta\to 0}\int_0^\infty dz\frac{1}{|\eta|}e^{-z/|\eta|}f(z) = f(0)$ yields the bosonic commutation algebra, while the fermionic anticommutation relations for $\eta\to\infty$ are trivially contained. If we set $x=y$, we arrive at a smeared anyonic Pauli principle in real space represented by

$$[\Psi^\dagger(x)]^2 = \int_0^\infty dz\frac{1}{|\eta|}e^{-z/|\eta|}\Psi^\dagger(y-z)\Psi^\dagger(x+z). \tag{16}$$

In one dimension, there exist ways to transform between different statistics regarding bosons, fermions, and spins, e.g., by bosonization, refermionization, and the Jordan-Wigner transformation [19,66]. Likewise here, there is a generalized Jordan-Wigner transformation from the present anyons to the bosons of the Lieb-Liniger model. Ultimately, this reflects the fact that the Fock space of anyons is naturally isomorphic to the one of the Lieb-Liniger model (if $\eta\neq\pm\infty$). To this end, consider the bosonic operators b_l with the algebra $[b_k,b^\dagger_l] = \delta(k-l)$ and $[b_k,b_l]=0$ with $k,l\in\mathbb{R}$. For $\eta\neq\pm\infty$, we define the generalized Jordan-Wigner transformation,

$$\tilde{a}(j) = \lim_{\epsilon\to 0^+} e^{i\int_{-\infty}^{j-\epsilon}dk\, b^\dagger_k b_k\phi_\eta(k-j)}b(j). \tag{17}$$

Calculating the algebra of $\tilde{a}$, we find $\tilde{a}_j\tilde{a}_k = \tilde{a}_k\tilde{a}_j e^{i\phi_\eta(k-j)}$ and $\tilde{a}_j\tilde{a}^\dagger_k = \tilde{a}^\dagger_k\tilde{a}_j e^{-i\phi_\eta(j-k)} + \delta(j-k)$, which is exactly the anyonic algebra described in Eq. (12). As an apparent peculiarity, we have $\tilde{a}^\dagger_k\tilde{a}_k = b^\dagger_k b_k$, which results, for $\eta>0$, in the same free Hamiltonian, Eq. (3), using either the bosonic or the anyonic description. One would naively expect that the transformation should generate the interacting Hamiltonian of Eq. (1) instead. However, since the theory of Leinaas and Myrheim is intrinsically constrained to the region $\mathcal{R}$, defined at the beginning of Sec. II, it makes sense that the transformation yields Eq. (3). The information about the interactions remains encoded in the boundary conditions rather than in the Hamiltonian.

V. SYSTEMS OF FINITE SIZE

When anyons are confined to the length L, one would expect the Dirichlet boundary conditions $\Psi(0,x_2,\ldots,x_n) = \Psi(x_1,\ldots,x_{n-1},L) = 0$ to quantize the allowed momenta, similar to the particle-in-a-box problem. In fact, the conditions translate to

$$\begin{aligned} \alpha(-k_1,\ldots,k_n) &= -\,\alpha(\boldsymbol{k}),\\ \alpha(k_1,\ldots,-k_n) &= -\,e^{2ik_nL}\alpha(\boldsymbol{k}). \end{aligned} \tag{18}$$

These constraints of Eq. (18) are only consistent with Eqs. (6) and (7) if the system of transcendental equations

$$Lk_j + \sum_{1\leqslant(i\neq j)\leqslant n}[\phi_\eta(k_i-k_j) - \phi_\eta(k_i+k_j)]/2 = \pi z_j \tag{19}$$

is fulfilled for j between 1 and n. Here, the z_j are positive integers. The momenta that solve Eq. (19) are discrete and readily numerically obtainable. In the context of the Lieb-Liniger model, these equations are very similar to the so-called logarithmic Bethe ansatz equations [59,67]. Note, however, that the Lieb-Liniger Bethe ansatz equations originally describe particles that are confined to a ring, while Eq. (19)

is adjusted to the case of particles in a box. Although no differences in the thermodynamic limit are to be expected, these Bethe ansatz equations should give more reasonable finite-size results for confined particles (see Refs. [51,52] for the discussion in the context of the Lieb-Liniger model).

