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Anisotropic chiral magnetic effect from tilted Weyl cones

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We determine the antisymmetric current-current response for a pair of (type-I) tilted Weyl cones with opposite chirality. We find that the dynamical chiral magnetic effect depends on the magnitude of the tilt and on the angle between the tilting direction and the wave vector of the magnetic field. Additionally, the chiral magnetic effect is shown to be closely related to the presence of an intrinsic anomalous Hall effect with a current perpendicular to the tilting direction and the electric field. We investigate the nonanalytic long-wavelength limit of the corresponding transport coefficients.

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Introduction. In classical electrodynamics, magnetic fields always induce currents that are perpendicular to the magnetic field direction due to the Lorentz force. However, in quantum electrodynamics, a current can also be generated in the same direction as the magnetic field. This was first realized for massless fermions in particle physics [1,2]. It is a consequence of the fact that quantum mechanically a magnetic field quenches the kinetic energy perpendicular to its direction and also spin polarizes the lowest Landau level. As a result, massless fermions only obtain a drift velocity along the magnetic field with an opposite sign for opposite chiralities. Inducing an imbalance between the two chiral species then gives a net current along the magnetic field direction known now as the chiral magnetic effect (CME).

Massless chiral fermions also occur as low-energy quasiparticles in the recently discovered Weyl (semi)metals [3–7]. These quasiparticles do not move at the speed of light, as in elementary-particle physics, but rather at the Fermi velocity. Additionally, the effective Weyl cones with different chirality are in a real material always connected by the full band structure, and hence electrons can be transported from one cone to another by applying both an electric and a magnetic field [8]. In particle physics, the same phenomenon occurs due to the breaking of chiral symmetry by quantum corrections. This breaking of chiral symmetry due to the renormalization of ultraviolet divergencies is called a chiral anomaly and causes the difference between the numbers of particles with positive and negative chirality to be no longer conserved [9–11].

The main difference with particle physics is that Lorentz invariance is not enforced in a condensed-matter material. This gives, besides a velocity that is smaller than the speed of light, also the possibility that Weyl nodes are separated in energy-momentum space. Splitting them in the momentum direction gives rise to a topological anomalous Hall effect [8], whereas splitting them in the energy direction is exactly the situation of most interest for the CME [2,12]. Indirect measurements of the chiral magnetic effect have recently been made by the observation of a negative magnetoresistance [13–16]. Another interesting possibility is tilting the Weyl cones, meaning that the slope of the dispersion relation is not the same in opposite directions [17,18]. Materials that exhibit such tilted Weyl cones are of type I if the tilt is relatively small and of type II if the cones are overtilted such that the electron and hole dispersions intersect the energy plane of the Weyl node itself [19–21]. Moreover, the tilt is affected and can even be generated by disorder and interaction effects [22–24]. It is thus of considerable interest to investigate what such a tilt does to the chiral magnetic conductivity of a Weyl (semi)metal.

The chiral magnetic conductivity is in principle a function of the wave number and frequency of the applied magnetic field [25]. When calculating the long-wavelength limit, the order of limits is crucial and we need to distinguish the case in which the Weyl nodes are located at the same energy and the case in which they are not [26–29]. Only when the chiral imbalance of the two Weyl nodes is exactly opposite to their energy separation, is there a vanishing current in the static limit [12,30,31].

Here, we reconsider these subtleties for a pair of type-I tilted Weyl cones. We first illustrate the short-wavelength physics involved by calculating the full frequency and wave-number dependence of the effective CME for a transverse electromagnetic wave propagating along the tilting direction. For arbitrary magnetic field directions we focus on the long-wavelength response. We find that the chiral magnetic conductivity is anisotropic and in general nonuniversal, even though the chiral anomaly is unmodified by the tilt. Our results for the homogeneous and static limit are summarized in Fig. 4.

Current-current response function. We consider a pair of Weyl cones with opposite chiralites $\pm$ that are doped with chemical potentials $\mu_\pm \equiv \mu \pm \mu_5$, defined with respect to the Weyl nodes, as depicted in Fig. 1. The chiral chemical potential μ_5 indicates a chiral population imbalance that can be created by applying an electric field pulse with a component parallel to an already present magnetic field. We also allow the Weyl nodes to be split up in energy, which we denote by $\Delta E \equiv E_+ - E_-$, and we comment on the effect of this later on. The topological anomalous Hall effect, however, is well understood and therefore not discussed throughout the following. Furthermore, we consider for simplicity cones with an isotropic Fermi velocity v_F, which is straightforwardly generalized to the anisotropic case.

