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Korshunov instantons in a superconductor at elevated bias current

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Even at zero temperature dissipation reduces quantum fluctuations and tends to localize particles. A notable exception is the nonlinear dissipation due to quasiparticle tunneling in a Josephson junction. It is well known that quasiparticle dissipation does not suppress tunneling of the superconducting phase difference between next-nearest metastable phase states even though tunneling to the nearest phase state is suppressed. The reason is that the dissipative action admits an instanton solution, the so-called Korshunov instanton. Here, we analyze this model at elevated bias current I. We find that besides the known regime where the logarithm of the tunneling rate scales as $I^{2/3}$ there is a novel regime with a scaling I^2. We argue that the increased tunneling rate that derives from the elevated bias current is favorable for experimental verification of the Korshunov instantons.

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I. INTRODUCTION

Dissipative effects in quantum mechanics have been the object of research for a long time. With their simple model of quantum dissipation, Caldeira and Leggett popularized a linear and analytically trackable model to effectively describe dissipation in quantum systems [1,2]. Despite the tremendous success of this simple model, it cannot describe all dissipative effects; for example, it does not describe quantization of the transported charge in a resistor built from a tunnel junction. To this end, the Ambegaokar-Eckern-Schön (AES) model was introduced, modeling a superconducting tunnel junction that is subject to quasiparticle tunneling [3]. The latter introduces a dissipative term in the action that is periodic in the superconducting phase difference with a period of 4π corresponding to the normal flux quantum. In Ref. [4], it was noted that due to quasiparticle tunneling the ground state is a superposition of the phase localized in the even or odd minima of the Josephson potential. Remarkably, these superpositions are immune to the quasiparticle dissipation and survive even in the limit of strong dissipation. As a consequence, the phase "particle" does not localize in one of the minima [5]. The action describing the quasiparticle tunneling admits instanton solutions, the so-called Korshunov instantons, that connect minima in the Josephson potential separated by 4π. This results in a coherent tunneling amplitude between states localized in next-nearest minima and, consequently, the formation of bands. However, the bandwidth is exponentially small in the damping parameter, and thus, experimental verification remains challenging.

In a different context, Korshunov instantons are important to understand the charging effects of metallic islands connected to reservoirs via tunnel junctions [6–10]. Recently, the charging energy of a normal island was measured as a function of the tunnel coupling [11]. However, to our knowledge, a direct observation of Korshunov instantons of the superconducting phase tunneling to the next-nearest minima is still missing. This observation would be interesting not only as an example in which strong dissipation does not completely suppress tunneling but also because Korshunov instantons are related to coherent, paired phase slips which are of interest, for example, for the realization of parity-protected qubits [12–15]. Additionally, the system is an interesting example of dissipative quantum mechanics with a multitude of different regimes that can be accessed by simply changing the bias current; the regimes cover coherent quantum dynamics, even in the presence of strong dissipation, the case of a special incoherent relaxation due to quasiparticle tunneling, and more conventional Ohmic relaxation [5,16].

In this work, we investigate the effect of an elevated bias current I on Korshunov instantons at zero temperature. This is relevant because increasing the bias current raises the tunneling rate $\Gamma_{4\pi}$ of the superconducting phase to the next-nearest minima of the potential that is the hallmark of the presence of strong quasiparticle dissipation and thus increases the chance of experimental verification of the theoretical results. We find that, apart from the low-bias regime with $\ln(\Gamma_{4\pi}) \propto I^{2/3}$, studied in Ref. [5], there is a novel regime at elevated bias current where $\ln(\Gamma_{4\pi}) \propto I^2$. At even higher bias current, the quasiparticle nature of the dissipation becomes irrelevant, and only tunneling to the next-nearest minima survives, which is described by the conventional Ohmic model of Caldeira and Leggett. We discuss the transition between the different regimes and propose an experimental method to measure the predicted decay rates.

This paper is organized as follows. In Sec. II, we introduce the setup and the theoretical model. In Sec. III, we provide a short introduction to the notation and the instanton method that we use throughout this work. In Sec. IV, we give a comprehensive derivation of the coherent tunneling amplitude before we proceed with the calculation of the incoherent tunneling rates in Sec. V. Note that these sections have some overlap with the work of Ref. [5]. Section V includes our main result of the scaling of the tunneling rate at elevated bias current. Moreover, we discuss the transition between the coherent, incoherent, and conventional Ohmic regimes. In Sec. VI, we propose a simple scheme to measure the incoherent tunneling rate before we end with our conclusions.

II. SETUP

For our analysis, we consider a current-biased tunnel junction between two superconducting leads that is intrinsically subject to quasiparticle tunneling that acts as a dissipative element. This can be described by the AES model. In the Euclidean (imaginary-time) path-integral formalism, its dimensionless action $S = S_c + S_\eta$ at zero temperature is given by [3]

$$S_c = \int_{-\infty}^{\infty} dt \left\{ \frac{\hbar C}{8e^2}\dot{\varphi}^2 - \frac{E_J}{\hbar}[1-\cos(\varphi)] + \frac{I\phi_0}{\hbar}\varphi \right\},$$
$$S_\eta = \frac{\hbar}{\pi e^2 R}\int_{-\infty}^{\infty} dt\, dt' \frac{\sin\{[\varphi(t)-\varphi(t')]/4\}^2}{(t-t')^2}, \qquad (1)$$

where φ is the superconducting phase difference across the Josephson junction and $\dot{\varphi} = d\varphi/dt$ is its derivative with respect to the imaginary time t. The first term, S_c, describes the coherent superconducting circuit consisting of a Josephson junction with Josephson energy $E_J = \phi_0 I_c/2\pi$, where I_c is the junction's critical current and $\phi_0 = 2e/h$ is the superconducting flux quantum. The capacitive energy due to the junction's capacitance C is given by $E_C = e^2/2C$. The second term, S_η (quasiparticle action), corresponds to the dissipation due to quasiparticle tunneling. Its magnitude is connected to the effective shunt resistor R. For small φ, the dissipative action can be expanded in a Taylor series, so that it reproduces the Ohmic action described by Caldeira and Leggett. However, this approach neglects the periodicity of S_η. The latter causes the action to stay invariant for a 4π phase shift corresponding to the tunneling of a normal flux quantum [17]. This refers to the fact that the quasiparticles are quantized single electrons and therefore do not feel a shift of a normal flux quantum.

In this work, we are interested in the regime where the dissipative action S_η dominates S_c, with $\hbar/4e^2R \gg (E_J/8E_C)^{1/2}$. Such a strong dissipation brings the system always into the semiclassical regime, so that an instanton analysis is applicable. Interestingly, the quasiparticle action by itself can admit instanton saddle points without an additional kinetic or potential term. Therefore, the solution of $\delta S_\eta/\delta\varphi = 0$, where $\delta S_\eta/\delta\varphi$ is the first variation of the quasiparticle action, is an approximative saddle point of the full action S. In Ref. [5], it was shown that a solution exists for this equation, the Korshunov instanton $\varphi_I(t) = 4\arctan(\Omega t)$, with arbitrary frequency Ω, that connects not neighboring minima of the Josephson potential but next-nearest minima. For vanishing bias current $I = 0$, it is this instanton of the dissipative action that results in a coherent tunnel element between minima shifted by 4π and leads to the formation of bands even in the presence of strong dissipation. However, the resulting bandwidth is small and difficult to tune, and the effect of the pure coherent tunneling therefore is difficult to measure. The situation can be changed by applying a bias current I. On the one hand, this destroys the bands, but on the other hand, it introduces a dissipative incoherent tunneling rate where a phase particle located in one of the minima tunnels by 4π to the next-nearest minimum. Additionally, the bias gives rise to "Ohmic" decay into the next minimum for which the quasiparticle action acts as a simple Ohmic shunt. Contrary to intuition, at low bias current, the 4π tunneling dominates the 2π tunneling; that is, the particle is more likely to tunnel to the next-nearest minimum than to the nearest minimum. While the 2π tunneling vanishes at zero bias, the 4π process transforms into the coherent tunneling element.

