Ising model in planar lacunary and fractal lattices: A path counting approach

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The method of counting loops for calculating the partition function of the Ising model on the two dimensional square lattice is extended to lacunary planar lattices, especially scale invariant fractal lattices, the Sierpiński carpets with different values of the scale invariance ratio and of the number of deleted sites. The critical temperatures T_c are exactly calculated for finite iteration steps k up to $k = 8$, for a range of the scale invariance ratio n from 3 to 1000 and of the number of deleted sites from $(n-2)^2$ to $(n-10)^2$. The critical parameters $v_c = \tanh(1/T_c)$ satisfy power laws of exponent -k, allowing the asymptotic extrapolation to the fractal limit $k \to \infty$. The extrapolated values of the critical temperature fit quite reliably power laws versus the fractal dimension. Results are compared to previous estimations obtained by numerical methods. Thermodynamical functions are also calculated and the fractal spectra of the Ising partition functions on several examples of Sierpiński carpets are illustrated.

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I. INTRODUCTION

The question of phase transitions on structures of noninteger dimension has a long history since the early work of Mandelbrot introducing fractals in physics. Among the many applications of fractals in most fields of physics, they have been used as models of structures of noninteger dimension for comparing scaling properties to results given by the usual theory of renormalization in which dimensionality plays a crucial role. Translational invariance is an important hypothesis of the renormalization approach, not satisfied by structures of noninteger dimension which are generally scale invariant rather than translationally invariant. Then unusual behavior may be expected.

The Sierpiński carpet [1], and its generalizations to any scale invariance ratio, is one of the simplest infinitely ramified fractal structures. An infinite ramification order is a mandatory condition to get a nonzero critical temperature to phase transitions [2,3]. An appropriate choice of the scale invariance ratio and of the number and positions of removed sites at each iteration allows one to approach any noninteger value comprised between 1 and 2 for the fractal dimension and a large range of topologies. Then the combination of Sierpiński carpets and the Ising model, one of the simplest models of phase transitions, has been extensively studied to investigate the properties of phases transitions in noninteger structures, even if recently other topologies have been also investigated [4].

Calculations involve different methods including the real space renormalization group (RSRG) [3,5–8]; high temperature expansions [8]; Monte-Carlo simulations either with the Metropolis or with the Wolf or the Swendsen-Wang algorithms combined with a finite size scaling analysis (MCFS) [9–18]; Monte-Carlo simulations combined with a renormalization group (MCRG) [19]; and Monte-Carlo simulations combined with short time dynamic scaling (MCSD) [20–23]. According to references, spins are located either on the center of the sites [12–18,21–23] or on the vertices of the lattice [2,3,5–12,20]. Some attempts have been done also on structures extending Sierpiński carpets in three dimensions: the Menger sponge and its generalizations to any scale invariance ratio [10,15]. One important goal was to compare the critical properties of the Ising model to those obtained by the analytical continuation of ε expansions of the renormalization approach in noninteger dimensions [24], critical exponents, and scaling relations.

One of the difficulties of the Monte-Carlo simulations is to obtain an accurate estimation of the critical temperature, which is necessary to perform the finite size scaling analysis. As fractals are not translationally invariant, the topology changes between two successive iterations (or segmentation steps) of the fractal structure. The mean number of nearest neighbors is not the same [16,25]) and consequently critical temperatures are also different, leading to "topological scaling corrections" [16]. Moreover, Pruessner *et al.* [17] questioned the relevance of finite size scaling performed on the successive segmentation steps of a single site, arguing that the size of the structure is not large enough to exceed the correlation length in the critical region. These authors developed an alternative approach based on the juxtaposition of several identical networks in the two dimensions of space to increase the size of the elementary cell before applying the usual periodic boundary conditions. For a scale invariance ratio $n = 3$ with a single centered removed site, the values of the critical temperatures obtained after this modification are close to those obtained previously, but significantly different according to their respective accuracy, 1.50 instead of 1.48. Table I summarizes the values from literature of the critical temperature of various Sierpiński carpets, with periodic boundary conditions (PBC) and spins located on the center of sites, which corresponds to the case treated in this paper. Estimations concentrate close to the same two previous significantly different values, either 1.48 [13,14,16,19] or 1.50 [17,21,23], leaving a debate opened.

To bring some light on this question, we calculate analytically the critical temperature of several Sierpiński carpets by a path counting method. We first remember the loops counting method on square lattices (lacunary or not) which has been introduced by Kac and Ward [26] as an alternative to the algebraic method of Onsager [27] to calculate the partition function of the two dimensional Ising on the square lattice. Then this method is applied to Sierpiński carpets with a central hole, but the method is suitable also for other topologies. This leads to the exact calculation of the critical temperatures for values of the segmentation ratio n from 3 to 1000, a

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TABLE I. Previous estimations of critical temperatures of $SC(n,p)$ with PBC and spins on the center of sites.

SC	Authors	T_c	Method	$k_{\max}$
	Bonnier *et al.* [12]	1.54	MCFS	3
	Monceau *et al.* [14]	1.482(15)	MCFS	7
	Carmona *et al.* [13]	1.481(1)	MCFS	7
$SC(3,1)$	Monceau *et al.* [16]	1.4795(5)	MCFS	8
	Pruessner *et al.* [17]	1.4992(11)	MCFS	6
	Hsiao *et al.* [19]	1.47946(16)	MCRG	8
	Bab *et al.* [21]	1.4945(50)	MCSD	6
	Bab *et al.* [23]	1.495(5)	MCSD	6
	Bonnier *et al.* [12]	1.25	MCFS	3
	Carmona *et al.* [13]	1.077(3)	MCFS	6
$SC(4,2)$	Monceau *et al.* [16]	<1.049	MCFS	6
	Monceau *et al.* [18]	1.13873(8)	MCFS	6
	Bab *et al.* [22]	1.10(1)	MCSD	6
	Bab *et al.* [23]	1.10(1)	MCSD	6
	Bonnier *et al.* [12]	2.06	MCFS	3
$SC(5,1)$	Monceau *et al.* [16]	2.0660(15)	MCFS	5
	Bab *et al.* [23]	2.067(2)	MCSD	6
	Monceau *et al.* [16]	<0.808	MCFS	5
$SC(5,3)$	Monceau *et al.* [18]	0.96143(11)	MCFS	5
	Bab *et al.* [23]	0.83(2)	MCSD	5
$SC(6,4)$	Bab *et al.* [23]	0.70(5)	MCST	4

number p of removed sites at each step from 1 to $n-2$ and up to to 8 iterations fractal steps. The full spectrum of the partition function on several Sierpiński carpets is also calculated and illustrated up to five iterations fractal steps. Finally thermodynamic functions of the Ising model on these structures are addressed.

Notations used in this paper are the following: $SQ(N)$ is the usual square lattice (without empty sites) of size $N\times N$. Parameters concerning the square lattice at the limit $N\to\infty$ will be indexed by sq. Generally the lacunary or fractal lattices investigated in this paper are obtained by the juxtaposition of N^2 identical patterns in a translationally invariant way in both directions of the plane. The reproduced pattern will be called the generating pattern. The number n is devoted to internal characteristics of the generating pattern. $SL(N,n)$ is the lacunary (nonrandom) square lattice of size $N\times N$ with periodic holes of size $(n-1)\times(n-1)$ separated by a single raw of occupied sites in both directions of the plane. Figure 1 shows the example of $SL(N,3)$; occupied sites are in gray, and the generating pattern is in dark gray.

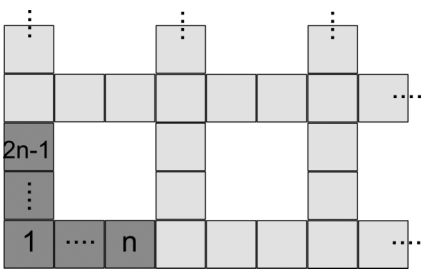


FIG. 1. $SL(N,n)$ for $n=3$; occupied sites are in gray, and the generating pattern is in dark gray.

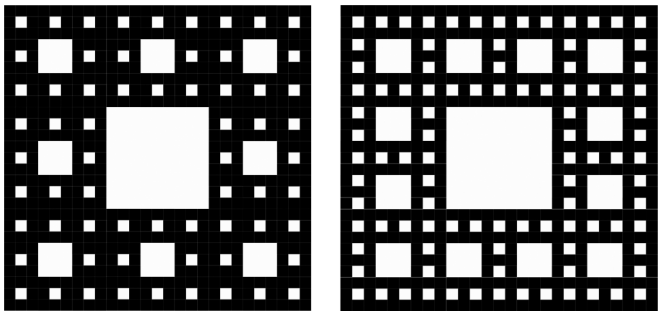


FIG. 2. The third iteration steps of $SC(3,1,3)$ (left) and $SM(3,3)$ (right).

