Corrections to the self-consistent Born approximation for Weyl fermions

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The average density of states of two- and three-dimensional Weyl fermions is studied in the self-consistent Born approximation (SCBA) and its corrections. The latter have been organized in terms of a $1/N$ expansion. It turns out that an expansion in terms of the disorder strength is not applicable, as previously mentioned by other authors. Nevertheless, the $1/N$ expansion provides a justification of the SCBA as the large N limit of Weyl fermions.

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I. INTRODUCTION

A very common and straightforward approach to the average one-particle Green's function of a disordered system of noninteracting electrons is the self-consistent Born approximation (SCBA). Numerous applications of SCBA-based techniques to low–dimensional systems of disordered electrons provided an excellent confirmation for a number of experimental observations [1–5]. However, the claim of a "failure of the SCBA" for two-dimensional (2D) Dirac particles by Aleiner and Efetov [6] has questioned whether this approach is applicable at all to two-band systems with spectral degeneracies in general. These authors argued that the self-energy diagrams of order g^2 in disorder strength dominate over the local density of states (DOS) predicted by the SCBA. Another approach to the average DOS, based on a bosonization concept, predicts that the DOS obeys a power law at the node with a disorder-dependent exponent [7], which also contradicts the nonvanishing DOS of the SCBA.

The SCBA and the power-law prediction for the average DOS were checked recently in numerical studies as well as by a functional renormalization group approach by Sbierski *et al.*, who found that there is no power law, and that the SCBA is only missing a factor of two in the logarithm of the DOS (i.e., the square root of SCBA must be taken) [9] in 2D. Moreover, in 3D, the critical disorder strength is twice as large for the SCBA, although the slope of the DOS agrees quite well [8,9]. These missing factors of two suggest that the corrections to the SCBA are of the same order as the SCBA itself. The renewed interest in the behavior of the average DOS of Weyl systems [8–12] suggests a clarification of the role of the traditional SCBA approach, which is based on a commonly accepted mean-field type of approximation. Since the SCBA is equivalent to a saddle-point approximation of a functional integral, these corrections are easily accessible from the fluctuations around the saddle point. The aim of the present paper is to analyze corrections to the SCBA for 2D and 3D disordered Weyl fermions in a systematic $1/N$ expansion [13–17].

II. THE MODEL

A generalization of the Weyl Hamiltonian to the one with N orbitals per site reads [18–20]:

$$H = \mathbf{1}_N \otimes i\partial\!\!\!/ + v \otimes \sigma_\nu, \tag{1}$$

where the Dirac contraction notation is $\partial\!\!\!/ = \sigma_1\partial_1 + \sigma_2\partial_2$ in 2D and $\partial\!\!\!/ = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3$ in 3D, respectively. In contrast to the 2D case, in 3D there is no Pauli matrix left, which anticommutes with the Dirac operator. The Pauli matrix σ_ν is either σ_0 (i.e., the 2×2 unit matrix) for a random scalar potential, or σ_3 for a random Dirac mass in 2D, and only σ_0 for a random scalar potential in 3D. Different physical realizations of Weyl electrons reveal different values of N, e.g., $N = 2$ for graphene, $N = 4$ for π-flux model in 2D, and $N = 8$ for π-flux model in 3D [10,11]. The random potential v represents a symmetric $N \times N$-matrix $(v_r^{ij})_{i,j=1,\ldots,N}$ with zero mean $\langle v_r^{ij}\rangle = 0$ and with the correlator

$$\langle v_r^{ij} v_{r'}^{kl}\rangle = \frac{g}{N}\delta_{il}\delta_{jk}\delta(r - r'). \tag{2}$$

We use the convention $\hbar v_F = 1$, where v_F denotes the Fermi velocity.

The DOS is the imaginary part of the diagonal element of the retarded Green's function $G(i\epsilon) = [\mathbf{1}_N \otimes G_0^{-1} + v \otimes \sigma_\nu]^{-1}$ ($\epsilon > 0$), where $G_0 = [i\epsilon\sigma_0 + i\partial\!\!\!/]^{-1}$ is the one-particle Green's function of the electron in a clean system:

$$\varrho(\epsilon) = -\frac{1}{\pi}\mathrm{Im}\mathrm{Tr}_{2N} G_{rr}(i\epsilon). \tag{3}$$

The operator Tr_{2N} denotes the trace with respect to the space of Pauli matrices and the N orbitals. The Green's function can be written as a functional integral:

$$G_{rr}^{ii} = -i\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{D}\varphi^\dagger\mathcal{D}\varphi\,\psi_r^i\psi_r^{i\dagger} e^{i\mathcal{S}_F + i\mathcal{S}_B}, \tag{4}$$

with the actions

$$\mathcal{S}_B = \varphi^\dagger\cdot[\mathbf{1}_N \otimes G_0^{-1} + v \otimes \sigma_\nu]\varphi \tag{5}$$

and

$$\mathcal{S}_F = \psi^\dagger\cdot[\mathbf{1}_N \otimes G_0^{-1} + v \otimes \sigma_\nu]\psi. \tag{6}$$

Here, φ represents a $2N$-component complex field and ψ a $2N$-component Grassmann field. Since $\epsilon > 0$, the convergence of the complex functional integral is guaranteed. The advantage of using complex and Grassmann fields is that the integral

$$\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{D}\varphi^\dagger\mathcal{D}\varphi\, e^{i\mathcal{S}_F + i\mathcal{S}_B} = 1 \tag{7}$$

is already normalized, whereas using only the complex or only the Grassmann part requires an extra normalization. This would create problems for the calculation of the average with respect to disorder.

Arranging bosonic and fermionic fields to a vector superfield $\Phi = (\varphi, \psi)^{\mathbf{T}}$ we can easily perform the disorder averaging (cf. Appendix A), and decouple by means of the matrix superfield $\hat{Q}$. This has the matrix structure

$$\hat{Q}_r = \begin{pmatrix} Q_r & \chi_r \\ \bar{\chi}_r & iP_r \end{pmatrix}, \tag{8}$$

with Q_r, P_r representing 2×2 matrices with commuting and $\chi_r, \bar{\chi}_r$ with anticommuting matrix elements. The average Green's function then becomes

$$\sum_{i=1}^{N} \langle G_{rr}^{ii}(i\epsilon) \rangle = -i\frac{N}{g}\sigma_\nu \int \mathcal{D}\hat{Q} P_r e^{-N\mathcal{S}[\hat{Q}]}, \tag{9}$$

with the effective action

$$\mathcal{S}[\hat{Q}] = \text{trg}\left\{\frac{1}{2g}\hat{Q}^2 + \log\left[\sigma_0 \otimes G_0^{-1} + \hat{Q}\Sigma_\nu\right]\right\}, \tag{10}$$

where $\Sigma_\nu = \sigma_0 \otimes \sigma_\nu$, and trg is the graded trace. Since the effective action does not depend on N, the integral in Eq. (9) suggests a saddle-point approximation for large N and a $1/N$-expansion. This will be discussed subsequently.

