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Quasiparticle relaxation in superconducting nanostructures

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We examine energy relaxation of nonequilibrium quasiparticles in "dirty" superconductors with the electron mean free path much shorter than the superconducting coherence length. Relaxation of low-energy nonequilibrium quasiparticles is dominated by phonon emission. We derive the corresponding collision integral and find the quasiparticle relaxation rate. The latter is sensitive to the breaking of time reversal symmetry (TRS) by a magnetic field (or magnetic impurities). As a concrete application of the developed theory, we address quasiparticle trapping by a vortex and a current-biased constriction. We show that trapping of hot quasiparticles may predominantly occur at distances from the vortex core, or the constriction, significantly exceeding the superconducting coherence length.

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I. INTRODUCTION

Current interest in the dynamics of Bogoliubov quasiparticles in superconductors is motivated in no small part by the efforts aimed at building a quantum computer. The actively explored "cavity QED" architecture relies on quantum coherence of qubits built of conventional superconductors [1]. Realization of the topological quantum computing requires coherence of devices made of proximitized semiconductor quantum wires brought into the p-wave superconducting state by applied magnetic field [2]. In any of the concepts, the presence of quasiparticles is detrimental to the coherence. The Q factors of the parts comprising the cavity-QED qubit are reduced by quasiparticles. They also are able to "poison" the Majorana states, which are central for the topological quantum computing.

The development of the qubit technology has advanced also the ability to monitor the quasiparticles population and dynamics. Time-resolved measurements performed with the transmon [3] and fluxonium qubits [4,5] allowed the experimentalists to measure the rates of quasiparticle trapping by a single vortex in a superconducting strip, to identify minute dissipative currents of quasiparticles across a Josephson junction (thus resolving a longstanding "$\cos\varphi$ problem" [6]), and to monitor the spontaneous temporal variations of the quasiparticle density.

Measurements [4,5] did confirm that at low temperatures (less than $\sim 0.1 T_c$ of Al) the quasiparticle density, albeit low, far exceeds the equilibrium value. Furthermore, statistics of temporal variations of the density substantially differs from thermal noise. Sources of excess quasiparticles remain unknown, and planting quasiparticle traps [7–9] remains a viable way of improving the device performance. Trap is a spatial region with a suppressed value of superconducting gap. Suppression may be achieved, e.g., through the proximity effect, or through local violation of time-reversal invariance (as it naturally happens in and around the core of a vortex). Energy loss in the trap (mostly due to phonon emission) prevents a quasiparticle from exiting into the region with the nominal gap value.

The importance of the quasiparticle energy relaxation in device applications, and the newly acquired ability of precise measurements [3–5] of the quasiparticles dynamics, prompts us to revisit the kinetic theory of quasiparticles interacting with phonons in a disordered superconductor. We derive the corresponding collision integrals and relaxation rates which then may be used in sophisticated phenomenological models of quasiparticle diffusion and trapping [9,10].

In considering the electron-phonon interaction in disordered metals, we follow the seminal works of Tsuneto [11] and Schmid [12], who established the correct form of the electron-phonon interaction in the limit of short electron mean free path, $ql \ll 1$ (here $\mathbf{q}$ is the phonon wave vector). We incorporate the electron-phonon interaction in the general framework of the Keldysh nonlinear sigma model. It allows us to consider on equal footing normal metals and superconductors, and it becomes especially convenient for describing the effect of breaking the time-reversal symmetry (TRS).

The kernel of the collision integral for electrons in normal metal which we find in the unified technique agrees with the earlier results [13,14] obtained diagrammatically; this kernel depends only on the energy transferred in the collision to a phonon. Considering the Bogoliubov quasiparticles, we are able to cast the result for the collision integral in the conventional terms of the quasiparticle energy distribution functions. In the presence of TRS, the corresponding kernel factorizes on two terms: the normal-state kernel and a combination of the Bogoliubov transformation parameters. Factorization takes place also if TRS is broken; in that case, the second factor is determined by the proper solution of the Usadel equation. In either of the two cases, the second factor depends separately on the initial and final energy of a quasiparticle.

The additional (compared to the normal state) energy dependence of the kernel affects the dependence of the quasiparticle relaxation rate on its energy. These rates, in turn, determine the effectiveness of trapping. As an example of application of the developed theory, we consider trapping of a quasiparticle by an isolated vortex and a current-biased constriction. In both cases there is a pattern of supercurrents, slowly decaying as a function of distance, $\sim r^{-1}$, from the vortex core, or $\sim r^{1-d}$ from a constriction with d-dimensional superconducting leads.

These supercurrents lead to a weak breaking of TRS and thus suppression of the energy gap and modification of the energy dependence of the density of states (DOS). Such a suppression allows quasiparticles to be trapped already very far from the vortex core or the constriction. For low enough phonon temperature and relatively "hot" quasiparticles this peripheral shallow trapping proves to be more efficient than the deep trapping by the core of the vortex, or the constriction. We discuss possible relation of the theory to experiments [3,15].

The paper is organized as follows: In Sec. II we review the theory of electron-phonon interactions in disordered normal metals. We derive the corresponding Keldysh nonlinear sigma model and use it to obtain the electron-phonon collision integral in the dirty limit. In Sec. III we generalize the sigma model on superconductors, including those with broken TRS, and in Sec. IV we derive the kinetic equation for the quasiparticle distribution. Section V is devoted to applications of the theory to trapping by vortex and current-biased constriction as well as discussion of the existing experiments. We summarize with a brief discussion in Sec. VI. Two appendices present an alternative derivation of the sigma model and summarize results for ultrasound attenuation.