For the analysis, we introduce the dimensionless parameters

$$j = I\phi_0/E_J, \quad \eta = \hbar/4e^2R, \quad \zeta = (E_J/8E_C)^{1/2}. \qquad (2)$$

The normalized bias current j gives a measure of how strongly the potential is tilted. For $j = 1$, the tilt due to the bias is so strong that the minima in the potential vanish. At this point, the particle classically slides down the potential landscape. The parameter ζ describes the ratio between the capacitive kinetic energy and the Josephson potential energy. Without dissipation, it describes the quantum uncertainty of the phase with $\delta\phi \propto \zeta^{-1}$. The parameter η describes the strength of dissipation. For large η, the dissipation is strong, and the phase becomes localized. Note that for $\eta \gg 1$, semiclassical methods are applicable even for $\zeta < 1$.

III. SADDLE-POINT APPROXIMATION

In this section, we concisely describe the instanton method for analyzing tunneling problems. In the following sections, coherent tunnel elements as well as incoherent tunneling rates will be calculated. Both can be accomplished by evaluating the imaginary-time path integral in the Gaussian approximation around a saddle point $\bar{\varphi}(t)$ of the action S. In general, the action admits different saddle points with different physical meanings. Given a saddle point, the imaginary-time propagator can be approximated as

$$G[\bar{\varphi}(t)] = \int_{\varphi\approx\bar{\varphi}} \mathcal{D}[\varphi]e^{-S_\mathcal{G}}. \qquad (3)$$

Here, $\bar{\varphi}$ is defined as the solution of $\delta S/\delta\varphi = 0$ with appropriate boundary conditions, $\mathcal{D}[\varphi]$ is the functional integration measure, and the subscript $\varphi \approx \bar{\varphi}$ indicates that the path integral should be evaluated in the Gaussian approximation around the extremum $\bar{\varphi}$.

The action $S_\mathcal{G}$ corresponds to S expanded to second order in the fluctuations deviating from the extremal path. In particular, we set

$$\varphi(t) = \bar{\varphi}(t) + \sum_n c_n \chi_n(t), \qquad (4)$$

with $n \in \mathbb{N}_0$. The approximated action $S_\mathcal{G}$ can be written as

$$S_\mathcal{G} = S_{\bar{\varphi}} + \sum_{n,n'} c_n c_{n'} \int dt\, \chi_n \frac{\delta^2 S}{\delta^2\varphi}[\bar{\varphi}]\,\chi_{n'} = S_{\bar{\varphi}} + \sum_n \Lambda_n c_n^2, \qquad (5)$$

where $S_{\bar{\varphi}}$ is the action directly evaluated at the extremal path $\bar{\varphi}$. For the second equality, we have assumed that the fluctuation modes χ_n are eigenfunctions of the second variation satisfying

$$\frac{\delta^2 S}{\delta\varphi^2}[\bar{\varphi}]\,\chi_n = \Lambda_n \chi_n, \qquad (6)$$

with eigenvalues Λ_n and normalized to $\int dt\, \chi_n(t)\chi_{n'}(t) = \delta_{n,n'}$. With this, the integration measure can be chosen to be $\mathcal{D}[\varphi] = \mathcal{N}\prod_n dc_n$, where $\mathcal{N}$ is a normalization constant. Every positive Λ_n leads to a Gaussian integral with the result

$$G[\bar{\varphi}] = \mathcal{N}\int \prod_n dc_n \exp\left[-S_{\bar{\varphi}} - \sum_n \Lambda_n c_n^2\right]$$
$$= \mathcal{N}\prod_n (\pi/\Lambda_n)^{-1/2} e^{-S_{\bar{\varphi}}}$$
$$= F e^{-S_{\bar{\varphi}}}. \qquad (7)$$

For an instanton solution $\bar{\varphi}$, we have to deal with a zero eigenvalue that cannot be treated by the simple Gaussian integration above. Handling it correctly [18,19] leads to the prefactor $F = \omega_0 A_1 A_2$, with [2]

$$A_1 = \sqrt{\frac{W}{2\pi}} \frac{\hbar\omega_0^2}{8E_C\sqrt{\Lambda_1\Lambda_2}}, \tag{8}$$

$$A_2 = \frac{8E_C \prod_{n=1}^{\infty} \Lambda_{n,0}^{1/2}}{\hbar\omega_0^2 \prod_{n=3}^{\infty} \Lambda_n^{1/2}}. \tag{9}$$

Here, the frequency $\omega_0 = (8E_J E_C)^{1/2}/\hbar$ denotes the plasma frequency, and the factor A_1 incorporates the product of the three lowest eigenvalues, including the zero eigenvalue. The zero mode is accounted for by the expression $W = \hbar \int dt\, \dot{\bar{\varphi}}^2/8E_C$. The factor A_2 includes the eigenvalues Λ_n with $n \geqslant 3$. Its leading behavior is determined by the asymptotics for $n \to \infty$. $\Lambda_{n,0}$ correspond to the fluctuations around the constant path $\varphi_0 = 0$. They enter the equation when fixing the normalization $\mathcal{N}$.

To conclude this section, we briefly discuss the applicability of the semiclassical approximation above. It corresponds to the method of steepest decent that is applicable as long as $S_{\bar{\varphi}}$ is much larger than 1. Additionally, within one potential well, the phase should be localized in the minimum. While this condition normally demands $E_J \gg E_C$, it is always fulfilled in the case of strong dissipation $\eta \gg 1$ as the dissipation localizes the phase difference across the Josephson junction.

IV. COHERENT TUNNELING

Coherent quantum tunneling describes the Hamiltonian evolution of a system that connects localized states separated by a classically inaccessible barrier. This unitary evolution leads to quantum superposition of the particle in different potential wells. In our case, the system is mainly localized in the minima of the Josephson potential, i.e., at $\phi \in 2\pi\mathbb{Z}$. This makes it possible to treat the minima of the cosine potential as sites of a linear lattice. The tunneling between different sites causes the formation of bands with a bandwidth Δ_I equivalent to twice the tunneling matrix element. The bandwidth can be expressed by the imaginary-time propagator evaluated at the so-called instanton φ_I. It is a saddle point of the action connecting two minima of the Josephson potential. It can be shown that the bandwidth is given by $\Delta_I = 4\hbar G[\varphi_I] = 4\hbar F_I e^{-S_I}$, where S_I is the action evaluated at the instanton saddle point φ_I and F_I originates from the Gaussian fluctuations around this instanton path [18].