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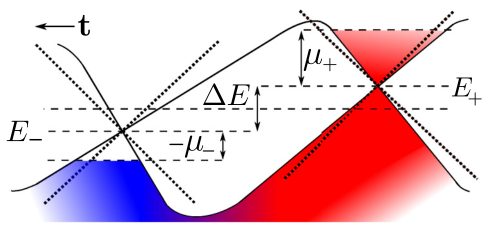


FIG. 1. Illustration of a band structure with two imbalanced Weyl cones at chiral chemical potentials $\mu_\pm$ and node energies $E_\pm$, both tilted in the same direction $\mathbf{t}$.

Tilting the cones in a direction $\mathbf{t}$ can be achieved in two distinct ways: Either we introduce a momentum-dependent chiral chemical potential $\mu_5(\mathbf{k}) \equiv \mu_5 - \hbar v_F \mathbf{k}\cdot\mathbf{t}$, or a momentum-dependent chemical potential $\mu(\mathbf{k}) \equiv \mu - \hbar v_F \mathbf{k}\cdot\mathbf{t}$, where $\hbar\mathbf{k}$ is the momentum. Only the latter replacement breaks inversion symmetry [32]. Physically, breaking inversion symmetry corresponds to tilting the two cones in the same direction (cf. Fig. 1), while inversion symmetry is preserved upon tilting the two cones in opposite directions. In this Rapid Communication we perform all calculations explicitly in the case that inversion symmetry is broken, and we comment on the other case in our discussion. Hence, the appropriate Hamiltonian reads ($\hbar = 1$)

$$H(\mathbf{k}) = (v_F\mathbf{k}\cdot\boldsymbol{\sigma} - \mu_5\sigma^0)\tau^z + (v_F\mathbf{k}\cdot\mathbf{t} - \mu)\tau^0\sigma^0, \tag{1}$$

where $\boldsymbol{\tau}$ are the Pauli matrices acting in orbital space and $\boldsymbol{\sigma}$ in spin space, complemented by the 2×2 unit matrices τ^0 and σ^0. The Hamiltonian has four distinct eigenvalues $\sigma E_{\mathbf{k}} + v_F\mathbf{k}\cdot\mathbf{t} - \mu_{\sigma'}$, with $E_{\mathbf{k}} \equiv v_F|\mathbf{k}|$ the dispersion relation of the massless fermions and $\sigma,\sigma' = \pm$. Here, we consider type-I (semi)metals, meaning that we restrict ourselves to $0 < t < 1$ for $t = |\mathbf{t}|$. For simplicity, we consider the two cones to have the same absolute value for the tilt t, but also this is easily generalized.

In order to calculate the response to an externally applied magnetic or electric field, we couple the fermions with charge $-e$ to an external vector potential $\mathbf{A}$ via the minimal coupling prescription $\mathbf{k} \to \mathbf{k} + e\mathbf{A}$. Next, we perform second-order perturbation theory in the external gauge field to obtain the current-current response function $\Pi^{ij}(\mathbf{q},\omega;\mathbf{t})$. In the process, the subtraction of the two Dirac seas of the cones leads to the elimination of a logarithmic ultraviolet divergence. In terms of the frequency $\omega^+ = \omega + i0$, the antisymmetric part of the retarded current-current response function $\Pi_l(\mathbf{q},\omega;\mathbf{t}) = \epsilon_{ijl}\Pi^{ij}(\mathbf{q},\omega;\mathbf{t})/2$ reads

$$i\Pi_l(\mathbf{q},\omega;\mathbf{t}) = \frac{e^2v_F^2}{2}\sum_{\sigma,\sigma',\sigma''}\sigma\int\frac{d^3\mathbf{k}}{(2\pi)^3}F_l^{\sigma'\sigma''}(\mathbf{k},\mathbf{q};\mathbf{t})\left[\frac{N_F(E_{\mathbf{k}} - \sigma'\mu_\sigma(\mathbf{k})) - \sigma'\sigma''N_F(E_{\mathbf{k+q}} - \sigma''\mu_\sigma(\mathbf{k+q}))}{\omega^+ - v_F\mathbf{q}\cdot\mathbf{t} + \sigma'E_{\mathbf{k}} - \sigma''E_{\mathbf{k+q}}}\right], \tag{2}$$