We have obviously $SQ(N)=SL(N,1)$. Sierpiński carpets of segmentation ratio n, with p^2 removed central subsites at each segmentation step, and after the kth iteration of this segmentation process, is noted $SC(n,p,k)$. The segmentation step can be omitted when not necessary for understanding: $SC(n,p)$. Figure 2 (left) illustrates the third segmentation step of $SC(3,1)$. We investigate also a different topology: a more lacunary fractal lattice with the same segmentation ratio $n=3$; the generating pattern is illustrated on Fig. 2 (right). It is not strictly scale invariant since it is not exactly the reproduction of the same pattern at each segmentation step; its fractal dimension is $\ln[(9+\sqrt{33})/2]/\ln 3\approx 1.8184$, it will be noted $SM(3,k)$ (as Sierpiński modified). Sites of the generating pattern of Sierpiński sets are numbered continuously, row by row, notwithstanding the empty sites, as illustrated in Fig. 3. The identity matrix of appropriate dimension is noted as I and d_h is the Hausdorff dimension. FBC means free boundary conditions.

II. PARTITION FUNCTION OF THE ISING MODEL AND CLOSED PATHS COUNTING

Generally, the partition function of the Ising model on any network is [28]

$$Z=\sum_{\{s_i\}}\prod_{i>j}e^{-\beta J s_i s_j},$$

where the sum runs over all spin configurations, and the product over all pairs of nearest neighbors spins. J characterizes the strength of the interaction between spins. It is an equivalent to the problem of counting closed paths (loops) on the network [26]:

$$Z=2^{\mathcal{N}}\cosh\left(\frac{1}{T}\right)^{\mathcal{N}_z}\sum_{l=0}^{\infty}g(l)v^l, \tag{1}$$

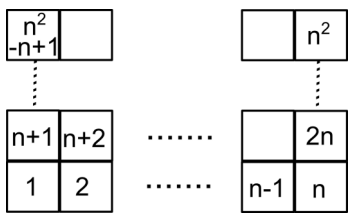


FIG. 3. The numbering of sites of a generating pattern of Sierpiński sets with a segmentation ratio n.

where $\mathcal{N}$ is the number of spins, $\mathcal{N}_z$ is the number of nearest neighbors pairs, T is the reduced temperature: $1/\beta J$, $v = \tanh(\beta J) = \tanh(1/T)$, and $g(l)$ is the number of loops of length l (counted as a multiple of the distance between two neighboring sites). In the next developments, we will call $\mathcal{P}(v) = \sum_{l=0}^{\infty} g(l) v^l$ the *partition polynomial*.

Kac and Ward [26] implemented this method to $SQ(N)$ in the following way (a variant using Pfaffians has also been proposed [29,30]). The basic feature is to attribute to each site a coefficient which connects the direction from where the path enters the site to the direction it goes out in order that the product of coefficients of all sites of a closed, non-self-intersecting path gives -1. The appropriate set of coefficients are 1 if the path continues in the same direction, $e^{\frac{i\pi}{4}}$ for a left turn, $e^{\frac{-i\pi}{4}}$ for a right turn, and 0 for the backward direction. On a finite lattice with FBC, this procedure ensures that, when performing the product over sites of all paths and the sum over all possible paths, each closed non-self-intersecting loop is counted the number of times it should be [31,32]. To implement this counting method on the whole lattice, four 4×4 matrices are introduced linking each site to its nearest neighbors. They contain the appropriate coefficients placed according to the directions "in" and "out," and they are indexed by the direction of the site from where the path is coming from, ordered as $-x$, $-y$, y, x:

$$M_{-x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{4}} & 1 \end{pmatrix}, \quad M_{-y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-\frac{i\pi}{4}} & 0 & 1 & e^{\frac{i\pi}{4}} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$M_{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{\frac{i\pi}{4}} & 1 & 0 & e^{-\frac{i\pi}{4}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{x} = \begin{pmatrix} 1 & e^{-\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

These matrices are themselves placed in a matrix $\mathcal{M}$ of all sites and the partition polynomial is

$$\mathcal{P}_N(v) = \sqrt{|I - v\mathcal{M}|}\,.$$

The spectrum of the partition function (including the root corresponding to the critical temperature) is given by the set of roots of this polynomial in the variable v.

This process may be extended to any planar lacunary square lattices, setting to 0 the 4×4 matrices corresponding to empty sites. We calculate here the partition function of $SL(N,n)$. The generating pattern has $2n-1$ occupied sites (Fig. 1); the dimension of the matrix $\mathcal{M}_{N,n}$ is $4N(2n-1)$. PBC make $\mathcal{M}_{N,n}$ cyclic which allows one to calculate $|I - v\mathcal{M}_{N,n}|$ as the product of N^2 $4(2n-1)\times4(2n-1)$ determinants according to a process which has been explained in many references for $n = 1$ [26,28,33]. Let us note

$$M'_{-x} = e^{\frac{ip\pi}{N}} M_{-x}, \quad M'_{-y} = e^{\frac{iq\pi}{N}} M_{-y},$$
$$M'_{y} = e^{-\frac{iq\pi}{N}} M_{y}, \quad M'_{x} = e^{-\frac{ip\pi}{N}} M_{x};$$

the corresponding determinant is the product over p and q varying from 1 to N of determinants of the following matrices:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n-1 \\ n \\ n+1 \\ n+2 \\ \vdots \\ 2n-2 \\ 2n-1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n-2 & 2n-1 \end{array} \\ \begin{pmatrix} 0 & M_{-x} & 0 & \dots & 0 & M'_x & M_{-y} & 0 & \dots & 0 & M'_y \\ M_x & 0 & M_{-x} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & M_x & 0 & \ddots & \ddots & & & & & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & \vdots & \ddots & \ddots & 0 & M_{-x} & \ddots & & & & \vdots \\ M'_{-x} & \vdots & & \ddots & M_x & 0 & 0 & \ddots & & & \vdots \\ M_y & \vdots & & & \ddots & 0 & 0 & M_{-y} & \ddots & & \vdots \\ 0 & \vdots & & & & \ddots & M_y & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & & & & & & \ddots & \ddots & 0 & M_{-y} \\ M'_{-y} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & M_y & 0 \end{pmatrix} \end{array}.$$

Given that their eigenvectors of eigenvalue λ are $V=(V_i)_{1\leqslant i\leqslant 2n-1}$ where the $(2n-1)$ four-vectors V_i (one by site of the generating pattern) have the following structure,

$$\text{for } 1<i\leqslant n, \qquad \text{for } n<i\leqslant 2n-1,$$

$$V_1=\begin{pmatrix}a\\b\\c\\d\end{pmatrix},\quad V_i=\begin{pmatrix}a\lambda^{(n-i+1)}e^{\frac{2i\pi p}{N}}\\0\\0\\d\lambda^{(i-1)}\end{pmatrix},\quad V_i=\begin{pmatrix}0\\b\lambda^{(2n-i)}e^{\frac{2i\pi q}{N}}\\c\lambda^{(i-n)}\\0\end{pmatrix},$$

and noting

$$M(N,p,q)=M'_{-x}+M'_{-y}+M'_{y}+M'_{x}\,, \tag{3}$$

the final product is

$$P_{N,n}(v)=\left[\prod_{p,q=1}^{N}|I-v^nM(N,p,q)|\right]^{\frac{1}{2}}$$
$$=\left[\prod_{p,q=1}^{N}(1+v^{2n})^2-2v^n(1-v^{2n})f(N,p,q)\right]^{\frac{1}{2}}, \tag{4}$$

with $f(N,p,q)=\cos\frac{2p\pi}{N}+\cos\frac{2q\pi}{N}$.

It should be noticed that $P_{N,n}(v)=P_{N,1}(v^n)$. The lacunary square lattice is equivalent to a filled square lattice in which the parameter v is replaced by v^n. That results from the structure of the generating pattern of the lacunary square lattice which funnels paths going through in a single direction, either Ox or Oy. Changes of direction can occur only on the first site of the pattern, as if there was a single site. Compared to a single site pattern, only the length of paths changes, not the topology. This equivalence will be exploited later for the calculation of the magnetization.

However, $P_{N,n}(v)$ is not exactly the partition polynomial $\mathcal{P}_{N,n}(v)$ of the Ising model on $SL(N,n)$. The process of indexing loops is rigorously demonstrated for FBC while PBC are necessary to calculate the determinant. PBC introduce loops looping the torus (Fig. 4) for which the product of all sites coefficients is 1 and not -1 as it should be.

These boundary effects vanish at the limit $(N\to\infty)$ giving the exact partition function of the infinite lattice, but they cannot be ignored when finite size lattices are concerned. For this reason, we will call $P_{N,n}(v)$ the *pseudopartition polynomial*. Additional terms necessary to obtain the exact partition polynomial $\mathcal{P}_{n,N}$ of the Ising model on $SQ(N)$ with PBC have been subsequently calculated [34]; however they will not play a role in the next developments of this paper.

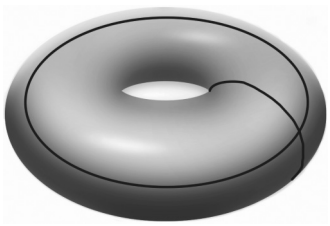


FIG. 4. Two paths looping the torus which are wrongly counted in PBC.

