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Creating a bosonic fractional quantum Hall state by pairing fermions

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We numerically study the behavior of spin-1/2 fermions on a two-dimensional square lattice subject to a uniform magnetic field, where opposite spins interact via an on-site attractive interaction. Starting from the noninteracting case where each spin population is prepared in a quantum Hall state with unity filling, we follow the evolution of the system as the interaction strength is increased. Above a critical value and for sufficiently low flux density, we observe the emergence of a twofold quasidegeneracy accompanied by the opening of an energy gap to the third level. Analysis of the entanglement spectra shows that the gapped ground state is the bosonic 1/2 Laughlin state. Our work therefore provides compelling evidence of a topological phase transition from the fermionic quantum Hall state at unity filling to the bosonic Laughlin state at a critical attraction strength of the order of the one-body spectrum linewidth.

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Topological phases of matter represent a subject of intense research [1–5], offering the prospect of realizing fault tolerant quantum computation [6]. The rapid growth of this field, since the first observation of the quantum Hall effect, is largely due to experimental progress in producing high quality and purity materials [2,3]. In this context, ultracold quantum gases, which feature clean, flexible, and well characterized environments, appear as promising systems. Furthermore, as they comprise electrically neutral particles that can be either fermionic or bosonic, atomic quantum gases may provide a new vista on the subject [7–10]. Notable achievements in this direction include the creation of Bose-Einstein condensates (BEC) in the lowest Landau level via fast rotation [11,12], the realization of the Harper-Hofstadter [13–15] and Haldane [16] models in optical lattices, and more recently the observation of chiral currents in atomic ladders [17–19], as well as the measurement of the second Chern number of a non-Abelian Yang monopole [20]. These experimental progresses were accompanied by numerous theoretical investigations of the topological phases that may appear in optical lattices [21–34].

A remarkable feature of ultracold gases is the ability to use tunable attractive interactions to convert, in real time, a pair of distinguishable fermions into a tightly bound bosonic molecule (BM) [35,36]. From the perspective of topological matter, this feature opens unique possibilities [37,38]. Suppose a two-dimensional (2D) spin-1/2 fermionic system is initially prepared in an integer quantum Hall (IQH) state for each spin component ($\uparrow$ and $\downarrow$), with a filling factor $\nu_\uparrow = \nu_\downarrow = n$. As the attraction strength is brought to the strong binding limit, each fermion pair forms a BM that carries twice the fermion (neutral) charge, and hence experiences twice the magnetic flux seen by a fermion. As a result, the final many-body system would be characterized by a filling factor $\nu_{\uparrow,\downarrow}/2$, provided all bosons occupy the lowest band.

Building upon this observation, Yang and Zhai pointed out, almost a decade ago, the possibility of using a Feshbach resonance to drive a transition between a fermionic IQH state and a bosonic fractional quantum Hall (FQH) Laughlin state [39]. Given their distinct topological natures, these phases must be separated by a quantum phase transition [40] whose critical behavior remains an important open question [41–43]. More recently, Ho proposed the use of a rapid sweep through this transition to *project* fermionic IQH states onto the BEC side of the Feshbach resonance in order to reveal the bosonic FQH structure of its center-of-mass wave function [44]. A key aspect concerns the value of the critical interaction, which is crucial to the prospect of observing this transition experimentally. Since this transition has not been supported so far by a microscopic description, the order of magnitude of this critical value remains an open question.

In this Rapid Communication, we numerically study the behavior of spin-1/2 fermions on a 2D lattice subject to a homogeneous magnetic field, with an attractive on-site interaction between the $\uparrow$ and $\downarrow$ spins. Using energy and particle entanglement spectroscopy on finite-size systems, we provide evidence for the existence of a topological phase transition from the fermionic IQH $\nu_{\uparrow,\downarrow} = 1$ state to the 1/2 bosonic Laughlin state above a critical attraction strength of the order of the linewidth of the one-body spectrum. The use of a lattice in our model (which leads to a finite one-body linewidth) is crucial in establishing the finite value of the critical interaction and clearly distinguishes our work from previous predictions [39,40,44] of the transition in continuum models.