A. Instanton action

We are going to determine the extremal action corresponding to an instanton that connects two minima of the Josephson potential. For this analysis, we are essentially following Ref. [5]. The saddle point equation $\delta S_\eta/\delta\varphi = 0$ reads

$$\frac{\delta S_\eta}{\delta\varphi}[\varphi_I] = \frac{2\eta}{\pi}\int dt' \frac{\sin\{[\varphi(t) - \varphi(t')]/2\}}{(t-t')^2} = 0. \tag{10}$$

An instanton solution to this equation is given by [5]

$$\varphi_I(t) = 4\arctan[\Omega(t-\tau)], \tag{11}$$

connecting a minimum of the cosine potential at $t = -\infty$ with a next-nearest-neighbor minimum shifted by 4π at $t = \infty$. It depends on the frequency Ω that determines how fast the phase flips. The solution φ_I is, in principle, a saddle point only of the quasiparticle action S_η and not of the full action S. However, in the case $\eta \gg \zeta$, the quasiparticle action dominates the saddle-point solution, and thus, even including the circuit action S_c in Eq. (10) changes the instanton only perturbatively. Therefore, inserting the quasiparticle instanton φ_I into the action S_c is justified, which corresponds to proceeding with first-order perturbation theory. We find as the resulting action $S_I(\Omega)$ on the instanton path

$$S_I(\Omega) = 4\pi\left(\eta + \frac{\hbar\Omega}{8E_C} + \frac{E_J}{\hbar\Omega}\right). \tag{12}$$

The action depends on Ω, so that we also need to extremize with respect to this parameter. We find a minimum of the action where Ω is equal to the plasma frequency ω_0 of the minimum, with $\Omega = \omega_0 = (8E_J E_C)^{1/2}/\hbar$. At this minimum the instanton action becomes

$$S_I = 4\pi(\eta + 2\zeta). \tag{13}$$

B. Instanton prefactor and result

The next step is the evaluation of the fluctuations to determine the prefactor F_I. The explicit action of the fluctuation operator on χ_n is given by

$$\frac{\delta^2 S_c}{\delta\varphi^2}[\varphi_I]\,\chi_n(t) = \left[-\frac{\hbar}{8E_C}\frac{\partial^2}{\partial t^2} + \frac{E_J}{\hbar}\cos(\varphi_I)\right]\chi_n(t), \tag{14}$$

$$\frac{\delta^2 S_\eta}{\delta^2\varphi}[\varphi_I]\,\chi_n(t) = \frac{\eta}{\pi}\int dt' \frac{\cos\{[\varphi_I(t) - \varphi_I(t')]/2\}}{(t-t')^2} \times [\chi_n(t) - \chi_n(t')], \tag{15}$$

where we separated the operator into circuit and dissipative contributions. By acting on χ_n, these operators define a stationary Schrödinger equation with a nonlocal potential. Here, the imaginary time plays the role of the spatial coordinate. The lower eigenvalues are determined mainly by the dissipative action corresponding to bounded states in the potential. However, for the high-energy modes, the kinetic-energy term dominates and gives rise to a continuum of states lying above the bounded spectrum. For ease of mode counting, we temporarily introduce a finite imaginary-time interval β with periodic boundary conditions, corresponding to nonzero temperatures. At the end, we send the interval to infinity again.

For low energies, only the dissipative action is relevant. The eigenvalue equation related to Eq. (15) is given explicitly as (for $\tau = 0$)

$$\frac{-2\Omega}{1+(\Omega t)^2}\left[\chi_n(t) - \int \frac{dt'}{\pi}\frac{\Omega\,\chi_n(t')}{1+(\Omega t')^2}\right] + \int \mathcal{P}\frac{dt'}{\pi}\frac{1}{(t-t')}\frac{d\chi_n(t')}{dt'} = \frac{\Lambda_n}{\eta}\chi_n(t), \tag{16}$$

where $\mathcal{P}$ denotes the Cauchy principle value. In general such an equation is hard to solve. However, we obtain a zero mode for each free parameter of the instanton solution (11), which in our case means the imaginary time τ and the frequency Ω. These zero modes generate a shift or dilation of the solution in imaginary time without changing the value of the action S_η. The zero modes can be found by taking the derivative of the instanton path with respect to the corresponding free parameters. We find

$$\chi_0 = N_0 \frac{d\varphi_I(t)}{d\tau} = \sqrt{\frac{2}{\pi}} \frac{\Omega^{1/2}}{1+(\Omega t)^2},$$
$$\chi_1 = N_1 \frac{d\varphi_I(t)}{d\Omega} = \sqrt{\frac{2}{\pi}} \frac{\Omega^{3/2} t}{1+(\Omega t)^2}, \tag{17}$$

both with eigenvalue $\Lambda_0 = \Lambda_1 = 0$; the normalization N_j is fixed by $N_j^2 \int dt\, \chi_j^2 = 1$.

It is well known that for Schrödinger-like equations the number of nodes in the eigenfunction can be associated with the size of the eigenvalue, where the eigenfunction with the lowest number of nodes corresponds to the lowest eigenvalue [20]. For higher modes, the zero modes should be modulated in order to obtain more nodes [19]. For $n = 2$, we obtain approximately [5]

$$\chi_2 = \left(\frac{2}{\beta}\right)^{1/2} \frac{\cos(\nu_1 t)(1 - t^2\Omega^2) + \sin(\nu_1 t) 2t\Omega}{1 + t^2\Omega^2}, \tag{18}$$

with the eigenvalue $\Lambda_2 = \eta\nu_1$, where $\nu_n = 2\pi n/\beta$ are the bosonic Matsubara frequencies. We incorporate the effect of S_c by performing lowest-order perturbation theory with

$$\Lambda_n = \int dt\, \chi_n \frac{\delta^2 S}{\delta\varphi^2}[\varphi_I]\, \chi_n. \tag{19}$$

We obtain $\Lambda_0 = 0$, $\Lambda_1 = \hbar\Omega^2/16E_C$, and $\Lambda_2 = E_J/\hbar = \hbar\Omega^2/8E_C$ for the lowest three eigenvalues determining A_1.

For the calculation of A_2 we first consider only the kinetic term in Eq. (14) and treat the rest as a perturbation. The eigenfunctions of the kinetic operator are given by $\chi_{2n} = (2/\beta)^{1/2}\sin(\nu_n t)$ and $\chi_{2n+1} = (2/\beta)^{1/2}\cos(\nu_n t)$, with eigenvalues $\Lambda_{2n} = \Lambda_{2n+1} = \hbar\nu_n^2/8E_C$. By treating the rest of the action in first-order perturbation theory, the eigenvalues at large n are given by

$$\Lambda_{2n-1} = \Lambda_{2n} = \int dt\, \chi_n \frac{\delta^2 S}{\delta\varphi^2}[\varphi_I]\, \chi_n$$
$$= \frac{\hbar}{8E_C}(\nu_n^2 + \omega_0^2) + \eta|\nu_n| - \eta\nu_1. \tag{20}$$

The term proportional to ω_0^2 originates from the fluctuations in the Josephson potential, while $\eta|\nu|$ is produced by the last term in Eq. (16). The n-independent offset $\eta\nu_1$ is generated by the first part of the first term in Eq. (16), whereas the integral without the principal part does not contribute for large n because it is exponentially suppressed by the factor $e^{-\nu_n}$. For the normalization of A_2 we also need the eigenvalues $\Lambda_{n,0}$ corresponding to the fluctuations around the constant path $\varphi_0 = 0$. These are given by

$$\Lambda_{2n-1,0} = \Lambda_{2n,0} = \frac{\hbar}{8E_C}(\nu_n^2 + \omega_0^2) + \eta|\nu_n| \tag{21}$$

and correspond to Λ_n in Eq. (20) without the offset $\eta\nu_1$.