A consequence is that the spectrum of the pseudopartition polynomial of $SL(N,n)$ is immediately deduced from the spectrum of the pseudopartition polynomial of $SQ(N)$. For any value of N, all roots v^n are located on two circles in the complex plane, centered, respectively, on -1 and 1 and of radius $\sqrt{2}$ [35]. Figure 5 shows these two circles and plots of the spectrum of the pseudopartition polynomial $P_5(v)$ (squares) and of the exact partition polynomial $\mathcal{P}_5(v)$ (triangles) of $SQ(5)$. The spectrum of $\mathcal{P}_N(v)$ converges toward the two circles when $N\to\infty$.

Figure 6 shows the plot of $P_{N,n}(v)$ versus v^n for $N=1$ to $N=7$. For all values of N, $P_{N,n}(v)$ vanishes for the same value of v^n. Indeed, all $P_{N,n}(v)$ share the common factor $P_{1,n}(v)$, the partition polynomial of a "single pattern lattice" (a "single site" lattice when $n=1$):

$$P_{1,n}(v)=1-2v^n-v^{2n}, \tag{5}$$

whose positive real root is $v_c=(\sqrt{2}-1)^{1/n}$, giving the critical temperature at the thermodynamic limit (this root is the same for all N; then also for the limit $N\to\infty$):

$$T_c=\frac{1}{\text{Atanh}(v_c)}=\frac{2}{\ln\left(\frac{1+(\sqrt{2}-1)^{\frac{1}{n}}}{1-(\sqrt{2}-1)^{\frac{1}{n}}}\right)}. \tag{6}$$

For $n=1$, $v_{csq}=\sqrt{2}-1$ gives the well known value:

$$T_{csq}=\frac{2}{\ln(1+\sqrt{2})}\approx 2.2692.$$

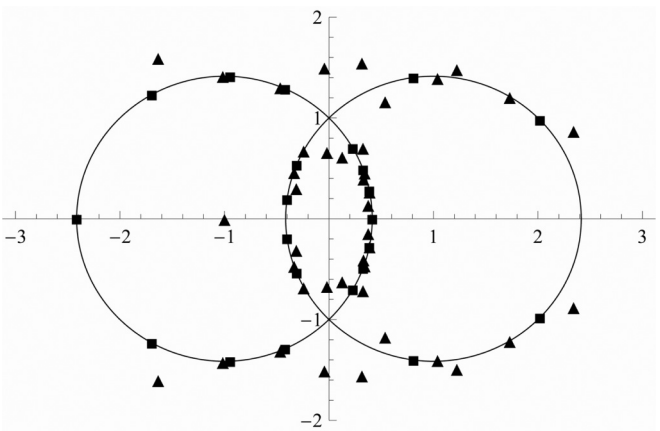


FIG. 5. Combined plots in the complex plane of the roots of $P_5(v)$ (squares), $\mathcal{P}_5(v)$ (triangles) and the two circles which contains the roots of $P_N(v)$, for all N.

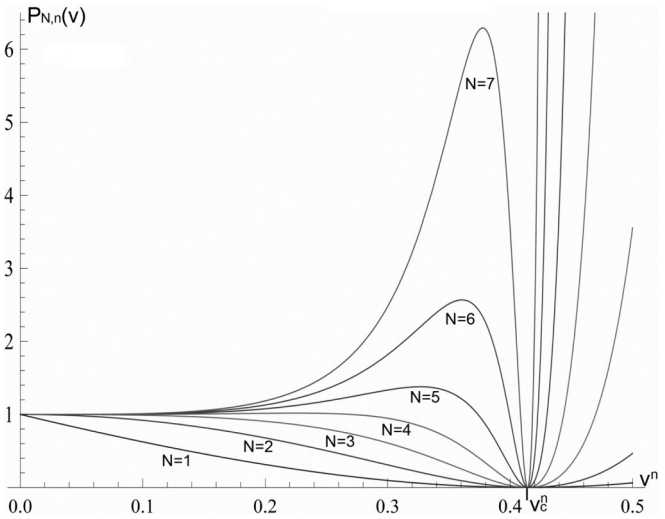


FIG. 6. The pseudopartition polynomial $P_{N,n}(v)$ of the Ising model on $SL(N,n)$ versus v^n for $1 \leqslant N \leqslant 7$.

Then a fast and easy way to calculate the critical temperature of any square lattice (lacunary or not) at the limit $N \to \infty$ is to find the real positive root of the pseudopartition polynomial of a single pattern lattice. The pseudopartition polynomial can be considered as a partition polynomial renormalized of the boundaries effects. This property will be used to calculate the critical temperature of Sierpiński carpets, which is the purpose of the next section.

III. PARTITION FUNCTION OF THE ISING MODEL ON $SC(n,p,k)$

Investigated lattices are composed by the juxtaposition of N^2 patterns $SC(n,p)$ in the two directions of the plane, with PBC. The generating pattern has $(n^2-p^2)^k$ occupied sites. The matrix $\mathcal{M}_{SC(n,p,k)}$ is obtained from the matrix $\mathcal{M}_{(nN)^{2k}}$ of the corresponding square lattice by setting to 0 the 4×4 matrices of each removed site. Lines and columns corresponding to these empty sites can be omitted, which reduces the dimension of the matrix to $4N^2(n^2-p^2)^k$. As for the square lattice, the determinant may be factorized by translational invariance in a product of N^2 determinants of dimension $4(n^2-p^2)^k$. Detailed calculations are performed for $SC(3,1,1)$; other values of n, p, and k are treated in a similar way.

A. The pseudopartition polynomial of $SC(3,1,1)$

In $SC(3,1,1)$ a single site is removed. Once reduced by translational invariance, the pseudopartition polynomial is the square root of the product of the N^2 $32N^2\times32N^2$ determinants of the following matrices (the site number 5, which is the single empty site, is omitted):

$$M_{N,SC(3,1,1)}(p,q) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & M_{-x} & M'_x & M_{-y} & 0 & M'_y & 0 & 0 \\ M_x & 0 & M_{-x} & 0 & 0 & 0 & M'_y & 0 \\ M'_{-x} & M_x & 0 & 0 & M_{-y} & 0 & 0 & M'_y \\ M_y & 0 & 0 & 0 & M'_x & M_{-y} & 0 & 0 \\ 0 & 0 & M_y & M'_{-x} & 0 & 0 & 0 & M_{-y} \\ M'_{-y} & 0 & 0 & M_y & 0 & 0 & M_{-x} & M'_x \\ 0 & M'_{-y} & 0 & 0 & 0 & M_x & 0 & M_{-x} \\ 0 & 0 & M'_{-y} & 0 & M_y & M'_{-x} & M_x & 0 \end{pmatrix} \end{array}, \tag{7}$$

leading to the pseudopartition polynomial

$$\begin{aligned} P_{N,SC(3,1,1)}(v) &= \prod_{p,q=1}^{N} |I - v.M_{N,SC(3,1)}(p,q)|^{\frac{1}{2}} \\ &= \prod_{p,q=1}^{N} \left[Q_1(v) + Q_2(v)\left(\cos\left(\frac{2\pi p}{N}\right) + \cos\left(\frac{2\pi q}{N}\right)\right) + Q_3(v)\cos\left(\frac{2\pi p}{N}\right)\cos\left(\frac{2\pi q}{N}\right) \right. \\ &\quad \left. + Q_4(v)\left(\cos\left(\frac{2\pi p}{N}\right)^2 + \cos\left(\frac{2\pi q}{N}\right)^2\right)\right]^{\frac{1}{2}}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} Q_1(v) &= (1+v^2)^2(1-2v^2+13v^4-8v^6+126v^8+108v^{10}+474v^{12}+248v^{14}+57v^{16}+6v^{18}+v^{20}), \\ Q_2(v) &= -4v^3(1-v^2)^2(1+3v^2)(1+v^2+2v^4)(1+2v^2+8v^4+4v^6+v^8), \\ Q_3(v) &= -4v^6(1-v^2)^4(1+v^2)(7+11v^2+13v^4+v^6), \\ Q_4(v) &= +4v^6(1-v^2)^5(1+v^2)(1+3v^2). \end{aligned}$$

The critical temperature is given by the real positive root of

$$P_{1,SC(3,1,1)}(v) = 1 - 4v^3 + 5v^4 - 16v^5 - 10v^6 - 20v^7 + v^8 - 24v^9 + 2v^{10} + v^{12}, \tag{9}$$

which is $v_c \approx 0.4960$ giving $T_c = 1/\mathrm{Atanh}(v_c) \approx 1.8384$.

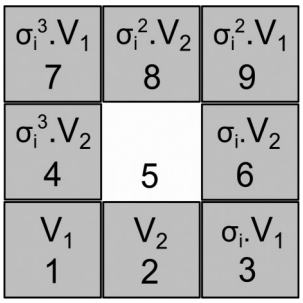


FIG. 7. $SC(3,1,1)$ with site numbers (down) and the corresponding part of the ith eigenvector of $M_{SC(3,1,1)}$ (up).

