Generating transverse response explicitly from harmonic oscillators

Yuan Yao,[1,2,*] Ying Tang,[1,†] and Ping Ao[1,3,‡]

[1]*Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*
[2]*Zhiyuan College, Shanghai Jiao Tong University, Shanghai 200240, China*
[3]*Key Laboratory of Systems Biomedicine Ministry of Education, Shanghai Center for Systems Biomedicine, Shanghai Jiao Tong University, Shanghai 200240, China*

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We obtain stochastic dynamics from a system-plus-bath mechanism as an extension of the Caldeira-Leggett (CL) model in the classical regime. An effective magnetic field and response functions with both longitudinal and transverse parts are exactly generated from the bath of harmonic oscillators. The effective magnetic field and transverse response are antisymmetric matrices: the former is explicitly time-independent corresponding to the geometric magnetism, while the latter can have memory. The present model can be reduced to previous representative examples of stochastic dynamics describing nonequilibrium processes. Our results demonstrate that a system coupled with a bath of harmonic oscillators is a general approach to studying stochastic dynamics, and provides a method to experimentally implement an effective magnetic field from coupling to the environment.

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I. INTRODUCTION

Quantifying dynamical response is essential to understanding stochastic systems. Recent studies on exploring a general framework of stochastic dynamics [1–5] find that the response function not only contains the symmetric longitudinal part representing dissipation [6], but also has an antisymmetric transverse part as well [1,7–10]. The transverse response has multiple corresponding physical phenomena, such as the geometric magnetism in classical chaotic systems [11], the magnetic field in vortex dynamics [12,13], and its counterpart in the quantum regime leads to the geometric (Berry's) phase [14,15]. For longitudinal responses, both Markovian and non-Markovian have been obtained from coupling between the system and the harmonic oscillator bath in the CL model [16,17], where the bath consists of a set of harmonic oscillators thereby having a wide application to the dissipative effects on a class of physical problems [18–20]. For transverse responses, the CL model has been studied by explicit introduction of external static magnetic field [21–24]. Nevertheless, to our knowledge, there has been no exact generation of the longitudinal and transverse responses simultaneously from coupling to a single bath of harmonic oscillators. This task is necessary, because, for example, the simultaneous generation of both responses affects the Landau-Zener transition probability in the manner that cannot be realized from an incoherent sum of purely longitudinal and purely transversal noises [25]. In addition, an explicit expression for transverse responses in terms of a bath of harmonic oscillators would allow us to experimentally implement and control an effective magnetic field by manipulating the coupling with bath similarly as that for dissipation [19].

In previous studies, the instantaneous transverse response can be described by Berry's geometric magnetic field [26–28]. When the time scale of interest is of the same order as or faster than the environment, the transverse responses may not be instantaneous [29,30], and then Berry's approach could not be applied due to the restriction of adiabatic condition. Besides, the ultrafast processes in these systems lead to the failure of Markov approximation [31,32]. Such memory effect has been widely observed such as in the electron gas [33–35] and the electrodynamics responses of the semiconductor in the Penn model [36]. Therefore the generation of the transverse response beyond adiabatic condition and Markov approximation remains elusive.

In this paper, we obtain exactly a stochastic dynamics in Eq. (16) where an effective magnetic field and time-dependent responses with both longitudinal and transverse parts are explicitly produced from the bath of harmonic oscillators without Markov approximation. As an extension of the original CL model, we consider a Hamiltonian (Lagrangian) with a general bilinear coupling that includes both position and momentum (velocity) variables between the system and the bath in high dimensional space, and therefore generally breaks the time-reversal symmetry. The effective magnetic field is time-independent corresponding to the geometric magnetism [11], and the time-dependent transverse response may have memory [33–35]. Thus the velocity-dependent coupling introduced in our model leads to new effects on the dynamics including the presence of geometric magnetism. Besides, the response function and the noise correlation function obey the fluctuation-dissipation theorem [37–40]. We further demonstrate that our model reproduces the previous stochastic dynamics describing nonequilibrium processes [1,41].

The paper is organized as follows. In Sec. II, we present the model of interest and review the influence functional approach. The effective stochastic dynamics of the subject system including equations of motion, response functions, and noise is derived by the influence functional method in Sec. III. Then, we also analyze several special cases in which our model reduces to previous representative examples. Finally, we conclude our results and discuss them by comparing with

*Present address: Institute for Solid State Physics, University of Tokyo, Kashiwa, Chiba 277-8581, Japan; smartyao@issp.u-tokyo.ac.jp
†jamestang23@gmail.com
‡aoping@sjtu.edu.cn

previous works in Sec. IV and append the detailed derivation of effective dynamics in Appendices.

II. MODEL AND GENERAL FRAMEWORK

In this part, we give the Hamiltonian of our system and the corresponding Lagrangian and introduce the influence functional method in a general manner [16,42].

A. Extended Caldeira-Leggett model

We consider a subject system interacting with a heat bath that can be described by the Hamiltonian

$$H = H_{\rm sys} + H_{\rm bath} + H_{\rm int}, \tag{1}$$

where the subscript "sys" and "int" denote the subject system and the interaction separately. The Hamiltonian of the system is $H_{\rm sys} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)$, where $\hbar$ is set as unit. The position variable of the system is denoted by an N-dimensional vector x. The system has one particle with mass m in the external potential field $V(x)$. The Hamiltonian of the bath is $H_{\rm bath}(R) = -\sum_j \frac{1}{2m_j}\frac{\partial^2}{\partial R_j^2} + \frac{1}{2}\sum_{k,j} V_{\rm bath}(R_k,R_j)$, which can have an infinite number of particles with the mutual interaction potential $V_{\rm bath}(R_k,R_j)$. The position variable of the jth particle with mass m_j is given by the real number R_j without loss of generality that general dimensions can be reached by choices of total oscillator numbers. The interaction between the system and bath is described by the Hamiltonian $H_{\rm int} = V_{\rm int}(x,-i\partial_x;R_j,-i\partial_{R_j})$, which can include different couplings for the system and the bath, such as the momentum-momentum coupling.

To achieve our purpose, we construct the Hamiltonian

$$\begin{aligned} H =& \left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right]^T \frac{\tilde m}{2m^2}\left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right] + V(x) \\ &+ \sum_j \left\{\left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right]^T \frac{W_j}{2mm_j}\left\{P_{R_j} - m^{-1}W_j^T\left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right]\right\} + \text{H.c.}\right\} \\ &+ \sum_j \left\{P_{R_j} - m^{-1}W_j^T\left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right]\right\}^T \frac{1}{2m_j}\left\{P_{R_j} - m^{-1}W_j^T\left[p_x + \sum_k \left(E_k R_k - m_k^{-1} W_k P_{R_k}\right)\right]\right\} \\ &+ \sum_j \frac{m_j}{2}\omega_j^2\left(R_j + \frac{C_j^T x}{m_j\omega_j^2}\right)^2, \end{aligned} \tag{2}$$

where x and p_x are the canonical coordinate and momentum of the subject system, and R_js and P_{R_j}s those for the jth harmonic oscillator of the heat bath. The coefficients C,E,W are real N-dimensional column vectors and the superscript T denotes transpose. Thus our analysis is valid for general N-dimensional space not restricted to two or three dimensions. The subscript "H.c." denotes Hermitian conjugation to make the Hamiltonian Hermitian in order to describe a closed quantum system. Besides,

$$\tilde m = m + \sum_j W_j W_j^T/m_j. \tag{3}$$

This mass matrix is positive definite and the additional term to the bare mass m is used to eliminate the mass shift [18].

By the Legendre transform, we get the Lagrangian of our model:

$$\begin{aligned} L =& \left[\frac{m}{2}\dot x^2 - V(x)\right] + \sum_j \left[\frac{m_j}{2}\dot R_j^2 - \frac{m_j}{2}\omega_j^2 R_j^2\right] - \Phi(x,\dot x) \\ &+ \sum_j [-(x^T C_j + \dot x^T E_j)R_j + \dot x^T W_j \dot R_j], \end{aligned} \tag{4}$$

where

$$\Phi(x,\dot x) = \sum_j \left[x^T\left(C_j C_j^T/2m_j\omega_j^2\right)x - \dot x^T\left(W_j W_j^T/2m_j\right)\dot x\right], \tag{5}$$

which is used to eliminate the potential shift in addition to the mass shift [18]. For later convenience, we introduce

$$L_{\rm sys}(x,\dot x) \doteq [m\dot x^2/2 - V(x) - \Phi(x,\dot x)] \tag{6}$$

with the subscript "sys" denoting the subject system.