where we defined a structure factor $F_l^{\sigma\sigma'}(\mathbf{k},\mathbf{q};\mathbf{t})$ by

$$\mathbf{F}^{\sigma\sigma'}(\mathbf{k},\mathbf{q};\mathbf{t}) \equiv \frac{\mathbf{k}}{|\mathbf{k}|} - \sigma\sigma'\frac{\mathbf{k}+\mathbf{q}}{|\mathbf{k}+\mathbf{q}|} - \sigma'\frac{\mathbf{q}(\mathbf{t}\cdot\mathbf{k}) - (\mathbf{q}\cdot\mathbf{t})\mathbf{k}}{|\mathbf{k}+\mathbf{q}||\mathbf{k}|}. \tag{3}$$

In Eq. (2) we denoted the Fermi-Dirac distribution by $N_F(x) \equiv (e^{x/k_BT} + 1)^{-1}$ and all three sums run over $\sigma,\sigma',\sigma'' = \pm$. Physically, the sum over σ accounts for the two cones, whereas the sums over σ' and σ'' account for the four possibilities for particle-hole pairs in a chiral cone consisting of two touching bands. In the limit $\mathbf{t} = \mathbf{0}$ the expression in Eq. (2) reduces to the well-known result for a three-dimensional chirally doped Weyl semimetal [33]. Including a tilt alters the energy-dispersion relation and yields an additional term in the interaction vertex, resulting in the last term in the structure factor in Eq. (3).

The antisymmetric part of the current-current response function in Eq. (2) is in general [34] spanned by a linear combination of the vectors $\mathbf{q}$ and $\mathbf{t}$, i.e., we can decompose it as

$$i\Pi_l(\mathbf{q},\omega^+;\mathbf{t}) = \sigma^{\text{CME}}(\mathbf{q},\omega)q_l + \sigma^{\text{AHE}}(\mathbf{q},\omega)\omega t_l. \tag{4}$$

As explicitly indicated, this gives rise to two distinct effects: a chiral magnetic effect and a tilt-induced planar intrinsic anomalous Hall effect (AHE) [35–37]. The corresponding currents read

$$\mathbf{J}^{\text{CME}}(\mathbf{q},\omega) = \sigma^{\text{CME}}(\mathbf{q},\omega)\mathbf{B}(\mathbf{q},\omega), \tag{5}$$

$$\mathbf{J}^{\text{AHE}}(\mathbf{q},\omega) = \sigma^{\text{AHE}}(\mathbf{q},\omega)\mathbf{t}\times\mathbf{E}(\mathbf{q},\omega), \tag{6}$$

in terms of the chiral magnetic and anomalous Hall conductivities $\sigma^{\text{CME}}(\mathbf{q},\omega)$ and $\sigma^{\text{AHE}}(\mathbf{q},\omega)$, respectively. The intimate relation between these two effects is even more clear in relativistic notation, where we have that $\Pi^{\kappa\nu} = i\epsilon^{\kappa\lambda\mu\nu}P_\lambda q_\mu$ and thus $J^\kappa = \Pi^{\kappa\nu}A_\nu = \epsilon^{\kappa\lambda\mu\nu}P_\lambda F_{\mu\nu}/2$, where $F_{\mu\nu}$ is the Faraday tensor and $P^\lambda = (\sigma^{\text{CME}},\sigma^{\text{AHE}}\mathbf{t})$ elegantly combines the two conductivities. Note that the gauge invariance of the result is then also manifest.

In the following, we discuss the tilt dependence of both effects separately. In principle, both $\sigma^{\text{CME}}(\mathbf{q},\omega)$ and $\sigma^{\text{AHE}}(\mathbf{q},\omega)$ depend on the angle between $\mathbf{q}$ and $\mathbf{t}$. In order to make analytic progress, however, we specialize to zero temperature and first consider as an illustrative example the propagation of a purely transverse electromagnetic wave (light) with $\mathbf{q} \parallel \mathbf{t}$ for arbitrary wave numbers and frequencies. This case corresponds to $\mathbf{B} \perp \mathbf{t}$, $\mathbf{E} \perp \mathbf{t}$, and $\mathbf{B} \perp \mathbf{E}$, as the magnetic field is given in momentum space by $\mathbf{B}(\mathbf{q},\omega) = i\mathbf{q}\times\mathbf{A}(\mathbf{q},\omega) = \mathbf{q}\times\mathbf{E}(\mathbf{q},\omega)/\omega$, and gives an effective CME response that, interestingly, is a combination of the chiral magnetic and anomalous Hall effects.