For small values of n and/or small values of k, $P_{1,SC(n,p,k)}$ may be calculated explicitly and roots are calculated numerically. For higher values of n or k, the spectrum may be deduced directly from the numerical evaluation of the eigenvalues of the exact matrix. But for very large values of n or k, the dimension of the matrix increases quickly. To get access to the spectrum of the partition function, the dimension of the matrix should be reduced and this can be achieved in two ways. First exploiting the square symmetry of the generating pattern, that is, its invariance under the group $D4$ (especially the rotation of angle $\pi/4$). Second, by removing the zero eigenvalues which come from dangling bonds of sites contiguous to empty sites. We will investigate successively these two ways.

B. Rotational invariance

The appropriate four dimensional representation of a rotation of $\pi/4$ is generated by a 4×4 matrix of the form

$$\sigma = \alpha \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{10}$$

which achieves a circular permutation of the four directions in the plane, with the constraint that $\sigma^4 = I$ leading to the condition $\alpha^4 = 1$. There are four possible values of α: $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = i$, $\alpha_4 = -i$ and four corresponding matrices σ_i, according to Eq. (10).

The eigenvectors of the matrix $M_{1,SC(n,p,k)}$ are composed of $(n^2-p^2)^k$ four-vectors numbered according to sites $V_1, V_2, \ldots, V_{n^2}$ (with the exclusion of empty sites) which can be associated four by four as the images of an initial eigenvector by the successive rotations σ_i, σ_i^2, σ_i^3. For $SC(3,1,1)$, this gives (Fig. 7)

$$V_i = \left(V_1, V_2, \sigma_i.V_1, \sigma_i^3.V_2, \sigma_i.V_2, \sigma_i^3.V_1, \sigma_i^2.V_2, \sigma_i^2.V_1\right).$$

Then, for $SC(n,p,k)$ (a single pattern lattice) with a total number of nonempty sites $(n^2-p^2)^k$, the $4(n^2-p^2)^k \times 4(n^2-p^2)^k$ determinant is transformed in the product of four $(n^2-p^2)^k \times (n^2-p^2)^k$ determinants. For $SC(3,1,1)$, we obtain the four matrices

$$M_{SC(3,1,1),i}(v) = \begin{pmatrix} M_x.\sigma_i + M_y.\sigma_i^3 & M_{-x} + M_{-y}.\sigma_i^3 \\ M_x + M_{-x}.\sigma_i & M_y.\sigma_i^2 \end{pmatrix}.$$

The pseudopartition polynomial of $SC(3,1,1)$ is

$$P_{1,SC(3,1,1)}(v) = \sqrt{\prod_{i=1}^{4} |I - vM_{SC(3,1,1),i}|}.$$

Polynomials $|I - vM_{SC(3,1,1),i}|$ are equal two by two:

$$|I - vM_{SC(3,1,1),1}| = |I - vM_{SC(3,1,1),3}| = Q^-_{SC(3,1,1)}(v),$$
$$|I - vM_{SC(3,1,1),2}| = |I - vM_{SC(3,1,1),4}| = Q^+_{SC(3,1,1)}(v),$$

with

$$Q^{\pm}_{SC(3,1,1)}(v) = 1 \pm \sqrt{2}v + (1 \pm \sqrt{2})v^2 \pm \sqrt{2}v^3 + (5 \pm 3\sqrt{2})v^4 \pm 2\sqrt{2}v^5 + v^6,$$

so that Eq. (9) can be written

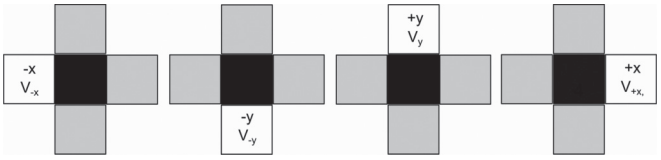
$$P_{1,SC(3,1,1)}(v) = Q^+_{SC(3,1,1)}(v).Q^-_{SC(3,1,1)}(v).$$

For all investigated lattices $SC(n,p,k)$, a single real root of the pseudopartition polynomial occurs between 0 and 1, which avoids any ambiguity to identify the critical temperature among all other roots, and it is always a root of Q^-.

C. Dangling bonds

To each site contiguous to an empty site, the direction of a path going to the empty site gives a zero eigenvalue corresponding to an eigenvector with all components equal to zero except the four components located on the position of the corresponding site. According to the position of the empty site relative to the position of the considered site: $-x$, $-y$, y, x, the four nonzero components of the eigenvectors are V_{-x}, V_{-y}, V_y, V_x, respectively, with their illustration below (the considered sites in black; occupied sites in gray; the empty sites in white):

$$\begin{pmatrix} 0 \\ e^{-\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} e^{\frac{i\pi}{4}} \\ 0 \\ -1 \\ e^{-\frac{i\pi}{4}} \end{pmatrix}, \quad \begin{pmatrix} e^{-\frac{i\pi}{4}} \\ -1 \\ 0 \\ e^{\frac{i\pi}{4}} \end{pmatrix}, \quad \begin{pmatrix} -1 \\ e^{\frac{i\pi}{4}} \\ e^{-\frac{i\pi}{4}} \\ 0 \end{pmatrix}.$$



Factorizing the part of the matrix corresponding to these eigenvectors reduces the dimension of the determinants (already reduced by rotational invariance) by two times the number of dangling bonds, that is, from $(n^2-p^2)^k$ to

$$(n^2-p^2)^k - 2p\frac{(n^2-p^2)^k - n^k}{n^2-p^2-n} + 2p^2\frac{(n^2-p^2)^{k-1} - n^{k-1}}{n^2-p^2-n}$$

for $n-p=2$, or

$$(n^2-p^2)^k - 2p\frac{(n^2-p^2)^k - n^k}{n^2-p^2-n}$$

for other values of $n-p$.

TABLE II. Critical temperatures of the Ising model on $SC(3,1,k)$ for all investigated values of k. Values which are between two columns correspond to evaluations obtained from crossing of Binder cumulants of two successive segmentation steps.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=\infty$
Monceau and Perreau [16]				1.486	1.482	1.480			1.4795
Carmona *et al.* [13]			1.724	1.590	1.538	1.511	1.497		1.481
Pruessner *et al.* [17]				1.5266	1.5081	1.4992			
This work	1.83842	1.65386	1.57659	1.52566	1.50446	1.49331	1.48719	1.48371	1.4803

IV. CRITICAL TEMPERATURES OF SIERPIŃSKI CARPETS

Critical temperatures of $SC(3,1)$ for finite values of k and their extrapolations to $k \to \infty$ from the literature (the few references which give the detailed values for each investigated finite k) and from this work are compared in Table II. Values obtained by Pruessner *et al.* [17] are the most accurate with a difference less than 0.27%. This is coherent since it is the single reference which uses multiple translationally invariant reproductions of a generating pattern. For $T_c(k=\infty)$, references obtaining results close to 1.48 [13,14,16,19] are more accurate that those obtaining results close from 1.50 [21,23].

Table IV in the Appendix gives all calculated critical temperatures for a wide range of values of n, p, and k. The corresponding values of $v_c - v_{csq}$ are plotted versus the segmentation step k for $p=n-2$, $p=n-4$, $p=n-6$, and $p=n-8$ in Fig. 8. $v_c - v_{csq}$ fits quite reliably a power law of exponent $-k$:

$$v_c - v_{csq} = v_c - \sqrt{2} + 1 = a(1+b^{-k}), \qquad (11)$$

where the parameters a and b depend on n and p and are listed in Table III. The least square linear fits are shown in dotted lines and have determination coefficients R^2 which differ from 1 by less than 10^{-4}. However, no such power law is satisfied by $SM(3,1,k)$; the strict scale invariance seems necessary to get Eq. (11).