The coupling in Eq. (4) is in the bilinear form including three types: position-position, velocity-position, and velocity-velocity, where the first variable and the second denote those of the subject system and the heat bath respectively. Due to simultaneous appearance of coupling C_js and E_js, there exists no canonical time-reversal symmetry for our model in Eq. (4). The coupling with velocity of the subject system can be physically realized in a black-body radiation field [17,43], a Josephson junction [44], and a superconducting quantum interference device (SQUID) [16,18]. Interacting with the dynamical radiation field, the momentum of the subject system is coupled with the bath oscillator creation operators [17,43]. Since the creation operator is a linear combination of both coordinate and momentum operators, the system momentum is coupled with bath oscillator coordinates and momentums simultaneously. In the original CL model [16], only the couplings C_j and E_j are separately taken to be nonzero. Therefore our model can be seen as an extension of the CL model.

B. Influence functional approach

The influence functional approach in the Feynman-Vernon theory [16,42] serves an appropriate and general starting point to obtain the equation of motion for the subject system. The density matrix of the total system in the coordinate representation is given by

$$\begin{aligned}\langle x,R|\rho(t)|x',R'\rangle \\ = \int dx_0 dx_0' dR_0 dR_0' K(x,R,t;x_0,R_0,0) \\ \times K^*(x',R',t;x_0',R_0',0)\langle x_0,R_0|\rho(0)|x_0',R_0'\rangle,\end{aligned} \quad (7)$$

where $\rho(0)$ is the initial density matrix, x and x' are position variables of the subject system, and R (R') is the abbreviation of the position variables for the set of bath particles $\{R_j\}$ ($\{R_j'\}$). The propagator

$$K(x,R,t;x_0,R_0,0) = \iint \mathcal{D}x\mathcal{D}R \exp i\mathcal{S}[x,R]$$

and

$$K^*(x',R',t;x_0',R_0',0) = \iint \mathcal{D}x'\mathcal{D}R' \exp -i\mathcal{S}[x',R'],$$

where $\mathcal{D}x\mathcal{D}R,\mathcal{D}x'\mathcal{D}R'$ denote the path measure [42,45], and the action function $\mathcal{S}[x] = \int_0^t L[x(s),\dot{x}(s),R(s),\dot{R}(s)]ds$.

The reduced density matrix for the subject system is obtained through tracing out the bath coordinates:

$$\rho_{\text{sys}}(x,x',t) \doteq \int dR dR' \delta(R-R')\langle x,R|\rho(t)|x',R'\rangle. \quad (8)$$

We further suppose that the subject system is initially decoupled with the bath:

$$\rho(0) = \rho_{\text{sys}}(0)\rho_{\text{bath}}(0), \quad (9)$$

where ρ_{sys} and ρ_{bath} are the density matrices of the subject system and the bath respectively. Then, the reduced density matrix for the subject system is

$$\rho_{\text{sys}}(x,x',t) = \int dx_0 dx_0' J(x,x',t;x_0,x_0',0)\rho_{\text{sys}}(x_0,x_0',0),$$

where the propagator of the reduced density matrix

$$J(x,x',t;x_0,x_0',0) = \iint \mathcal{D}x\mathcal{D}x'\mathcal{I}[x,x'] \exp i(\mathcal{S}_{\text{sys}}[x]\text{-}\mathcal{S}_{\text{sys}}[x'])$$

with $\mathcal{S}_{\text{sys}}[x] = \int_0^t L_{\text{sys}}[x(s),\dot{x}(s)]ds$ denoting action of the subject system. The influence functional is

$$\begin{aligned}\mathcal{I}[x,x'] = \int dR_0 dR_0' \rho_{\text{bath}}(R_0,R_0',0) \iint \mathcal{D}R\,\mathcal{D}R' \\ \times \exp i(\mathcal{S}_{\text{int}}[x,R] - \mathcal{S}_{\text{int}}[x',R'] \\ + \mathcal{S}_{\text{bath}}[R] - \mathcal{S}_{\text{bath}}[R']),\end{aligned} \quad (10)$$

where $\mathcal{S}_{\text{int}}$ and $\mathcal{S}_{\text{bath}}$ are actions for the Lagrangian of coupling and bath in Eq. (4) separately. The influence functional contains the full information of the bath's effect on the subject system. We also take the initial density matrix of the bath $\rho_{\text{bath}}(0)$ so that the ith bath particle is initially at equilibrium with velocity-position coordinates $(W_i^T\dot{X}_0/m_i, -C_i^T X_0/(m_i\omega_i^2))$, where X_0 is the initial coordinate of the subject system [20,46,47]. Otherwise, there would be additional artificial terms in the effective equation of motion [46] (cf. Appendix C). The equilibrium temperature of the bath is $1/(k_B\beta)$ with k_B denoting Boltzmann constant.

III. EFFECTIVE STOCHASTIC DYNAMICS

In the following, we discuss the effective stochastic dynamics of the subject system. The response functions and noise are derived and analyzed in both the discrete spectrum and the continuum limit.

A. Equation of motion

To derive the equation of motion, we first add the total differentiation term

$$\Delta L = \sum_j \frac{d}{dt}(-\dot{x}^T W_j R_j) \quad (11)$$

to Eq. (4), and such addition will not change the equations of motion in both the classical and the quantum-mechanical regimes. Then, the total Lagrangian becomes

$$\begin{aligned}L = L_{\text{sys}}(x,\dot{x}) + \sum_j \left(\frac{m_j}{2}\dot{R}_j^2 - \frac{m_j}{2}\omega_j^2 R_j^2\right) \\ - \sum_j (x^T C_j + \dot{x}^T E_j + \ddot{x}^T W_j)R_j.\end{aligned} \quad (12)$$

The form of Eq. (12) is the same as the original CL model except for that R_js are coupling with the combination of x, $\dot{x}$, and $\ddot{x}$ in our model. Despite of this difference, the result of influence functional of CL model [16] can be still applied to our model (cf. Appendix A).

We next transform to the center of mass coordinate $X \doteq (x+x')/2$ and the relative coordinate $Y \doteq x - x'$ for the subject system following the previous path integral approach [48] to dissipative systems. Then, the influence functional for Eq. (4) is explicitly calculated out:

$$\mathcal{I}[X+Y/2, X-Y/2] = \exp(i\text{Im} + \text{Re}), \quad (13)$$

where

$$\begin{aligned}\text{Im} = -\int_0^t d\tau \int_0^\tau ds\, Y_\tau^T \Big[R(\tau-s)\dot{X}_s + B\dot{X}_\tau + \Delta m \ddot{X}_\tau \\ - \sum_j (C_j C_j^T/m_j\omega_j^2)X_\tau - f(\tau)\Big]\end{aligned} \quad (14)$$

and

$$\text{Re} = -\int_0^t d\tau \int_0^t ds\, Y_\tau^T G(\tau-s)Y_s/2. \quad (15)$$

The terms $R(\tau-s)$ and $G(\tau-s)$ are given in Eqs. (18) and (22). $f(\tau)$ is eliminated by the choice of the initial equilibrium distribution of the bath at arbitrary temperature in Appendix C. The mass shift $\Delta m=\sum_j(W_jW_j^T/m_j)$ and the frequency shift by $-\sum_j(C_jC_j^T/m_j\omega_j^2)$ are eliminated by $\Phi(x,\dot{x})$ defined by Eq. (5).