II. ELECTRON-PHONON INTERACTIONS IN DISORDERED NORMAL METALS

A. Interaction vertex

The theory of electron-phonon interactions in normal disordered metals has had a long and, at times, controversial history. Early considerations were based on the Fröhlich Hamiltonian [16], which assumes screened Coulomb interactions between electron density and induced lattice charge, $e\rho_0 \,\mathrm{div}\, \mathbf{u}$, created by phonon displacement $\mathbf{u}(\mathbf{r},t)$. Here $e\rho_0$ is the uniform lattice charge density. Due to global neutrality it is exactly equal to the electron density:

$$\rho_0 = \int^{p_F} \frac{d^d\mathbf{p}}{(2\pi)^d} = \int_0^{\epsilon_F} d\epsilon\, \nu(\epsilon) = \frac{v_F p_F \nu}{d}, \tag{1}$$

where normal metal DOS is $\nu(\epsilon) = (\epsilon/\epsilon_F)^{d/2-1}\nu$ and $\nu = \nu(\epsilon_F)$. While perfectly legitimate in the clean case, the Fröhlich Hamiltonian misses an important piece of the physics in the "dirty" limit $ql \ll 1$, where q is phonon wave number and l is electron elastic mean free path.

As was first realized by Pippard [17], phonons not only deform the lattice but also displace impurities, transforming formerly static impurity potential $U_{\mathrm{imp}}(\mathbf{r})$ into the dynamic one, $U_{\mathrm{imp}}(\mathbf{r}) \to U_{\mathrm{imp}}(\mathbf{r}+\mathbf{u}(\mathbf{r},t))$. Colloquially, this leads to the electron density being dragged along with the lattice displacement and providing a perfect compensation for the induced lattice charge $e\rho_0 \,\mathrm{div}\, \mathbf{u}$. In other words, the displaced impurity potential provides fast elastic relaxation of the electron distribution around the Fermi surface locally deformed by phonons. These ideas were put on a quantitative basis by Tsuneto [11] and Schmid [12], who showed that in the limit $ql \ll 1$ the Fröhlich Hamiltonian should be substituted by another effective electron-phonon interaction vertex:

$$iS_{e:ph} = \int dt \sum_{\mathbf{p},\mathbf{q}} \bar{\psi}\left(\mathbf{p}+\frac{\mathbf{q}}{2},t\right)\Gamma_{\mu\nu}(\mathbf{p})i\mathbf{q}^{\mu}\mathbf{u}^{\nu}_{\mathbf{q},t}\psi\left(\mathbf{p}-\frac{\mathbf{q}}{2},t\right), \tag{2}$$

where $\bar{\psi}$ and ψ are electrons creation and annihilation operators and $\Gamma_{\mu\nu}(\mathbf{p})$ is the traceless tensor

$$\Gamma_{\mu\nu}(\mathbf{p}) = \mathbf{p}_\mu \mathbf{v}_\nu - \frac{p_F v_F}{d}\,\delta_{\mu\nu}. \tag{3}$$

Notice that, in view of Eq. (1), the last term here represents the Fröhlich coupling $-\nu^{-1}(\rho_0 \mathrm{div}\,\mathbf{u})(\bar{\psi}\psi)$. Upon averaging over the Fermi surface it is exactly compensated by the first term in Eq. (3), i.e., $\int d\Omega_{\mathbf{p}}\Gamma_{\mu\nu}(\mathbf{p}) = 0$. This property is a result of the perfect screening of the bare Coulomb interactions. The latter is a good approximation as long as phonon frequencies are much smaller than electronic plasma frequency. The remaining coupling is of a quadrupole nature, as seen from Eqs. (2) and (3). This leads to a significantly weaker electron-phonon coupling than the one inferred from the Fröhlich term [18]. A number of subsequent studies [13,14,19] reaffirmed validity of the Schmid coupling (2), (3) from various perspectives.

The most straightforward way to derive Eqs. (2), (3) [11,20] is by performing a unitary transformation, which yields a Hamiltonian in the co-moving reference frame, where the impurity potential is static. We shall not repeat this derivation here. Instead, we accept Eqs. (2), (3) as a starting point and derive an effective nonlinear sigma model which incorporates electron-phonon interaction in the Schmid form. In Appendix A we provide an alternative derivation of the sigma model, which proceeds in the laboratory reference frame and deals with a dynamic random potential $U_{\mathrm{imp}}(\mathbf{r}+\mathbf{u}(\mathbf{r},t))$ and strong Coulomb interactions between electron density and induced lattice charges. We show that it brings the same effective sigma model, justifying the use of the effective electron-phonon vertex in the Schmid form (2), (3).

B. Nonlinear sigma model

We now perform the standard [21,22] averaging over the static disorder and introduce the nonlocal field $Q_{t,t'}(\mathbf{r})$ to split the emerging four-fermion term. The resulting action, including electron-phonon coupling, Eqs. (2), (3), is now quadratic in the fermionic fields which may be integrated out in the usual way, leading to

$$iS = -\frac{\pi\nu}{4\tau}\,\mathrm{Tr}\{Q^2\} + \mathrm{Tr}\log\left\{G_0^{-1} + \frac{i}{2\tau}Q + \Gamma_{\mu\nu}\partial^{\mu}\mathbf{u}^{\nu}\right\}, \tag{4}$$

where the inverse *bare* electron Green function is given by $G_0^{-1} = i\partial_t + \nabla^2/2m + \mu \approx i\partial_t + i\mathbf{v}_\mu\partial^\mu$.

From this point on, one proceeds along the standard root of deriving Keldysh nonlinear sigma model [21,22]. To this end one passes to the Keldysh 2×2 structure, by splitting the contour on forward and backward branches and performing Keldysh rotation. Upon this procedure the fields acquire the matrix structure, e.g., $\mathbf{u} \to \hat{\mathbf{u}} = \mathbf{u}^{\alpha}\hat{\gamma}^{\alpha}$, where $\alpha = cl,q$ denotes classical and quantum Keldysh components and $\hat{\gamma}^{cl} = \hat{\sigma}^0, \hat{\gamma}^q = \hat{\sigma}^1$ are the two vertex matrices in the Keldysh space.