Once a and b determined, the critical temperature for $k \to \infty$ can be extrapolated from Eq. (11): $v_c(\infty) = v_{csq} + a$. The values are listed in the last column of Table III. Figure 9 (left) shows the plots of $\ln[T_c(\infty)]$ versus d_h for $p=n-2$, $n-4$, $n-6$, and $n-8$. The curves are approximately linear giving

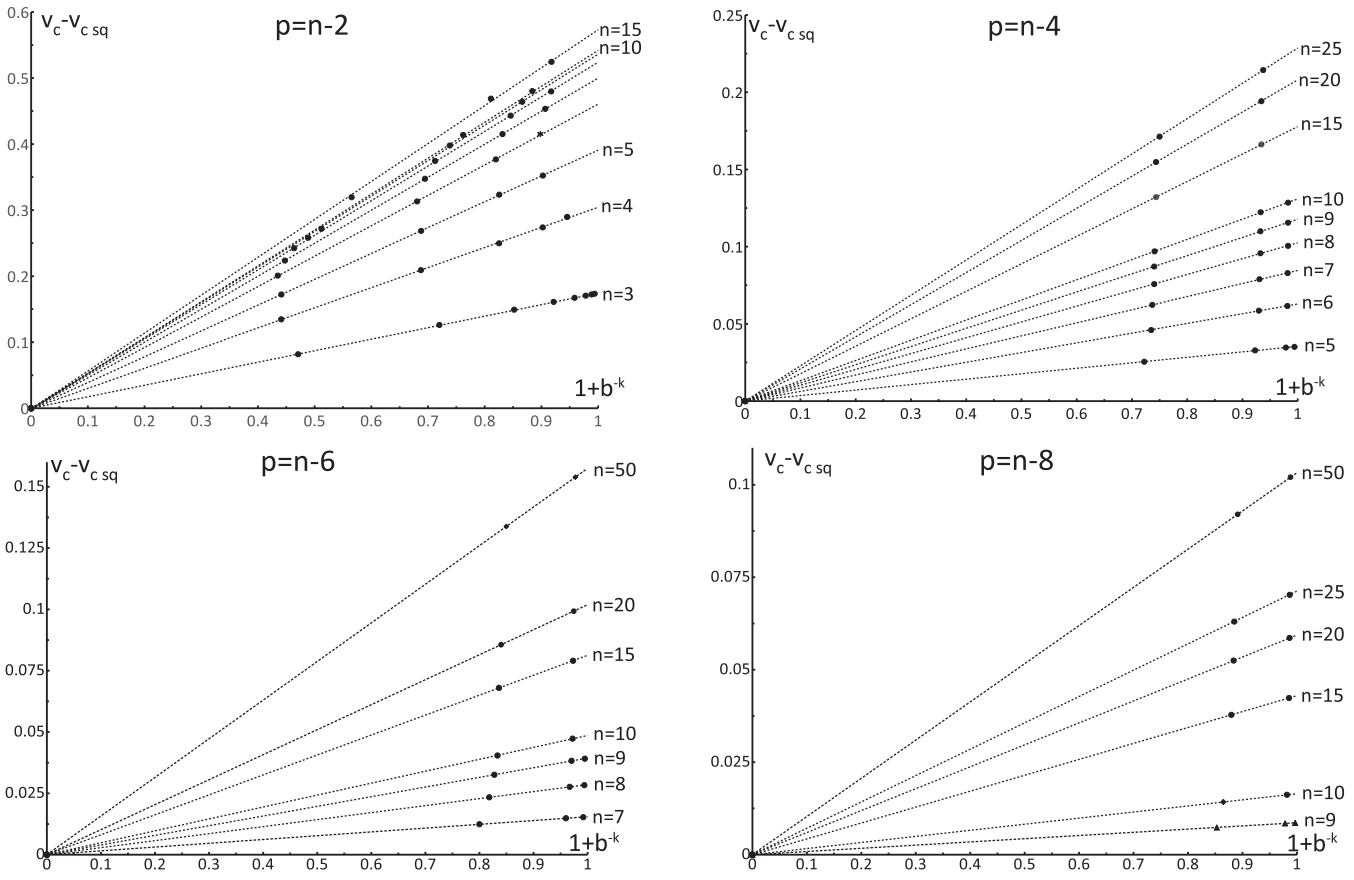


FIG. 8. $v_c - v_{csq}$ versus $(1+b^{-k})$ for $SC(n,p,k)$. Slopes a and values of b are given in Table III.

TABLE III. Values of a, b of Eq. (11) and of $T_c(k=\infty)$.

$p=n-2$				$p=n-4$			
n	a	b	$T_c(\infty)$	n	a	b	$T_c(\infty)$
3	0.1747	1.892	1.4803	5	0.0355	3.60	2.0647
4	0.3013	1.789	1.1131	6	0.0628	3.77	1.9264
5	0.3909	1.790	0.8985	7	0.0846	3.80	1.8258
6	0.4579	1.770	0.7453	8	0.1023	3.85	1.7496
7	0.4996	1.810	0.6451	9	0.1177	3.85	1.6869
8	0.5204	1.865	0.5905	10	0.1309	3.86	1.6356
9	0.5365	1.96	0.5438	15	0.1778	3.90	1.4690
10	0.5501	2.00	0.4990	20	0.2080	3.90	1.3725
15	0.3828	2.35	0.4459	25	0.2286	4.00	1.3107
$p=n-6$				$p=n-8$			
n	a	b	$T_c(\infty)$	n	a	b	$T_c(\infty)$
7	0.0154	5.00	2.1767	9	0.0086	6.8	2.2168
8	0.0286	5.50	2.1021	10	0.0165	7.4	2.1703
9	0.0394	5.80	2.0441	15	0.0430	8.3	2.0253
10	0.0486	6.00	1.9965	20	0.0594	8.6	1.9429
15	0.0813	6.10	1.8405	25	0.0712	8.7	1.8867
20	0.1019	6.25	1.7513	50	0.1033	9.2	1.7454
50	0.1575	6.65	1.5384				

the following power laws:

$$\begin{aligned}
&\text{for} \quad p=n-2, \quad T_c(\infty)\approx 2.48\,10^{-3}\,(30.0)^{d_h},\\
&\text{for} \quad p=n-4, \quad T_c(\infty)\approx 0.184\,(3.36)^{d_h},\\
&\text{for} \quad p=n-6, \quad T_c(\infty)\approx 0.400\,(2.30)^{d_h},\\
&\text{for} \quad p=n-8, \quad T_c(\infty)\approx 0.566\,(1.95)^{d_h}.
\end{aligned} \tag{12}$$

Figure 9 (right) shows the plot of the logarithm of v_c versus $1/n$ for $SC(n,n-2,k)$. For $k=2$, the plot is approximately linear, at least for large values of n. A linear fit gives

$$\ln(v_c)\approx \frac{-0.8132}{n} \quad (R^2=0.9998).$$

However, such a linear approximation does not hold for $k\neq 2$ or $p\neq n-2$.

V. SPECTRA OF THE PARTITION FUNCTION OF SIERPIŃSKI CARPETS

Not only the critical parameter v_c but also the full spectrum of the pseudopartition function may be calculated. Several examples of spectra of Sierpiński carpets are plotted in the complex plane on Fig. 10. They may be compared to other fractal spectra obtained in hierarchical models [36].

Figure 11 shows a magnification of the area of the spectrum located in the vicinity of the critical parameter v_c for $SC(3,1)$ at steps $k=5$ and $k=6$. It has been suggested (especially for lattices with random impurities [35,37–39]) that the topology of this area of the spectrum may be related to the critical behavior of thermodynamic functions. Several similar patterns occur at different scales, but with differences which are stable with the segmentation step. The scale invariance is not rigorous since the patterns do not clearly converge towards an exact scale invariant structure for $k\to\infty$.

Figure 10 shows also that coherently, when the structure becomes more dense, and the proportion of voids decreases, from $p=n-2$ to $p=n-8$, the spectrum aggregates along the two circles of Fig. 5.

VI. THERMODYNAMICS

A. Internal energy and specific heat on $SL(\infty,n)$

The generating pattern of $SL(N,n)$ contains $2n-1$ sites and $2n$ bonds. The total number of sites is $\mathcal{N}=(2n-1)N^2$ and from (4), the pseudopartition function of $SL(N,n)$ is

$$\begin{aligned}
Z &= 2^{\mathcal{N}}\cosh(\beta J)^{2nN^2}P_{N,n}(v)=\frac{2^{\mathcal{N}}P_{N,n}(v)}{(1-v^2)^{nN^2}}\\
&= 2^{\mathcal{N}}\prod_{p,q=1}^{N}\left[R_1(n,v)-R_2(n,v)\left(\cos\frac{2p\pi}{N}+\cos\frac{2q\pi}{N}\right)\right]^{\frac{1}{2}},
\end{aligned} \tag{13}$$

where

$$R_1(n,v)=\frac{(1+v^{2n})^2}{(1-v^2)^{2n}} \quad \text{and} \quad R_2(n,v)=\frac{2v^n(1-v^{2n})}{(1-v^2)^{2n}}.$$