With the eigenvalues at hand we are in the position to evaluate A_2. Evaluating the infinite product ratio (9), we can write in our regime $\eta \gg \zeta$

$$A_2 = \frac{\eta^2}{\zeta^2}. \tag{22}$$

Using the results (19) in Eqs. (8), (22), and the zero-mode normalization $W_I = \pi\hbar\Omega/E_C$, the final expression for the bandwidth is given by [21]

$$\Delta_I = 4\frac{\eta^2\hbar\Omega}{\zeta^{3/2}} e^{-4\pi(\eta+2\zeta)}. \tag{23}$$

V. INCOHERENT TUNNELING

Switching to a finite bias current j, we render the minima in the Josephson potential unstable. Considering the Hamiltonian time evolution in this system, we cannot treat the minima of the Josephson potential as sites with a single level of a tight-binding model as for the case of coherent tunneling. The evolution brings the initial state into a superposition of excited states of the neighboring minimum. Only strong dissipation then localizes these states again in the local minimum. Such an evolution is called incoherent tunneling. For intermediate evolution times, this can be approximated as an exponential relaxation out of the original well and can be expressed by an imaginary part of the energy when starting in a single minimum. For this problem, the important object is not the instanton trajectory but the bounce φ_B. This is a cyclic trajectory connecting the minimum to a turning point and going back to the starting point, as shown in the lower plot of Fig. 1. It can be shown [18] that in this case, the incoherent decay rate $\Gamma_{4\pi}$ is given by $G[\varphi_B] = F_B e^{-S_B}$; here, S_B is the action S evaluated at the bounce trajectory, and in Eq. (8) we have to replace Λ_1 by $|\Lambda_0|$ because of an occurring negative eigenvalue of the second variation (see below).

A. Bounce action

We start this section with a discussion of the bounce action S_B. In principle, as the quasiparticle action dominates, it is justified to find a saddle point of only the quasiparticle action and treat the circuit action in perturbation theory, as in the case of the instanton. However, there is also a bounce solution that is mainly determined by the circuit action. It corresponds to the tunneling of the phase difference through the barrier between the origin and the nearest-neighbor minimum of the Josephson potential. In Fig. 2, this is indicated by the arrow labeled 2π. For such a trajectory, we can expand the quasiparticle action to second order, so that it reproduces conventional Ohmic dissipation. Therefore, we call the decay due to this bounce solution in the following Ohmic decay. It results at low temperatures in the decay rate $\Gamma_{2\pi} \propto j^{4\pi\eta-1}$ [16]. For small currents, this rate is lower than the rate of decay to the next-nearest minimum caused by the quasiparticle action. While it accounts for a 2π phase slip, the quasiparticle bounce corresponds to a paired 4π phase slip into the next-nearest minimum. Therefore, both processes can physically be distinguished and should be individually considered. In the following, we calculate the dominating rate of decay due to the quasiparticle tunneling.

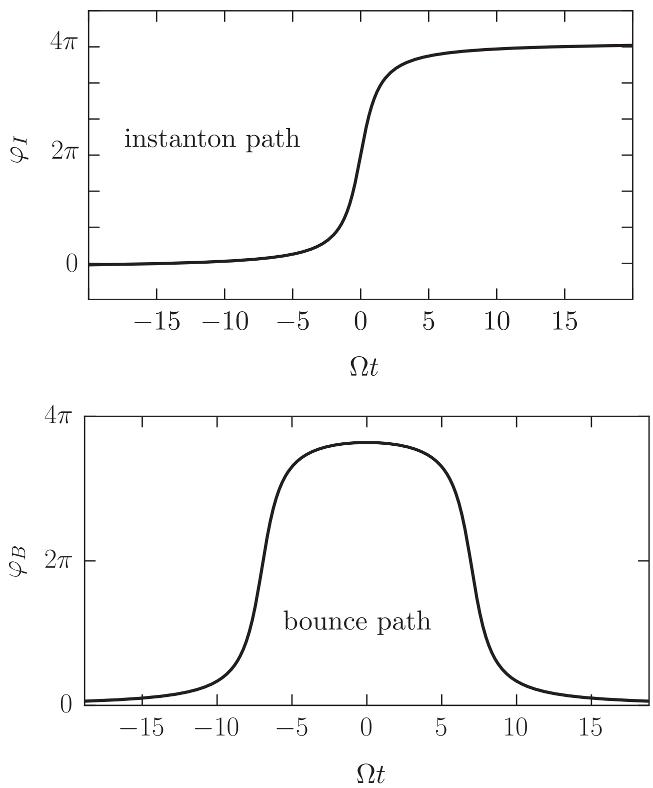

FIG. 1. The top panel shows the instanton path φ_I connecting the minima of the Josephson potential at $\varphi = 0$ and at $\varphi = 4\pi$. The instanton solution corresponds to coherent tunneling of the phase difference. The bottom panel shows the bounce path φ_B, a closed trajectory connecting the origin with itself via a fast penetration of the potential barrier. In our case, it consists of a superposition of an instanton shifted by $\tau/2$ in imaginary time with an anti-instanton shifted by $-\tau/2$. For the plot, we have chosen the value $\Omega\tau = 20$. It is related to the incoherent decay out of the potential minimum at the origin; see Sec. V.

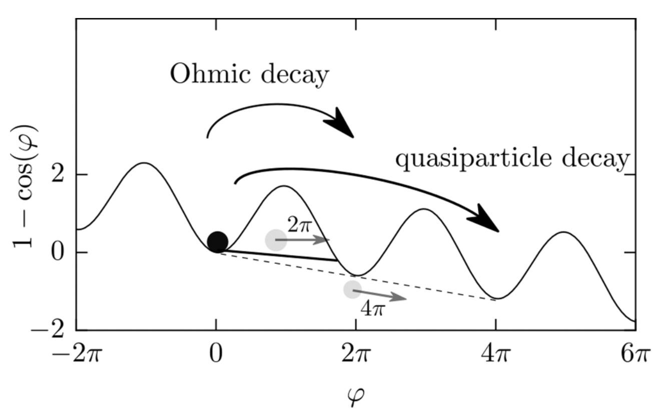

FIG. 2. The Josephson potential biased by a current $j = 0.1$. The black circle corresponds to the phase difference localized at the origin. Conventional Ohmic decay tunnels the phase through the potential barrier to the next minimum, as shown by the black line. The decay due to the quasiparticle tunneling is only slightly influenced by the potential and directly goes to the minimum shifted by 4π, indicated by the dashed line.