Effective chiral magnetic effect for a transverse wave with $\mathbf{q} \parallel \mathbf{t}$. In the above case, the total current along the magnetic field is determined by the effective CME conductivity $\sigma_\perp^{\text{CME}}(\mathbf{q},\omega) \equiv iq^l\Pi_l(\mathbf{q},\omega^+;\mathbf{t})/q^2 = \sigma^{\text{CME}}(\mathbf{q},\omega) + \sigma^{\text{AHE}}(\mathbf{q},\omega)\omega t/q$, with $q = |\mathbf{q}|$. The details of the

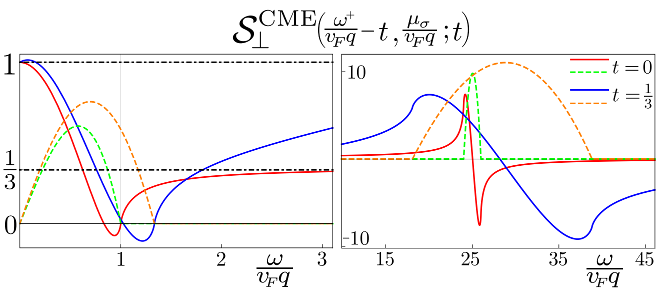


FIG. 2. Plot of the real (solid lines) and imaginary (dashed lines) part of $\mathcal{S}_{\perp}^{\rm CME}$ for $2\mu_\sigma/v_F q = 25$ and for $t = 0$ (red, green) and $t = 1/3$ (blue, orange). The left plot shows the behavior for small $\omega/v_F q$, whereas the right plot shows the resonance for larger values of $\omega/v_F q$. The black dotted-dashed lines indicate the static and homogeneous limit for $t = 0$.

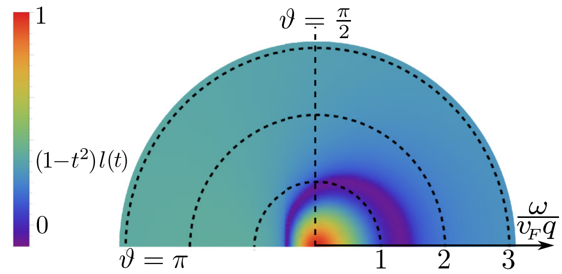


FIG. 3. Anisotropic behavior of Re[$\mathcal{S}^{\rm CME}$] for $t = 1/2$ as a function of the angle ϑ and the radial coordinate $\omega/v_F q$. In the static (small radius) and homogeneous (large radius) limit, we obtain the isotropic results 1 and $(1 - t^2)l(t) \simeq 0.3$.

calculation can be found in the Supplemental Material [38]. Ultimately we find for the effective chiral magnetic conductivity

$$\sigma_{\perp}^{\rm CME}(q,\omega) = \frac{e^2}{4\pi^2}\sum_{\sigma=\pm}\sigma\mu_\sigma \mathcal{S}_{\perp}^{\rm CME}\left(\frac{\omega^+}{v_F q} - t, \frac{\mu_\sigma}{v_F q}; t\right). \quad (7)$$

The dimensionless function $\mathcal{S}_{\perp}^{\rm CME}(x,y;t)$ captures all frequency, wave number, and tilt dependence of the conductivity. It is given by

$$\mathcal{S}_{\perp}^{\rm CME}(x,y;t) = \frac{1-x^2}{2(1+xt)} - \sum_{\sigma,\sigma'=\pm}\sigma K_{\sigma'}(x,y;t)H_{\sigma\sigma'}(x,y;t), \quad (8)$$

in terms of the dimensionless functions

$$K_\sigma(x,y;t) \equiv \left(\frac{1-x^2}{16y}\right)\left[\left(\frac{2\sigma y + t + x}{1+xt}\right)^2 - 1\right], \quad (9)$$

$$H_{\sigma\sigma'}(x,y;t) \equiv \log\left(1 + \frac{2y}{(1-\sigma\sigma' t)(\sigma' x - \sigma)}\right). \quad (10)$$