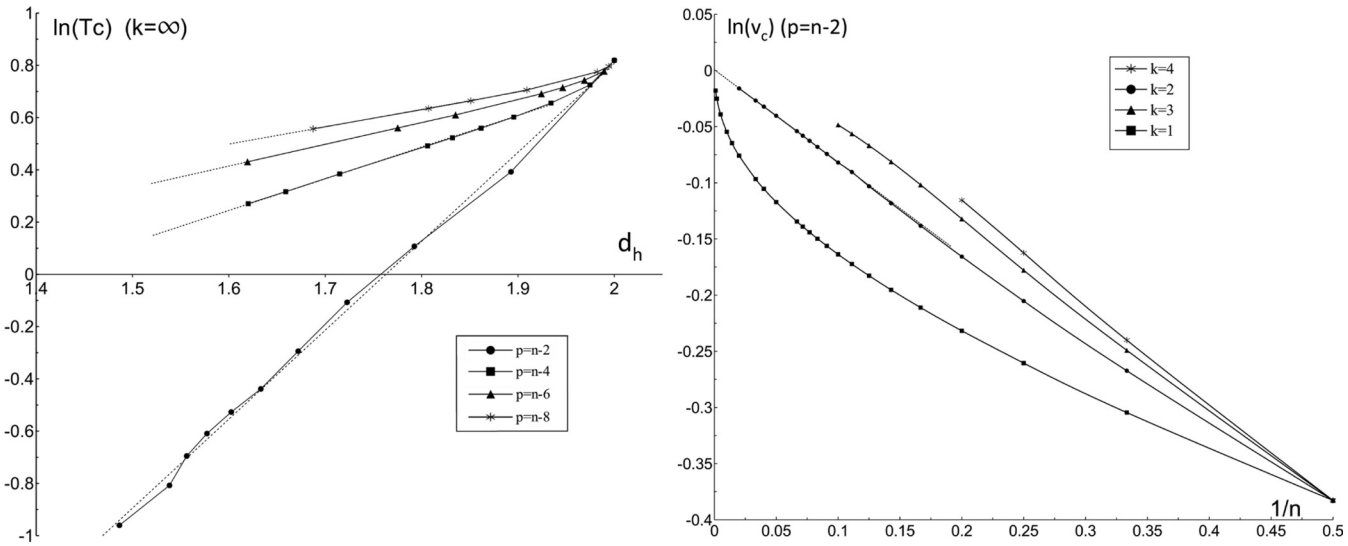


FIG. 9. Plots of $\ln[T_c(\infty)]$ versus d_h (right) and $\ln(v_c)$ versus $1/n$ (for $p=n-2$) (left). Dotted lines are linear fits.

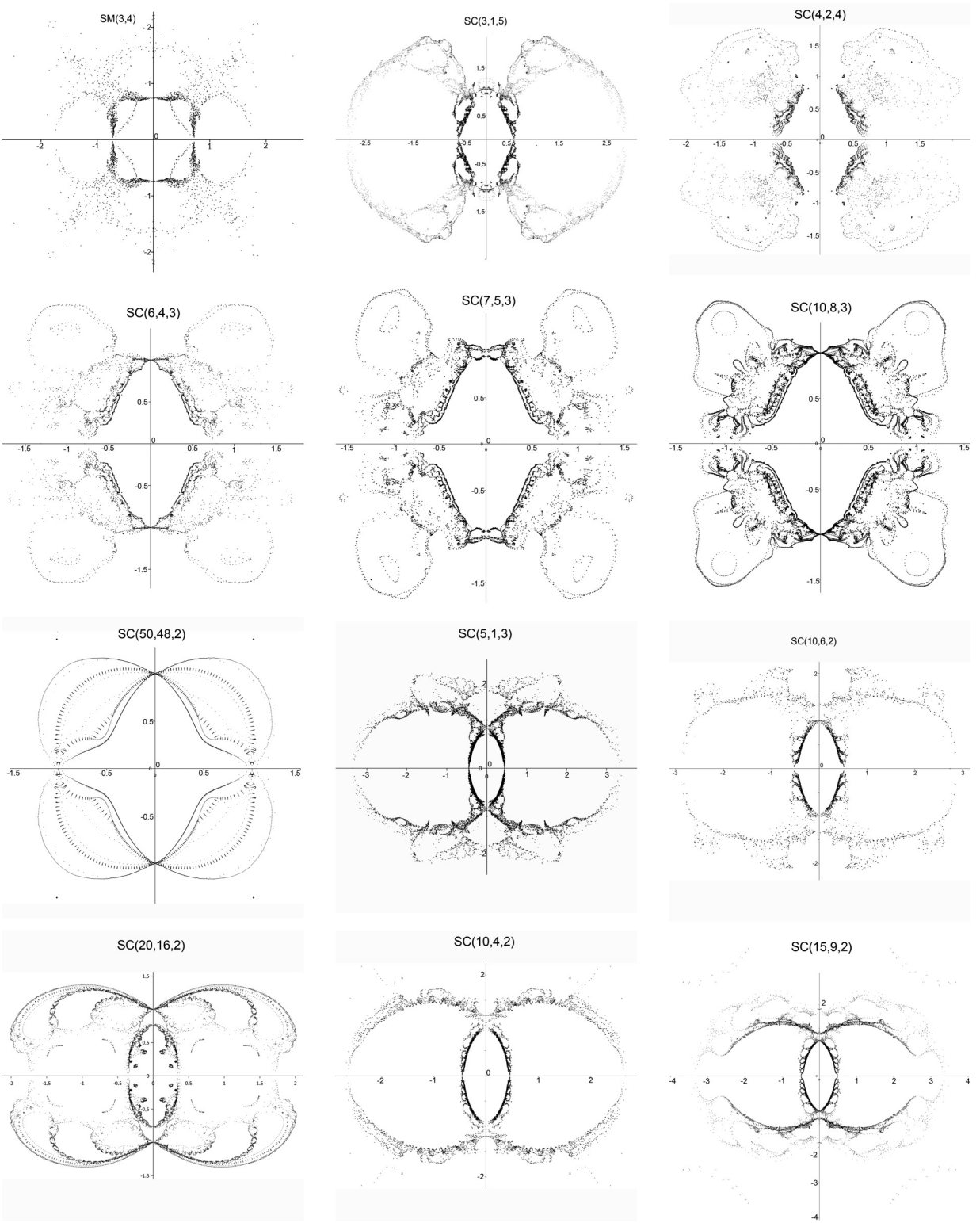


FIG. 10. Spectra of several examples of Sierpiński carpets.

Calculations of the filled square Ising model are extended in a straight line to the lacunary square lattice. The free energy F is obtained taking the logarithm of (13), dividing by $\mathcal{N}$, and, at the limit $N \to \infty$, replacing the double sum by a double integral:

$$\beta F = -\ln 2 - \frac{1}{8(2n-1)\pi^2} \iint_0^{2\pi} \ln\left[R_1(n,v) - R_2(n,v)(\cos x + \cos y)\right] dx\,dy. \tag{14}$$

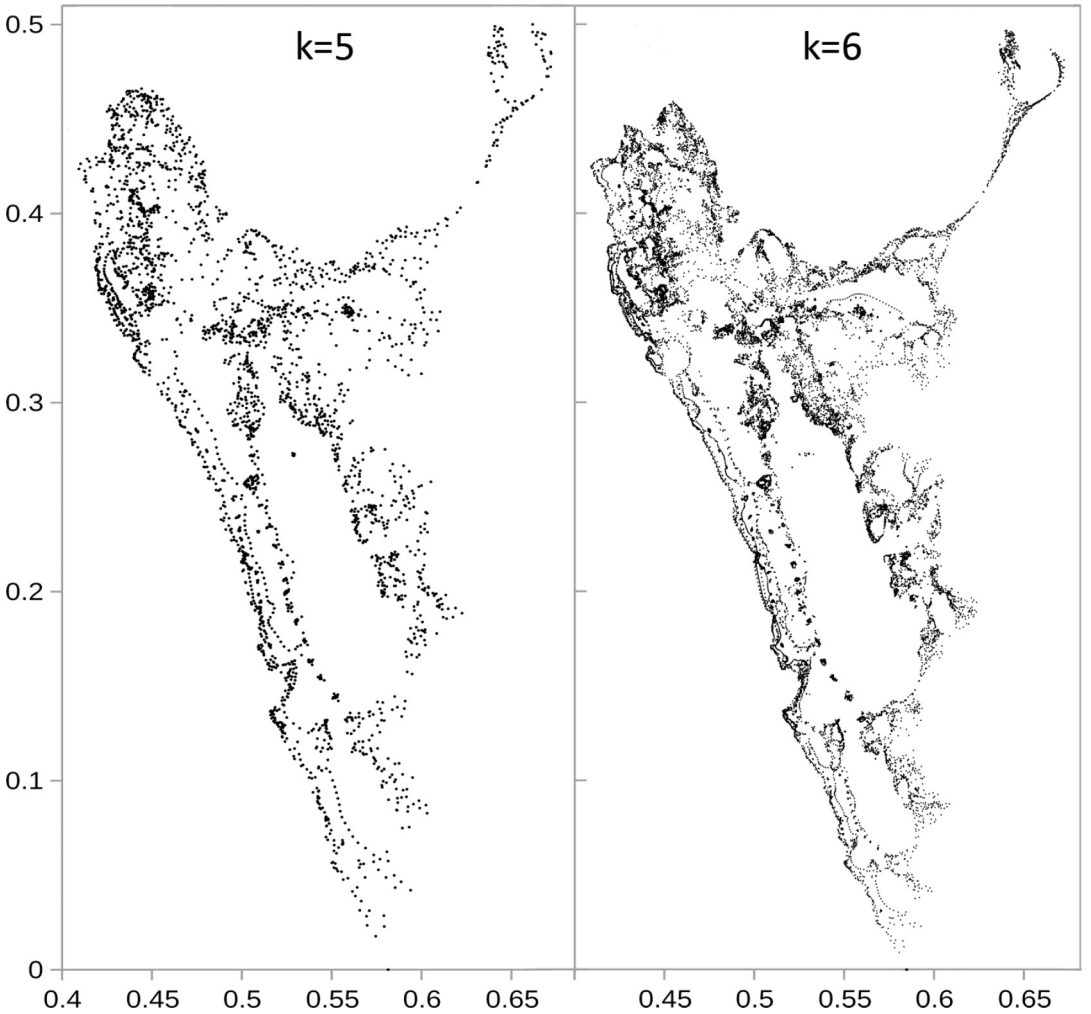


FIG. 11. The vicinity of the real axis in the critical region of the spectrum of $SC(3,1,k)$.