In the classical regime, the dominant contribution in the propagator of the density operator comes from the path with $x(t)$ close to $x'(t)$ [49], which means $|x-x'|\ll|x+x'|$, i.e., $|Y|\ll|X|$. Thus we expand $J[X,Y]$ to the order of Y^2 and integrate out Y. Then we achieve one of the main results of this paper, the equation of motion for the subject system:

$$m\ddot{X}_\tau=-\nabla V(X_\tau)-\int_0^\tau ds\,R(\tau-s)\dot{X}_s-B\dot{X}_\tau+F_{\rm rand}(\tau),\tag{16}$$

where the time-independent effective magnetic field is given by the $N\times N$ dimensional matrix:

$$B=\sum_j\frac{\left(E_jW_j^T-W_jE_j^T\right)}{m_j},\tag{17}$$

the detailed derivations and formulations of which are given in Appendix B.

B. Response functions and noise

The $N\times N$-dimensional response matrix $R(\tau-s)$ can be decomposed as

$$R(\tau-s)=R_l(\tau-s)+R_t(\tau-s),\tag{18}$$

where the subscripts "l" and "t" denote "longitudinal" and "transverse" separately. The longitudinal and the transverse responses take the form of

$$R_l(\tau-s)=\sum_j\frac{\cos\omega_j(\tau-s)}{m_j\omega_j^2}\big[\omega_j^2E_jE_j^T+\left(C_j-\omega_j^2W_j\right)\times\left(C_j-\omega_j^2W_j\right)^T\big],\tag{19}$$

$$R_t(\tau-s)=\sum_j\frac{\sin\omega_j(\tau-s)}{m_j\omega_j}\big[\left(\omega_j^2W_j-C_j\right)E_j^T-E_j\left(\omega_j^2W_j-C_j\right)^T\big].\tag{20}$$

The random force $F_{\rm rand}$ obeys a Gaussian distribution with $\langle F_{\rm rand}(\tau)\rangle=0$ and

$$\left\langle F_{\rm rand}(\tau)F_{\rm rand}^T(s)\right\rangle=G(\tau-s).\tag{21}$$

The $N\times N$ dimensional noise correlation matrix is

$$G(\tau-s)=\sum_j\frac{1}{2}\omega_j\coth\frac{\beta\omega_j}{2}R^{(j)}(\tau-s),\tag{22}$$

where $R^{(j)}(\tau-s)$ denote the jth term in the summation of $R(\tau-s)$.

From Eqs. (17), (19), and (20), the quantities B, $R_l(\tau-s)$, and $R_t(\tau-s)$ can be independent with each other. Specifically, we can choose coupling constants C, E, and W to make several of B, $R_l(\tau-s)$, and $R_t(\tau-s)$ nonzero. We list three typical cases in Table I.

TABLE I. The response functions under three different choices on the coupling coefficients in the discrete spectral case.

	B	$R_l(\tau-s)$	$R_t(\tau-s)$
$E=0$, $C-\omega^2W\neq0$	zero	nonzero	zero
$W=0$, $CE^T-EC^T\neq0$	zero	nonzero	nonzero
$C-\omega^2W=0$, $EW^T-WE^T\neq0$	nonzero	nonzero	zero

Note that for the discrete spectral density if $R_l(\tau-s)$ vanishes, B and $R_t(\tau-s)$ are zero as well. However, when the bath consists of infinite many harmonic oscillators, these three response functions can be nonzero alone [47].

We make several remarks here. First, both the effective magnetic field B and the transverse response $R_t(\tau-s)$ are antisymmetric matrices, and can induce a curl flux. When $C=0$, $R_t(\tau-s)=-\sum_j\omega_jB_j\sin\omega_j(\tau-s)$, where B_j denotes the jth term in the summation of B. Second, as indicated in Eq. (17), the effective magnetic field B is generally independent of spectral functions for R_l,R_t, and thus it can still exist with a fast oscillating bath. In addition, we have imposed no condition for the time scale of interest and thus the quantum phase induced by the closed motion in this effective magnetic field exposes the geometrical property [11,50].

The noise correlation function and the velocity response function in Eq. (22) obeys the fluctuation-dissipation theorem [38,39]. It can be described in the integral form:

$$G(\tau-s)=\frac{1}{\pi}\left[\int_0^{+\infty}d\omega\chi''_{FFl}(\omega)\coth\frac{\beta\omega}{2}\cos\omega(\tau-s)+\int_0^{+\infty}d\omega\chi''_{FFt}(\omega)\coth\frac{\beta\omega}{2}\sin\omega(\tau-s)\right],\tag{23}$$

where the symmetric matrix $\chi''_{FFl}(\omega)$ and antisymmetric matrix $\chi''_{FFt}(\omega)$ correspond to the longitudinal and the transverse response functions separately. The correlation matrix $G(\tau-s)$ is symmetric under the combination of the matrix transpose and the time reversal. Equation (23) has also been validated in a matrix approach by the first-order theory of irreversible thermodynamics [37]. In high-temperature regime $\beta\omega_j\ll1$ for all j, it reduces to the fluctuation-dissipation theorem of the second type [39]: $G(\tau-s)=R(\tau-s)/\beta$. In frequency space, it takes the form of

$$\left\langle\tilde{F}_{{\rm rand},\omega}[\omega]\tilde{F}^T_{{\rm rand},\omega}[\omega']\right\rangle=\tilde{R}[\omega]\,\delta(\omega+\omega^{'})/(\pi\beta),\tag{24}$$

which is the Nyquist theorem [51].

C. Continuous bath spectrum limit

To show that Eq. (16) can reduce to representative examples of stochastic dynamics, we consider continuous frequency distribution for bath oscillators: $\sum_j\to\int d\omega\,\rho_D(\omega)$, where $\rho_{\rm D}(\omega)=\sum_j\delta(\omega-\omega_j)$ is the frequency density. Then, the longitudinal and the transverse response functions are

TABLE II. Four choices on the coupling coefficients in Eq. (4) leading to the same effective dynamics, Eq. (33). C-bath denotes that $C \neq 0$ with other coupling coefficients vanishing, and the same notation goes for each choice.

	$J_l(\omega)$	$\Phi(x,\dot{x})$
C-bath [16]	$\rho_D(\omega)C(\omega)C^T(\omega)/(m\omega)$	$\int d\omega\, \rho_D(\omega)x^T CC^T x/(2m\omega^2)$
E-bath [16]	$\rho_D(\omega)\omega E(\omega)E^T(\omega)/m$	0
W-bath	$\rho_D(\omega)\omega^3 W(\omega)W^T(\omega)/m$	$-\int d\omega\, \rho_D(\omega)\dot{x}^T WW^T\dot{x}/(2m)$
CW-bath	$\rho_D(\omega)[C(\omega)-\omega^2 W(\omega)]$ $\cdot[C(\omega)-\omega^2 W(\omega)]^T/(m\omega)$	$-\int d\omega\rho_D(\omega)\dot{x}^T WW^T\dot{x}/(2m)$ $+\int d\omega\rho_D(\omega)x^T CC^T x/(2m\omega^2)$

denoted by

$$R_l(\tau-s) \doteq \int_0^{+\infty} d\omega[J_l(\omega)\cos\omega(\tau-s)]/\omega, \quad (25)$$

$$R_t(\tau-s) \doteq \int_0^{+\infty} d\omega[J_t(\omega)\sin\omega(\tau-s)]/\omega, \quad (26)$$

where we introduce the spectral functions $J_l(\omega)$ and $J_t(\omega)$. By Eqs. (19) and (20) with the coupling constants valued at $\omega = \omega_j$,

$$J_l(\omega) \doteq \frac{\rho_D(\omega)}{m\omega}[(C(\omega)-\omega^2 W(\omega))(C(\omega)-\omega^2 W(\omega))^T + \omega^2 E(\omega)E^T(\omega)], \quad (27)$$

$$J_t(\omega) \doteq \frac{\rho_D(\omega)}{m}[E(\omega)(C(\omega)-\omega^2 W(\omega))^T - (C(\omega)-\omega^2 W(\omega))E^T(\omega)], \quad (28)$$

where we treat these spectral functions as smooth functions. Since we are interested in the long-time behavior of the effective dynamics, it is sufficient to expand them to the lowest order of the frequency ω:

$$J_l(\omega) = (2S\omega_c/\pi)(\omega/\omega_c)^{\kappa_l}\exp(-\omega/\omega_c), \quad (29)$$

$$J_t(\omega) = A\omega(\omega/\omega_c)^{\kappa_t}\exp(-\omega/\omega_c), \quad (30)$$

where ω_c is a cutoff frequency and $\kappa_{l,t}$ is the corresponding lowest order. From Eq. (19), the friction matrix S and matrix A are symmetric and antisymmetric separately, i.e., $S = S^T$ and $A = -A^T$. These spectral functions determine the equation of motion since they give the response functions by Eqs. (25) and (26):

$$R_l(\tau-s) = \mathrm{Re}\left\{\frac{2S\omega_c}{\pi}\frac{\Gamma(\kappa_l)}{[1-i\omega_c(\tau-s)]^{\kappa_l}}\right\} \quad (31)$$

and

$$R_t(\tau-s) = \mathrm{Im}\left\{A\omega_c\frac{\Gamma(\kappa_t+1)}{[1-i\omega_c(\tau-s)]^{(\kappa_t+1)}}\right\}, \quad (32)$$

where $\Gamma(x)$ is Γ function and various diffusion properties related with the longitudinal response $R_l(\tau-s)$ have been investigated in the quantum Brownian system [52–56].