VI. APPLICATION

Equipped with the developed formalism, we next consider observables of experimental interest. First, we calculate the spectrum of two confined anyons numerically by solving Eq. (19). The result is depicted in Fig. 2(a). The anyonic spectra interpolate between the familiar bosonic and fermionic particle-in-a-box spectra for positive η. For instance, setting $E_0 = \hbar^2\pi^2/(2mL^2)$, the bosonic level with an energy of $2E_0$ continuously evolves to the fermionic level with $5E_0$. At negative η, a two-anyon bound state forms with an energy proportional to $-\eta^2$ in the infinite-size limit $L \to \infty$. Energetically higher anyonic bound states correspond to kinetic excitations of this composite anyon in analogy to the behavior of a single particle in a box. Some anyonic levels refuse to form bound states and instead converge to fermionic energies as $\eta \to -\infty$. These levels ensure that the finite-size spectrum coherently converges to the infinite-size spectrum. Energy spectra could be a viable observable in systems with few anyons, such as interacting cold-atom chains [19,20,40], and are detectable by spectroscopic techniques.

Turning to systems containing many anyons, as possibly being the case in solid state systems, unavoidable level broadening renders an accurate measurement of the discrete spectrum unfeasible. Yet, the momentum distribution could uncover the character of the anyons [19]. We depict the momentum density n_k at zero temperature in Fig. 2(b). This function gives the number of anyons with momentum between k_1 and k_2 by $\int_{k_1}^{k_2} dk\, n_k$. For bosons and fermions, it is proportional to the Bose-Einstein and Fermi-Dirac distribution, respectively. Anyons with a positive statistical parameter transform these distributions into each other, still preserving a sharply defined chemical potential reflected by a discontinuity in n_k. This has to be seen in contrast to the behavior of a Tomonaga-Luttinger liquid. The depicted momentum density is well known from the Lieb-Liniger model [45]. Because of the difference between Eqs. (4) and (5), however, the momentum density is not the physically measurable one of the interacting bosons [61].

If the spectral properties of a system are inaccessible, the statistics is still inferable via local properties, e.g., the finite-size density fluctuations [20,48]. While bosons condense to the middle of the system, fermions distribute equally spaced (by Pauli repulsion), resulting in oscillations of the particle density. Figure 2(c) depicts the scenario for four anyons in the ground state. Unbound anyons suppress the fermionic peaks and broaden the bosonic one, which is characteristic of intermediate statistics [20,48].

VII. THE QUANTUM-STATISTICAL CONDENSATE

For $\eta < 0$, the anyonic ground state is a cluster of the form $\mu_j = i\frac{\eta}{2}(n_\mu - 2j + 1)$ as depicted in Fig. 1. We call this cluster the quantum-statistical condensate since, in the anyonic picture, its origin is solely based on the quantum statistics. It can be conceived as a single composite anyon and corresponds to the bound state of bosons that forms in the Lieb-Liniger model [54,55] for an attractive interaction. Therefore, its local density is similar to the one of a single quantum particle, which, in turn, is the same as the one of the Bose-Einstein condensate [cf. Fig. 2(c)]. In fact, the Bose-Einstein condensate can be interpreted as the limit of the quantum-statistical condensate as $\eta \to 0^-$. Besides this, both condensates differ profoundly: Bosons condense into their single-particle ground state, but anyons into an inseparable many-body ground state.

Let us derive further characteristics of the quantum-statistical condensate. First, we obtain its ground-state energy

$$\epsilon_{\mathrm{GS}} = -\frac{\hbar^2}{24m}\eta^2(n-1)n(n+1) \qquad (20)$$

by Eq. (14). The proportionality to n^3 reveals an exceptional stability of the condensate [68]. Let us for a moment regard charged anyons exhibiting Hubbard repulsion, which has an associated energy proportional to n^2. Then, providing a sufficiently large number of anyons, the negative statistical energy outperforms the positive one created by charge repulsion.