Differentiating a first time, we obtain the internal energy $U = \frac{\partial \beta F}{\partial \beta}$. Let us note R_1' and R_2', the derivatives of R_1 and R_2 with respect to v:

$$U = -\frac{1}{8(2n-1)\pi^2}\frac{dv}{d\beta}\iint_0^{2\pi}\frac{R_1' - R_2'(\cos x + \cos y)}{R_1 - R_2(\cos x + \cos y)}dxdy = -\frac{(1-v^2)}{128}\left[\frac{R_2'}{R_2} + \frac{2}{\pi}\left(\frac{R_1'}{R_1} - \frac{R_2'}{R_2}\right)K\left(\frac{2R_2}{R_1}\right)\right]$$
$$= \frac{n}{(2n-1)}\left[\frac{(v^{2n+2} + 3v^{2n} - 3v^2 - 1)}{2v(1-v^{2n})} + \frac{(1-v^2)(v^{4n} - 6v^{2n} + 1)}{\pi v(1-v^{4n})}K\left(\frac{4v^n(v^{2n}-1)}{(v^{2n}+1)^2}\right)\right],$$

where K is the elliptic integral of the first kind.

Differentiating a second time, we obtain the specific heat $C = -\beta^2\frac{\partial^2 \beta F}{\partial \beta^2}$:

$$C = \frac{n\beta^2}{2(2n-1)\pi}\left[\frac{(2(v^2+1)(v^{6n} - 7v^{4n} + 7v^{2n} - 1) + n(v^2-1)(v^{6n} + 11v^{4n} + 11v^{2n} + 1))}{v^2(v^{2n}-1)^2(v^{2n}+1)}K\left(\frac{4v^n(1-v^{2n})}{(v^{2n}+1)^2}\right)\right.$$
$$\left. - \frac{n(v^2-1)(v^{2n} - 2v^n - 1)^2}{v^2(v^{2n}-1)^2}E(\frac{4v^n(1-v^{2n})}{(v^{2n}+1)^2}) - \frac{\pi(4(n-1)v^{2n+2} + v^{4n+2} - 3v^{4n} - 4(n-1)v^{2n} + 3v^2 - 1)}{v^2(v^{2n}-1)^2}\right], \quad (15)$$

where E is the elliptic integral of the second kind.

The internal energy U and the specific heat C are plotted in Fig. 12 for $n = 1$ to 5 and $n = 10$.

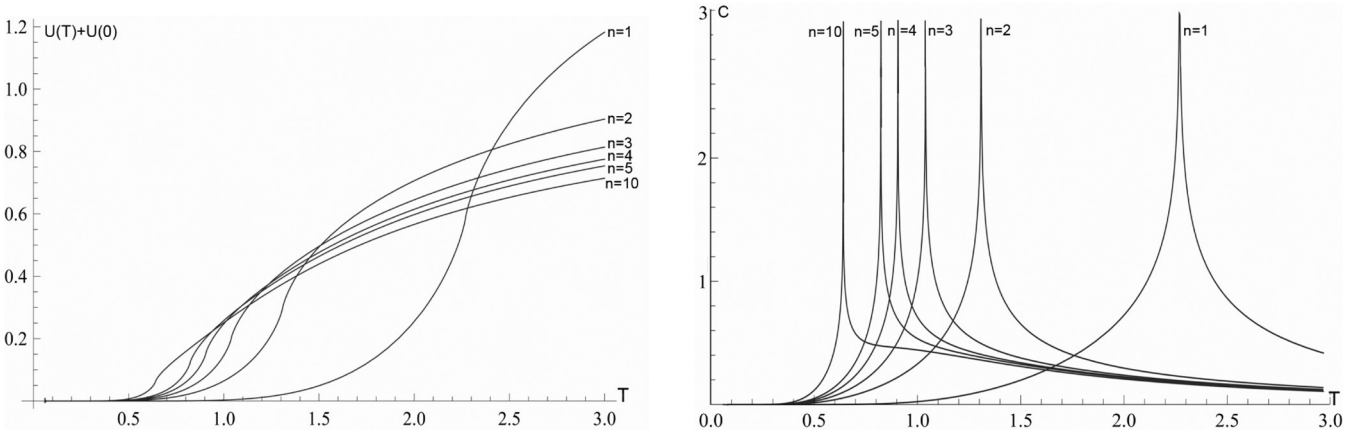


FIG. 12. The internal energy and specific heat of $SL(\infty,n)$ versus temperature for $n = 1$ to 5 and 10.

B. Internal energy and specific heat on Sierpiński carpets

The generating pattern of $SC(3,1,2)$ has 8 sites and 14 bonds; then the pseudopartition function of $SC(3,1,1)$ is

$$Z = 2^{8N^2} \frac{P_{N,SC(3,1,1)}(v)}{(1-v^2)^{7N^2}}.$$

From (8) and taking the limit as $N \to \infty$, the free energy per site is

$$\beta F = -\ln 2 - \frac{1}{64\pi^2} \iint_0^{2\pi} \ln[R_1(v) + R_2(v)(\cos x + \cos y) + R_3(v)\cos x \cos y + R_4(v)(\cos^2 x + \cos^2 y)]dxdy, \quad (16)$$

where $R_i(v) = \frac{Q_i(v)}{(1-v^2)^{14}}$

Calculations have been performed also for the second segmentation steps $SC(3,1,2)$ and $SM(3,2)$. Their pseudopartition polynomials have nine terms and their degrees are, respectively, 84 and 64. They are explicitly given as Supplemental Material [40]. The internal energy and specific heat of $SC(3,1,1)$, $SC(3,1,2)$, and $SM(3,2)$ are plotted in Fig. 13.

C. Magnetization of the Ising model on $SL(\infty,n)$

The magnetization m of the square lattice without an external field has been explicitly calculated in [41]. The method has been subsequently refined [42,43] and adapted to the loops counting method we are using in this paper [34]. The basic principles is to evaluate the asymptotic limit of average correlations between two spins when their distance become infinite. The calculation is generally performed within a row of spins:

$$m = \lim_{k\to\infty} \langle s_1, s_{k+1} \rangle.$$

In terms of the matrices $\mathcal{M}$ involved in the calculations of the partition function, correlations within one row of sites are calculated from matrices elements of $(I - v\mathcal{M})^{-1}$.

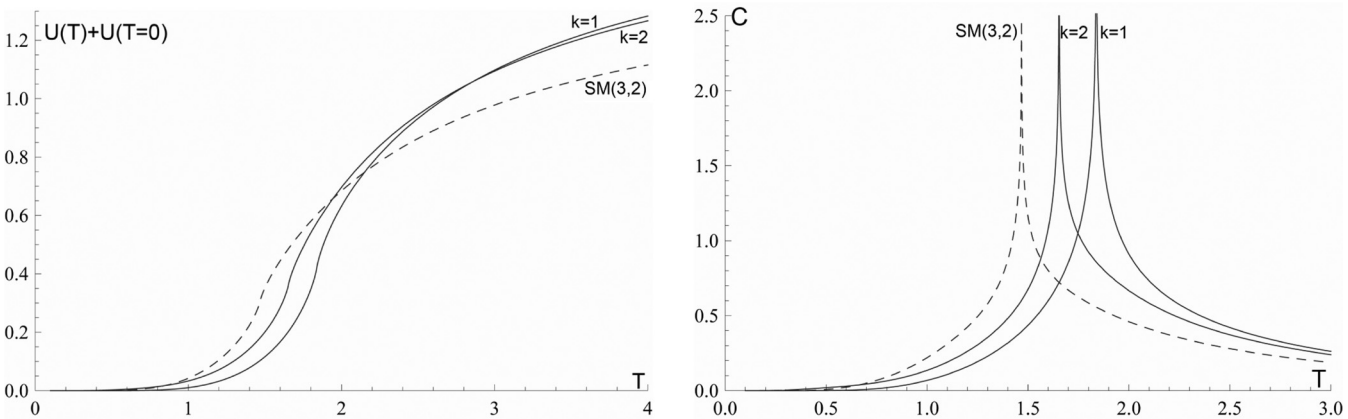


FIG. 13. The internal energy and specific heat of $SC(3,1,1)$, $SC(3,1,2)$, and $SM(3,2)$ versus temperature.