The analytical solution to the saddle-point problem of S_η that fulfills the boundary conditions of the bounce is not known. However, we can construct an asymptotic saddle point by adding an instanton shifted by $\tau/2$ in imaginary time with an anti-instanton shifted by $\tau/2$ in the other direction, resulting in the bounce path $\varphi_B = \varphi_I(t + \tau/2) - \varphi_I(t - \tau/2)$. This trajectory has the free parameters Ω and τ, where the first describes how fast the phase switches in imaginary time and the second indicates how long it stays in the shifted minimum before it returns. In the limit $\Omega\tau \to \infty$, φ_B becomes an exact saddle point of the dissipative action S_η. Evaluating the whole action S for this trajectory corresponds to first-order perturbation theory in the circuit action S_c. This approach leads to a bounce action $S_B(\Omega,\tau)$ still depending on the two free parameters of the bounce. To find the approximate saddle point, we need to extremize with respect to these parameters. We find two distinct regimes: the first one corresponds to the regime found in Ref. [5] that is valid as long as Ω stays approximately constant. We denote the regime at small bias current $j < (\zeta/2\eta)^{1/2}$ by (i). In this regime, we find

$$\tau^{(\mathrm{i})} = 2\left(\frac{2\hbar\eta}{jE_J\omega_0^2}\right)^{1/3}, \quad \Omega^{(\mathrm{i})} = \omega_0, \tag{24}$$

resulting in the action

$$S_B^{(\mathrm{i})} = 4\pi[2\eta + 4\zeta - 3(2\eta j^2\zeta^2)^{1/3}]. \tag{25}$$

At elevated bias currents $(\zeta/2\eta)^{1/2} < j < j_{\mathrm{crit}} \approx 0.2$, we find a second novel regime in which the frequency Ω starts to decay $\propto j^{-2}$. In this regime, we have to minimize both parameters Ω and τ (see the Appendix for more information). The resulting saddle-point solution is given by

$$\tau^{(\mathrm{ii})} = \frac{2}{j\Omega^{(\mathrm{ii})}}, \quad \Omega^{(\mathrm{ii})} = \frac{E_J}{\hbar\eta j^2}, \tag{26}$$

with

$$S_B^{(\mathrm{ii})} = 8\pi\eta(1 - j^2). \tag{27}$$

As $j \lesssim 0.2$, the term $8\pi\eta$, which is the quasiparticle action contribution of two infinitely separated instantons, always dominates. This is in agreement with our assumption that the dissipative term approximately determines the saddle point. If we exceed the critical current j_{crit}, the extremum for $S_B(\Omega,\tau)$ is found at $\Omega = 0$, and therefore, the bounce of the dissipative action S_η approaches the constant solution φ_0 that stays in the minimum of the Josephson potential. In Fig. 3, we compare Eqs. (25) and (27) to the value of $S_B(\Omega,\tau)$ at the saddle point that we obtained numerically.

B. Bounce prefactor and result

To find the 4π tunneling rate, the remaining task is to calculate the prefactor F_B that represents the quantum fluctuations on top of the bounce path. The procedure is similar to the calculations for the instanton, however with some complications added. First of all, we have to evaluate the fluctuation operator at the bounce trajectory so that the eigenvalue equation does not take the simple form (16). We can

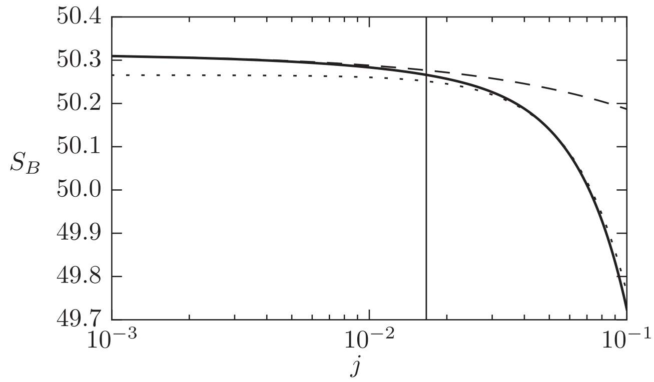

FIG. 3. The value of the bounce action $S_B(\Omega,\tau)$ at the saddle point plotted versus the bias current j for $\eta = 2$ and $\zeta = 10^{-3}$. The solid black line corresponds to the saddle point obtained numerically, while the dashed line corresponds to $S_B^{(i)}$ [Eq. (25)] with a $j^{2/3}$ dependence. The dotted line depicts the action $S_B^{(ii)}$ [Eq. (27)] with a j^2 dependence. The solid vertical line marks the crossover between regimes (i) and (ii) at $j = (\zeta/2\eta)^{1/2} \approx 0.016$. We observe that the validity of the solution $S_B^{(i)}$ breaks down for elevated bias currents, and the action changes its behavior from a $j^{2/3}$ dependence to a j^2 dependence.

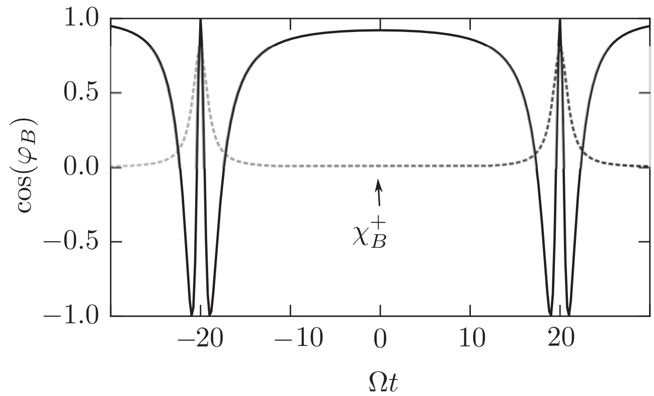

FIG. 4. The solid line shows the effective potential $\cos(\varphi_B)$ of the circuit action S_c for the fluctuations around the bounce path (for $\Omega\tau = 40$). The potential consists of two double wells at $\pm\tau/2$. The dashed line corresponds to the even eigenmode χ_B^+ of the fluctuation operator. It is approximately given by a superposition of the (shifted) instanton eigenmodes $\chi_0(t \pm \tau/2)$, where the lighter part of the curve corresponds to $\chi_0(t + \tau/2)$ and the darker part corresponds to $\chi_0(t - \tau/2)$.

approximate the exact fluctuation operator of S_η by the form

$$
\begin{aligned}
&-\frac{2\Omega}{1+\Omega^2(t-\tau/2)^2}\left[\chi_{B,n}(t) - \int \frac{dt'}{\pi}\frac{\Omega\,\chi_{B,n}(t')}{1+\Omega^2(t'-\tau/2)^2}\right]\\
&-\frac{2\Omega}{1+\Omega^2(t+\tau/2)^2}\left[\chi_{B,n}(t) - \int \frac{dt'}{\pi}\frac{\Omega\,\chi_{B,n}(t')}{1+\Omega^2(t'+\tau/2)^2}\right]\\
&+\int \mathcal{P}\frac{dt'}{\pi}\frac{1}{(t-t')}\frac{d\chi_{B,n}(t')}{dt'} = \frac{\Lambda_{B,n}}{\eta}\chi_{B,n}(t), \qquad (28)
\end{aligned}
$$

valid for $\Omega\tau \to \infty$. It corresponds to the instanton eigenvalue equation (16) with a potential at each position $\pm\tau/2$ of the constituting instantons. For large $\Omega\tau$, the potentials are well separated, so that the eigenmodes are expected to be superpositions of the instanton eigenmodes.