One then realizes that the soft diffusive modes of the action are described by the manifold $\hat{Q}^2 = 1$ and therefore one can write $\hat{Q} = \hat{\mathcal{R}}^{-1}\hat{\Lambda}\hat{\mathcal{R}}$, where $\hat{\Lambda}$ is the Green function in coinciding spatial points,

$$\hat{\Lambda} = \frac{i}{\pi\nu}\sum_{\mathbf{p}} \hat{G}_0(\mathbf{p},\epsilon) = \begin{pmatrix} 1 & 2F_\epsilon \\ 0 & -1 \end{pmatrix}, \quad (5)$$

and F_ϵ is a distribution function. Rotation matrices, $\hat{\mathcal{R}}^{-1}$, belong to an appropriate symmetry group. One then introduces *dressed* Green function $\hat{G} = (\hat{G}_0^{-1} + \frac{i}{2\tau}\hat{\Lambda})^{-1}$ and rewrites the action (4) as

$$iS = \mathrm{Tr}\log\{1+\hat{G}\hat{\mathcal{R}}[\hat{G}_0^{-1},\hat{\mathcal{R}}^{-1}] + \hat{G}\,\hat{\mathcal{R}}\,\Gamma_{\mu\nu}\,\partial^\mu\hat{\mathbf{u}}^\nu\,\hat{\mathcal{R}}^{-1}\}. \quad (6)$$

Finally, one expands the logarithm here to the lowest nonvanishing orders. This way one obtains the standard nonlinear sigma-model action (first neglecting the electron-phonon Γ term):

$$iS_{\hat{Q}} = -\frac{\pi\nu}{4}\,\mathrm{Tr}\{D\,(\partial_{\mathbf{r}}\hat{Q})^2 - 4\partial_t\hat{Q}\}, \quad (7)$$

where $D = v_F^2\tau/d$ is the diffusion constant and d is the dimensionality of the electron system. We focus now on the phonon-induced term. It is easy to see that the first order in Γ term vanishes due to the fact that the integral over the Fermi surface $\int d\Omega_{\mathbf{p}}\Gamma_{\mu\nu}(\mathbf{p}) = 0$. It is this point, where the Schmid coupling, Eqs. (2), (3), is qualitatively different from the Frölich one (the latter would bring the first order $\mathrm{Tr}\{(\rho_0 \mathrm{div}\,\hat{\mathbf{u}})\hat{Q}\}$ term). Going to the second order in Γ, one finds:

$$iS_{\hat{Q},\mathbf{u}} = -\tfrac{1}{2}\,\mathrm{Tr}\{\hat{G}\,\hat{\mathcal{R}}\,\Gamma_{\mu\nu}\,\partial^\mu\hat{\mathbf{u}}^\nu\,\hat{\mathcal{R}}^{-1}\hat{G}\,\hat{\mathcal{R}}\,\Gamma_{\eta\lambda}\,\partial^\eta\hat{\mathbf{u}}^\lambda\,\hat{\mathcal{R}}^{-1}\}. \quad (8)$$

We use now

$$\hat{G}_{\mathbf{p}} = \tfrac{1}{2}\,G^R_{\mathbf{p}}\,(1+\hat{\Lambda}) + \tfrac{1}{2}\,G^A_{\mathbf{p}}\,(1-\hat{\Lambda}), \quad (9)$$

along with

$$\sum_{\mathbf{p}} \mathbf{p}_\mu \mathbf{v}_\nu G^R_{\mathbf{p}}\,\mathbf{p}_\eta \mathbf{v}_\lambda G^A_{\mathbf{p}} = \frac{2\pi\nu\tau p_F^2 v_F^2}{d(d+2)}(\delta_{\mu\nu}\delta_{\eta\lambda} + \delta_{\mu\eta}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\eta}) \quad (10)$$

to find

$$iS_{\hat{Q},\mathbf{u}} = \frac{\pi\nu D\,p_F^2}{4}\,\mathrm{Tr}\{[\hat{Q}\,,\partial^\mu\hat{\mathbf{u}}^\nu][\hat{Q}\,,\partial^\eta\hat{\mathbf{u}}^\lambda]\}\,\Upsilon_{\mu\nu,\eta\lambda}, \quad (11)$$

where

$$\Upsilon_{\mu\nu,\eta\lambda} = \frac{1}{d+2}\left[\delta_{\mu\eta}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\eta} - \frac{2}{d}\,\delta_{\mu\nu}\delta_{\eta\lambda}\right]. \quad (12)$$

The local vertex (11) is the leading term describing interaction of phonons with the electronic degrees of freedom in disordered metals, in the $ql \ll 1$ limit. The naive deformation potential term $S \propto \rho_0\mathrm{Tr}\{\hat{Q}\,\mathrm{div}\,\hat{\mathbf{u}}\}$ is absent due to the perfect screening manifested in the traceless form of the electron-phonon vertex (3). See also Appendix A for more discussion of this issue. The second order cross term between the two terms in the logarithm in Eq. (6) leads to $S \propto \rho_0\tau D\mathrm{Tr}\{\nabla^2\hat{Q}\,\mathrm{div}\,\hat{\mathbf{u}}\}$. It is of the order $(ql) \ll 1$ of the leading term (11) and thus should not be kept within the accuracy of the adopted approximations.

The effective electron-phonon sigma model, Eqs. (7) and (11), should be supplemented with the standard phonon action. In the Keldysh technique it is given by

$$iS_{\mathbf{u}} = i\,\frac{\rho_m}{2}\sum_{\mathbf{q},\omega,j}\bar{\mathbf{u}}^{\mu,\alpha}_{\mathbf{q},\omega}\big[\omega^2 - \big(\omega^{(j)}_{\mathbf{q}}\big)^2\big]\hat{\sigma}^1_{\alpha\beta}\eta^{(j)}_{\mu\nu}(\mathbf{q})\,\mathbf{u}^{\nu,\beta}_{\mathbf{q},\omega}, \quad (13)$$

where ρ_m is the material mass density, $j = l,t$ labels longitudinal and transversal polarizations encoded by the projectors

$$\eta^{(l)}_{\mu\nu}(\mathbf{q}) = \frac{q_\mu q_\nu}{q^2}; \quad \eta^{(t)}_{\mu\nu}(\mathbf{q}) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad (14)$$

and $\omega^{(j)}_{\mathbf{q}} = v_j q$ is the acoustic phonons dispersion with the speed of sound $v_{l,t}$. Indexes $\alpha,\beta = cl,q$ and Pauli matrix $\hat{\sigma}^1$ act in the 2×2 Keldysh space. We will need an imaginary part of the corresponding retarded propagator:

$$\begin{aligned}\mathrm{Im}\,U^R_{\nu\mu}(\mathbf{q},\omega) &= \mathrm{Re}\big\langle \mathbf{u}^{\nu,cl}_{\mathbf{q},\omega}\bar{\mathbf{u}}^{\mu,q}_{\mathbf{q},\omega}\big\rangle \\ &= \sum_j \frac{\eta^{(j)}_{\nu\mu}(\mathbf{q})}{\rho_m}\,\frac{\pi}{2\omega^{(j)}_{\mathbf{q}}} \\ &\quad\times\big[\delta\big(\omega - \omega^{(j)}_{\mathbf{q}}\big) - \delta\big(\omega + \omega^{(j)}_{\mathbf{q}}\big)\big]. \quad (15)\end{aligned}$$