The expression for the conductivity in Eq. (7) has a nontrivial dependence on the wave number q and frequency ω of the externally applied field. In fact, it is a function of the fraction $\omega/v_F q$, giving a different result in the homogeneous limit and the static limit. Indeed, in the static limit ($\omega/v_F q \to 0$), we find the well-known [8,19] universal result $e^2\mu_5/2\pi^2$, whereas in the homogeneous limit ($\omega/v_F q \to \infty$), we find the tilt-dependent result

$$\lim_{\substack{q\to 0\\ \omega\to 0}} \sigma_{\perp}^{\rm CME}(q,\omega) \to \left[1 - 2l(t) + tl(t)\frac{\omega}{v_F q}\right]\frac{e^2\mu_5}{2\pi^2}, \quad (11)$$

in terms of the function

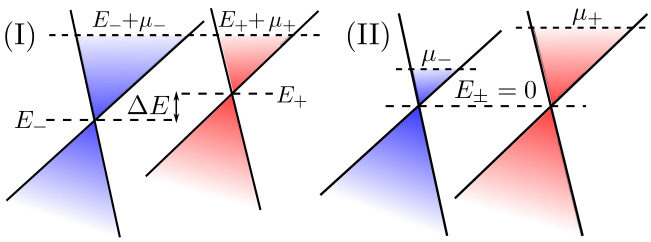
$$l(t) \equiv \frac{1}{2t^3}\log\left(\frac{1+t}{1-t}\right) - \frac{1}{t^2} \overset{t\to 0}{=} \frac{1}{3}. \quad (12)$$

We thus obtain the result $e^2\mu_5/6\pi^2$ for the homogeneous limit of Eq. (11) if $t = 0$ [29]. The function $l(t)$ diverges upon taking the limit $t \to 1$. The physical reason for this divergence is that then the cones are tilted up to the point that the density of states becomes infinite, thus resulting in an infinite conductivity. In fact, the conductivity in Eq. (11) is due to the presence of the in-plane anomalous Hall effect formally always infinite in the homogeneous limit $\omega/v_F q \to \infty$. Note, however, that for light propagation, we have that ω/q is equal to the speed of light in the material.

We plot the full dependence of the real and imaginary part of $\mathcal{S}_{\perp}^{\rm CME}(\omega^+/v_F q - t, \mu_\sigma/v_F q, t)$ on $\omega/v_F q$ for a fixed value of $\mu_\sigma/v_F q$ and different values of the tilt t in Fig. 2. When $t = 0$, the real part interpolates between the value 1 in the static limit and $1/3$ in the homogeneous limit [33,39]. Additionally, there is a resonance at $\omega = 2\mu_\sigma$, after which the conductivity goes to zero as $1/\omega$. This resonance is effectively shifted to infinity in the homogeneous limit. For a nonzero tilt t, the static limit remains unchanged and in the homogeneous limit the real part of the conductivity diverges as $tl(t)\omega/v_F q$, rather than becoming constant as in the case of zero tilt. The resonance at $\omega = 2\mu_\sigma$ remains present at nonzero tilt but becomes broader, as its width is now set by $2\mu_\sigma/(1 \pm t)$. Note that the conductivity $\sigma_{\perp}^{\rm CME}(q,\omega)$ depends in a highly nonlinear way on the chiral imbalance μ_5. Theoretically, this implies that the CME is not fully determined by the triangle diagram of the chiral anomaly. This is only true in the long-wavelength limit [40,41].

At this point it is important to discuss why the conductivity is finite in the static limit in equilibrium. In deriving Eq. (2) a logarithmic divergence was avoided by a cancellation of the Dirac-sea contributions of the two cones. This cancellation is correct up to a constant, which is proportional to the energy separation $\Delta E = E_+ - E_-$ of the Weyl nodes [42–44]. Hence, our answers for the static limit and Eq. (11) only apply when the energy separation between the nodes is zero. If that is not the case, then the true equilibrium situation corresponds to the situation where the chiral imbalance is exactly canceled by the energy separation between the Weyl nodes, i.e., $2\mu_5 = \mu_+ - \mu_- = -\Delta E$. Using this renormalization condition, we find that the chiral magnetic conductivity is zero in equilibrium, as expected. We will follow the same procedure when we consider a general angle between the externally applied magnetic field and the tilt direction.