For example, for the low-energy eigenvalues needed in the factor $A_{B,1}$, we can make the ansatz of the even and odd superpositions of the shifted instanton zero modes

$$
\begin{aligned}
\chi_B^\pm &= \frac{1}{(N_B^\pm)^{1/2}}(\chi_{0,+} \pm \chi_{0,-})\\
&= \frac{1}{(2\pi N_B^\pm)^{1/2}}\left[\frac{\Omega^{1/2}}{1+\Omega^2(t+\tau)^2} \pm \frac{\Omega^{1/2}}{1+\Omega^2(t-\tau)^2}\right],
\end{aligned} \qquad (29)
$$

where $\chi_{0,\pm} = \chi_0(t \pm \tau/2)$ and the new normalization is given by $N_B^\pm = [2 \pm 8/(4 + \Omega^2\tau^2)]^{1/2}$.

By comparing (29) with the derivative of the bounce with respect to the imaginary time we see that the odd superposition indeed corresponds to the real zero mode. This zero mode generates a shift of the whole bounce trajectory in imaginary time. Moreover, approximately [up to $O((\Omega\tau)^{-4})$], the even superposition is a zero mode of the quasiparticle action, too. It is a so-called breathing mode and generates a shift of the two instanton parts of the bounce in two different directions, changing the size of the bounce. It can also be obtained by taking the derivative of the bounce with respect to τ. Counting the numbers of nodes, we recognize that the zero mode χ_B^- has one node, while the even mode χ_B^+ has no nodes. Therefore, the even eigenvalue has to be negative. For a negative eigenvalue the naive Gaussian fluctuation approximation breaks down. However, it is this negative eigenvalue that gives rise to the imaginary part of the energy that corresponds to the decay rate [18].

The degeneracy between the even and odd modes is lifted if we perturbatively consider the fluctuations of the circuit action. However, here, we cannot take the simple approach as in Eq. (19) for the instanton. We encounter the problem that we do not know the eigenfunctions accurately enough for this treatment. The eigenfunctions of the quasiparticle action are a good approximation of the real eigenmodes away from the position $\pm\tau/2$ of the instantons. However, close to these positions the eigenmodes are subject to "fast" modulations that are not included in lowest-order perturbation theory.

In Fig. 4, we show the Josephson potential for the fluctuations around the bounce and the even mode χ_B^+ to visualize the problem. The eigenfunction χ_B^+, plotted with the dashed line, is clearly not a ground state for the potential close to the points with $t = \pm\tau/2$. The missing fast modulations are irrelevant for the quasiparticle action but change the contribution by the Josephson potential already on the order of ζ. However, the splitting between the even and odd modes is of the order of $(\Omega\tau)^{-2}$, and thus, we have to apply a modified procedure.

The idea is to directly calculate the splitting $\Delta\Lambda_B$ between the two lowest eigenvalues $\Lambda_{B,0}$ and $\Lambda_{B,1}$ instead of finding their absolute values. Knowing that $\Lambda_{B,1} = 0$ for the exact solution of the problem, we obtain $\Lambda_{B,0} = -\Delta\Lambda_B$. It is possible to calculate $\Delta\Lambda_B$ without accurate knowledge of the wave functions close to the instanton position. For that we define $T_{\rm kin} = -(\hbar/8E_C)(\partial/\partial t)^2$, $V_\pm = E_J\cos[\varphi_I(t \pm \tau/2)]/\hbar$, and $V_{\rm pert} = V_0 - V^+ - V^-$, with $V_0 = E_J\cos(\varphi_B)$ being the Josephson potential evaluated at the bounce. We can rewrite the circuit fluctuation operator as

$$\frac{\delta^2 S_c}{\delta\varphi^2}[\varphi_B] = T_{\text{kin}} + V^+ + V^- + V_{\text{pert}}. \tag{30}$$

In the expression

$$\begin{aligned}\Delta\Lambda_B &= \int dt[\chi_B^-(T_{\text{kin}} + V^+ + V^- + V_{\text{pert}})\chi_B^- \\ &\quad - \chi_B^+(T_{\text{kin}} + V^+ + V^- + V_{\text{pert}})\chi_B^+] \\ &= 2\int dt\, \chi_{0,+}(V^- + V^+ + 2V_{\text{pert}})\chi_{0,-} \\ &= \frac{4E_J}{\hbar(\Omega\tau)^2}\end{aligned} \tag{31}$$

for the first-order perturbation, we make use of the fact that $(T_{\text{kin}} + V^{\pm})\chi_{0,\pm} = 0$ for the zero mode. This removes the terms $V^{\pm}\chi_{0,\pm}^2$ that are localized in the dangerous region around the instanton position. Additionally, for the second equality, we have left out terms proportional to $V^{\mp}\chi_{0,\pm}^2$ that are higher order in $\Omega\tau$.

For the modes with more than one node ($n > 1$), the accuracy of the conventional perturbation theory is sufficient. By using the odd superposition of the shifted $\chi_{1,\pm}$ instanton eigenmodes, we can estimate the third eigenvalue as $\Lambda_{B,2} = \hbar\Omega^2/16E_C$. The expression for the normalization due to the zero mode reads $W_B = 2\pi\hbar\Omega/E_C$. As a result, we obtain the prefactor

$$A_{B,1} = 2\sqrt{\frac{E_J}{\hbar\Omega}}\Omega\tau. \tag{32}$$

In order to calculate $A_{B,2}$, we still have to determine the higher eigenvalues corresponding to $n \to \infty$. The high-energy eigenmodes are still approximately given by the eigenfunctions of the kinetic operator. We obtain the corresponding eigenvalues by inserting the second variation (28) of the bounce into expression (20). We find the result

$$\Lambda_{B,2n-1} = \Lambda_{B,2n} = \frac{\hbar}{8E_C}\left(\nu_n^2 + \omega_0^2\right) + \eta|\nu_n| - 2\eta\nu_1, \tag{33}$$

where the factor 2 in the last term not present in Eq. (20) originates from the fact that there are two instantons contributing to the bounce. Plugging (33) into (9) yields (for $\eta \gg \zeta$)

$$A_{B,2} = (A_2)^2 = \frac{\eta^4}{\zeta^4}. \tag{34}$$

With the results (33) and (34) and the ones in Sec. V A, we are in the position to evaluate the decay rate for the two regimes identified above. For low bias current $j < (\zeta/2\eta)^{1/2}$, the rate is given by [21]

$$\Gamma_{4\pi}^{(\text{i})} = \frac{\Delta_I^2\zeta^{-1/2}}{8\hbar^2}\left(\frac{2\hbar\eta}{j\omega_0^2E_J}\right)^{1/3} e^{12\pi(2\eta j^2\zeta^2)^{1/3}}. \tag{35}$$

For elevated currents with $(\zeta/2\eta)^{1/2} < j < 0.2$, the decay rate is given by

$$\Gamma_{4\pi}^{(\text{ii})} = 4\omega_0\frac{\eta^{7/2}}{\zeta^4}e^{-8\pi\eta(1-j^2)}. \tag{36}$$