The corresponding Keldysh component is given by the fluctuation-dissipation relation: $U^K_{\nu\mu} = \mathcal{B}_\omega(U^R_{\nu\mu} - U^A_{\nu\mu})$, where $\mathcal{B}_\omega = \coth(\omega/2T)$ is the bosonic distribution function.

The effective action, Eqs. (7), (11), and (13) with the vertices defined in Eqs. (12) and (14) serves as the starting point for investigating the kinetics of electrons and phonons. We relegate the evaluation of the ultrasonic attenuation to Appendix B, where we reaffirm the known results [12,19,20,23] obtained by different techniques and proceed to study the electron kinetics.

C. Electron-phonon collision integral

To derive the collision integral for electron-phonon interactions one first integrates over the phonon displacements $\mathbf{u}(\mathbf{r},t)$ with the help of Eq. (15) to obtain the collision action from Eq. (11):

$$S_{\mathrm{coll}} = \frac{\pi\nu D p_F^2}{4}\,\mathrm{Tr}\{\hat{Q}_{\epsilon-\omega,\epsilon'-\omega}\hat{\gamma}^\alpha\,\hat{Q}_{\epsilon',\epsilon}\hat{\gamma}^\beta\}\,U^{\alpha\beta}_{\nu\lambda}\,\mathbf{q}^\mu\mathbf{q}^\eta\,\Upsilon_{\mu\nu,\eta\lambda}, \quad (16)$$

where $U^{\alpha\beta}_{\nu\lambda} = U^{\alpha\beta}_{\nu\lambda}(\mathbf{q},\omega)$ and summation over $\epsilon,\epsilon',\omega,\mathbf{q}$ are understood. One now looks for the stationary point equation for the action $S_Q + S_{\mathrm{coll}}$. Its Keldysh (1,2) component constitutes the kinetic equation [21,22] for the distribution function F_ϵ in Eq. (5),

$$\partial_t F_\epsilon - \nabla_{\mathbf{r}}[D\nabla_{\mathbf{r}}F_\epsilon] = -2I_{\mathrm{coll}}[F_\epsilon(\mathbf{r},t)], \quad (17)$$

where the collision integral is given by

$$I_{\mathrm{coll}}[F_\epsilon(\mathbf{r},t)] = -\frac{1}{2\pi\nu}\left\langle\left(\frac{\delta i S_{\mathrm{coll}}}{\delta\hat{Q}_{\epsilon\epsilon}(\mathbf{r})}\right)^{(1,2)}\right\rangle_{\hat{Q}}. \quad (18)$$

The variational derivative here ought to be restricted to the sigma-model target space, $\hat{Q}^2 = 1$. A way to ensure this is to use parametrization $\hat{Q} \to e^{-\hat{W}/2}\hat{Q}e^{\hat{W}/2} \approx \hat{Q} + \frac{1}{2}[\hat{Q},\hat{W}]$ and expand the action (16) to the linear order in $\hat{W}_{\epsilon\epsilon}$. Here $\hat{W}$'s are infinitesimal generators of the symmetry transformations. Because of the local nature of the vertex in Eq. (16), the $\hat{Q}$

integration may be substituted by the stationary point: $\hat{Q} \to \hat{\Lambda}$. This way one obtains:

$$I_{\rm coll}[F_\epsilon] = i\frac{Dp_F^2}{8}\sum_{\mathbf{q},\epsilon'}[\hat{\gamma}^\beta\hat{\Lambda}_{\epsilon'}\hat{\gamma}^\alpha\hat{\Lambda}_\epsilon - \hat{\Lambda}_\epsilon\hat{\gamma}^\beta\hat{\Lambda}_{\epsilon'}\hat{\gamma}^\alpha]^{(1,2)} \times U^{\alpha\beta}_{\nu\lambda}(\mathbf{q},\epsilon-\epsilon')\mathbf{q}^\mu\mathbf{q}^\eta\Upsilon_{\mu\nu,\eta\lambda} = \frac{1}{4}\sum_{\epsilon'}M_{\epsilon,\epsilon'}\mathcal{I}[F];$$

$$\mathcal{I}[F] = -1 + F_\epsilon F_{\epsilon'} + \mathcal{B}_{\epsilon-\epsilon'}[F_\epsilon - F_{\epsilon'}]. \tag{19}$$

The phonon matrix element in the collision integral (19) in normal metal is found to depend only on the energy difference, $M_{\epsilon,\epsilon'} = M^N_{\epsilon-\epsilon'}$, where

$$M^N_\omega = 2Dp_F^2\sum_{\mathbf{q}}\text{Im}\big[U^R_{\nu\lambda}(\mathbf{q},\omega)\big]\mathbf{q}^\mu\mathbf{q}^\eta\Upsilon_{\mu\nu,\eta\lambda} = M^{(l)}_\omega + M^{(t)}_\omega, \tag{20}$$

and the superscripts $j = l,t$ stand for longitudinal and transverse modes, respectively. Expressing the fermion and boson distributions in terms of the respective occupation numbers, $F_\epsilon = 1 - 2f_\epsilon$ and $\mathcal{B}_\omega = 1 + 2N_\omega$, we may bring $I_{\rm coll}[F_\epsilon]$ to the standard "in minus out" form,

$$I_{\rm coll}[f_\epsilon] = \int\frac{d\epsilon'}{2\pi}M^N_\omega[N_\omega f_{\epsilon'}(1-f_\epsilon) - (1+N_\omega)f_\epsilon(1-f_{\epsilon'})], \tag{21}$$

where $\omega = \epsilon - \epsilon'$ and $I_{\rm coll}[f_\epsilon] = 0$ in equilibrium.