The system of interest here is described by the Fermi-Hubbard model with minimal coupling to a gauge field via Peierl's substitution. The Hamiltonian reads

$$\mathcal{H} = -t \sum_{\langle \mathbf{r},\mathbf{r}'\rangle,\sigma=\uparrow,\downarrow} (e^{i\varphi_{\mathbf{r}\mathbf{r}'}} c^\dagger_{\mathbf{r},\sigma} c_{\mathbf{r}',\sigma} + \text{H.c.}) + U \sum_{\mathbf{r}} n_{\mathbf{r},\uparrow} n_{\mathbf{r},\downarrow}, \tag{1}$$

where $\langle \mathbf{r},\mathbf{r}'\rangle$ denotes neighboring lattice sites $\mathbf{r} = (x,y)$, $c_{\mathbf{r},\sigma}$ ($c^\dagger_{\mathbf{r},\sigma}$) annihilates (creates) a fermion with spin σ at site

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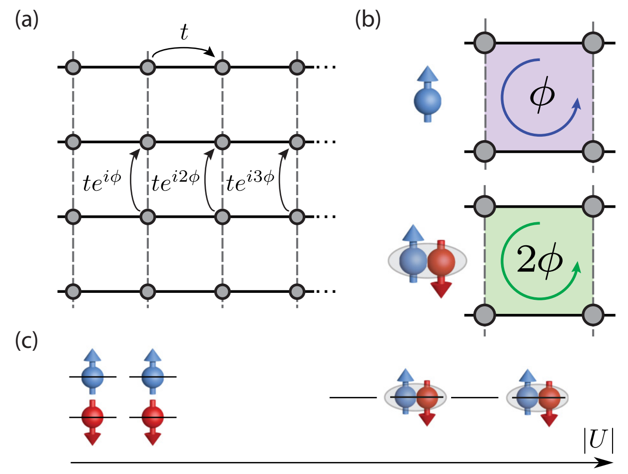

FIG. 1. (a) Schematic description of the HH model on a square lattice with the Landau gauge in the y direction. Each plaquette is pierced by a flux $\phi = 2\pi\alpha$, where α is the flux density defined in the main text. The phase ϕ is proportional to $\varphi_{\mathbf{rr}'}$ in Eq. (1). For simplicity, we omit the minus sign in the tunneling. (b) The flux seen by a BM is twice the one of a single fermion. (c) Possible scenario of a phase transition induced by an attractive on-site interaction between the fermions. At small $|U|$, the system is described by the $\nu_{\uparrow,\downarrow} = 1$ IQH state. At large $|U|$, fermions bind into BMs and the system forms the 1/2 Laughlin state.

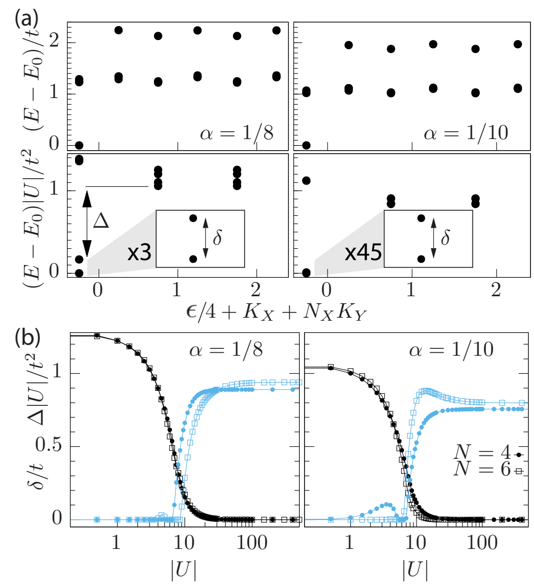

FIG. 2. (a) Energy spectrum of $2N = 6$ fermions for flux densities $\alpha = 1/8$ (left column) and $\alpha = 1/10$ (right column) and different interaction strengths $U = -t$ (upper panel) and $U = -30t$ (lower panel). The GS is not degenerate for $U = -t$ while it is twofold quasidegenerate for $U = -30t$. The even states under spin inversion have an energy offset $|U|$ and thus are not visible on the graphs. (b) Evolution of the first (δ) and second (Δ) energy gaps, in black and blue, respectively, with increasing interaction strength $|U|$ for $2N = 4$ and $2N = 6$.