The scattering rate η is related to the nontrivial saddle point of $\mathcal{S}[\hat{Q}]$, which is a solution of the saddle-point equation $\delta\mathcal{S} = 0$. Thus, the field can be written as $\hat{Q} = \hat{Q}_0 + \hat{Q}'$ with the saddle point

$$\hat{Q}_0 = i\eta\Sigma_\nu. \tag{11}$$

For convenience, we rename the integration field $\hat{Q}' \to \hat{Q}$. In terms of the Green's function we get

$$\sum_{i=1}^{N} \langle G_{rr}^{ii} \rangle = -iN\frac{\eta}{g}\sigma_0 + \sum_{i=1}^{N} \delta G_{rr}^{ii}, \tag{12}$$

where the first term represents the uniform saddle-point contribution through the scattering rate and the second term represents the correction due to quantum fluctuations around the saddle point

$$\sum_{i=1}^{N} \delta G_{rr}^{ii} = -i\frac{N}{g}\sigma_\nu \int \mathcal{D}\hat{Q}_r P_r e^{-N\mathcal{S}[\hat{Q}]}, \tag{13}$$

with the shifted action

$$\mathcal{S}[\hat{Q}] = \text{trg}\left\{\frac{1}{2g}(\hat{Q} + i\eta\Sigma_\nu)^2 + \log[\bar{G}^{-1} + \hat{Q}\Sigma_\nu]\right\}. \tag{14}$$

The inverse average Green's function reads $\bar{G}^{-1} = \sigma_0 \otimes [iz\sigma_0 + i\partial\!\!\!/]$, $z = \epsilon + \eta$. From Eqs. (3) and (12) the saddle-point approximation of the DOS becomes

$$\varrho_{\text{SCBA}} = \frac{2N}{\pi}\frac{\eta}{g}, \tag{15}$$

regardless of the model dimension d. The behavior of the scattering rate η does, however, crucially depend on d.

III. SADDLE-POINT ANALYSIS AND FLUCTUATIONS

To obtain the effective action in the limit of slowly varying quantum fields, we expand Eq. (14) in powers of fluctuations $\hat{Q}$ around the saddle point. The small expansion parameter is $1/N$, since only the prefactor of the action depends on N in Eq. (13). Therefore, we can employ a saddle-point approximation, which leads to the saddle-point condition:

$$\eta = g\int \frac{d^dq}{(2\pi)^d}\frac{\eta}{\eta^2 + q^2}. \tag{16}$$

Solutions of this equation are described in the literature [1,2,21,22]. While in 2D, they predict an exponentially small but nonvanishing scattering rate for any value of the disorder strength g,

$$\eta_{2d} \sim \Lambda e^{-2\pi/g}. \tag{17}$$

In 3D, the nonvanishing scattering rate emerges only if the disorder strength becomes larger than a critical value,

$$g_c \sim \frac{2\pi^2}{\Lambda}, \tag{18}$$

where Λ represents a UV-cutoff of the order of inverse lattice constant, giving for small values of g:

$$\eta_{3d} \sim g\left(\frac{2\pi}{g_c}\right)^2 \theta(g - g_c). \tag{19}$$

The expansion of the logarithm around this nontrivial vacuum reads

$$\mathcal{S}[\hat{Q}] = \text{trg}\left\{\frac{1}{2g}\hat{Q}^2 - \frac{1}{2}[\bar{G}\hat{Q}\Sigma_\nu]^2 - \sum_{n\geq 3}\frac{(-1)^n}{n}[\bar{G}\hat{Q}\Sigma_\nu]^n\right\}. \tag{20}$$

The third term represents a perturbation to the scattering rate beyond the Gaussian approximation. Because of the structure of the matrix $\bar{G}$, all sectors of our theory, bosonic and fermionic, have the same propagators; i.e., there is no supersymmetry breaking. In Gaussian order of P_r, we get

$$\mathcal{S}_G[P] = \text{tr}\left\{\frac{1}{2g}P^2 - \frac{1}{2}\bar{G}P\sigma_\nu\bar{G}P\sigma_\nu\right\}, \tag{21}$$

with the Hermitean matrix field P, which can be represented as

$$P = \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & P_0 - P_3 \end{pmatrix} = P_\alpha\sigma_\alpha \tag{22}$$

with real P_α. The summation convention is used in Eq. (21) for $\alpha = 0,1,2,3$, and σ_α denote the Pauli matrices. This enables us to perform the trace in the first term immediately:

$$\text{tr}\frac{1}{2g}P^2 = \frac{1}{g}P\cdot P = \frac{1}{g}\int\frac{d^dq}{(2\pi)^d}P_q\cdot P_{-q}, \tag{23}$$

where the vector P is assembled from elements of the matrix P_α. Second term reads after transforming it into Fourier representation

$$\frac{1}{2}\text{tr}\bar{G}P\sigma_\nu\bar{G}P\sigma_\nu = \int\frac{d^dq}{(2\pi)^d}P_q^\alpha P_{-q}^\beta \Gamma_{2|\alpha\beta}^{(\nu)}(q), \tag{24}$$

where $\nu = 0$ denotes the random scalar potential and $\nu = 3$ a random gap. The explicit expression and evaluation of the two-point vertex function $\Gamma_2^{(\nu)}$ are given in Appendix B. It turns out that the inverse effective propagators do not have zero modes.

This reflects the absence of a broken continuous symmetry. For vanishing momenta and frequencies, the effective action becomes

$$\mathcal{S}_G[\hat{Q}] = \mathrm{M}^{(\nu)}_{aa} \int d^d r \left[Q^a_r Q^a_r + 2\chi^a_r \bar{\chi}^a_r + P^a_r P^a_r \right]. \tag{25}$$

In 2D, the solution $\eta = 0$ of Eq. (16) is always unstable (cf. Appendix B), and for $\eta > 0$ the mass matrices are

$$\mathrm{M}^{(0)}_{2\mathrm{D}} \sim \begin{pmatrix} \frac{c_\Lambda}{2\pi} & 0 & 0 & 0 \\ 0 & \frac{1}{g} + \frac{c_\Lambda}{2\pi} & 0 & 0 \\ 0 & 0 & \frac{1}{g} + \frac{c_\Lambda}{2\pi} & 0 \\ 0 & 0 & 0 & \frac{2}{g} \end{pmatrix}, \quad c_\Lambda = \frac{\Lambda^2}{\eta^2 + \Lambda^2} \tag{26}$$

for a random scalar and

$$\mathrm{M}^{(3)}_{2d} \sim \begin{pmatrix} \frac{2}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{g} - \frac{c_\Lambda}{4\pi} & 0 & 0 \\ 0 & 0 & \frac{1}{g} - \frac{c_\Lambda}{4\pi} & 0 \\ 0 & 0 & 0 & \frac{c_\Lambda}{2\pi} \end{pmatrix} \tag{27}$$

for a random gap. In the 3D case, the mass matrix for the random potential reads

$$\mathrm{M}^{(0)}_{3\mathrm{D}} \sim \begin{pmatrix} \frac{\eta}{4\pi} & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{28}$$

with the matrix element λ given in Appendix B, Eq. (B6).