D. Representative stochastic dynamics

The first example is to reproduce the original CL model [16]. If we choose the ohmic bath for $J_l(\omega)\,(\kappa_l = 1)$: $J_l(\omega) = (2S\omega/\pi)\exp(-\omega/\omega_c)$, and $J_t(\omega) = 0$, our model when $\omega_c \to +\infty$ produces the Langevin equation describing the quantum Brownian motion [16] by Eqs. (31) and (32):

$$m\ddot{X} = -\nabla_X V - S\dot{X} + F_{\mathrm{rand}}(\tau), \quad (33)$$

where the noise correlation function

$$\langle F_{\mathrm{rand}}(\tau)F_{\mathrm{rand}}^T(s)\rangle = \int_0^{+\infty}(S\omega/\pi)\coth(\beta\omega/2)\cos\omega(\tau-s).$$

In the high-temperature regime $\beta\omega \ll 1$, it becomes $\langle F_{\mathrm{rand}}(\tau)F_{\mathrm{rand}}^T(s)\rangle = (2S/\beta)\delta(\tau-s)$. Furthermore, there are many choices on the coupling coefficients C, E, and W in our model that can give the same spectral functions $J_{l,t}(\omega)$ above. Thus they lead to the same equation of motion, Eq. (33). We list four choices in Table II. The redundant degrees of freedom of choosing the coupling coefficients to generate Eq. (33) has also been discussed [18,57].

Furthermore, Eq. (16) can reduce to the stochastic dynamics for nonequilibrium processes [1]. If we choose the ohmic bath for $J_l(\omega)\,(\kappa_l = 1)$ and $J_t(\omega)\,(\kappa_t = 1)$: $J_l(\omega) = (2S\omega)/\pi\exp(-\omega/\omega_c)$, $J_t(\omega) = A\omega^2/\omega_c\exp(-\omega/\omega_c)$, by Eqs. (31) and (32), we obtain both of the longitudinal and the transverse responses:

$$\int_0^{\tau} ds\, R_l(\tau-s)\dot{X}_s = S\dot{X}_\tau, \quad (34)$$

$$\int_0^{\tau} ds\, R_t(\tau-s)\dot{X}_s = A\dot{X}_\tau. \quad (35)$$

We now have different ways to choose the coupling coefficients. For example, let C, E be nonzero and W zero, and then $B = 0$. The equation of motion takes the form of

$$m\ddot{X} = -\nabla_X V - (S+A)\dot{X} + F_{\mathrm{rand}}(\tau), \quad (36)$$

with the noise correlation $\langle F_{\mathrm{rand}}(\tau)F_{\mathrm{rand}}^T(s)\rangle = (2S/\beta)\delta(\tau-s)$ in the high-temperature regime. It has the same form as the Klein-Kramers equation in previous works [1]. This equation in three dimension also corresponds to the classical equation of motion for a charged particle in superfluid with vortices [12].

IV. CONCLUSIONS AND DISCUSSIONS

We compare our results with previous studies. First, the effect of magnetic field in the system-plus-bath model has been discussed [21–24]. However, the magnetic field there is put by hand an external field or a vector potential in the Hamiltonian, rather than explicitly generated by bath oscillators. Second, generating the transverse response from the system-plus-bath mechanism has attracted sustained theoretical interest [11,24,58–60]. However, those previous attempts typically require the adiabatic approximation that the system variables change very slowly compared with the bath variables, while ideal adiabaticity is difficult to achieve in real experiments. Third, all of the four couplings in Eq. (4) have been taken into account [57,61], but it is restricted to one-dimensional space. Therefore Eq. (16) that includes the effective magnetic field and the transverse response with memory has not been exactly obtained from the harmonic oscillator bath before. Moreover, the simultaneous occurrences of the momentum-momentum W_j and the coordinate-coordinate C_j couplings have been applied to the quantum frustration where two ensembles of independent harmonic oscillators are responsible for these two different couplings separately [62]. Such setting can be reduced to by our model in Eq. (4) if $C_j = 0$ for a set of js while $W_j = 0$ for the rest of js. Therefore our model can produce the phase diagram triggered by competing dissipative processes of two noncommutating operators $[x, p_x] \neq 0$ [63,64]. Besides this generalization, the inclusion of transverse responses is expected to enrich the possible phases in addition to the previous picture of longitudinal responses alone.

To conclude, we have obtained a stochastic dynamics with an effective magnetic field, the longitudinal and the transverse response functions exactly from bath of harmonic oscillators in the influence functional approach. Two representative examples of previous stochastic dynamics in classical regime has been reproduced from our model. The physical effects of longitudinal and transverse response functions in subohmic, ohmic, and superohmic baths will be classified in our future work. Experimental implementation on the effective magnetic field requires further investigations. Generalization of our model to the system with non-trivial topology [65] and quantum regime remain to be explored since the addition of total derivatives in Eq. (11) can change quantum-mechanical quantities, such as ground-state degeneracies in systems with nontrivial topology [50].

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APPENDICES

In the following, we append the results used in the previous sections, including the influence functional formulation and the effective dynamics.

APPENDIX A: DERIVATION ON INFLUENCE FUNCTIONAL

We first review the quantum influence functional of Caldeira-Leggett model, which corresponds to Eq. (4) with $C_j = \mathfrak{C}_j$, $E_j = 0$, and $W_j = 0$. In the center of mass coordinate and the relative coordinate $X \doteq (x + x')/2$, $Y \doteq x - x'$, the influence functional is [16,20]

$$\begin{aligned}\mathcal{I}_c[X + Y/2, X - Y/2] &= \exp \int_0^t d\tau \int_0^t ds \, -iY_\tau^T[R_c(\tau - s)\dot{X}_s + h_c(\tau - s)X_s] \\ &\times \exp \int_0^t d\tau \int_0^t ds \, -Y_\tau^T \frac{G_c(\tau - s)}{2} Y_s, \end{aligned} \tag{A1}$$

where we have defined

$$R_c(\tau - s) = 0, \tag{A2}$$

$$h_c(\tau - s) = \sum_j 2\theta(\tau - s)\left\{-\frac{\sin \omega_j(\tau - s)}{2m_j\omega_j}\mathfrak{C}_j\mathfrak{C}_j^T\right\}, \tag{A3}$$

$$G_c(\tau - s) = \sum_j \left[\mathfrak{C}_j\mathfrak{C}_j^T \frac{\cos \omega_j(\tau - s)}{2m_j\omega_j}\right] \coth \frac{\beta\omega_j}{2}. \tag{A4}$$

We can generalize this result of the Caldeira-Leggett model to our model by the following substitutions:

$$\mathfrak{C}_j \to 1, \quad X \to \tilde{X}_j \text{ and } Y \to \tilde{Y}_j, \tag{A5}$$

where $\tilde{X}_j \doteq (\tilde{x}_j + \tilde{x}'_j)/2$ and $\tilde{Y}_j \doteq (\tilde{x}_j - \tilde{x}'_j)$, with $\tilde{x}'_j$ defined similarly to $\tilde{x}_j$ by

$$\tilde{x}'_j \doteq C_j^T x' + E_j^T \dot{x}' + W_j^T \ddot{x}'', \tag{A6}$$

$$\tilde{x}_j \doteq C_j^T x + E_j^T \dot{x} + W_j^T \ddot{x}. \tag{A7}$$

The coupling between this transformed system coordinate $\tilde{x}_j$ and the bath coordinates has the same bilinear from as that of the original CL model [16], and thus we can directly apply their influence functional approach.