$\mathbf{r}$, $n_{\mathbf{r},\sigma} = c^{\dagger}_{\mathbf{r},\sigma} c_{\mathbf{r},\sigma}$, and $\varphi_{\mathbf{rr}'} = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A} \cdot d\mathbf{l}$ are Aharonov-Bohm phases derived from the coupling to the underlying vector potential $\mathbf{A}$ [see Fig. 1(a)]. Fermions of opposite spin interact through an attractive on-site interaction of strength $U \leqslant 0$. Without interactions, the spectrum of this model is the Hofstadter butterfly [23].

We consider $2N$ fermions (N per spin state) distributed over N_s sites of the square lattice and experiencing a total magnetic flux $N_\phi \Phi_0$, where Φ_0 is the flux quantum. We define the flux density $\alpha = \frac{N_\phi}{N_s}$, meaning that the flux per lattice site is $\alpha\Phi_0$. We choose $\alpha = 1/q$ with q an integer, such that each magnetic unit cell is composed of q lattice sites, and is characterized by a single-particle spectrum with q bands. The filling factor per spin state of the lowest band therefore reads $\nu_{\uparrow,\downarrow} = N/N_\phi$. We allow the interaction strength $|U|$ to take arbitrarily large values and therefore take into account all q bands. While this restricts the numerically accessible system sizes to $2N \leqslant 8$, it has the crucial advantage of including band mixing and band dispersion effects, thus capturing all features of the Harper-Hofstadter (HH) model. Our aim is to study the nature of the many-body ground state (GS) as a function of flux density and interaction strength. We focus on the situation where the number of fermions per spin state is equal to the number of flux quanta ($N = N_\phi$), such that the filling factor per spin state is $\nu_{\uparrow,\downarrow} = 1$.

In order to build an intuition about the system's possible behavior, let us consider first the low flux limit $\alpha \ll 1$, where the HH model is similar to the continuum case. There, one can anticipate that a system initially prepared in the $\nu_{\uparrow,\downarrow} = 1$ IQH state could evolve into the 1/2 Laughlin state [39]. Indeed, for increasing attraction, $\uparrow$ and $\downarrow$ fermions will form pairs of increasing binding energy and decreasing pair size. For sufficiently large attraction, each fermion pair, whose size becomes smaller than any other length scale, acts as a BM experiencing a flux density $\tilde{\alpha} = 2\alpha$. Therefore, the emerging many-body system—composed of N BMs—could feature a filling factor $\tilde{\nu} = \frac{1}{\tilde{\alpha}}\frac{N}{N_s} = \frac{\nu_{\uparrow,\downarrow}}{2} = 1/2$ of the lowest band. This scenario, depicted in Fig. 1, remains to be revealed via a microscopic description in both the continuum and lattice systems.

In lattice systems, it was shown that the 1/2 Laughlin state may form in the HH model at low flux ($\tilde{\alpha} \ll 1$) for both hard-core and finite repulsive interactions [45,46]. Therefore, in the strong binding $|U|/t \gg 1$ and low flux $\tilde{\alpha} \ll 1$ limits, the Laughlin state represents a reasonable candidate GS of the Hamiltonian [Eq. (1)]. However, our model comes with an additional difficulty: The composite nature of the bosons is associated with the energy scale $|U|$, whose competition with the other energy scales could favor competing phases. In the following, we describe our analysis of the microscopic Hamiltonian [Eq. (1)].