A. Corrections to the DOS: Weak disorder

The calculation of the DOS corrections can be organized in terms of a $1/N$ expansion, which is obtained by rescaling the field $\hat{Q}$ with $\sqrt{N}$. This absorbs the prefactor N in the exponent of Eq. (13) into the quadratic order of the expansion in Eq. (20) and produces powers of $1/\sqrt{N}$ for higher order terms. A further simplification comes from the assumption of weak disorder; i.e., $g \ll \Lambda^2$. In this case, only the smallest diagonal element of the mass matrices in Eqs. (26)–(28) dominates the Gaussian fluctuations. Taking the momentum dependence to leading order in a gradient expansion into account, the corresponding excitation mode becomes

$$\Pi^{(\nu)}_{\nu\nu}(q) \sim \frac{12\pi}{g} \frac{\eta^{4-d}}{q^2 + m_d^2}, \tag{29}$$

where the masses are $m_2^2 = 12\eta^2$ in 2D and $m_3^2 = 6\eta^2$ in 3D (cf. Appendix B). Since the supersymmetry remains unbroken, the correlation function of the Grassmann field $\bar{\chi}\chi$ and the Hermitean field Q are given by the same expression. In position space, the correlator Eq. (29) decays exponentially (in 2D it is proportional to the modified Bessel function of second kind $K_0[m_2|r - r'|]$, in 3D to $\exp[-m_3|r - r'|]/|r - r'|$) and can be crudely approximated by the Dirac delta function

$$\Pi^{(\nu)}_{\nu\nu}(r,r') \sim \delta(r - r'). \tag{30}$$

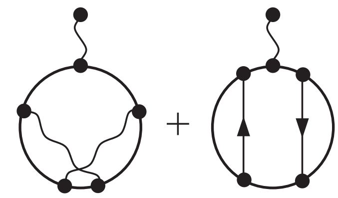

FIG. 1. Leading corrections of the DOS to the SCBA in the $1/N$ expansion.

Then, the first nonvanishing correction to the average one-particle Green's function (cf. Fig. 1) is of the order $1/N^2$ and reads

$$\varrho = \varrho_{\mathrm{SCBA}} + \frac{\varrho_{\mathrm{SCBA}}}{N^2}\left(\frac{2}{3} - \frac{g}{2\pi} - \frac{g^3}{4\pi^3} \right) + O(N^{-2}) \tag{31}$$

in 2D and

$$\varrho = \varrho_{\mathrm{SCBA}} + \frac{\varrho_{\mathrm{SCBA}}}{N^2}\left[\frac{\pi^2}{2}\left(1 - \frac{5\pi}{19}\right) - \frac{g\eta}{2\pi} - \frac{g^2\eta^2}{8} \right] + O(N^{-2}) \tag{32}$$

in 3D for the DOS of Eq. (3). Details of the $1/N$ expansion are presented in Appendices C and D.

Our calculation identifies the "sunrise" (or maximally crossed) diagrams as dominant. This gives corrections in each order $1/N$ with a polynomial in g. In other words, for each order $1/N$ there are g-independent contributions to DOS. Since, however, an n-order correction goes proportionally to N^{1-n}, the series converges rapidly.

B. Corrections to the DOS: Strong disorder

In the regime of large g values, we can neglect the gradient terms in the correlators. In 2D and for scalar disorder, the correlator of the fields P_0 then reads

$$\langle P_{0r} P_{0r'} \rangle \sim \frac{\pi}{g} \delta(r - r'), \tag{33}$$

which is needed in order for DOS to have a finite trace, while that of $P_{i=1,2}$ reads

$$\langle P_{ir} P_{ir'} \rangle \sim \frac{\pi}{2\pi + g c_\Lambda} \delta(r - r'), \tag{34}$$

which is negligible in comparison to the correlator of the fields P_3:

$$\langle P_{3r} P_{3r'} \rangle \sim \tfrac{1}{4} \delta(r - r'). \tag{35}$$

For the case of the random mass disorder, the situation is analogous, with the interchanging role of the fields P_0 and P_3. The dominant contribution comes from the diagrams with the loops which couple to the external field via P_0 channel and internal coupling of P_3 fields. In Appendix E we obtain for 2D,

$$\varrho = \varrho_{\mathrm{SCBA}} + \frac{\varrho_{\mathrm{SCBA}}}{(4N)^2}\left(\frac{3}{2} - \frac{2\pi}{g} \right) + O(N^{-2}), \tag{36}$$

i.e., the correction is positive. In 3D, the correlator of P_0 in strong disorder limit reads

$$\langle P_{0r} P_{0r'} \rangle \sim \frac{2\pi}{g\eta} \delta(r - r'), \tag{37}$$

while that of $P_{i=1,2,3}$ is

$$\langle P_{ir} P_{ir'} \rangle \sim \frac{1}{g\Lambda} \delta(r - r'), \tag{38}$$

cf. Eq. (B6), and since $\Lambda \gg \eta$, they are parametrically smaller and can be neglected in crudest approximation. We get

$$\varrho = \varrho_{\text{SCBA}} - \frac{\varrho_{\text{SCBA}}}{N^2} \left[\frac{\pi^3}{g\eta} + \frac{(2\pi)^2}{g^2\eta^2} - \frac{(2\pi)^5}{8g^3\eta^3} \left(1 - \frac{5\pi}{19} \right) \right] + O(N^{-2}), \tag{39}$$

i.e. the corrections are negative for large g.

IV. DISCUSSION

Our analysis of the perturbative expansion for the average DOS in terms of disorder strength g clearly indicates that this expansion cannot be organized in a systematic way in powers of g. This result seems to support the claim of Aleiner and Efetov [6] of a "failure of the SCBA". However, using N copies of Weyl fermions, as described by the model in Eqs. (1) and (2), provides a systematic $1/N$ expansion, which reveals that the SCBA has only corrections of order $1/N$. Moreover, the corrections up to order $1/N$ in Eqs. (31), (32), (36), and (39) give an enhancement of the DOS in comparison to the SCBA, in agreement with the numerical results found by Sbirski *et al.* [8,9], who got a doubling of the SCBA values in 2D [9]. The $1/N$ expansion suggests that this doubling is specific for a single-component Weyl fermion. For models with larger N, e.g., for different versions of the π-flux model [10,11], the DOS corrections to the SCBA become virtually negligible and the SCBA is exact in the limit $N \to \infty$. Finally, we don't find a shift of the critical disorder strength g_c in the $1/N$ expansion for the appearance of a nonzero DOS in 3D Weyl fermions, which was predicted in Refs. [8,9].

The behavior of the leading order correction as a function of the disorder strength depends crucially on the spatial dimension of the system. For instance, in zero dimension (random matrix model) the saddle point condition reads $g = \eta^2$, which results in a g-independent DOS.

V. CONCLUSION

Our extended Weyl-fermion model with N orbitals per site gives in the $N \to \infty$ limit for the average DOS the SCBA result and a systematic $1/N$ expansion for the corrections to the SCBA at finite N. Each term in the $1/N$ expansion depends on the disorder parameter g, which can be expanded for weak disorder as a power series of g or for strong disorder as a power series of $1/g$. This result demonstrates the reliability of the SCBA and the existence of a systematic expansion for disordered Weyl fermions at the node.