Then the influence functional of our model can be expressed as

$$\mathcal{I}_c[\tilde{X} + \tilde{Y}/2, \tilde{X} - \tilde{Y}/2] = \exp(i\,\mathrm{Im} + \mathrm{Re}), \tag{A8}$$

where

$$\mathrm{Im} = \sum_j \int_0^t d\tau \int_0^t ds \, \tilde{Y}_{\tau,j}^T \left\{2\theta(\tau - s)\frac{\sin \omega_j(\tau - s)}{2m_j\omega_j}\right\} \tilde{X}_{s,j} \tag{A9}$$

and

$$\begin{aligned}\mathrm{Re} = \sum_j \int_0^t d\tau \int_0^t ds \, \tilde{Y}_{\tau,j}^T \frac{1}{2} \\ \times \left\{-\frac{\cos \omega_j(\tau - s)}{2m_j\omega_j}\right\} \coth \frac{\beta\omega_j}{2} \tilde{Y}_{s,j}.\end{aligned} \tag{A10}$$

We first simplify the notation in the following manner:

$$\mathrm{Im} = \int_0^t d\tau \int_0^t ds \tilde{Y}_\tau^T \left\{2\theta(\tau - s)\frac{\sin \omega(\tau - s)}{2m\omega}\right\} \tilde{X}_s \tag{A11}$$

and

$$\mathrm{Re} = \int_0^t d\tau \int_0^t ds \tilde{Y}_\tau^T \frac{1}{2}\left\{-\frac{\cos \omega(\tau - s)}{2m\omega}\right\} \coth \frac{\beta\omega}{2} \tilde{Y}_s, \tag{A12}$$

where we have suppressed the subscripts "j"s and its summation $\sum_j$ just for convenience and we will restore them in the end.

We first evaluate the imaginary part Im by inserting the replacements in Eq. (A5) for $\tilde{Y}_\tau$:

$$\mathrm{Im} = \int_0^t d\tau \int_0^t ds\big[\ddot{Y}_\tau^T W + \dot{Y}_\tau^T E + Y_\tau^T C\big]\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\tilde{X}_s = I_1 + I_2 + I_3. \tag{A13}$$

We have defined

$$\begin{aligned}
I_1 &\doteq \int_0^t d\tau \int_0^t ds \ddot{Y}_\tau^T W\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\tilde{X}_s = \int_0^t d\tau \int_0^t ds Y_\tau^T W \frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg]\tilde{X}_s,\\
I_2 &\doteq \int_0^t d\tau \int_0^t ds \dot{Y}_\tau^T E\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\tilde{X}_s = \int_0^t d\tau \int_0^t ds - Y_\tau^T E \frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\tilde{X}_s\\
I_3 &\doteq \int_0^t d\tau \int_0^t ds Y_\tau^T C\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\tilde{X}_s,
\end{aligned} \tag{A14}$$

where we have integrated it by part and omitted the surface terms dependent on $Y(t)$ and $Y(0)$ since these terms contributes impulse forces proportional to $\delta(\tau)$ or $\delta(\tau-t)$ in the final equation of motion. We discard these surface terms because we are interested in the time $0 \ll \tau \ll t$.

We sum up $I_{1,2,3}$:

$$\begin{aligned}
\mathrm{Im} = &\int_0^t d\tau \int_0^t ds Y_\tau^T \bigg\{W\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg] - E\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\\
&+ C\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\bigg\}[C^T X_s + E^T\dot{X}_s + W^T\ddot{X}_s]\\
= &I_4 + I_5 + I_6,
\end{aligned} \tag{A15}$$

where we have further defined

$$\begin{aligned}
I_4 \doteq &\int_0^t d\tau \int_0^t ds Y_\tau^T\bigg\{W\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg] - E\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg] + C\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\bigg\}C^T X_s\\
= &\int_0^t d\tau \int_0^t ds Y_\tau^T W\, 2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}C^T\dot{X}_s + \int Y_\tau^T W 2\theta(\tau)\frac{\cos\omega\tau}{2m}C^T X_0\\
&- \int_0^t d\tau \int_0^t ds Y_\tau^T E\, 2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}C^T\dot{X}_s - \int Y_\tau^T E\, 2\theta(\tau)\frac{\sin\omega\tau}{2m\omega}C^T X_0\\
&- \int_0^t d\tau \int_0^t ds Y_\tau^T CC^T\frac{\cos\omega(\tau-s)}{2m\omega^2}\,2\theta(\tau-s)\dot{X}_s - \int_0^t d\tau Y_\tau^T CC^T\, 2\theta(\tau)\frac{\cos\omega\tau}{2m\omega^2}X_0\\
&+ \int_0^t d\tau \int_0^t ds Y_\tau^T CC^T\, 2\delta(\tau-s)\frac{1}{2m\omega^2}X_s,
\end{aligned} \tag{A16}$$

$$\begin{aligned}
I_5 \doteq &\int_0^t d\tau \int_0^t ds Y_\tau^T\bigg\{W\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg] - E\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg] + C\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\bigg\}E^T\dot{X}_s\\
= &\int_0^t d\tau \int_0^t ds Y_\tau^T\bigg\{W\, 2\delta(\tau-s)\frac{1}{2m} - W\, 2\theta(\tau-s)\frac{\omega\sin\omega(\tau-s)}{2m}\\
&- E\, 2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m} + C\, 2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg\}E^T\dot{X}_s,
\end{aligned} \tag{A17}$$

and

$$\begin{aligned}
I_6 \doteq &\int_0^t d\tau \int_0^t ds Y_\tau^T\bigg\{W\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg] - E\frac{d}{d\tau}\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg] + C\bigg[2\theta(\tau-s)\frac{\sin\omega(\tau-s)}{2m\omega}\bigg]\bigg\}W^T\ddot{X}_s\\
= &\int_0^t d\tau \int_0^t ds Y_\tau^T\bigg\{W\, 2\delta'(\tau-s)\frac{1}{2m} - W\, 2\theta(\tau-s)\frac{\omega^2\cos\omega(\tau-s)}{2m} - E\, 2\delta(\tau-s)\frac{1}{2m}\\
&+ E\, 2\theta(\tau-s)\frac{\omega\sin\omega(\tau-s)}{2m} + C\, 2\theta(\tau-s)\frac{\cos\omega(\tau-s)}{2m}\bigg\}W^T\dot{X}_s\\
&- \int_0^t d\tau Y_\tau^T\bigg\{-W\, 2\theta(\tau)\frac{\omega\sin\omega\tau}{2m} + E\, 2\theta(\tau)\frac{\cos\omega\tau}{2m} + C\bigg[2\theta(\tau)\frac{\sin\omega\tau}{2m\omega}\bigg]\bigg\}W^T\dot{X}_0.
\end{aligned} \tag{A18}$$

We have set the initial values for $X(s=0)$ and $\dot{X}(s=0)$ as X_0 and $\dot{X}_0$, and omitted the term proportional to $Y(\tau=0)$ in the last line above. These terms will contribute to the equation of motion by an external force $f(\tau)$ [cf. Eq. (B5)], which is a finite force at any time τ. We have also done the integral by part, and neglect the surface terms dependent on $Y(0)$ and $Y(t)$.