Energy spectra. For our numerical calculations, we apply periodic boundary conditions in both directions and choose the number of sites in each direction such that the aspect ratio is close to one. The momenta (K_x, K_y) are defined by the translations of the magnetic unit cell on the lattice [47]. The Hamiltonian is block-diagonal in these quantum numbers as well as in the total magnetization S_z and spin inversion parity ϵ for $S_z = 0$. We performed the exact diagonalization of the Hamiltonian of Eq. (1) at $\nu_{\uparrow,\downarrow} = 1$ up to $q = 10$ using these symmetries. Our results are plotted in Fig. 2 for $\alpha = 1/8$ and

$\alpha = 1/10$. When $U = 0$, the fermions completely fill up the lowest band of the HH model and realize the $\nu_{\uparrow,\downarrow} = 1$ IQH state. It is characterized by a single GS in the $(K_x, K_y) = (0,0)$ sector separated by a large gap δ from higher energy states. A moderate interaction ($U = -t$) does not change this picture [Fig. 2(a)]. Conversely, when $|U|/t = 30$ [Fig. 2(a)], the gap δ is small compared to the energy difference between the second and third lowest energy eigenstates Δ. This twofold quasidegeneracy is characteristic of the $\tilde{\nu} = 1/2$ Laughlin state on a torus (here the lattice with periodic boundary conditions). The quasidegenerate GSs are found in the $(K_x, K_y) = (0,0)$, $S_z = 0$ sector, and $\epsilon = (-1)^N$ spin inversion parity sector, which is consistent with this interpretation. We computed these spectra for various interaction strengths U and the results are summarized in Fig. 2(b). The gap δ is seen to persist for attractive interactions as strong as $U \simeq -10t$, a regime at which Δ starts to increase. $\Delta|U|$ eventually saturates at $|U|/t \simeq 30$. Note that for $\alpha > 1/8$, we do not observe any twofold quasidegeneracy at any value of U even for system sizes as large as $2N \leqslant 8$, which is probably due to a larger lattice effect. This is reminiscent of the bosonic calculations, where the Laughlin state is not observed for $\tilde{\alpha} \gtrsim 1/3$ [45,46]. Interestingly, the closing of the IQH gap δ and opening of the Laughlin gap Δ seem to coincide for all system sizes considered here, suggesting a direct phase transition between the two phases. The critical value of U shows very little dependence with system size and flux density and is of the order of the one-body linewidth (approximately $8t$). In the continuum, given the unbounded nature of the Landau level spectrum, a careful study including multiple Landau levels would be necessary to probe this transition. We computed the formation cost of a BM in the lowest Landau level [48]. This energy can be roughly related to the lower bound for U_c, the critical value of U. It suggests an increasing value of U_c with system size, and thus possibly an infinite value in the thermodynamic limit. This raises the question of the fate of this transition in the continuum, which we do not address here.

As complementary evidence to the GS degeneracy, we also checked the degeneracy of the quasihole excitations [48]. However, in a finite system, a charge density wave (CDW) can have the same low energy spectrum as the Laughlin state (and likewise concerning their respective excitations). The precise nature of the phase can be determined using entanglement spectroscopy [49,50], which is the purpose of the following section.

Entanglement spectroscopy. In order to establish the nature of the twofold GS at large $|U|$, we use the particle entanglement spectrum (PES) [50]. The entanglement spectrum was originally introduced to extract the edge spectrum from the GS wave function [49]. The PES uses a similar method to reveal the nature of bulk quasihole excitations and was crucial to fully establish the emergence of FQH states in Chern insulators [28]. Unlike the number of quasihole states in the energy spectrum, it can distinguish between FQH and CDW states in Chern insulators [51] and identify superfluid phases in the HH model [52]. The PES is the spectrum of $-\log \rho_A$, where $\rho_A = \mathrm{Tr}_B \rho$ is the reduced density matrix obtained by tracing over $N_B \equiv 2N - N_A$ particles, the labels A and B referring to two complementary parts of the whole system. Thus, ρ_A commutes with the total momentum $(K_{x,A}, K_{y,A})$ and the spin