Employing Eqs. (12), (15), and (20) one finds:

$$M^{(j)}_\omega = \frac{b_j\pi Dp_F^2}{\rho_m}\sum_{\mathbf{q}}\frac{q^2}{\omega^{(j)}_{\mathbf{q}}}\delta\big(\omega - \omega^{(j)}_{\mathbf{q}}\big) = \frac{b_j\pi\Omega_d}{(2\pi)^d}\frac{Dp_F^2\omega^d}{\rho_m v_j^{d+2}}, \tag{22}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of S_{d-1} unit sphere and d is the effective phonon dimensionality. The coefficients are given $b_t = (d-1)/(d+2)$; $b_l = 2b_t/d$ (in these instances d is the dimensionality of the electron system).

In fact, Eqs. (20), (22) should include a dressed phonon propagator rather than the bare one. Such dressing leads to: (i) a renormalization of sound velocities, (ii) a finite lifetime of the phonon modes given by the ultrasound attenuation coefficient, reviewed in Appendix B. The former effect is accommodated by using the correct (i.e., renormalized) values of v_l and v_t in the phonon propagator. The latter leads to the broadening of the delta function in Eq. (22) into the Lorentzian of the width $\gamma_{\mathbf{q}}$ (see Appendix B), which has a negligible effect as long as $\gamma_{\mathbf{q}} \ll \omega_{\mathbf{q}}$.

These results for the normal metal were derived in Refs. [12–14] using diagrammatic techniques. Here we reproduced them through the sigma-model technique, which is much more suitable for treating the superconducting case, considered below. We notice that $M^{(l)}_\omega$ of Eq. (22) is factor $(v_l/v_F)^2 \ll 1$ smaller than that of Ref. [18] for $\omega < v_l^2/D$. The latter was obtained with the Fröhlich coupling (i.e., disregarding impurities shifting with the lattice deformations). The two approaches give comparable results for the longitudinal phonons at $\omega \approx v_l/l$ where they both match with the clean limit expectation $M^{(l)}_\omega \propto v_F p_F^2\omega^{d-1}/(\rho_m v_l^{d+1})$. The transversal phonons give the dominant contribution to the collision integral in the disordered limit $\omega < v_t/l$, since typically $v_t < v_l$. However, in the opposite clean limit, the transversal matrix element is $M^{(t)}_\omega \propto Dp_F^2\omega/(\rho_m v_t^3 l^2)$ [14] and is less important than the longitudinal one.

For comparison, the electron-electron collision integral may be written in the form of Eq. (21), with the bosonic occupation number $N_\omega = \omega^{-1}\int d\epsilon'' f_{\epsilon''}(1 - f_{\epsilon''-\omega})$ and a different matrix element given by: $M^N_\omega \to M^{e:e}_\omega \propto \omega^{d/2-1}D^{-d/2}/\nu$ [24]. As a result the ratio of electron-electron and phonon matrix elements in normal metals is

$$\frac{M^{e:e}_\omega}{M^N_\omega} \propto \frac{M}{m}\left(\frac{v_j^2}{D\omega}\right)^{d/2+1} \propto \left(\frac{m}{M}\right)^{d/2}\left(\frac{1}{\omega\tau}\right)^{d/2+1}, \tag{23}$$

where M is the ion mass and we used that $\rho_m \propto M\nu p_F^2/m$ and $v_j^2 \propto v_F^2 m/M$. Therefore electron-electron relaxation in normal metals dominates for the energy transfer $\omega < \tau^{-1}(m/M)^{d/(d+2)}$.

III. DISORDERED SUPERCONDUCTORS

A. Sigma model

The nonlinear sigma model is readily extended to disordered superconductors [22,25]. It is written in terms of the local pair correlation function $\check{Q}_{t,t'}(\mathbf{r}) \propto \langle\Psi(\mathbf{r},t)\Psi^\dagger(\mathbf{r},t')\rangle$, where $\Psi(\mathbf{r},t)$ is the four component spinor in the Nambu and Keldysh subspaces. As a result $\check{Q}_{t,t'}(\mathbf{r})$ is a 4×4 matrix, as well as the matrix in the time t,t' space. It satisfies the nonlinear condition $\check{Q}^2 = 1$. Its dynamics is governed by the action:

$$iS_{\check{Q}:\check{\Delta}} = -\frac{\pi\nu}{8}\text{Tr}\{D(\partial_{\mathbf{r}}\check{Q})^2 - 4\check{\mathcal{T}}_3\partial_t\check{Q} + 4i\check{\Delta}\check{Q}\}, \tag{24}$$

where $\check{\Delta}(\mathbf{r},t) = \Delta(\mathbf{r},t)\gamma^{cl}\otimes\hat{\tau}^+ - \overline{\Delta}(\mathbf{r},t)\gamma^{cl}\otimes\hat{\tau}^-$ is the order parameter matrix. To discuss broken TRS later on, we have also included a vector potential through the long derivative:

$$\partial_{\mathbf{r}}\check{Q} = \nabla_{\mathbf{r}}\check{Q} + i[\mathbf{A}\check{\mathcal{T}}_3,\check{Q}]. \tag{25}$$

Hereafter $\hat{\tau}^{0,1,2,3}$ are Pauli matrices in the Nambu space and $\check{\mathcal{T}}_3 = \hat{\gamma}^{cl}\otimes\hat{\tau}^3$. Here the operation Tr involves trace in 4×4 Nambu-Keldysh space, as well as trace in time (or equivalently energy) space and the spatial integration.