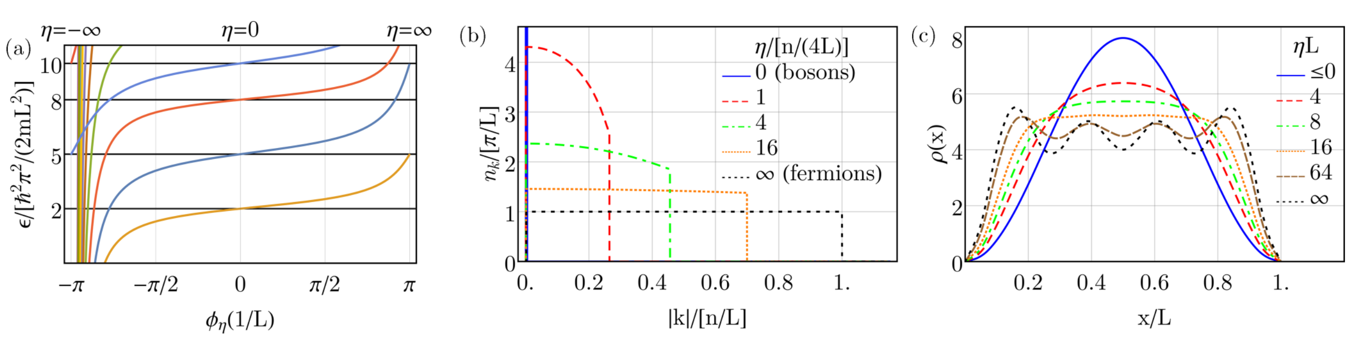


FIG. 2. Observables for confined anyons. The anyonic properties for $0 < \eta < \infty$ continuously interpolate between bosons ($\eta = 0$) and fermions ($\eta = \infty$). The statistical condensate forms at $\eta < 0$. (a) Discrete energy spectrum of two anyons. To depict the full range of η, we plot against $\phi_\eta(1/L)$. The two-anyon bound state and its excitations emerge for negative η. (b) Momentum density at zero temperature in the limit of infinitely many particles (numerical calculations for $n = 512$ anyons, where the curves almost converge to the limit $n \to \infty$). (c) Finite-size oscillations of the particle density ρ for four anyons.

The quantum-statistical condensate is hence stable against the introduction of charge. We conjecture that this property could lead to anyon superconductivity [69–71]. In fact, in the context of the Lieb-Liniger model, a phase of pairwisely bound bosons has been predicted [59].

A cluster behaves as an individual anyon, the energy of which separates into a kinetic and an internal part. Additionally, by Eq. (11), clusters acquire different statistical phases than their constituents. For instance, clusters of two anyons behave as anyons with the statistical phase $2\phi_\eta + \phi_{2\eta}$ (see Appendix D for details). In the vocabulary of topological field theories for two-dimensional anyons [9,16,72], the formation of clusters is linked to anyon fusion.

VIII. CONCLUSIONS

On the basis of the general assumptions of Leinaas and Myrheim [1], we derive an exact quantum many-body formalism for one-dimensional anyons including the exact wave functions, the second quantization, and the momentum discretizing equations for anyons in a box. The formalism is based on the equivalence to the Lieb-Liniger model of locally interacting bosons for which an interpretation in the anyonic context is established. We numerically calculate characteristic observables, namely, the energy spectrum, the momentum statistics, and the finite-size density fluctuations. For a negative statistical parameter, anyons attract each other with a force purely induced by their quantum statistics and form the quantum-statistical condensate. This genuine quantum many-body phase is shown to be more robust than the Bose-Einstein condensate. In particular, the statistical condensate is stable against the introduction of charged anyons. The clusters themselves should be conceived as individual anyons and obtain a different statistical phase than their constituents. Our work shows that one-dimensional anyons exhibit original and interesting physics even in the absence of spatial exchange. Furthermore, it emphasizes the link between anyons and interacting bosons and thereby opens possibilities of synthesizing either physical system by its formally equivalent partner.