We sum up $I_{4,5,6}$ to get Im:

$$\mathrm{Im} = I_4 + I_5 + I_6 = -\int_0^t d\tau \int_0^\tau ds\, Y_\tau^T \left[R(\tau - s)\dot{X}_s + B\dot{X}_\tau + \Delta m \ddot{X}_\tau - \sum_j \frac{C_j C_j^T}{m_j \omega_j^2} X_\tau - f(\tau) \right], \tag{A19}$$

where explicit expressions of the terms $R(\tau - s)$, B, Δm, $f(\tau)$ are given by Eqs. (19), (20), and (17),

$$f(\tau) = \sum_j \left[\frac{\omega_j^2 W_j - C_j}{m_j \omega_j^2} \cos \omega_j \tau - \frac{E_j \sin \omega_j \tau}{m_j \omega_j} \right] C_j^T X_0 + \left[\frac{\omega_j^2 W_j - C_j}{m_j \omega_j} \sin \omega_j \tau + \frac{E_j \cos \omega_j \tau}{m_j} \right] W_j^T \dot{X}_0, \tag{A20}$$

and $G(\tau - s)$ by Eq. (22).

Next, we evaluate the real part "Re" in the influence functional. We use integration by part, and neglect the surface terms dependent on $Y(0)$ and $Y(t)$:

$$\begin{aligned}
\mathrm{Re} &= \int_0^t d\tau \int_0^t ds \tilde{Y}_\tau^T \frac{1}{2}\left[-\frac{\cos \omega(\tau - s)}{2m\omega} \right] \coth \frac{\beta\omega}{2} \tilde{Y}_s \\
&= \int_0^t d\tau \int_0^t ds \left[-\omega^2 Y_\tau^T W + Y_\tau^T C \right] \frac{1}{2}\left[-\frac{\cos \omega(\tau - s)}{2m\omega} \right] \coth \frac{\beta\omega}{2} [C^T Y_s - W^T \omega^2 Y] + \int_0^t d\tau \int_0^t ds\, \omega Y_\tau^T E \frac{1}{2} \\
&\quad \times \left[-\frac{\cos \omega(\tau - s)}{2m\omega} \right] \coth \frac{\beta\omega}{2} E^T \omega Y_s + \int_0^t d\tau \int_0^t ds [-\omega^2 Y_\tau^T W + Y_\tau^T C] \frac{1}{2}\left[+ \frac{\sin \omega(\tau - s)}{2m\omega} \right] \coth \frac{\beta\omega}{2} E^T \omega Y_s \\
&\quad + \int_0^t d\tau \int_0^t ds\, \omega Y_\tau^T E \frac{1}{2}\left[-\frac{\sin \omega(\tau - s)}{2m\omega} \right] \coth \frac{\beta\omega}{2} \left[C^T Y_s - W^T \omega^2 Y_s \right].
\end{aligned} \tag{A21}$$

It can be further simplified as

$$\mathrm{Re} = \int_0^t d\tau \int_0^t ds - Y_\tau^T \frac{G(\tau - s)}{2} Y_s. \tag{A22}$$

As a result, the influence functional takes the form of

$$\begin{aligned}
\mathcal{I}_c[X + Y/2, X - Y/2] &= \exp(i\mathrm{Im} + \mathrm{Re}) \\
&= \exp\left\{ -i \int_0^t d\tau \int_0^\tau ds\, Y_\tau^T \left[R(\tau - s)\dot{X}_s + B\dot{X}_\tau + \Delta m \ddot{X}_\tau - \sum_j \frac{C_j C_j^T}{m_j \omega_j^2} X_\tau - f(\tau) \right] \right\} \\
&\quad \times \exp\left\{ \int_0^t d\tau \int_0^t ds - Y_\tau^T \frac{G(\tau - s)}{2} Y_s \right\}.
\end{aligned} \tag{A23}$$

APPENDIX B: DERIVATION ON THE EQUATION OF MOTION

To get the equation of motion, we put the influence functional into the propagator of the density operator:

$$J[X,Y] = \int DX\, DY \exp\left(\frac{i}{\hbar} \{ \mathcal{S}_{\mathrm{sys}}[X + Y/2] - \mathcal{S}_{\mathrm{sys}}[X - Y/2] \} \right) \mathcal{I}[X + Y/2, X - Y/2].$$

We next expand the functionals to the first order of Y_τ, and explicitly do the functional integral of Y_τ without the surface terms:

$$\begin{aligned}
J[X] = \int DX \exp \int d\tau_1 \int d\tau_2 \Bigg(-\frac{1}{2} \Bigg[&(m + \Delta m)\ddot{X}_{\tau_1} + \nabla\big(V(X_{\tau_1}) + \Phi(X_{\tau_1}, \dot{X}_{\tau_1})\big) \\
&+ \partial_{\dot{X}_{\tau_1}} \Phi\big(X_{\tau_1}, \dot{X}_{\tau_1}\big) + \int ds\, R(\tau_1 - s)\dot{X}_s - \sum_j \frac{C_j C_j^T}{m_j \omega_j^2} X_{\tau_1} + B\dot{X}_{\tau_1} - f(\tau_1) \Bigg]^T G^{-1}(\tau_1 - \tau_2) \Bigg[(m + \Delta m)\ddot{X}_{\tau_2} \\
&+ \nabla\big(V(X_{\tau_2}) + \Phi(X_{\tau_2}, \dot{X}_{\tau_2})\big) + \partial_{\dot{X}_{\tau_2}} \Phi\big(X_{\tau_2}, \dot{X}_{\tau_2}\big) + \int ds\, R(\tau_2 - s)\dot{X}_s - \sum_j \frac{C_j C_j^T}{m_j \omega_j^2} X_{\tau_2} + B\dot{X}_{\tau_1} - f(\tau_2) \Bigg] \Bigg),
\end{aligned} \tag{B1}$$

where $\Phi(x,\dot{x})$ is supposed to eliminate the mass shift and the potential shift [18]. Here, $G^{-1}(\tau - s)$ is defined as

$$\int_0^t G^{-1}(\tau - w)G(w - s)dw = \delta(\tau - s). \tag{B2}$$

This gives the equation of motion of the subject system in the classical regime:

$$\begin{aligned} m\ddot{X}_\tau &= \left(-\Delta m\ddot{X}_\tau - \partial_{\dot{X}_\tau}\Phi(X_\tau,\dot{X}_\tau)\right) - \nabla\left(V(X_\tau) + \Phi(X_\tau,\dot{X}_\tau) - \sum_j \frac{X_\tau C_j C_j^T X_\tau}{2m_j\omega_j^2}\right) \\ &\quad - \int_0^\tau ds\, R(\tau - s)\dot{X}_s - B\dot{X}_\tau + f(\tau) + f_{\text{rand}}(\tau). \end{aligned} \tag{B3}$$

The random force $f_{\text{rand}}(\tau)$ has the following correlation:

$$\left\langle f_{\text{rand}}(\tau)f_{\text{rand}}^T(s)\right\rangle = G(\tau - s) \tag{B4}$$

where $\langle \dots \rangle$ is the statistical average over the system ensemble.

To eliminate the mass shift Δm and the frequency shift, we set the function $\Phi(x,\dot{x})$ by Eq. (5):

$$\Phi(x,\dot{x}) = \sum_j \left(x^T \frac{C_j C_j^T}{2m_j\omega_j^2}x - \frac{1}{2}\dot{x}^T\frac{W_j W_j^T}{m_j}\dot{x}\right).$$

Then, the equation of motion takes the form as

$$m\ddot{X}_\tau = -\nabla V(X_\tau) - \int_0^t ds\, R(\tau - s)\dot{X}_s - B\dot{X}_\tau + f(\tau) + f_{\text{rand}}(\tau). \tag{B5}$$