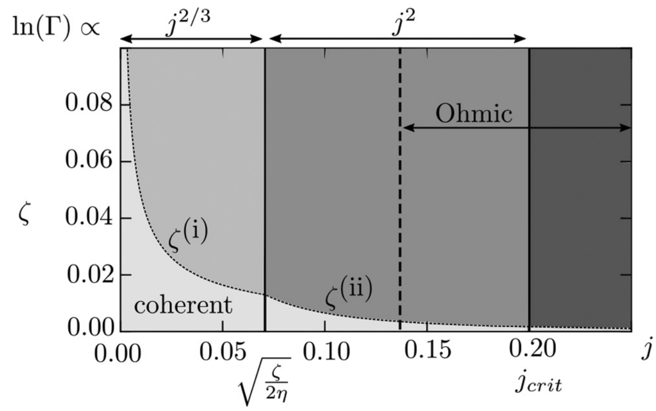

FIG. 5. The crossovers between the different regimes. The axes are the bias current j and ζ (for fixed $\eta/\zeta = 100$). From light to dark color we go from the coherent regime to regime (i) with a scaling of the decay rate $\ln\Gamma_{4\pi} \propto j^{2/3}$ followed by the regime (ii) with $\ln\Gamma_{4\pi} \propto j^2$ and finally end up with Ohmic dissipation. The dashed black curve marks the crossover to the coherent regime. For very small ζ, there is no crossover to the $j^{2/3}$ regime. For approximately $j > 0.14$ (indicated by the dashed vertical line) the rate of 2π decay generated by the Ohmic bounce becomes larger than the rate of the 4π quasiparticle decay. However, the two processes can be distinguished, so that the quasiparticle decay can still be measured. Above j_{crit}, the bounce of the quasiparticle action approaches the constant solution $\varphi_0 = 0$, and only the 2π process, for which the quasiparticle dissipation is approximately Ohmic, survives. For larger ζ than shown, the coherent regime vanishes already for a small bias current, while the other crossover lines do not depend on ζ.

The crossover from the result (35) to (36) that we describe in more detail below and the decay rate (36) at elevated bias current are the main results of the present work.

C. Regimes and crossovers

In this section, we discuss the crossovers between the regimes identified above. Without bias current, the system forms bands due to dissipation-mediated coherent tunneling. We call this regime the "coherent regime" (see Fig. 5). The amplitude $\Delta_I/2$ then defines a tunneling matrix element for a 4π phase slip. Increasing the bias current j, more than a single state in the well separated by 4π becomes energetically accessible, and the coherent tunneling transforms into an incoherent relaxation. A quantitative criterion for the crossover from the coherent to the incoherent regime can be defined by $(\tau^{(\text{i,ii})})^2 > \delta\tau^2 \approx [\partial^2 S_B(\Omega,\tau)/\partial\tau^2]^{-1}$. This gives an estimate of whether we can treat the position τ of the bounce as a classical variable or whether quantum fluctuations have to be taken into account. As long as the quantum fluctuations of τ are smaller than the optimal separation between the instantons $\tau^{(\text{i,ii})}$, the bounce and therefore incoherent tunneling are an appropriate description. If the fluctuations in τ increase, the system is more accurately described by a gas of individual instantons giving rise to coherent tunneling elements. Depending on the parameters, tuning j up leads in general to a crossover of the action to regime (i) with a scaling of $\ln\Gamma \propto j^{2/3}$ and then to regime (ii) with a scaling $\propto j^2$. However, for $\zeta < 0.012$, regime (i) is never realized, and the system directly crosses over from the coherent regime to regime (ii). From the crossover criterion above, we obtain the approximate expressions for the crossover (at fixed η/ζ),

$$\zeta^{(\mathrm{i})} = \frac{1}{24\pi(10 + 2\eta/\zeta)^{1/3} j^{2/3}} \tag{37}$$

in the regime $j < (\zeta/2\eta)^{1/2}$ and

$$\zeta^{(\mathrm{ii})} = \frac{\eta/\zeta}{\pi[24 + (48\eta/\zeta - 80)j^2]} \tag{38}$$

in the regime $(\zeta/2\eta)^{1/2} < j < 0.2$. At $j \approx 0.14$ the rate of 2π phase slips $\Gamma_{2\pi}$ generated by the Ohmic bounce solution is of the same order as the quasiparticle decay $\Gamma_{4\pi}$. However, the two processes are physically distinguishable, so they can be individually measured (see also below). At a bias current j above j_{crit}, the bounce connecting two minima separated by 4π vanishes, such that only Ohmic dissipation is present in this regime.

VI. MEASUREMENT

As demonstrated above, Josephson junctions with strong quasiparticle dissipation admit many interesting properties that can be the subject of an experimental investigation. The simplest approach to observe the effects of the special (nonlinear) form of the dissipation due to quasiparticle tunneling is to measure the incoherent tunneling. Measuring the coherent tunneling directly is challenging due to the small bandwidth exponentially suppressed in η without any additional tuning parameter. Therefore, we propose to measure paired phase-slip events and compare the resulting rates to expression (35) or (36).

The key idea for the experimental observation of the paired phase slips is to raise the bias as much as possible, i.e., smaller than j_{crit} but still in its vicinity, in order to increase the rate of paired phase slips. An important requirement for the experimental setup in order to be able to operate at elevated bias current is the possibility to distinguish between double and single phase slips. The reason is that at elevated bias current, the rate of unpaired 2π phase slips can already dominate the rate of paired 4π phase slips. Additionally, even if we can distinguish between the two processes, we need to make sure that the 4π process can be uniquely associated with the periodic quasiparticle tunneling, while the 2π process is caused solely by the conventional Ohmic tunneling. The latter process does not necessarily end up in the nearest minimum. If the momentum, i.e., the kinetic energy, of the phase difference is too large it may not be retrapped after the tunneling, but it can classically go on over the next potential hill and end up in the following minimum. Especially, after the point at which the Ohmic tunneling rate $\Gamma_{2\pi}$ exceeds the quasiparticle rate $\Gamma_{4\pi}^{(i)}$ or $\Gamma_{4\pi,2}^{(ii)}$, it is not possible to distinguish between the two processes anymore. Therefore, it is best to keep the capacitance C small, so that the dissipation always brings the Ohmic phase slips to rest in the next minimum. Additionally, it is advantageous to use small $\zeta \ll 1$ because it allows us to consider systems with smaller η without making the ratio η/ζ too large. Smaller η then keeps the exponential suppression of the phase-slip rate low.

An approach that can satisfy the above requirements is to include the Josephson junction in a loop with inductance L or, alternatively, build an asymmetric superconducting quantum interference device so that one Josephson junction serves as an inductance (see Refs. [22,23] for a recent experimental setup). With a magnetic bias, it is possible to add an external flux φ_{ex} in the loop that takes the role of the bias current. Placing the circuit in a transmission line, the number of flux quanta in the loop can be measured nondestructively by a flux-dependent shift of the transmission phase of the input and output signals into the transmission line. This flux-dependent shift in the transmission phase directly indicates when a 2π or a 4π event has happened. By recording these events over a given measurement time, the resulting rates can be compared with the results (35) or (36). Theoretically, the setup corresponds to introducing the additional term $S_L = \int dt\, \phi_0^2(\varphi - \varphi_{\mathrm{ex}})^2/(8\pi^2 L)$ to the circuit action S_c with induction L. In this setup, the bias current is given by the term linear to φ with $I = \hbar\phi_0\varphi_{\mathrm{ex}}/4\pi^2 L$. The additional quadratic contribution $\propto \varphi^2$ simply changes the bias current according to $j \mapsto j - \hbar\phi_0^2/\pi L E_J$. This takes care of the fact that a quadratic potential needs already an external flux of $\varphi_{\mathrm{ex}} = 4\pi$ until the minimum at $\varphi = 0$ becomes unstable for the Korshunov decay channel, while without the quadratic confinement an infinitesimal bias is already enough to render the minimum unstable. A similar setup has been used in Ref. [23] to measure the interference between phase slips in two parallel nanowires. It indicates that the quasiparticle tunneling in nanowires is strong, and therefore, these wires are a potential candidate for such an experimental setup. Note that there are also other potential experimental probes that can detect changes in flux. For example, a flux-dependent absorption process could be used to measure the rate of paired phase slips [24,25].