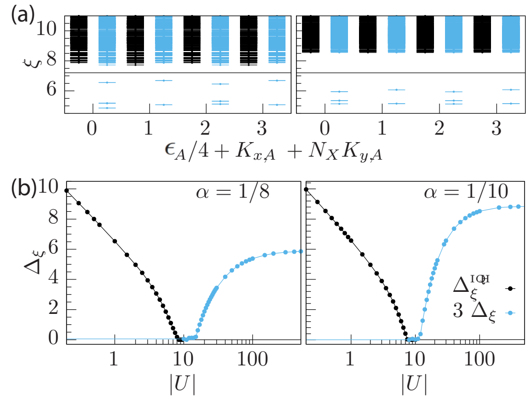

FIG. 3. (a) Entanglement spectrum for $2N = 8$ fermions, $|U| = 30t$ and $\alpha = 1/8$ (left) and $\alpha = 1/10$ (right), with a particle partition $N_A = 4$ in the sector $S_z^A = 0$. The low entanglement energy levels are all even (blue) under spin inversion which reveals the existence of a pairing gap. The horizontal black line indicates the midgap energy. The number of states below this line is 20 (6,4,6,4), which is the expected counting for the Laughlin state. (b) Evolution of the entanglement gap of the IQH ($\Delta_\xi^{\mathrm{IQH}}$) and Laughlin ($\Delta_\xi$) states with increasing pairing strength $|U|$ for $2N = 6$ and $N_A = 3$, $\alpha = 1/8$ (left) and $\alpha = 1/10$ (right).

$\mathbf{S}_A$ of the subsystem A, as well as the spin inversion parity ϵ_A when $S_{z,A} = 0$. When the GS is almost twofold degenerate, we consider $\rho = \frac{1}{2}\sum_{i=1}^{2} |\Psi_i\rangle\langle\Psi_i|$, where $|\Psi_1\rangle, |\Psi_2\rangle$ are the two GSs. Generically, the PES has low entanglement levels separated from higher entanglement levels by an entanglement gap, which is infinite for model FQH states, but finite in the HH model [52]. The number of levels below the entanglement gap is related to the number of quasihole states and is a fingerprint of a given topological phase.

We computed the PES at small and large $|U|$, starting respectively from the unique and the twofold degenerate GS. We focused on the fluxes $\alpha = 1/8$ and $\alpha = 1/10$ for which quasidegeneracies were observed. Here, we provide evidence that the large $|U|$ phase shows all features of the Laughlin state from the PES perspective. Additionally, we use the entanglement gap as a witness of the phase transition. Further details of the PES analysis are provided in Ref. [48].

The nature of the phase at large $|U|$ can be settled by investigating the $N_A = 4$ partition. Indeed, it is the smallest partition with multiple (two) BMs in the subsystem A, and the PES will reveal their interaction. Figure 3(a) shows the PES obtained at strong attraction ($|U| = 30t$) for $2N = 8$ with a particle partition $N_A = 4$ in the sector $S_z^A = 0$. Below the entanglement gap, there are 20 states (6,4,6,4), which is the expected counting for the $\tilde{\nu} = 1/2$ Laughlin state, as known from the generalized exclusion principle [53]. This constitutes strong evidence of the emergence of Laughlin state behavior in this system. Given the large Hilbert space dimension and the smallness of the energy gap, we used a truncated Hilbert space (corresponding to the projection onto the space with at least three tightly bound BMs) to obtain the eigenvectors of these two systems before computing their PES (see the Supplemental Material [48] for a quantitative justification).

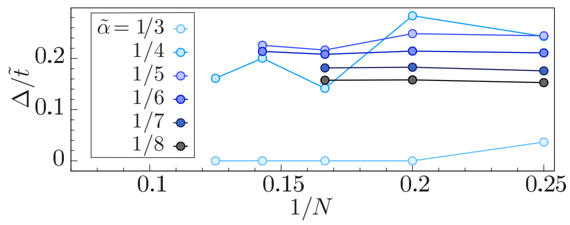

FIG. 4. Scaling of the energy gap above the twofold quasidegenerate GS of the hard-core bosonic model Eq. (2) with inverse particle number for various flux densities $\tilde{\alpha}$. There is no quasidegeneracy for $\tilde{\alpha} > 1/4$. The finite-size effects are important for $\tilde{\alpha} = 1/4$, but become smaller for $\tilde{\alpha} < 1/4$, suggesting a finite gap in the thermodynamic limit.