Angle dependence of the chiral magnetic effect. We define ϑ to be the angle between $\mathbf{q}$ and $\mathbf{t}$, such that $\mathbf{q}\cdot\mathbf{t} = qt\cos\vartheta \equiv qt_\parallel$. For arbitrary angles ϑ we cannot perform the necessary integrals analytically for all wave numbers q and frequencies ω. However, we can investigate the tilt dependence of the long-wavelength limit of the conductivity for arbitrary angles. To do so, we take the limit $q \to 0$ in the integrand of Eq. (2),



$\sigma^{\rm CME}(q,\omega)$	$2\mu_5 = -\Delta E$ (I)	$\Delta E = 0$ (II)
$\lim\limits_{q\to 0}\lim\limits_{\omega\to 0}$	0	$\frac{e^2\mu_5}{2\pi^2}$
$\lim\limits_{\omega\to 0}\lim\limits_{q\to 0}$	$[1-(1-t^2)l(t)]\frac{e^2\Delta E}{4\pi^2}$	$(1-t^2)l(t)\frac{e^2\mu_5}{2\pi^2}$

FIG. 4. Results for the long-wavelength limit of the chiral magnetic conductivity. We show the cases (I) where $2\mu_5 = -\Delta E$ and (II) where the Weyl node separation is zero, i.e., $\Delta E = 0$. The zero-tilt results are [28,29] (bottom, left) $e^2\Delta E/6\pi^2$, and (bottom, right) $e^2\mu_5/6\pi^2$. The response in the second row is sometimes referred to as gyrotropy.

while keeping $\omega/v_F q$ fixed. Keeping in mind that we are not considering a possible topological contribution to the anomalous Hall effect, we find in general the interesting relation $\sigma^{\rm AHE}(\mathbf{q},\omega) = \sigma^{\rm CME}(\mathbf{q},\omega)/v_F(1-t^2)$ with

$$\sigma^{\rm CME}(\mathbf{q},\omega) = \frac{e^2\mu_5}{2\pi^2}\mathcal{S}^{\rm CME}\left(\frac{\omega^+}{v_F q} - t_\parallel;\mathbf{t}\right). \tag{13}$$

The dimensionless function $\mathcal{S}^{\rm CME}(x;\mathbf{t})$ is given by

$$\mathcal{S}^{\rm CME}(x;\mathbf{t}) = \frac{(1-t^2)(1-x^2)}{N^2(x,\mathbf{t})}\left[1 + \frac{(x+t_\parallel)H(x;\mathbf{t})}{2N(x;\mathbf{t})}\right], \tag{14}$$

in terms of $N(x;\mathbf{t}) \equiv \sqrt{(1-t^2)(1-x^2)+(x+t_\parallel)^2}$, $H(x;\mathbf{t}) \equiv \sum_{\sigma=\pm}\sigma\log[(x-\sigma)M_\sigma(x;\mathbf{t})]$, and finally also $M_\sigma(x;\mathbf{t}) \equiv [1+N(x;\mathbf{t})+t_\parallel x](1-\sigma t_\parallel) - (t^2-t_\parallel^2)(1+\sigma x)$ [38]. In Fig. 3 we show the resulting angular dependence of the conductivities. In the static limit we have that the chiral magnetic conductivity is equal to $e^2\mu_5/2\pi^2$ and is independent of the tilt [19]. In the homogeneous limit we find the result $(1-t^2)l(t)e^2\mu_5/2\pi^2$ for all angles [45]. Again, we need to add an appropriate renormalization constant for $\Delta E \neq 0$ such that the chiral magnetic current is zero in equilibrium. Using this subtraction procedure, which in particle physics amounts to adding a Bardeen counterterm [44], we find a general answer that depends on μ_5 and ΔE and modifies the results in the homogeneous limit. We displayed the final results for the special cases of equilibrium and zero energy separation between the cones in Fig. 4.