For $SQ(N,n)$, using the translational invariance [44] and the further reduction already used for the calculation of the partition polynomial, the matrix involved is the inverse of the matrix of Eq. (3), replacing $\frac{2\pi p}{N}$ and $\frac{2\pi q}{N}$, respectively, by x and y, $M(N,p,q) \to M(x,y)$ (expressions are taken for $N \to \infty$):

$$\Delta(v,x,y)[I - v^n M(x,y)]^{-1} = \begin{pmatrix} \gamma_1(v,x,y) & \gamma_{2-}(v,-x,y) & \gamma_{2+}(v,-x,-y) & \gamma_3(v,-x,-y) \\ \gamma_{2+}(v,-y,x) & \gamma_1(v,y,x) & \gamma_3(v,-y,x) & \gamma_{2-}(v,-y,-x) \\ \gamma_{2-}(v,y,x) & \gamma_3(v,y,-x) & \gamma_1(v,-y,x) & \gamma_{2+}(v,y,-x) \\ \gamma_3(v,x,y) & \gamma_{2+}(v,x,y) & \gamma_{2-}(v,x,-y) & \gamma_1(v,-x,y) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} \Delta(v,x,y) &= |I - v^n M(x,y)| \\ &= (1+v^{2n})^2 - 2v^n(1-v^{2n})(\cos x + \cos y), \\ \gamma_1(v,x,y) &= 1 + v^{2n} - v^n(1-v^{2n})e^{ix} - 2v^n \cos y, \\ \gamma_{2\pm}(v,x,y) &= e^{\pm\frac{i\pi}{4}} v^n (e^{ix} - v^n - v^n e^{ix+iy} - v^{2n} e^{iy}), \\ \gamma_3(v,x,y) &= 2v^{2n} e^{ix} \sin y. \end{aligned} \tag{18}$$

From these expressions, similar calculations as for $n = 1$ [34,42], with the substitution $v \to v^n$, leads to the magnetization:

$$\begin{aligned} m &= \left[1 - \frac{(1-v^{2n})^4}{16v^{4n}}\right]^{\frac{1}{8}} \\ &= \left[\frac{(1-v^{2n})^2 P_{1,n}(v) P_{1,n}(-v)}{16v^{4n}}\right]^{\frac{1}{8}}, \end{aligned} \tag{19}$$

which is plotted on Fig. 14 for $n = 1$ to 5 and $n = 10$.

When taking the first term of the expansion of m in the vicinity of the critical parameter $v_c = (\sqrt{2}-1)^{\frac{1}{n}}$, the magnetization writes

$$m^8 = 4n\sqrt{2}(\sqrt{2}+1)^{\frac{1}{n}}(v - v_c) + O(v - v_c)^2. \tag{20}$$

It would be tempting to extend Eq. (19) to Sierpiński carpets by substituting $P_{1,SC(n,p,k)}(v)$ to $P_{1,n}(v)$ and adapting the powers of other factors accordingly. Unfortunately, the passage from the first line to the second in (19) implicitly involves the invariance under Kramers-Wannier's transformation [45] which is still valid for the lacunary square lattice provided the change $v \to v^n$, but does not hold for $SC(n,p,k)$. The calculation of the magnetization of Sierpiński carpets requires one to start back from the inversion of the matrix (7) which goes beyond the scope of this paper.

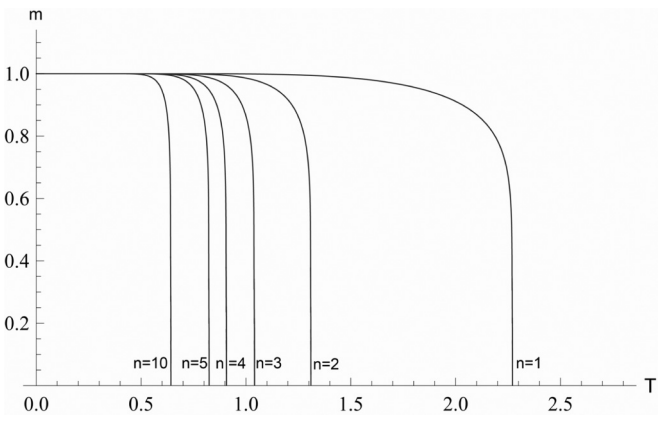


FIG. 14. The magnetization of the lacunary square lattice $SL(\infty,n)$, for $n = 1$ to 5 and $n = 10$.

VII. DISCUSSION AND CONCLUSION

At this step, we are facing the problem of taking the thermodynamic limit in fractal systems. Sierpiński carpets are obtained by the iteration of a segmentation rule step by step (which is noted k in this paper). But at each finite step, reproducing the generating pattern periodically N^2 times restores an artificial translational invariance. In a strict meaning, the thermodynamic limit should be taken setting $k \to \infty$. That is done in this paper in Secs. IV and V for the calculation of critical temperatures and spectra. But for a finite segmentation step k, calculations of thermodynamic functions (Sec. VI) have been done by setting $N \to \infty$. In this way, basic symmetries are the same as the square lattice and the critical exponents are the same, which obviously results from Eqs. (16) and (20). This is a general characteristic of lattices which have translational invariance, whatever the period, that the critical behavior is the same as the usual square lattice as previously pointed out in periodic layered lattices [46]. Moreover, the amplitude of the logarithmic divergence of the specific heat decreases as the size of the generating pattern (the period) increases and/or the proportion of voids increases, a situation noticed analogously in lattices with random impurities [47].

To compare critical exponents with simulations as done for critical temperatures in Table II, scale invariance should be implemented in order to get a fully solvable model of the Ising ferromagnetism on fractal lattices at the limit $k \to \infty$. Presently, no recursion formula linking the matrices of two successive segmentation steps is available. This question is also linked to relations between the topology of the spectrum in the vicinity of the critical region and the critical behavior of thermodynamic functions. Both questions are still open for further investigations.

ACKNOWLEDGMENT

I thank Prof. B. McCoy for fruitful discussions and suggestions.

APPENDIX: CRITICAL TEMPERATURES OF $SC(n,p,k)$

Table IV gives all calculated critical temperatures for a wide range of values of n, p, and k.

TABLE IV. The critical temperatures of Sierpiński carpets $SC(n,p,k)$ for $3 \leqslant n \leqslant 1000$ and $1 \leqslant p \leqslant n-2$.

$p = n-2$						$p = n-4$						$p = n-6$				
n	d_h	T_c				n	d_h	T_c				n	d_h	T_c		
		$k=1$	$k=2$	$k=3$	$k=4$			$k=1$	$k=2$	$k=3$	$k=4$			$k=1$	$k=2$	$k=3$
4	1.7925	1.6213	1.3689	1.2502	1.1845	5	1.9746	2.1193	2.0790	2.0690	2.0667	7	1.9894	2.1942	2.1798	2.1774
5	1.7227	1.4875	1.19857	1.0579	0.9882	6	1.9343	2.0097	1.9469	1.9323		8	1.9690	2.1310	2.1072	2.1036
6	1.6720	1.3946	1.0832	0.9309	0.8429	7	1.8957	1.9286	1.8512	1.8334		9	1.9464	2.0803	2.0499	2.0454
7	1.6332	1.3252	0.9997	0.8431	0.7556	8	1.8617	1.8656	1.7773	1.7571		10	1.9243	2.0386	2.0031	
8	1.6025	1.2709	0.9363	0.7793	0.6939	9	1.8320	1.8145	1.7180	1.6956		15	1.8352	1.9016	1.8506	
9	1.5773	1.2267	0.8825	0.7308		10	1.8062	1.7721	1.6688	1.6447		20	1.7752	1.8213	1.7623	
10	1.5563	1.1899	0.8461	0.6926		11	1.7150	1.6310	1.5080			50	1.6194	1.6246	1.5510	
11	1.5384	1.1585	0.8126	0.6621		20	1.6590	1.5479	1.4156			100	1.5330	1.5123		
12	1.5229	1.1314	0.7846			25	1.6201	1.4908	1.3531			500	1.3989	1.3103		
13	1.5093	1.1076	0.7599			50	1.5211	1.3430				1000	1.3593	1.2389		
14	1.4972	1.0864	0.7387			100	1.4472	1.2253								
15	1.4864	1.0675	0.7197	0.5787		500	1.3340	0.9812								
20	1.4456	0.9954	0.6515			1000	1.3007	0.9474								
25	1.4180	0.9460	0.6069			$p = n-8$						$p = n-10$				
30	1.3976	0.9091	0.5748			n	d_h	T_c				n	d_h	T_c		
50	1.3492	0.8195	0.5007					$k=1$	$k=2$	$k=3$	$k=4$			$k=1$	$k=2$	$k=3$
70	1.3229	0.7693				10	1.9823	2.1834	2.1721			200	1.5606	1.7277		
100	1.2988	0.7222				15	1.9093	2.0524	2.0286			500	1.4804	1.6363		
200	1.2607	0.6449				20	1.8510	1.9769	1.9467							
500	1.2227	0.5642				25	1.8072	1.9253	1.8910							
1000	1.2005	0.5150				50	1.6874	1.7932								
						100	1.5932	1.6895								
SM	1.8928	1.8184	1.4677	1.2593	1.1687	500	1.4448	1.50266								
						1000	1.4008	1.4355								

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