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APPENDIX A: NOTIONS OF INTERMEDIATE STATISTICS IN ONE SPATIAL DIMENSION

There exists a variety of formalisms describing particles of intermediate statistics in one dimension, which are expected to be applicable to different physical situations. Although they differ in their phenomenology, these particles are all called anyons. For clarity, we briefly introduce some prominent theories that are applicable to one spatial dimension.

If the occupation number of a single-particle quantum state is restricted to maximally assume a given integer, the particles can be described as parafermions, which are closely related to Potts and clock models [26,27] and Gentile statistics [25]. Such particles are, among others, expected to exist as magnetic excitations [28,29]. Another kind of intermediate statistics considers the representations of the local current algebra (the commutation relations between the particle density and the particle currents in all spatial dimensions) [5,30] or the quantization of the algebra of allowed observables of indistinguishable particles. The latter has been applied to superconducting vortices [31] and two-dimensional anyons effectively confined to one dimension by a strong magnetic field [32]. Yet another notion of anyons in one dimension can be derived from Haldane’s generalization of the Pauli principle [33], which is, for instance, applicable to spinon excitations in spin chains. In this approach, the single-particle Hilbert space dimension depends on the total number of particles in the system. Finally, the term anyons is used in one dimension to describe low-energy quasiparticle excitations of interacting fermionic systems [34] linked to the Calogero-Sutherland model [35,39], the Haldane-Shastry chain, and the fractional excitations in Tomonaga-Luttinger liquids [19,36–38]. It is known that these particles (considering each channel separately in the case of a Tomonaga-Luttinger liquid) break time-reversal symmetry on the fundamental level of their operator algebra, reflected by an asymmetric momentum distribution [19,40].

APPENDIX B: DETAILS ON THE CONSTRUCTION OF THE SECOND QUANTIZATION

In this Appendix, we give details on the construction of the second quantization formalism in the main text. First, we define the Fock space based on the valid wave functions in momentum space, given by linear combinations of the basis functions in Eq. (9), together with an auxiliary vacuum state, and construct the creation and annihilation operators in momentum space.

The n-particle anyonic Hilbert space $\mathcal{H}_n$ in a real-space representation is formed by the wave functions described in Eq. (9) acting on the n dimensional configuration space $\mathcal{R}_n = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 < x_2 < \cdots < x_n\}$.

These functions form an orthonormal basis. It is worth noting in this regard, that although we generally would take an overcomplete set of basis functions into account by this approach, the constraints of Eq. (6), however, filter out a complete set of mutually orthogonal functions if only properly ordered vectors $\boldsymbol{k}$ are considered. For most of the pairings, the orthogonality can be deduced by noticing that eigenfunctions of the Hermitian Hamiltonian with different eigenvalues are orthogonal. It is worth mentioning here that the scalar product in a real-space representation is an integral restricted to the configuration space as opposed to running over $\mathbb{R}^n$. For two wave functions $\Psi_1, \Psi_2 : \mathcal{R}_n \to \mathbb{C}$ their scalar product is

$$(\Psi_1, \Psi_2) = \int_{\boldsymbol{x}\in\mathcal{R}_n} \Psi_1^*(\boldsymbol{x})\Psi_2(\boldsymbol{x}). \quad \text{(B1)}$$