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APPENDIX A: HUBBARD-STRATONOVICH TRANSFORMATION

The graded trace of a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A1}$$

with quadratic matrices A, B, C, D reads

$$\text{Trg} M = \text{Tr}[A - D]. \tag{A2}$$

The graded determinant of the matrix M reads

$$\text{detg} M = \frac{\det A}{\det D} \det[1 - BD^{-1}CA^{-1}]. \tag{A3}$$

The ensemble average of the Green's function reads

$$\langle G_{rr} \rangle = \mathcal{N}^{-1} \int_{-\infty}^{\infty} dv \mathcal{P}(v) G_{rr}, \tag{A4}$$

$$\mathcal{N} = \int_{-\infty}^{\infty} dv \mathcal{P}(v), \quad \mathcal{P}(v) = e^{-\frac{N}{2g}\text{tr}v^2}, \tag{A5}$$

where the integrals are the functional ones. Consider the combination of v-dependent terms in the exponent of the integral:

$$\begin{aligned} &\frac{N}{2g}\text{tr}v^2 - i\Phi^\dagger \cdot [\mathbf{1}_2 \otimes v \otimes \sigma_\nu]\Phi \\ &= \frac{N}{2g} v_{ab}^2 - i v_{ab} [\varphi_a^\dagger \sigma_\nu \varphi_b + \psi_a^\dagger \sigma_\nu \psi_b] \\ &= \frac{N}{2g} v_{ab}^2 - i v_{ab} \Phi_a^\dagger \Sigma_\nu \Phi_b, \end{aligned}$$

where $\Sigma_\nu = \sigma_0 \otimes \sigma_\nu$. Here, the summation convention is understood. The integration over v_{ab} can be performed after completing the square and shifting the potential matrix elements as

$$v_{ab} \to v_{ab} - i \frac{g}{N} \Phi_a^\dagger \Sigma_\nu \Phi_b. \tag{A6}$$

What remains is the interaction term

$$-\frac{g}{2N} \text{trg}\big(\Sigma_\nu \Phi_a \Phi_a^\dagger \Sigma_\nu \Phi_b \Phi_b^\dagger\big), \tag{A7}$$

which gives the full action

$$\bar{\mathcal{S}}_{BF}[\Phi^\dagger, \Phi] = i\Phi^\dagger \cdot \mathbf{1}_N \otimes G_0^{-1} \Phi - \frac{g}{2N} \text{trg}[\Sigma_\nu \Phi_a \Phi_a^\dagger]^2. \tag{A8}$$

The interaction term is then decoupled by means of a Hubbard-Stratonovich transformation:

$$\begin{aligned} &\bar{\mathcal{S}}_{BF}[Q, \Phi^\dagger, \Phi] \\ &= \frac{N}{2g} \text{trg}\left[\hat{Q} - i \frac{g}{N} \Sigma_\nu \Phi_a \Phi_a^\dagger \right]^2 - i\Phi^\dagger \cdot \mathbf{1}_N \otimes G_0^{-1} \Phi \\ &\quad + \frac{g}{2N} \text{trg}[\Sigma_\nu \Phi_a \Phi_a^\dagger]^2, \end{aligned} \tag{A9}$$

where we shifted the matrix field $\hat{Q}$ exploiting the "translational invariance" of the corresponding functional integral measure. The element $\sim(\Sigma_\nu \Phi_a \Phi_a^\dagger)^2$ vanishes, but the form of Eq. (A9) is useful in order to recognize the structure of the integrand. For every copy it can be expressed in terms of the matrix field $\hat{Q}$ as follows:

$$\psi_r \psi_r^\dagger = i\frac{1}{g}\sigma_\nu\left[iP - i\frac{g}{N}\sigma_\nu \Phi_{2a}\Phi_{2a}^\dagger - iP\right]_r. \tag{A10}$$

Inserting this expression into the functional integral, we notice that the integration over the term $P - (g/N)\sigma_\nu \Phi_{2a}\Phi_{2a}^\dagger$ can be performed independently and in the position space, since the term with $\hat{Q} - i(g/N)\Sigma_\nu \Phi_a \Phi_a^\dagger$ does not possess any gradients and therefore is already diagonal. The contribution from this term is zero. Then combining Eq. (A9) and Eq. (A10) we get

$$\langle G_{rr}^{ii}\rangle = -\frac{i}{g}\sigma_\nu \int \mathcal{D}\hat{Q} P_r \int \mathcal{D}\Phi^\dagger \mathcal{D}\Phi e^{-\bar{\mathcal{S}}_{BF}[\hat{Q},\Phi^\dagger,\Phi]}, \tag{A11}$$

at which point the integration over vector fields can be carried out. Rising the graded determinant into the exponent we acquire the log term in Eq. (9).

APPENDIX B: EFFECTIVE PROPAGATORS AND CORRELATION FUNCTIONS

Below, we always send the UV-cutoff of radial integrals to infinity if the dimensional analysis points out their convergence. In the infrared, the divergences are cut off by the scattering rate η. The two-point vertex functions which appear in Eq. (24) read

$$\Gamma_{2|\alpha\beta}^{(\nu)}(q) = \frac{1}{2}\mathrm{Tr}\int \frac{d^d p}{(2\pi)^d}\frac{[-iz + \not{p}]\sigma_\alpha\sigma_\nu[-iz + \not{q} + \not{p}]\sigma_\beta\sigma_\nu}{[z^2+p^2][z^2+(p+q)^2]}. \tag{B1}$$

To calculate the contributions to the mass we set $\epsilon = 0$ and $p = 0$. We first neglect all terms under the integral which are not rotationally invariant:

$$\Gamma_{2|\alpha\beta}^{(\nu)} = \frac{1}{2}\mathrm{Tr}\int \frac{d^d p}{(2\pi)^d}\frac{-\eta^2\sigma_\alpha\sigma_\nu\sigma_\beta\sigma_\nu + \frac{p^2}{d}\sigma_{i=1,\dots,d}\sigma_{\alpha=0,\dots,3}\sigma_\nu\sigma_i\sigma_\beta\sigma_\nu}{[\eta^2+p^2]^2}, \tag{B2}$$

where the factor $1/d$ in front of the second part appears due to the angular average. Since the product of any two or three Pauli matrices is a Pauli matrix again, the trace in Eq. (B2) is nonzero only for $\alpha = \beta$. Therefore, the inverse propagator is diagonal in both 2D and 3D. Below we evaluate Eq. (B2) for all combinations of external indices α,β and use the following shorthand $\sigma_i\sigma_\nu = \zeta\sigma_\nu\sigma_i$ for all i. In 2D $\zeta = (-)+1$, if σ_ν (anti)commutes with σ_i, in 3D $\zeta = +1$. The trace of the term proportional to η^2 gives $\mathrm{Tr}\sigma_\alpha\sigma_\nu\sigma_\beta\sigma_\nu = 2\zeta\delta_{\alpha\beta}$. The second part has to be evaluated for different index combinations separately. In 2D: 1) $\alpha = \beta = 1,2$ is zero because of the matrix product property $\sigma_i\sigma_\alpha\sigma_i = \sigma_\alpha(\sigma_{i=\alpha} - \sigma_{i\neq\alpha})\sigma_i = 0$, i is summed over; 2) $\alpha = \beta = 0$: $\mathrm{Tr}[-\eta^2\sigma_\nu\sigma_\nu + p^2/2\sigma_i\sigma_\nu\sigma_i\sigma_\nu] = 2(\zeta p^2 - \eta^2)$; 3) $\alpha = \beta = 3$, i.e., σ_ν commutes with σ_3 for both disorder types: $\mathrm{Tr}[-\eta^2\sigma_3\sigma_\nu\sigma_3\sigma_\nu + p^2/2\sigma_i\sigma_3\sigma_\nu\sigma_i\sigma_3\sigma_\nu] = -2(\eta^2 + \zeta p^2)$. With the help of the saddle-point condition Eq. (16), cf. Ref [21], the elements of the mass matrix become

$$\mathrm{M}_{00}^{(\nu)} = \frac{1}{g} - \int \frac{d^2p}{(2\pi)^2}\frac{-\eta^2+\zeta p^2}{[p^2+\eta^2]^2} = \begin{cases} \frac{\Lambda^2}{2\pi(\eta^2+\Lambda^2)}, & \nu = 0,\ \zeta = +1 \\ \frac{2}{g}, & \nu = 3,\ \zeta = -1 \end{cases}, \tag{B3}$$