The electron-phonon interactions are given by Eq. (11) (with factor 1/2 to compensate for the Nambu doubling of the degrees of freedom), where the displacement field $\check{\mathbf{u}}$ is proportional to $\hat{\tau}^0$ matrix in the Nambu space. The corresponding collision action, obtained by integrating out the phonon degrees of freedom, is given by Eq. (16) (again with factor 1/2). Its variation over $\check{Q}$ leads to the collision integral in the form of Eq. (19). The major difference of the superconducting case is that the $\check{\Lambda}_\epsilon$ matrices in Eq. (19) are rotated in Nambu space, as explained below.

Taking variation of the effective action (24), (11) with respect to the $\check{Q}$ as explained after Eq. (18), one obtains the saddle point Usadel equation [22,26]

$$\{\check{\mathcal{T}}_3\partial_t,\check{Q}\}_+ - \hat{\partial}_{\mathbf{r}}(D\check{Q}\,\hat{\partial}_{\mathbf{r}}\check{Q}) - i[\check{\Delta},\check{Q}] = \frac{1}{\pi\nu}\frac{\delta S_{\rm coll}}{\delta\check{Q}}. \tag{26}$$

We look for a solution of this equation $\check{Q} = \check{\Lambda}$ in the standard form respecting causality:

$$\check{\Lambda} = \begin{pmatrix} \hat{\Lambda}^R & \hat{\Lambda}^K \\ 0 & \hat{\Lambda}^A \end{pmatrix}_K, \quad (27)$$

with retarded, advanced, and Keldysh components being matrices in the Nambu subspace. The nonlinear constraint $\check{\Lambda}^2 = 1$ is resolved as

$$\hat{\Lambda}^R\hat{\Lambda}^R = \hat{\Lambda}^A\hat{\Lambda}^A = \hat{1}, \quad \hat{\Lambda}^K = \hat{\Lambda}^R\hat{F} - \hat{F}\hat{\Lambda}^A, \quad (28)$$

where $\hat{F}$ is a distribution matrix in the Nambu space, which may be written as [27] $\hat{F} = F^L_{\epsilon\epsilon'}(\mathbf{r})\hat{\tau}^0 + F^T_{\epsilon\epsilon'}(\mathbf{r})\hat{\tau}^3$. Here $F^{L,T}$ are longitudinal (odd with respect to energy permutation) and transverse (even in energy permutation) components of the quasiparticle distribution function. These two are responsible for the transport of energy and charge correspondingly. [The conventional distribution functions $F^{L,T}_\epsilon(\mathbf{r},t)$) are obtained by Wigner transformation with $(\epsilon + \epsilon')/2 \to \epsilon$ and $\epsilon - \epsilon' \to t$.] Since the transversal component usually decays fast to zero, we shall primarily focus only on the long-lived longitudinal component of the nonequilibrium quasiparticle distribution and often omit the superscript for brevity $F^L_\epsilon(\mathbf{r},t) = F_\epsilon(\mathbf{r},t)$. In thermal equilibrium $F^L_\epsilon = \tanh \epsilon/2T$, while $F^T_\epsilon = 0$.

The nonlinear constraints $(\hat{\Lambda}^{R(A)})^2 = \hat{1}$, Eq. (28), may be explicitly resolved in the Nambu space by the angular parametrization [22,28]:

$$\hat{\Lambda}^R(\mathbf{r},\epsilon) = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta\; e^{i\chi} \\ -\sinh\vartheta\; e^{-i\chi} & -\cosh\vartheta \end{pmatrix}_N = \hat{V}^{-1}\hat{\tau}^3\hat{V};$$

$$\hat{\Lambda}^A(\mathbf{r},\epsilon) = \begin{pmatrix} -\cosh\bar{\vartheta} & -\sinh\bar{\vartheta}\; e^{i\bar{\chi}} \\ \sinh\bar{\vartheta}\; e^{-i\bar{\chi}} & \cosh\bar{\vartheta} \end{pmatrix}_N = -\hat{\bar{V}}^{-1}\hat{\tau}^3\hat{\bar{V}}, \quad (29)$$

where $\vartheta(\mathbf{r},\epsilon)$ and $\chi(\mathbf{r},\epsilon)$ are *complex*, coordinate-, and energy-dependent angles. Here

$$\hat{V}_\epsilon(\mathbf{r}) = e^{\frac{\vartheta}{2}\hat{\tau}^1} e^{-i\frac{\chi}{2}\hat{\tau}^3}; \quad \hat{\bar{V}}_\epsilon(\mathbf{r}) = e^{\frac{\bar{\vartheta}}{2}\hat{\tau}^1} e^{-i\frac{\bar{\chi}}{2}\hat{\tau}^3}. \quad (30)$$

Notice that in the presence of the phase χ the matrix $\hat{\bar{V}}_\epsilon$ is not a complex conjugate of $\hat{V}_\epsilon$. The full saddle point $\check{\Lambda}$ matrix (27), (28) then acquires the form

$$\check{\Lambda}(\mathbf{r},\epsilon) = \check{U}^{-1}_\epsilon(\mathbf{r})\,\hat{\tau}^3 \otimes \hat{\sigma}^3\,\check{U}_\epsilon(\mathbf{r}), \quad (31)$$

where $\hat{\sigma}^3$ is the Keldysh space matrix and

$$\check{U} = \begin{pmatrix} \hat{V} & \hat{V}\hat{F} \\ 0 & -\hat{\bar{V}} \end{pmatrix}_K; \quad \check{U}^{-1} = \begin{pmatrix} \hat{V}^{-1} & \hat{F}\hat{\bar{V}}^{-1} \\ 0 & -\hat{\bar{V}}^{-1} \end{pmatrix}_K. \quad (32)$$

The expectation value of the order parameter satisfies the self-consistency equation, obtained by variation of $S_{\check{Q},\check{\Delta}} - \frac{i\nu}{2\lambda}\mathrm{Tr}\{\check{\Delta}\hat{\sigma}^1 \otimes \hat{\tau}^0\check{\Delta}\}$ over the quantum component Δ^q. This leads to (we assume $F^T = 0$):

$$\Delta = \frac{\lambda}{4}\int_{-\omega_D}^{\omega_D} d\epsilon\, F^L_\epsilon\,[\sinh\vartheta + \sinh\bar{\vartheta}], \quad (33)$$

where λ is the BCS interaction constant and ω_D is the Debye frequency cutoff.