APPENDIX C: CHOICES ON THE INITIAL BATH DENSITY OPERATOR

To eliminate the force $f(\tau)$ in Eq. (B5), we can choose a proper initial distribution of the bath oscillators. We want to find an initial state of the heat bath that is locally coupled with the system at the starting point, and the force f is combined into the random force before the bath is traced out. We achieve this by solving out the part of the Euler-Lagrange equation, which contains an expected random force. The general bilinear Lagrangian in Eq. (4) gives the equations of motion:

$$m\ddot{X} + W\ddot{R} + \frac{WW^T}{m}\ddot{X} - E\dot{R} = -\nabla V - \frac{CC^T}{m\omega^2}X - CR \tag{C1}$$

and

$$m\ddot{R} + W^T\ddot{X} = -m\omega^2 R - C^T X - E^T\dot{X}, \tag{C2}$$

where $E = D + U$. For convenience, we have omitted the subscript "j" and the summation $\sum_j$. To get the form of the random force, it is convenient to solve this set of linear differential equations by Laplace transform techniques:

$$m\ddot{X} + \nabla V + \frac{CC^T}{m\omega^2}X - \frac{W}{m}(C^T X + E^T\dot{X}) = (\omega^2 W - C)R + E\dot{R}, \tag{C3}$$

where R has the image function solved by Eq. (C2):

$$\tilde{R}[p] = \frac{1}{m(\omega^2 + p^2)} \times [m(\dot{R}_0 + pR_0) + (E^T + pW^T)X_0 + W^T\dot{X}_0 - (C^T + pE^T + p^2W^T)\tilde{X}[p]], \tag{C4}$$

with $R_0 \equiv R(\tau = 0)$. Then, put the object function $R(\tau)$ of Eq. (C4) into Eq. (C3), we obtain the following equation:

$$\begin{aligned} m\ddot{X} &= -\nabla V - \frac{EW^T - WE^T}{m}\dot{X} + f_{\text{rand}}(R_0,\dot{R}_0) + f(\tau) - \int_0^\tau ds\frac{\cos\omega(\tau - s)}{m\omega^2}[(C - \omega^2 W)(C - \omega^2 W)^T + \omega^2 EE^T]\dot{X}(s) \\ &\quad - \int_0^\tau ds\frac{\sin\omega(\tau - s)}{m\omega}[(\omega^2 W - C)E^T - E(\omega^2 W - C)^T]\dot{X}(s), \end{aligned} \tag{C5}$$

where the velocity response parts are exactly the same those in the equation of motion, Eq. (B5), and the forces $f_{\text{rand}}(R_0,\dot{R}_0)$ and $f(\tau)$ are

$$f_{\text{rand}}(R_0,\dot{R}_0) = \left[(\omega^2 W - C)\frac{\sin\omega\tau}{\omega} + E\cos\omega\tau\right]\dot{R}_0 + [(\omega^2 W - C)\cos\omega\tau - E\omega\sin\omega\tau]R_0 \tag{C6}$$

and

$$f(\tau) = \left[(\omega^2 W - C)\frac{\sin\omega\tau}{\omega} + E\cos\omega\tau\right]\frac{W^T}{m\omega}\dot{X}_0 + [(\omega^2 W - C)\cos\omega\tau - E\omega\sin\omega\tau]\frac{C^T}{m\omega^2}X_0. \tag{C7}$$

1. High-temperature regime

To eliminate the force $f(\tau)$ in high-temperature regime, we adopt the method in Refs. [20,46]. The force $f(\tau)$ can be combined with $f_{\text{rand}}(\tau)$ above in the following way:

$$F_{\text{rand}} \doteq f(\tau) + f_{\text{rand}}(R_0,\dot{R}_0) = \left[(\omega^2 W - C)\frac{\sin\omega\tau}{\omega} + E\cos\omega\tau\right]\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right) + [(\omega^2 W - C)\cos\omega\tau - E\omega\sin\omega\tau]\left(R_0 + \frac{C^T}{m\omega^2}X_0\right). \tag{C8}$$

After applying the substitution

$$\dot{R}_0 \to \dot{R}_0 + \frac{W^T}{m}\dot{X}_0, \quad R_0 \to R_0 + \frac{C^T}{m\omega^2}X_0 \tag{C9}$$

to the equipartition theorem at high temperature, we get another set of equipartition theorem [66]:

$$\begin{aligned}\left\langle\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)^T\right\rangle &= 0;\\ \left\langle\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)^T\right\rangle &= \frac{1}{m\beta}I,\\ \left\langle\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)^T\right\rangle &= \frac{1}{m\omega^2\beta}I,\end{aligned} \tag{C10}$$

where "0" and "I" are zero matrix and identity matrix, respectively.

This equipartition theorem implies that the bath oscillators are initially at thermodynamic equilibrium with the coordinate $(-C^T X_0/(m\omega^2), -W^T\dot{X}_0/m)$ in position-velocity space and we can set the classical phase space distribution function as $\rho^{\text{cl}}_{\text{bath}}(R_0,P_0) = \exp(-\beta[(P_0 + W^T\dot{X}_0)^2/(2m) + m\omega^2(R_0 + C^T X_0/(m\omega^2))^2/2])$ with $P_0 \equiv m\dot{R}_0$. Then, the newly defined random force F_{rand} has $\langle F_{\text{rand}}(\tau)\rangle = 0$, and after the summation $\sum_j$

$$\left\langle F_{\text{rand}}(\tau)F^T_{\text{rand}}(s)\right\rangle = \sum_j \frac{1}{\beta}\left\{\frac{\cos\omega_j(\tau - s)}{m_j\omega_j^2}\left[\left(C_j - \omega_j^2 W_j\right)\left(C_j - \omega_j^2 W_j\right)^T + \omega_j^2 E_j E_j^T\right] + \frac{\sin\omega_j(\tau - s)}{m_j\omega_j}\left[\left(\omega_j^2 W_j - C_j\right)E_j^T - E_j\left(\omega_j^2 W_j - C_j\right)^T\right]\right\} = \lim_{\beta\to 0} G(\tau - s), \tag{C11}$$

which tells that $F_{\text{rand}}(\tau)$ obeys a Gaussian distribution and has zero mean value

$$\langle F_{\text{rand}}\rangle = 0, \quad \left\langle F_{\text{rand}}(\tau)F^T_{\text{rand}}(s)\right\rangle = \left\langle f_{\text{rand}}(\tau)f^T_{\text{rand}}(s)\right\rangle = \lim_{\beta\to 0} G(\tau - s) \tag{C12}$$

in the classical high temperature limit. Finally, the equation of motion is

$$m\ddot{X}_\tau = -\nabla V(X_\tau) - B\dot{X}_\tau - \int_0^\tau R(\tau - s)\dot{X}_s + F_{\text{rand}}(\tau), \tag{C13}$$

with the noise term $F_{\text{rand}}(\tau)$ satisfying Eq. (C12).

2. Generalization to arbitrary temperature

To eliminate the force $f(\tau)$ in Eq. (B5) at arbitrary temperature, we apply the following initial density operator for the bath:

$$\rho_{\text{bath}}(R_1,R_0;t=0) = \frac{1}{2}\left\{\int_{R_0}^{R_1 - \beta W^T\dot{X}_0/m} + \int_{R_1}^{R_0 - \beta W^T\dot{X}_0/m}\right\}\mathcal{D}R(\tau)\exp\left\{\int_0^\beta -\left[\frac{1}{2}m(\dot{R}(\tau) + W^T\dot{X}_0/m)^2 + \frac{1}{2}m\omega^2\left(R(\tau) + \frac{C^T X_0}{m\omega^2} + \frac{W^T\dot{X}_0\tau}{m}\right)^2\right]d\tau\right\}, \tag{C14}$$

where we have defined the imaginary time path integral by $t \to -i\tau$ and we have already absorbed the renormalization constant into the measure $\int \mathcal{D}R(\tau)$. The time derivatives above are for the imaginary time derivatives, e.g., $\dot{R} \equiv dR/d\tau$ and $\dot{X}_0 \equiv dX/d\tau|_{\tau=0}$. To see how this initial density operator works, we can evaluate the mean value of any quantity in the form as $N(\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2))$:

$$\begin{aligned}
&\langle N(\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2))\rangle_\tau \\
&= \iint dR_1 dR_0 \cdot \delta(R_1 - R_0)\cdot \rho_{\text{bath}}(R_1,R_0;t=0)\cdot N)\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2)) \\
&= \iint dR_1 dR_0 \cdot \delta(R_1 - R_0)\int dR_\beta \cdot \delta(R_1 - (R_\beta + W^T\dot{X}_0\beta/m))\int_{R_0}^{R_\beta}\mathcal{D}R(\tau)\exp\left\{\int_0^\beta -\left[\frac{1}{2}m(\dot{R}(\tau) + W^T\dot{X}_0/m)^2\right.\right. \\
&\quad \left.\left. + \frac{1}{2}m\omega^2\left(R(\tau) + \frac{C^T X_0}{m\omega^2} + \frac{W^T\dot{X}_0\tau}{m}\right)^2\right]\right\} N(\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2)),
\end{aligned} \tag{C15}$$