$$\mathrm{M}_{\alpha\alpha=1,2}^{(\nu)} = \frac{1}{g} + \int \frac{d^2p}{(2\pi)^2}\frac{\zeta\eta^2}{[p^2+\eta^2]^2} = \frac{1}{g} + \frac{\zeta}{4\pi} = \begin{cases} \frac{1}{g} + , & \nu = 0,\ \zeta = +1 \\ \frac{1}{g} - \frac{\Lambda^2}{4\pi(\eta^2+\Lambda^2)}, & \nu = 3,\ \zeta = -1 \end{cases}, \tag{B4}$$

$$\mathrm{M}_{33}^{(\nu)} = \frac{1}{g} - \int \frac{d^2p}{(2\pi)^2}\frac{-\eta^2-\zeta p^2}{[p^2+\eta^2]^2} = \begin{cases} \frac{2}{g}, & \nu = 0,\ \zeta = +1 \\ \frac{\Lambda^2}{2\pi(\eta^2+\Lambda^2)}, & \nu = 3,\ \zeta = -1 \end{cases}. \tag{B5}$$

In 3D, the evaluation differs technically in that respect, that there is no Pauli matrix which anticommutes with the kinetic energy operator $-i\not{\partial}$. Second term is for 1) $\sigma_\alpha \neq 0$, $\sigma_\beta \neq 0$: $\mathrm{Tr}\sigma_\alpha\sigma_i\sigma_\beta\sigma_i = -2\delta_{\alpha\beta}$; 2) $\alpha = \beta = 0$, $\mathrm{Tr}\sigma_0\sigma_i\sigma_0\sigma_i = 6$. The vertex is a diagonal matrix $\Gamma_{2|\alpha\beta}(0) = \delta_{\alpha\beta}\Gamma_{2|\alpha}(0)$, with elements

$$\Gamma_0(0) = -\int \frac{d^3p}{(2\pi)^3}\frac{\eta^2-p^2}{[p^2+\eta^2]^2}, \quad \text{and}$$

$$\Gamma_{i=1,2,3}(0) = -\frac{1}{3}\int \frac{d^3p}{(2\pi)^3}\frac{3\eta^2+p^2}{[\eta^2+p^2]^2}.$$

All elements of the mass matrix are then massive:

$$M_0 = \frac{1}{g} - \Gamma_0(0) = 2\int \frac{d^3p}{(2\pi)^3}\frac{\eta^2}{[p^2+\eta^2]^2} = \frac{\eta}{4\pi},$$
$$M_{i=1,2,3} = \frac{1}{g} + \frac{1}{3}\int \frac{d^3p}{(2\pi)^3}\frac{3\eta^2+p^2}{[\eta^2+p^2]^2} = \lambda. \tag{B6}$$

APPENDIX C: DETAILS OF THE PERTURBATIVE CORRECTIONS TO THE DOS

Main corrections to the DOS are calculated as

$$\delta G_{rr}^{ii} \sim -i\sqrt{\frac{N}{g}}\sigma_\nu \int \mathcal{D}\hat{Q} e^{-\mathcal{S}_G} P_r^i(1 + \mathcal{S}_p + \cdots), \tag{C1}$$

where $\mathcal{S}_G$ represents the full Gaussian action and

$$\mathcal{S}_p = N\sum_{n\geqslant 3}\frac{(-1)^n}{n}\left(\frac{g}{N}\right)^{\frac{n}{2}}\mathrm{trg}[\bar{G}\hat{Q}\Sigma_\nu]^n, \tag{C2}$$

where the fields are again rescaled as $\hat{Q} \to \sqrt{g/N}\hat{Q}$. It is obvious from Eq. (C1) that only terms with an odd power of fields P contribute to the DOS. To order g^3 the relevant contributions are

$$\mathcal{S}_p \sim -\frac{N}{3}\left(\frac{g}{N}\right)^{\frac{3}{2}} \mathrm{trg}[\bar{G}\hat{Q}\Sigma_\nu]^3 - \frac{N}{5}\left(\frac{g}{N}\right)^{\frac{5}{2}} \mathrm{trg}[\bar{G}\hat{Q}\Sigma_\nu]^5. \tag{C3}$$

After performing the graded trace and retaining only contributions with an odd number of field factors third order term becomes:

$$-i\frac{N}{3}\left(\frac{g}{N}\right)^{\frac{3}{2}} \mathrm{tr}\{[\bar{G}P\sigma_\nu]^3 - 3\bar{G}P\sigma_\nu\bar{G}\bar{\chi}\sigma_\nu\bar{G}\chi\sigma_\nu\}, \tag{C4}$$

and five-field term becomes

$$i\frac{N}{5}\left(\frac{g}{N}\right)^{\frac{5}{2}} \mathrm{tr}\{[\bar{G}P\sigma_\nu]^5 - 5[\bar{G}P\sigma_\nu]^3\bar{G}\bar{\chi}\sigma_\nu\bar{G}\chi\sigma_\nu\} \tag{C5a}$$

$$+iN\left(\frac{g}{N}\right)^{\frac{5}{2}} \mathrm{tr}\{\bar{G}P\sigma_\nu(\bar{G}\bar{\chi}\sigma_\nu[\bar{G}Q\sigma_\nu]^2\bar{G}\chi\sigma_\nu + [\bar{G}\bar{\chi}\sigma_\nu\bar{G}\chi\sigma_\nu]^2)\}. \tag{C5b}$$

Here, the fermionic fields are normally ordered to guarantee for the positive sign of contractions. The contribution from Eq. (C4) reads:

$$\delta G^{(1)}_{rr} = -\frac{g}{3}\sigma_\nu\sigma_a \sum_{r_1,r_2,r_3} [\langle P^a_r P^\alpha_{r_1} P^\beta_{r_2} P^\gamma_{r_3}\rangle_G - 3\langle P^a_r P^\alpha_{r_1}\rangle_G \langle \bar{\chi}^\beta_{r_2}\chi^\gamma_{r_3}\rangle_G]\Gamma^{(\nu)}_{3|\alpha r_1,\beta r_2,\gamma r_3}, \tag{C6}$$

where the contraction brackets represent functional integration over the Gaussian action. The third order virtual fermion loop reads

$$\Gamma^{(\nu)}_{3|\alpha r_1,\beta r_2,\gamma r_3} = \mathrm{Tr}\sigma_\alpha\sigma_\nu\bar{G}_{r_1r_2}\sigma_\beta\sigma_\nu\bar{G}_{r_2r_3}\sigma_\gamma\sigma_\nu\bar{G}_{r_3r_1}, \tag{C7}$$

and is invariant under cyclic index permutations. Because of this cyclicity, all three pairwise contractions of fields P contribute equally after index relabeling:

$$\langle P^a_r P^\alpha_{r_1} P^\beta_{r_2} P^\gamma_{r_3}\rangle_G = 3\langle P^a_r P^\alpha_{r_1}\rangle_G \langle P^\beta_{r_2} P^\gamma_{r_3}\rangle_G, \tag{C8}$$

and since bosonic and fermionic correlators are the same, this DOS correction vanishes as a whole. Diagrammatically, this equation is shown in Fig. 2. This result is nothing but the manifestation of the linked-cluster theorem and has a very simple meaning, namely it postulates the vanishing of the leading order "rainbowlike" corrections, which are already accounted in the saddle-point equation.