VII. CONCLUSION

In conclusion, we have shown that the coherent dissipation due to quasiparticle tunneling over a Josephson junction in a superconductor can be probed by the measurement of 4π phase-slip events. These 4π phase slips are caused by Korshunov instantons probing the specifics of the nonlinear dissipation due to quasiparticles. We have identified a novel regime at elevated bias current that leads to a substantially increased rate of 4π phase slips. This is important as the low rate is one of the main reasons why paired phase slips are challenging to measure. We have discussed the different crossovers between the coherent regime and the incoherent regimes. In addition, we have proposed a measurement scheme for the detection of the paired phase slips; fixing the bias current slightly below a critical current $j_{\mathrm{crit}} \approx 0.2$ and working with a small capacitance C, corresponding to a large charging energy, offer the best chance to observe paired phase slips due to the increased rates. We hope that our analysis helps to guide the experimental effort to directly observe Korshunov instantons as paired phase slips of the superconducting phase.

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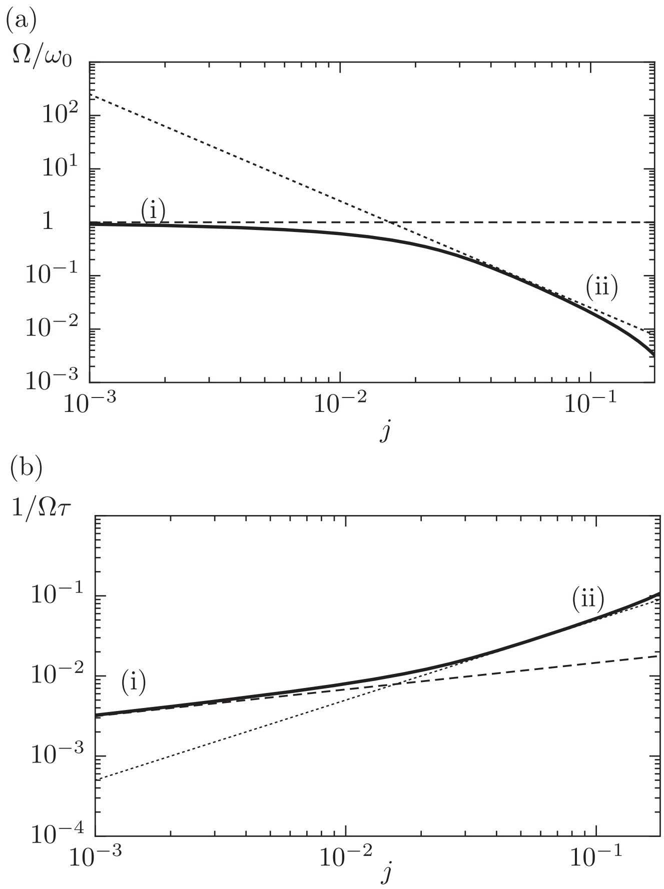

FIG. 6. Double logarithmic plots of the extremal parameters (a) Ω/ω_0 and (b) $1/\Omega\tau$ as a function of the dimensionless bias current j for the parameters $\eta = 2$ and $\zeta = 10^{-3}$. The thick black lines correspond to the result that is obtained by simply extremizing the whole action (A1) with respect to both parameters Ω and τ numerically. The dashed line represents regime (i), where $\Omega = \omega_0$ is constant. The dotted line represents regime (ii), where Ω decays to zero.

APPENDIX: EXTREMIZING THE ACTION IN DIFFERENT REGIMES

In this Appendix, we provide details for the extremizing the action S_B to find regime (i), corresponding to Eqs. (24) and (25), and regime (ii), corresponding to Eqs. (26) and (27). The action S_B evaluated at the bounce trajectory and consistently expanded for large $\Omega\tau$ up to second order reads

$$S_B \approx 8\pi(\eta + E_J/\hbar\Omega + \hbar\Omega/8E_C) - 4\pi E_J \tau j/\hbar - \frac{32\pi(\eta + 4E_J/\hbar\Omega + \hbar\Omega/8E_C)}{\Omega^2\tau^2}. \quad \text{(A1)}$$

In Fig. 6, we show an example of the resulting optimal parameters calculated by a numerical optimization of the action above with respect to Ω and τ. It clearly shows two distinct regimes with different power-law behaviors. The first regime corresponds to regime (i) with a constant Ω, while the second regime corresponds to regime (ii) with decaying Ω. The crossover is numerically found to be at $j_c \simeq (\zeta/2\eta)^{1/2}$ (see below).

An analytic expression valid in regime (i) can be found by assuming $\Omega = \omega_0$ and optimizing (A1) with respect to the single parameter τ. In this case, only the two last terms in Eq. (A1) contribute. This yields Eqs. (24) and (25) for the optimal point.

By increasing the bias current, the assumption $\Omega = \omega_0$ fails to hold as the inverse size of the instanton Ω starts to decline with increasing bias current j. As a result the term $8\pi E_J/\hbar\Omega$ starts to become relevant. The point at which this happens can be estimated by comparing it to one of the last two terms, e.g., $E_J/\hbar\omega_0 \simeq E_J \tau j_c/\hbar$. With $\tau \simeq (\hbar\eta/jE_J\omega_0^2)^{1/3}$ [from (24)], we obtain the estimate for the crossover current $j_c \simeq (\zeta/\eta)^{1/2}$ as before.

So for $j \gg j_c$, the parameters τ and Ω in the action have to be simultaneously optimized. Not all terms of the action (A1) are relevant. In the first term, we can neglect the term proportional to Ω as $\Omega \ll \omega_0$. In the last term, only the term proportional to η is relevant as $\eta \gg \zeta$. Thus, the effective action in regime (ii) reads

$$S_B \approx 8\pi(\eta + E_J/\hbar\Omega) - 4\pi E_J \tau j/\hbar - 32\pi\eta/\Omega^2\tau^2. \quad \text{(A2)}$$

Extremizing this action with respect to the parameters Ω and τ is straightforward and leads to the results of Eq. (26). Inserting the optimized parameters into Eq. (A2) yields the simple expression for the action

$$S_B \approx 8\pi\eta(1 - j^2), \quad \text{(A3)}$$

which is equivalent to (27). For bias currents $j > 0.1$, the accuracy of (27) can be increased by including small corrections to the j^2 dependence with first-order perturbation theory. This corresponds to inserting the optimized values Ω and τ from Eq. (26) into the full action (A1).

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