where $\langle\cdots\rangle_\tau$ means the quantity is averaged on the imaginary time and the derivatives in it are imaginary time derivatives. Then we transform the functional integral $R(\tau)$ to $r(\tau)$ by $r(\tau) \equiv R(\tau) + C^T X_0/(m\omega^2) + \tau W^T\dot{X}_0/m$. In this coordinate,

$$\begin{aligned}
\langle N(\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2))\rangle &= \iint dR_1 dR_0\, \delta(R_1 - R_0)\int dR_\beta\, \delta(R_1 - (R_\beta + W^T\dot{X}_0\beta/m)) \\
&\quad \times \int_{r_0}^{r_\beta}\mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2} + \frac{1}{2}m\omega^2 r(\tau)^2\right]\right\} N(\dot{r}_0, r_0),
\end{aligned} \tag{C16}$$

where $r_0 \equiv R_0 + C^T X_0/(m\omega^2)$ and $r_\beta \equiv R_\beta + C^T X_0/(m\omega^2) + W^T\dot{X}_0\beta/m$. Therefore

$$\begin{aligned}
\langle N(\dot{R}_0 + W^T\dot{X}_0/m, R_0 + C^T X_0/(m\omega^2))\rangle &= \iint dR_1 dr_0\, \delta(R_1 - r_0 + C^T X_0/(m\omega^2))\int dr_\beta\, \delta(R_1 - r_\beta + C^T X_0/(m\omega^2)) \\
&\quad \times \int_{r_0}^{r_\beta}\mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2} + \frac{1}{2}m\omega^2 r(\tau)^2\right]\right\} N(\dot{r}_0, r_0) \\
&= \int dr_0 \int_{r_0}^{r_0}\mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2} + \frac{1}{2}m\omega^2 r(\tau)^2\right]\right\} N(\dot{r}(0), r(0)).
\end{aligned} \tag{C17}$$

Then we set the function $N(\dot{r}_0, r_0)$ to be the following quantities and transform back to the real time

$$\begin{aligned}
\left\langle\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)^T\right\rangle_t &= i\langle\dot{r}_0 r_0^T\rangle_\tau; \\
\left\langle\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)\left(\dot{R}_0 + \frac{W^T}{m}\dot{X}_0\right)^T\right\rangle_t &= -\langle\dot{r}_0 \dot{r}_0^T\rangle_\tau, \\
\left\langle\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)\left(R_0 + \frac{C^T}{m\omega^2}X_0\right)^T\right\rangle_t &= \langle r_0 r_0^T\rangle_\tau,
\end{aligned} \tag{C18}$$

and we will calculate them one by one. To do so, we will use the Matsubara representation:

$$r(\tau) = \sum_n r_n \exp(-i\Omega_n\tau), \tag{C19}$$

where $\Omega_n \equiv 2n\pi/\beta$. In the Matsubara modes,

$$\begin{aligned}
i\langle\dot{r}_0 r_0^T\rangle_\tau &= \int dr_0\int_{r_0}^{r_0}\mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2} + \frac{1}{2}m\omega^2 r(\tau)^2\right]\right\}(i\,\dot{r}(0)r(0)^T) \\
&= \int\mathcal{D}(r_n, r_{-n})\exp\left[\beta\sum_n -\frac{m(\Omega_n^2 + \omega^2)}{2}r_n r_{-n}\right]\sum_k(\Omega_k r_k r_{-k}^T) = \sum_k \frac{\Omega_k}{m\beta(\Omega_k^2 + \omega^2)}I = 0,
\end{aligned} \tag{C20}$$

$$\begin{aligned}
-\langle\dot{r}_0 \dot{r}_0^T\rangle_\tau &= \int dr_0\int_{r_0}^{r_0}\mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2} + \frac{1}{2}m\omega^2 r(\tau)^2\right]\right\}(-\dot{r}(0)\dot{r}(0)^T) \\
&= \int\mathcal{D}(r_n, r_{-n})\exp\left[\beta\sum_n -\frac{m(\Omega_n^2 + \omega^2)}{2}r_n r_{-n}\right]\sum_k(-\Omega_k^2 r_k r_{-k}^T) = \sum_k \frac{-\Omega_k^2}{\beta m(\Omega_k^2 + \omega^2)}I = \frac{\omega}{2m}\coth\frac{\beta\omega}{2}I,
\end{aligned} \tag{C21}$$

and

$$\begin{aligned}\langle r_0 r_0^T\rangle_\tau &= \int dr_0 \int_{r_0}^{r_0} \mathcal{D}r(\tau)\exp\left\{\int_0^\beta -\left[\frac{m\dot{r}(\tau)^2}{2}+\frac{1}{2}m\omega^2 r(\tau)^2\right]\right\}(r(0)r(0)^T)\\ &= \int \mathcal{D}(r_n,r_{-n})\exp\left[\beta\sum_n -\frac{m(\Omega_n^2+\omega^2)}{2}r_n r_{-n}\right]\sum_k (r_k r_{-k}^T)\\ &= \sum_k \frac{1}{\beta m(\Omega_k^2+\omega^2)}I = \frac{1}{2m\omega}\coth\frac{\beta\omega}{2}I,\end{aligned} \tag{C22}$$

where the summations over the Matsubara modes Ω_k can be found in one of the condensed matter physics textbooks [50]. From the above results, we can obtain

$$\begin{aligned}\left\langle\left(\dot{R}_0+\frac{W^T}{m}\dot{X}_0\right)\left(R_0+\frac{C^T}{m\omega^2}X_0\right)^T\right\rangle &= 0;\\ \left\langle\left(\dot{R}_0+\frac{W^T}{m}\dot{X}_0\right)\left(\dot{R}_0+\frac{W^T}{m}\dot{X}_0\right)^T\right\rangle &= \frac{\omega}{2m}\coth\frac{\beta\omega}{2}I,\\ \left\langle\left(R_0+\frac{C^T}{m\omega^2}X_0\right)\left(R_0+\frac{C^T}{m\omega^2}X_0\right)^T\right\rangle &= \frac{1}{2m\omega}\coth\frac{\beta\omega}{2}I,\end{aligned} \tag{C23}$$

which are valid for any temperature and they can be reduced to the results in the high temperature regime Eq. (C10) by $\beta\omega \ll 1$. After putting them into $\langle F_{\rm rand}(\tau)F_{\rm rand}^T(s)\rangle$, we get

$$\begin{aligned}\langle F_{\rm rand}(\tau)F_{\rm rand}^T(s)\rangle &= \sum_j\left[(\omega_j^2 W_j - C)\frac{\sin\omega_j\tau}{\omega}+E_j\cos\omega_j\tau\right]\left[(\omega_j^2W_j-C_j)\frac{\sin\omega_j s}{\omega_j}+E_j\cos\omega_j s\right]^T\frac{\omega_j}{2m_j}\coth\frac{\beta\omega_j}{2}\\ &\quad+\left[(\omega_j^2W_j-C_j)\cos\omega_j\tau - E_j\omega_j\sin\omega_j\tau\right]\left[(\omega_j^2W_j-C_j)\cos\omega_j s - E\omega_j\sin\omega_j s\right]^T\frac{\coth(\beta\omega_j/2)}{2m_j\omega_j}\\ &= \sum_j G_j(\tau-s),\end{aligned} \tag{C24}$$

where $G_j(\tau-s)$ is exactly the noise correlation function in Eq. (22). Thus the equation of motion is

$$m\ddot{X}_\tau = -\nabla V(X_\tau) - \sum_j B_j\dot{X}_\tau - \int_0^\tau \sum_j R_j(\tau-s)\dot{X}_s + F_{\rm rand}(\tau), \tag{C25}$$

where

$$\langle F_{\rm rand}(\tau)\rangle = 0, \tag{C26}$$

$$\langle F_{\rm rand}(\tau)F_{\rm rand}^T(s)\rangle = \sum_j G_j(\tau-s). \tag{C27}$$

Therefore this equation of motion holds at arbitrary temperature.

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