A similar line of reasoning reveals the mutual annihilation of all rainbow- and "bulgelike" DOS corrections to order g^2

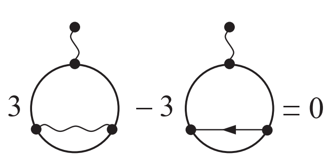

FIG. 2. Compensation of the leading order corrections from "rainbow" diagrams as explained in the main text. Wavy lines denote contractions of bosonic fields P while straight lines denote the contractions of Grassmann variables $\bar{\chi}\chi$.

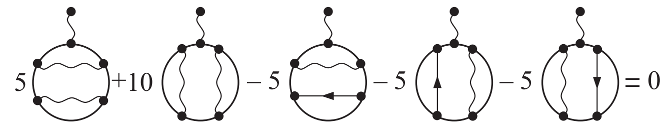

FIG. 3. Partial compensation of the second order "rainbow" and "bulge" corrections arising from Eq. (C5a) as explained in the main text.

encoded in Eq. (C5a) as depicted in Fig. 3. Five nonvanishing pairwise contractions

$$\langle P^a_r P^\alpha_{r_1} P^\beta_{r_2} P^\gamma_{r_3} P^\iota_{r_4} P^\tau_{r_5}\rangle_G = 5\langle P^a_r P^\alpha_{r_1}\rangle_G \langle P^\beta_{r_2} P^\iota_{r_4}\rangle_G \langle P^\gamma_{r_3} P^\tau_{r_5}\rangle_G \tag{C9}$$

generate the so-called sunrise diagrams shown in Fig. 1 on the left. The factorization of the four-fermion term from Eq. (C5b) is not unique and yields two contributions

$$\langle P^a_r P^\alpha_{r_1} \bar{\chi}^\beta_{r_2}\chi^\gamma_{r_3}\bar{\chi}^\iota_{r_4}\chi^\tau_{r_5}\rangle_G = \langle P^a_r P^\alpha_{r_1}\rangle_G \langle \bar{\chi}^\beta_{r_2}\chi^\gamma_{r_3}\rangle_G \langle \bar{\chi}^\iota_{r_4}\chi^\tau_{r_5}\rangle_G - \langle P^a_r P^\alpha_{r_1}\rangle_G \langle \bar{\chi}^\beta_{r_2}\chi^\tau_{r_5}\rangle_G \langle \bar{\chi}^\iota_{r_4}\chi^\gamma_{r_3}\rangle_G, \tag{C10}$$

where the minus sign in front of the second term is due to the odd number of Grassmannian permutations. Because of the sublying supersymmetry, this negative term gets totally annihilated by the term

$$\langle P^a_r P^\alpha_{r_1} \bar{\chi}^\beta_{r_2} Q^\gamma_{r_3} Q^\iota_{r_4}\chi^\tau_{r_5}\rangle_G = \langle P^a_r P^\alpha_{r_1}\rangle_G \langle \bar{\chi}^\beta_{r_2}\chi^\tau_{r_5}\rangle_G \langle Q^\iota_{r_4} Q^\gamma_{r_3}\rangle_G. \tag{C11}$$

This is shown diagrammatically in Fig. 4. Hence, the rainbow-like contributions get annihilated to this order too. The positive contraction from Eq. (C10) gives rise to the nonvanishing correction to the DOS in form of a "bulge" diagram, shown in Fig. 1 on the right. This is a rather remarkable unexpected result, since, naively, diagrams of that type are considered to be one particle reducible. This misapprehension roots in the formal similarity of this diagrammatic approach to that of the nonlocal self-energy of interacting systems which employs a slightly different version of the linked-cluster theorem. The nonvanishing terms to order g^2 are

$$\delta G^{(2)}_{rr} = \frac{g^2}{N}\sigma_\nu\sigma_a \sum_{r_1r_2r_3r_4r_5} \Gamma^{(\nu)}_{5|\alpha r_1,\beta r_2,\gamma r_3,\iota r_4,\tau r_5} \times [\langle P^a_r P^\alpha_{r_1}\rangle_G \langle P^\beta_{r_2} P^\iota_{r_4}\rangle_G \langle P^\gamma_{r_3} P^\tau_{r_5}\rangle_G + \langle P^a_r P^\alpha_{r_1}\rangle_G \langle \bar{\chi}^\beta_{r_2}\chi^\gamma_{r_3}\rangle_G \langle \bar{\chi}^\iota_{r_4}\chi^\tau_{r_5}\rangle_G], \tag{C12}$$

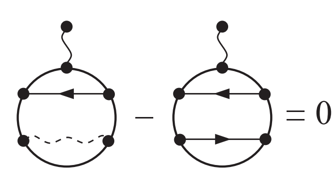

FIG. 4. Partial compensation of the second order "rainbow" corrections arising from Eq. (C5b) as explained in the main text. The dashed wavy line denotes the contraction of the fields Q. Together with Fig. 4, the total annihilation of the "rainbow" corrections, already included in the saddle-point equation is insured.

with the fifth order virtual fermion loop

$$\Gamma^{(\nu)}_{5|\alpha r_1,\beta r_2,\gamma r_3,\iota r_4,\tau r_5} = \mathrm{Tr}\sigma_\alpha\sigma_\nu\bar{G}_{r_1r_2}\sigma_\beta\sigma_\nu\bar{G}_{r_2r_3}\sigma_\gamma\sigma_\nu\bar{G}_{r_3r_4}\sigma_\iota\sigma_\nu\bar{G}_{r_4r_5}\sigma_\tau\sigma_\nu\bar{G}_{r_5r_1}. \tag{C13}$$

In ultraweak limit, the correlators are replaced by delta functions. The detailed evaluation of this correction in ultraweak disorder limit is given in Appendix D.

APPENDIX D: EVALUATION OF THE PERTURBATIVE CORRECTIONS: WEAK DISORDER LIMIT

In ultraweak disorder limit, the correction to the Green's function which arise from diagrams depicted in Fig. 1 read

$$\delta G^{(2)}_{rr} \sim \frac{g^2}{N}\sigma_0\mathrm{Tr}\sum_{r_1,r_2}\big[\bar{G}_{rr_1}\bar{G}_{r_1r_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r_2}\bar{G}_{r_2r} + \bar{G}_{rr_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r}\big]. \tag{D1}$$

First term is harmless: each of the two bulges can be expressed using the saddle-point condition as $\bar{G}_{rr} = -i\eta/g$, while the remaining loop converges in both dimensions.

$$\begin{aligned}\mathcal{D}_1 &= \mathrm{Tr}\sum_{r_1,r_2}\bar{G}_{rr_1}\bar{G}_{r_1r_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r_2}\bar{G}_{r_2r}\\ &= -\frac{\eta^2}{g^2}\mathrm{Tr}\int\frac{d^dq}{(2\pi)^d}\bar{G}^3(q)\\ &= 2i\frac{\eta^3}{g^2}\int\frac{d^dq}{(2\pi)^d}\frac{3q^2-\eta^3}{[q^2+\eta^2]^3},\end{aligned} \tag{D2}$$

which leads to

$$\mathcal{D}_1 = \frac{i}{2\pi}\frac{\eta^{d-1}}{g^2}. \tag{D3}$$

To the contrary, the evaluation of the first contribution is technically more demanding. Transforming the loop into the Fourier space we get

$$\begin{aligned}\mathcal{D}_2 &= \mathrm{Tr}\sum_{r_1,r_2}\bar{G}_{rr_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r_1}\bar{G}_{r_1r_2}\bar{G}_{r_2r}\\ &= \mathrm{Tr}\int\frac{d^dp\,d^dq}{(2\pi)^{2\mathrm{D}}}\bar{G}(q+p)\bar{G}(q)\\ &\quad\times\int\frac{d^dk}{(2\pi)^d}\bar{G}(k-p)\bar{G}(k)\bar{G}(k).\end{aligned} \tag{D4}$$