The Fock space is constructed by $\mathcal{F} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n$. Here, $\mathcal{H}_0$ is the auxiliary vacuum space, isomorphic to $\mathbb{C}$. The scalar product of Eq. (B1) is extended to act on $\mathcal{F}$ by demanding $(\Psi,\Psi') = 0$ if Ψ and Ψ' are states with a different particle number. We now define the action of the linear creation operators $a^\dagger_{\boldsymbol{\mu}} : \mathcal{H}_n \to \mathcal{H}_{n+1}$ for a cluster $\boldsymbol{\mu}$ as in Eq. (10) of the main text by its action on the basis function and linear continuation to all states, i.e.,

$$a^\dagger_{\boldsymbol{\mu}} \Psi_{O(\mathcal{D})} = \sqrt{M(\boldsymbol{\mu})+1}\, e^{i\Phi^{\boldsymbol{\mu}}_\eta(\mathcal{D})} \Psi_{O(\{\boldsymbol{\mu}\}\cup\mathcal{D})}, \tag{B2}$$

where $M(\boldsymbol{\mu})$ is the number of clusters $\boldsymbol{\mu}$ in $\mathcal{D}$. It is straightforward to show by virtue of Eq. (9) that $a^\dagger_{\boldsymbol{\mu}}$ is well defined, i.e., Eq. (9) results in a unique representation of $a^\dagger_{\boldsymbol{\mu}}$ in the given basis functions. Its adjoint counterpart, the annihilation operator, is correspondingly defined employing the scalar product

$$\left(\Psi_1, a^\dagger_{\boldsymbol{\mu}} \Psi_2\right) = \left(a_{\boldsymbol{\mu}} \Psi_1, \Psi_2\right). \tag{B3}$$

These definitions result in the operator algebra described in the main text [see Eqs. (11) and (12)], by standard algebraic manipulations, i.e.,

$$a^\dagger_{\boldsymbol{\mu_1}} a^\dagger_{\boldsymbol{\mu}_2} = e^{i\varphi^{\boldsymbol{\mu_1},\boldsymbol{\mu_2}}_\eta} a^\dagger_{\boldsymbol{\mu}_2} a^\dagger_{\boldsymbol{\mu_1}}. \tag{B4}$$

Applied to clusters of single particles, we have

$$\begin{aligned} a^\dagger_p a^\dagger_q &= e^{i\phi_\eta(p-q)} a^\dagger_q a^\dagger_p, \\ a_p a^\dagger_q &= e^{-i\phi_\eta(p-q)} a^\dagger_q a_p + \delta(p-q). \end{aligned} \tag{B5}$$

Finally, we want to mention that the anyonic algebra for any values of the statistical parameter but $\eta = \pm\infty$ results in bosonic relations for equal momentum: $[a_p, a_p] = 0$, and, if $p \to q$, $[a_p, a^\dagger_q] \to \delta(p-q)$. However, for $\eta = \pm\infty$, we, by notation, strictly set $e^{-i\phi_{\pm\infty}(p-q)} = -1$, which results in the familiar fermionic anticommutation algebra.

APPENDIX C: DERIVATION OF EQ. (14)—CLUSTER ENERGY

The eigenvalues of Eq. (3) determine, as usual, the energy of an eigenstate. Inserting an eigenstate [Eq. (9)] defined by a vector $\boldsymbol{k}$ into Eq. (3), the eigenenergy assumes the familiar form $\epsilon = \frac{\hbar^2}{2m} \sum_{j=1}^{n_k} k_j^2$, where n_k is the number of elements of $\boldsymbol{k}$. To derive the energy of a single cluster, Eq. (14), we insert the general form of a momentum vector describing a single cluster,

$$\boldsymbol{\mu} = \left(K_\mu + i\frac{\eta}{2}(n_\mu - 2j + 1) \right)\Big|_{j=1}^{n_\mu}, \tag{C1}$$

into the mentioned equation to obtain

$$\begin{aligned} \epsilon_{\boldsymbol{\mu}} = \frac{\hbar^2}{2m} \Bigg(& n_\mu K_\mu^2 + i\eta K_\mu \sum_{j=1}^{n_\mu} (n_\mu - 2j + 1) \\ & - \frac{\eta^2}{4} \sum_{j=1}^{n_\mu} (n_\mu - 2j + 1)^2 \Bigg), \end{aligned} \tag{C2}$$