The entanglement gap Δ_ξ can be used to monitor the phase transition, providing complementary evidence to the energy spectra. In Fig. 3(b) we plot Δ_ξ as a function of the interaction strength for $2N = 6$ fermions and $N_A = 3$, which corresponds to a reasonable computation time. It reveals the transition from an IQH entanglement gap to a Laughlin entanglement gap. Remarkably, the transition point is found for $|U| \simeq 10\,t$, in agreement with the result extracted from the energy spectra [Fig. 2(b)].

System size effects. In order to characterize the phase transition from a fermionic IQH state to the candidate bosonic phases, it is necessary to reliably extrapolate our results to the thermodynamic limit, which requires large N values. However, the spinful fermionic model Eq. (1) limits us to systems with only a few pairs of fermions. As a compromise, we focus here on the large U limit, where the model Eq. (1) can be mapped onto the following hard-core bosonic Hamiltonian,

$$\mathcal{H}_{\text{bos}} = -\tilde{t} \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} (e^{2i\varphi_{\mathbf{r}\mathbf{r}'}} a_{\mathbf{r}}^\dagger a_{\mathbf{r}'} + \text{H.c.}), \tag{2}$$

where $\tilde{t} = \frac{t^2}{|U|}$ and $a_{\mathbf{r}}^\dagger$ ($a_{\mathbf{r}}$) creates (annihilates) a hard-core boson at site $\mathbf{r}$. This Hamiltonian allows us to evaluate finite-size effects by simulating larger systems as well as to investigate a larger range of $\tilde{\alpha}$. In this limit, we will thus treat the bosonic model Eq. (2) with N bosons at flux density $\tilde{\alpha} = 2\alpha$ as the analog of the model Eq. (1) with $2N$ fermions at flux density α. The $1/|U|$ scaling of the gap in the large $|U|$ limit [Fig. 2(b)] supports this approximation for $|U|/t > 50$. We computed the low energy spectrum of the Hamiltonian Eq. (2) for $1/8 \lesssim \tilde{\alpha} \lesssim 1/3$ and all numerically accessible particle numbers, extending by up to four bosons the computations in Ref. [45]. For $\tilde{\alpha} \lesssim 1/4$, we found that the GS is twofold quasidegenerate. The gap Δ between the second and third lowest energy states is shown in Fig. 4, and displays a smooth behavior as a function of N for $\tilde{\alpha} < 1/4$, which suggests a finite value at the thermodynamic limit. As expected, the finite-size effects decrease as $\tilde{\alpha}$ approaches the continuous limit $\tilde{\alpha} \ll 1$ and they appear to still be important at $\tilde{\alpha} = 1/4$. For $\tilde{\alpha} \leqslant 1/4$, the PES analysis yields a number of states below the entanglement gap that is the one expected for the Laughlin state, in agreement with our findings on the fermionic model at even N_A.

Conclusion. In this Rapid Communication, we showed that attractive interactions between opposite spin fermions trigger a phase transition between a $\nu_{\uparrow,\downarrow} = 1$ fermionic IQH state and a bosonic Laughlin state in the Harper-Hofstadter model. The transition occurs for a critical interaction strength of the order of the full linewidth of the one-body spectrum. The extrapolation of this result to the continuum is unclear since the Landau level spectrum is unbounded, calling for an in-depth numerical study of this case. Our work demonstrates that dynamical fermion pairing, as realized in quantum gas experiments, opens unique possibilities for the exploration of topological matter. Combined with the techniques developed for the realization of topological matter with ultracold atoms, this feature might provide a different pathway to create and probe nontrivial topological phases [54].

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