Integrals over q and k can be carried out separately using Feynman representation of the fraction product:

$$\frac{1}{A^{1+n}B} = \int_0^1 dx\frac{(n+1)(1-x)^n}{[(1-x)A+xB]^{2+n}}. \tag{D5}$$

The q integral reads

$$\begin{aligned}I_1 &= \int\frac{d^dq}{(2\pi)^d}\bar{G}(q+p)\bar{G}(q)\\ &= \int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{[\not{q}+\not{p}-i\eta][\not{q}-i\eta]}{[x(q+p)^2+(1-x)q^2+\eta^2]^2}.\end{aligned} \tag{D6}$$

By shifting $q_i \to q_i - xp_i$ the denominator becomes rotationally invariant, which enables us to drop all odd powers of q_i in the numerator, getting

$$I_1 = \int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{q^2-\eta^2-x(1-x)p^2-i\eta\not{p}(1-2x)}{[q^2+\eta^2+x(1-x)p^2]^2}. \tag{D7}$$

One recognizes that the term with $1-2x$ vanishes after integration over x: Since $1-2x = \frac{d}{dx}x(1-x)$ and the remaining expression depends only on $x(1-x)$ we get

$$\begin{aligned}&\int_0^1 dx f[x(1-x)]\frac{d}{dx}[x(1-x)]\\ &= \int_0^1 dx\frac{d}{dx}F[x(1-x)] = F[0]-F[0] = 0,\end{aligned} \tag{D8}$$

where $F(x)$ is the indefinite integral of $f(x)$. Since I_1 is symmetric under $p \to -p$, we can omit all odd powers of p in the integral over k. In 2D, the remaining integral I_1 was computed in Ref. [23]. Assuming a very large upper cutoff and using the saddle-point condition we get:

$$I_1^{2\mathrm{D}} \sim \frac{1}{g} - \frac{1}{2\pi}\sqrt{\frac{4+t^2}{t^2}}\mathrm{atanh}\sqrt{\frac{t^2}{4+t^2}}\bigg|_{t=p/\eta}. \tag{D9}$$

The evaluation in 3D takes a few computational lines more: Splitting the integrand in divergent and convergent parts

$$\begin{aligned}I_1^{3\mathrm{d}} = \int_0^1 dx\int\frac{d^3q}{(2\pi)^3}\bigg[&\frac{1}{q^2+\eta^2+x(1-x)p^2}\\ &-2\frac{\eta^2+x(1-x)p^2}{[q^2+\eta^2+x(1-x)p^2]^2}\bigg],\end{aligned} \tag{D10}$$

we can perform q integral in the convergent part. We continue by adding and subtracting $1/g$ to the divergent part and using the saddle point equation:

$$\begin{aligned}I_1^{3\mathrm{d}} &= \frac{1}{g} - \frac{1}{4\pi}\int_0^1 dx\sqrt{\eta^2+x(1-x)p^2} + \frac{1}{2\pi^2}\int_0^1 dx\\ &\quad\times\int_0^\infty dq\bigg[\frac{q^2}{q^2+\eta^2+x(1-x)p^2} - \frac{q^2}{q^2+\eta^2}\bigg].\end{aligned} \tag{D11}$$

The divergent contribution in the remaining q integral cancels, hence the integral can be carried out using the residue theorem:

$$I_1^{3\mathrm{d}} = \frac{1}{g} + \frac{\eta}{4\pi} - \frac{\eta}{2\pi}\int_0^1 dx\sqrt{1+x(1-x)t}\bigg|_{t=p/\eta}. \tag{D12}$$

The indefinite integral over x is known, putting the boundaries and simplifying the expression we finally get

$$I_1^{3\mathrm{d}} \sim \frac{1}{g} - \frac{\eta}{8\pi}\frac{4+t^2}{t}\mathrm{atan}\left(\frac{t}{2}\right)\bigg|_{t=p/\eta}. \tag{D13}$$

Second integral can be evaluated in a similar fashion: Using the Feynman parametrization we get

$$I_2 = \int \frac{d^dq}{(2\pi)^d}\bar{G}(q-p)\bar{G}(q)\bar{G}(q) = 2\int_0^1 dx(1-x)\int \frac{d^dq}{(2\pi)^d}\frac{[\not{q}-\not{p}-i\eta][\not{q}-i\eta][\not{q}-i\eta]}{[(1-x)q^2+x(q-p)^2+\eta^2]^3} \tag{D14}$$

$$= -2i\eta\int_0^1 dx(1-x)\int \frac{d^dq}{(2\pi)^d}\frac{3q^2-\eta^2-x(1-x)p^2}{[q^2+\eta^2+x(1-x)p^2]^3}. \tag{D15}$$

Power counting indicates that the integral over q converges in both dimensions. The symmetrization of the denominator is achieved by shifting $q_i \to q_i + xp_i$, when we dropped odd powers of q and p and regrouped x-dependent factors at p^2 using the fact that the integral operator $\int_0^1 dx$ does not change under substitution $x \to 1-x$. This leads in 2D to

$$I_2^{2D} = -\frac{i}{2\pi\eta}\int_0^1 dx\frac{1-x}{1+x(1-x)t^2}\bigg|_{t=p/\eta} = -\frac{i}{\pi\eta}\frac{1}{\sqrt{t^2(4+t^2)}}\text{atanh}\sqrt{\frac{t^2}{4+t^2}}\bigg|_{t=p/\eta}. \tag{D16}$$

In 3D we analogously get

$$I_2^{3D} = -\frac{i}{2\pi}\int_0^1 dx\frac{1-x}{\sqrt{1+x(1-x)t^2}}\bigg|_{t=p/\eta} = -\frac{i}{2\pi t}\text{atan}\left(\frac{t}{2}\right)\bigg|_{t=p/\eta}. \tag{D17}$$

Taking the trace over the Dirac space becomes trivial and gives a factor of two. In 2D, we obtain with Eqs. (D9) and (D16)

$$\mathcal{D}_2^{2d} = -2i\frac{\eta}{\pi}\int\frac{d^2t}{(2\pi)^2}\frac{\text{atanh}\sqrt{\frac{t^2}{t^2+4}}}{t\sqrt{t^2+4}}\left[\frac{1}{g}-\frac{1}{2\pi}\sqrt{\frac{t^2+4}{t^2}}\text{atanh}\sqrt{\frac{t^2}{t^2+4}}\right]. \tag{D18}$$