In the absence of the vector potential, i.e., with unbroken TRS, substituting Eqs. (29) into the retarded and advanced components of the Usadel equation (26) one finds for the Nambu angle:

$$\epsilon = \Delta\coth\vartheta = \Delta\coth\bar{\vartheta}. \quad (34)$$

For $|\epsilon| > \Delta$ one thus finds that $\vartheta(\epsilon)$ is real and

$$\cosh\vartheta = \frac{\epsilon}{\xi_\epsilon}; \quad \sinh\vartheta = \frac{\Delta}{\xi_\epsilon}; \quad \xi_\epsilon \equiv \mathrm{sgn}(\epsilon)\sqrt{\epsilon^2 - \Delta^2}. \quad (35)$$

Within the energy gap, $|\epsilon| < \Delta$, the angle is $\vartheta = -i\pi/2 + \theta$, with real θ. For all energies the following symmetry relation holds $\vartheta(-\epsilon) = -\bar{\vartheta}(\epsilon)$. The local DOS is expressed through the Nambu angle as

$$\nu(\epsilon) = \frac{\nu}{2}\,\mathrm{Re\,tr}\{\hat{\tau}^3\hat{\Lambda}^R\} = \nu\mathrm{Re}\cosh\vartheta(\epsilon) = \nu\,\frac{\epsilon}{\xi_\epsilon}\,\Theta(|\epsilon| - \Delta), \quad (36)$$

where Θ is the step function.

B. Superconductors with broken TRS

In many cases of the practical interest the vector potential (and hence the phase χ) changes slowly on the scale of the superconducting coherence length. In these cases one may disregard the gradient terms in the action (24) and write it as:

$$iS^{(0)}_{\check{Q},\check{\Delta}} = -\frac{\pi\nu}{8}\,\mathrm{Tr}\left\{-\frac{\gamma}{2}[\check{\mathcal{T}}_3,\check{Q}]^2 + 4i\epsilon\check{\mathcal{T}}_3\check{Q} + 4i\check{\Delta}\check{Q}\right\}, \quad (37)$$

where $\gamma = 2D\mathbf{A}^2$ is the energy scale associated with the local breaking of TRS. For a vortex $A = 1/(2r)$, where r is distance from the core, and thus $\gamma = \frac{1}{2}\Delta(\xi/r)^2$, where $\xi = \sqrt{D/\Delta}$ is the coherence length. For a thin film of width $d < \xi$ in a parallel magnetic field $H_\parallel$ one finds $\gamma = \frac{1}{6}D(H_\parallel d)^2$.

Taking retarded and advanced components of the Usadel equation (26) without gradients [or equivalently, substituting the saddle point ansatz (27)–(30) into the action (37) and taking variation over the complex Nambu angles ϑ, $\bar{\vartheta}$], one finds the saddle point condition:

$$\epsilon = \Delta\coth\vartheta - i\gamma\cosh\vartheta = \Delta\coth\bar{\vartheta} + i\gamma\cosh\bar{\vartheta}. \quad (38)$$

Its solution is depicted in Fig. 1 and admits an important symmetry relation:

$$\vartheta(-\epsilon) = -\bar{\vartheta}(\epsilon). \quad (39)$$

The local DOS is expressed through the Nambu angle as

$$\nu(\epsilon) = \frac{\nu}{4}\,\mathrm{tr}\{\hat{\tau}^3\hat{\Lambda}^R - \hat{\Lambda}^A\hat{\tau}^3\} = \frac{\nu}{2}[\cosh\vartheta(\epsilon) + \cosh\bar{\vartheta}(\epsilon)]; \quad (40)$$

it is shown in Fig. 2. Within the energy gap, $|\epsilon| < \epsilon_g$, DOS is zero, i.e., $\mathrm{Re}[\cosh\vartheta] = 0$ and thus the angle is $\vartheta = -i\pi/2 + \theta$, with real θ. This brings $\epsilon = \Delta\tanh\theta - \gamma\sinh\theta$. The right hand side of the latter condition reaches maximum at $\cosh\theta = (\Delta/\gamma)^{1/3}$. Substituting this back into Eq. (34) one finds for the energy gap [29]

$$\epsilon_g = (\Delta^{2/3} - \gamma^{2/3})^{3/2} \approx \Delta\left(1 - \frac{3}{2}\left(\frac{\gamma}{\Delta}\right)^{2/3}\right), \quad (41)$$

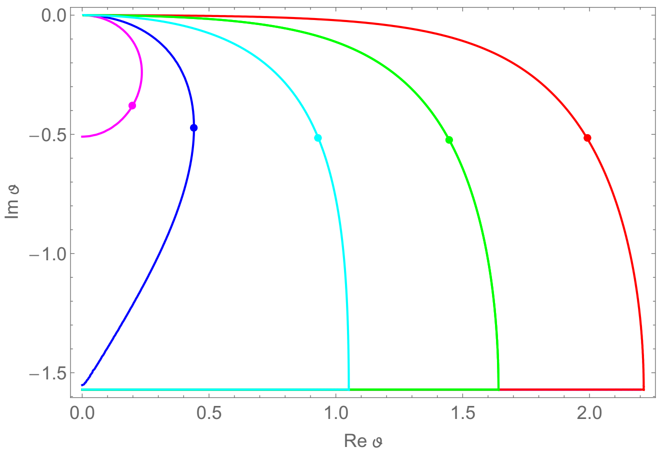

FIG. 1. Complex plane of $\vartheta(\epsilon)$ for $\gamma/\Delta_0 = 0.01; 0.05; 0.2; 0.456; 0.49$ from right to left. In the gapped case (i.e., $\gamma/\Delta_0 < 0.456$) $\vartheta(0) = -i\pi/2$ and there is a cusp at $\epsilon = \epsilon_g$; eventually $\vartheta(\infty) \to 0$. Full dots indicate $\epsilon = \Delta$, notice $\mathrm{Im}\vartheta(\Delta) \to -i\pi/6$ as $\gamma \to 0$.

where the last approximate relation holds for $\gamma \ll \Delta$. The gap closes at $\gamma = \Delta$. Immediately above the gap, $\epsilon \gtrsim \epsilon_g$, DOS takes the form:

$$\nu(\epsilon) = \nu\sqrt{\frac{2}{3}}\left(\frac{\Delta}{\gamma}\right)^{2/3}\sqrt{\frac{\epsilon-\epsilon_g}{\Delta}}. \tag{42}$$

At $\epsilon \approx \Delta$ it reaches its maximum value $\nu(\Delta) \approx \frac{\sqrt{3}}{4}\nu(4\Delta/\gamma)^{1/3}$ and merges with the BCS result $\nu(\epsilon) = \nu\epsilon/\sqrt{\epsilon^2-\Delta^2}$ at $\epsilon - \Delta \propto \Delta^{1/3}\gamma^{2/3}$, see Fig. 2.