Frequency dependence of the anomalous Hall effect. As advertised, the tilt induces another interesting effect, namely, a planar intrinsic anomalous Hall effect with a current given by Eq. (6) that is perpendicular to both the external electric field and the tilting direction. Apart from the long-wavelength limit following from Eq. (7), we are also able to obtain the full frequency dependence of the homogeneous anomalous Hall conductivity as [38]

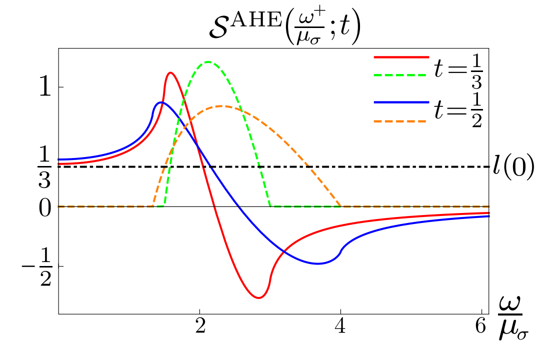


FIG. 5. Plot of the real (solid lines) and imaginary (dashed lines) parts of the function $\mathcal{S}^{\rm AHE}(\omega^+/\mu_\sigma;t)$ for $t = 1/3$ (red, green), and $t = 1/2$ (blue, orange) [35]. The black dotted-dashed line indicates the tilt-independent limit.

$$\sigma^{\rm AHE}(\mathbf{0},\omega) = \frac{e^2}{4\pi^2 v_F}\sum_{\sigma=\pm}\sigma\mu_\sigma\mathcal{S}^{\rm AHE}\left(\frac{\omega^+}{\mu_\sigma};t\right), \tag{15}$$

in terms of the dimensionless function

$$\mathcal{S}^{\rm AHE}(y;t) = -\frac{1}{2t^2} + \sum_{\sigma,\sigma'}\sigma' L_\sigma(y;t)H_{\sigma\sigma'}(0,1/y;t), \tag{16}$$

where $L_\sigma(y;t) \equiv [y^2t^2 - (2-\sigma y)^2]/16yt^3$ and again we encounter the functions $H_{\sigma\sigma'}(x,y;t)$ from Eq. (10). This result was recently obtained in a different way both analytically [35,36] and numerically [37]. In the zero-frequency limit the conductivity reduces to $\sigma^{\rm AHE}(\mathbf{0},0) = l(t)e^2\mu_5/2\pi^2 v_F$, which corresponds exactly to the slope of the linear divergence in Eq. (11). We plot the dependence of the real and imaginary part of $\mathcal{S}^{\rm AHE}(\omega^+/\mu_\sigma;t)$ on ω/μ_σ in Fig. 5 for several magnitudes of the tilt. Again, we observe a resonance behavior around $\omega = 2\mu_\sigma$, similar to the one in Fig. 2, because the current response in that figure is dominated by the AHE at large frequencies.

Discussion. We have shown that the electric and magnetic response of a pair of tilted Weyl cones is in general nonuniversal and depends on the magnitude of the tilt and on the angle between the tilt direction and the wave vector of the magnetic field. However, the chiral anomaly is due to the lowest Landau level, which only obtains a change of slope due to a tilting of the cones [2]. Hence, we expect the chiral anomaly to be unmodified and thus isotropic. Using the relation between the current-current correlation function and the triangle diagram in the static (adiabatic) limit, we find for the time derivative of the chiral number density $n_5 \equiv n_+ - n_-$,

$$\frac{dn_5}{dt} = \lim_{\omega\to 0}\frac{e^2}{2\pi^2}\mathcal{S}^{\rm CME}\left(\frac{\omega^+}{v_F q} - t_\parallel;\mathbf{t}\right)\mathbf{E}\cdot\mathbf{B}. \tag{17}$$

In the static limit we have $\mathcal{S}^{\rm CME}(\omega^+/v_F q - t_\parallel;\mathbf{t}) \to 1$, such that we indeed find an unmodified chiral anomaly.

Additionally, we showed that the chiral magnetic effect is closely related to an in-plane tilt-induced anomalous Hall effect, for which we calculated the dynamical conductivity.

We have also performed all these calculations in the case that inversion symmetry is not broken, corresponding to tilting the Weyl cones in opposite directions. An important consequence is that in the long-wavelength limit the anomalous Hall effect becomes proportional to 2μ, instead of $2\mu_5$, i.e., $\sigma^{\mathrm{AHE}}(\mathbf{0},0) = l(t)e^2\mu/2\pi^2 v_F$. The chiral magnetic effect, however, remains proportional to $2\mu_5$ due to Bloch's theorem [46].

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