Extracting from the saddle-point Eq. (16) the fitting expression

$$\frac{1}{2}\log\left[1+\frac{\Lambda^2}{\eta^2}\right] = f_{2d} \equiv \frac{2\pi}{g}, \tag{D19}$$

we obtain with high accuracy

$$\begin{aligned} -i\frac{\eta}{\pi^2 g}\int_0^{\Lambda/\eta} dt\frac{\text{atanh}\sqrt{\frac{t^2}{t^2+4}}}{\sqrt{t^2+4}} &\sim -i\frac{1}{\pi^2}\frac{\eta}{g}\frac{f_{2d}^2}{2} = -i2\frac{\eta}{g^3},\\ i\frac{4\eta}{(2\pi)^3}\int_0^{\Lambda/\eta} dt\frac{1}{t}\left[\text{atanh}\sqrt{\frac{t^2}{t^2+4}}\right]^2 &\sim i\frac{4\eta}{(2\pi)^3}\left[\frac{1}{2}+\frac{f_{2d}^3}{3}\right] = i\frac{\eta}{g}\left[\frac{4}{3g^2}+\frac{2g}{(2\pi)^3}\right], \end{aligned} \tag{D20}$$

where in the second equality the saddle-point condition is used. Counting $\mathcal{D}_1^{2d}$ and $\mathcal{D}_2^{2d}$ together we finally get the correction to the

$$\delta G_{rr}^{ii} \sim -i\frac{\varrho_{SCBA}}{2N^2}\left[\frac{2}{3}-\frac{g}{2\pi}-\frac{g^3}{4\pi^3}\right]\sigma_0, \tag{D21}$$

and from here the DOS correction given in Eq. (31). In 3D, the remaining integral reads

$$\begin{aligned} \mathcal{D}_2^{3d} &= -\frac{i\eta^3}{g\pi}\int\frac{d^3t}{(2\pi)^3}\frac{1}{t}\text{atan}\left(\frac{t}{2}\right) + \frac{i\eta^4}{8\pi^2}\int\frac{d^3t}{(2\pi)^3}\frac{4+t^2}{t^2}\left[\text{atan}\left(\frac{t}{2}\right)\right]^2\\ &= -\frac{i}{g}\frac{\eta^3}{2\pi^3}\int_0^{\Lambda/\eta} dt\,t\,\text{atan}\left(\frac{t}{2}\right) + \frac{i\eta^4}{(2\pi)^4}\int_0^{\Lambda/\eta} dt(4+t^2)\left[\text{atan}\left(\frac{t}{2}\right)\right]^2. \end{aligned} \tag{D22}$$

From the saddle-point Eq. (16) we get the fitting polynomial

$$\frac{\Lambda}{\eta} - \text{atan}\left(\frac{\Lambda}{\eta}\right) = f_{3d} \equiv \frac{2\pi^2}{g\eta}. \tag{D23}$$

Fitting integrals in Eq. (D22) with different powers of the polynomial f_{3D}, we obtain with excellent accuracy

$$I_2^{3D} \sim -\frac{i}{g}\frac{\eta^3}{2\pi^3}\frac{\pi}{4}f_{3D}^2 + \frac{i\eta^4}{(2\pi)^4}\left(\pi^2 f_{3D} + \frac{5\pi}{19}f_{3D}^3\right) = -i\frac{\eta}{g}\left[\frac{\pi^2}{2g^2}\left(1-\frac{5\pi}{19}\right) - \frac{\eta^2}{8}\right]. \tag{D24}$$

Counting $\mathcal{D}_1^{3\mathrm{D}}$ and $\mathcal{D}_2^{3\mathrm{D}}$ together we eventually obtain

$$\delta G_{rr}^{ii} \sim -i\frac{\varrho_{\mathrm{SCBA}}}{2N^2}\left[\frac{\pi^2}{2}\left(1-\frac{5\pi}{19}\right)-\frac{g\eta}{2\pi}-\frac{g^2\eta^2}{8}\right]\sigma_0, \tag{D25}$$

which upon taking the trace over the Dirac space and the imaginary part yields the correction in Eq. (32).

APPENDIX E: EVALUATION OF THE PERTURBATIVE CORRECTIONS: STRONG DISORDER LIMIT

Here we get

$$\delta G_{rr}^{(2)} \sim \frac{\pi g}{16N}\sigma_0 \mathrm{Tr}\sum_{r_1,r_2}\left[\bar{G}_{rr_1}\sigma_3\bar{G}_{r_1r_1}\sigma_3\bar{G}_{r_1r_2}\sigma_3\bar{G}_{r_2r_2}\sigma_3\bar{G}_{r_2r}+\bar{G}_{rr_1}\sigma_3\bar{G}_{r_1r_2}\sigma_3\bar{G}_{r_2r_1}\sigma_3\bar{G}_{r_1r_2}\sigma_3\bar{G}_{r_2r}\right]. \tag{E1}$$

The evaluation of the first contribution is entirely analogous to the weak disorder case, we get

$$\mathcal{D}_1 = \frac{\pi g}{16N}\mathrm{Tr}\sum_{r_1,r_2}\bar{G}_{rr_1}\sigma_3\bar{G}_{r_1r_1}\sigma_3\bar{G}_{r_1r_2}\sigma_3\bar{G}_{r_2r_2}\sigma_3\bar{G}_{r_2r} = i\frac{\varrho_{\mathrm{SCBA}}}{(8N)^2}. \tag{E2}$$

Second contribution reads

$$\mathcal{D}_2 = \mathrm{Tr}\int\frac{d^2p\,d^2q}{(2\pi)^4}\bar{G}(q)\sigma_3\bar{G}(q+p)\sigma_3\int\frac{d^2k}{(2\pi)^2}\bar{G}(k+p)\sigma_3\bar{G}(k)\bar{G}(k)\sigma_3. \tag{E3}$$

The presence of the σ_3 matrix which anticommutes with the Dirac Hamiltonian changes the sign of the q integral:

$$\int\frac{d^2q}{(2\pi)^2}\bar{G}(q)\sigma_3\bar{G}(q+p)\sigma_3 = -\left(\frac{1}{g}-\frac{1}{2\pi}\frac{t}{\sqrt{4+t^2}}\mathrm{atanh}\sqrt{\frac{t^2}{4+t^2}}\right)+i\frac{\not{t}}{\pi}\frac{1}{t\sqrt{4+t^2}}\mathrm{atanh}\sqrt{\frac{t^2}{4+t^2}}, \tag{E4}$$

where again $t = p/\eta$ and $\not{t} = t_i\sigma_{i=1,2}$. Second integral becomes

$$\int\frac{d^2k}{(2\pi)^2}\bar{G}(k+p)\sigma_3\bar{G}(k)\bar{G}(k)\sigma_3 = \frac{2i-\not{t}}{2\pi\eta}\left(\frac{1}{4+t^2}-\frac{t}{(4+t^2)^{3/2}}\mathrm{atanh}\sqrt{\frac{t^2}{4+t^2}}\right), \tag{E5}$$

which eventually leads to

$$\mathcal{D}_2 = -i\frac{\varrho_{\mathrm{SCBA}}}{2\pi N}\int_0^{\Lambda/\eta}dt\left(\frac{t}{4+t^2}-\frac{t^2}{(4+t^2)^{3/2}}\mathrm{atanh}\sqrt{\frac{t^2}{4+t^2}}\right) = -i\frac{\varrho_{\mathrm{SCBA}}}{2\pi N}\left[\frac{2\pi}{g}-\frac{1}{2}\left(\frac{2\pi}{g}\right)^2\right]. \tag{E6}$$

The integrals can be evaluated analytically. Adding the contributions from all diagrams and extracting the DOS we finally get

$$\varrho \sim \varrho_{\mathrm{SCBA}} + \frac{\varrho_{\mathrm{SCBA}}}{(4N)^2}\left(\frac{3}{2}-\frac{2\pi}{g}\right). \tag{E7}$$

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