where n_μ is the number of elements of $\boldsymbol{\mu}$. Using $\sum_{j=1}^{n_\mu} j = \frac{n_\mu(n_\mu+1)}{2}$ and $\sum_{j=1}^{n_\mu} j^2 = \frac{1}{6} n_\mu(n_\mu+1)(2n_\mu+1)$, we see that the imaginary part vanishes and obtain Eq. (14) with

$$\epsilon_{\boldsymbol{\mu}} = \frac{\hbar^2}{2m} \left(n_\mu K_\mu^2 - \frac{1}{12}\eta^2 (n_\mu - 1) n_\mu (n_\mu + 1) \right). \tag{C3}$$

APPENDIX D: INTERPRETATION OF ANYON CLUSTERS AS INDIVIDUAL ANYONS

We want to show how the exchange phase of clusters can be interpreted as the statistical phase of a composite species of anyons reaching further than the interpretation supported by Eq. (11). To this end, we consider two clusters of anyons $\boldsymbol{\mu_1} = (K_1 + i\eta/2, K_1 - i\eta/2)$ and $\boldsymbol{\mu_2} = (K_2 + i\eta/2, K_2 - i\eta/2)$, the cluster structures of which are depicted in Fig. 1(b). We introduce the center-of-mass coordinates $X_1 = (x_1 + x_2)/2$ and $X_2 = (x_3 + x_4)/2$, as well as the relative coordinates $Z_1 = (x_2 - x_1)/2$ and $Z_2 = (x_4 - x_3)/2$. Under the assumption that the two clusters are sufficiently far away from each other, i.e., $X_2 - X_1 \to \infty$ and Z_1, Z_2 finite, we obtain

$$\begin{aligned} &\Psi_{(\mu_1,\mu_2)}(X_1, X_2, Z_1, Z_2) \\ &\quad \propto [e^{2i(K_1X_1 + K_2X_2)} + e^{i\varphi^{\boldsymbol{\mu_1},\boldsymbol{\mu_2}}_\eta} e^{2i(K_2X_1 + K_1X_2)}] e^{\eta(Z_1+Z_2)}. \end{aligned} \tag{D1}$$

This wave function resembles a wave function of two composite anyons with an altered statistical phase of $\varphi^{\boldsymbol{\mu_1},\boldsymbol{\mu_2}}_\eta$, especially if we recall that Z_1 and Z_2 are of the order of $1/\eta$. This can be physically interpreted as the fusion of anyons to clusters which themselves behave as a composite anyon species. Interestingly, the combined statistical phase is

$$\varphi^{\boldsymbol{\mu_1},\boldsymbol{\mu_2}}_\eta = 2\phi_\eta(K_2 - K_1) + \phi_{2\eta}(K_2 - K_1), \tag{D2}$$

where ϕ_η is the statistical phase defined in Eq. (8). This has an appealing geometric interpretation, which we depict in Fig. 3.

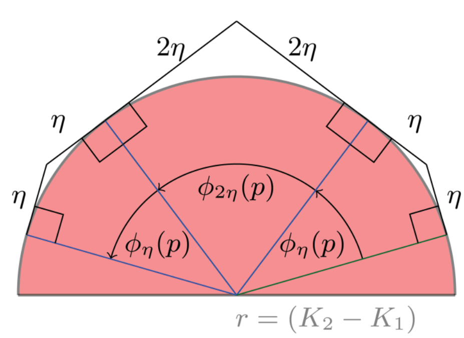

FIG. 3. Geometric interpretation of the statistical phase of two clusters, each consisting of two anyons. The radius of the circle denotes the relative momentum between clusters $K_2 - K_1$. The statistical angle is obtained by adding up three summands. Two of these summands are the normal statistical angle ϕ_η, and the third summand is the statistical angle of the doubled statistical parameter $\phi_{2\eta}$. The statistical parameter η appears in the lengths of the drawn tangential segments.

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