At $T = 0$ the self-consistency relation (33) takes the form:

$$\Delta = \frac{\lambda}{2}\,\mathrm{Re}\int_0^{\omega_D} d\epsilon \sinh\vartheta = \frac{\lambda}{2}\,\mathrm{Re}\int d\vartheta\,\frac{d\epsilon}{d\vartheta}\sinh\vartheta, \tag{43}$$

where according to Eq. (34) $d\epsilon/d\vartheta = -\Delta\sinh^{-2}\vartheta - i\gamma\sinh\vartheta$ and the last integral runs along the contour depicted in Fig. 1. Performing the elementary integration one

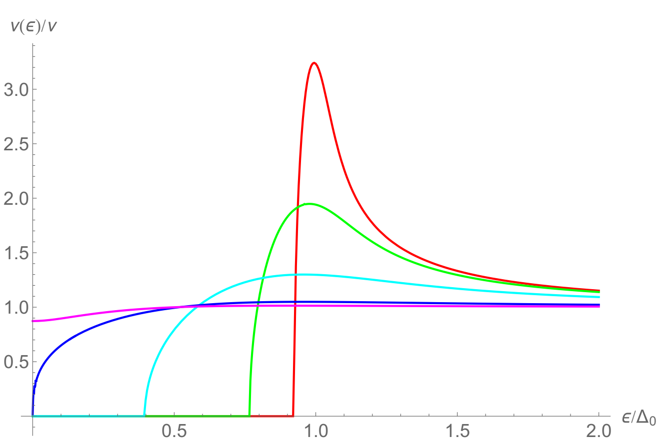

FIG. 2. DOS as a function of energy for the same values of TRS breaking parameter γ as in Fig 1.

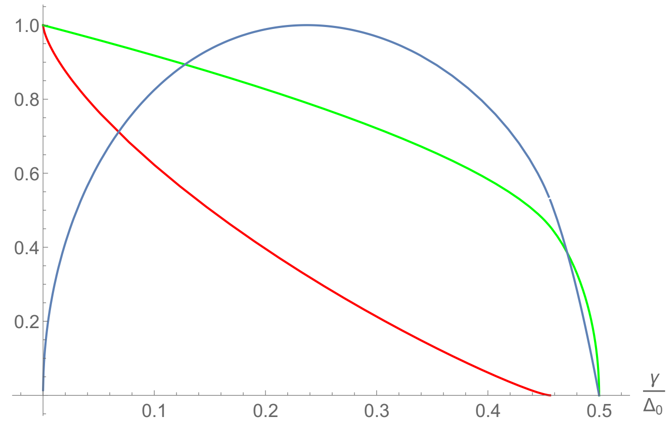

FIG. 3. Order parameter Δ/Δ_0 (green) and gap energy ϵ_g/Δ_0 (red) as functions of TRS breaking parameter γ/Δ_0. The blue line is the supercurrent density $j_s(\gamma)$ (in arbitrary units), given by Eq. (63).

finds [29]

$$\ln\frac{\Delta_0}{\Delta} = \begin{cases}\pi\gamma/(4\Delta); & \gamma \leqslant \Delta,\\ g(\gamma/\Delta); & \gamma > \Delta,\end{cases} \tag{44}$$

$$g(x) = \ln(x+\sqrt{x^2-1}) - \frac{1}{2x}\sqrt{x^2-1} + \frac{x}{2}\arcsin x^{-1},$$

where Δ_0 is the order parameter at $\gamma = 0$. Since $g(x) \to \ln(2x)$ at $x \to \infty$, the self-consistency condition looses a nontrivial solution at $\gamma \geqslant \Delta_0/2$. On the other hand, the gap closes at $\gamma = \Delta = e^{-\pi/4}\Delta_0 \approx 0.456\Delta_0$. Therefore in the narrow range $0.456 < \gamma/\Delta_0 < 0.5$ the order parameter is finite, while where is no gap in DOS, Fig. 3. This is the phenomenon of gapless superconductivity.

For $\gamma \ll \Delta$ one finds from Eq. (44) $\Delta \approx \Delta_0 - \frac{\pi}{4}\gamma$. The linear in γ suppression of the order parameter may be also found from Ginzburg-Landau equation for $T \lesssim T_c$. Notice that this suppression of the order parameter is parametrically weaker than suppression of the gap, Eq. (41). Therefore for the weak breaking of TRS, $\gamma \ll \Delta_0$, one may drop the distinction between Δ and Δ_0.

IV. KINETICS OF QUASIPARTICLES

A. Kinetic equation

The kinetic equations are given by the (1, 2) Keldysh component of the Usadel equation (26). Employing Wigner representation and projecting onto $\hat{\tau}^0$ and $\hat{\tau}^3$ Nambu components, one obtains equations for the longitudinal $F^L_\epsilon(\mathbf{r},t) = -F^L_{-\epsilon}(\mathbf{r},t)$ and the transversal $F^T_\epsilon(\mathbf{r},t) = F^T_{-\epsilon}(\mathbf{r},t)$ distribution functions:

$$\frac{\nu(\epsilon)}{\nu}\,\partial_t F^L_\epsilon - \nabla_{\mathbf{r}}\big[D^L(\epsilon)\nabla_{\mathbf{r}} F^L_\epsilon\big] = -2I^L_{\mathrm{coll}}, \tag{45}$$

$$\frac{\nu(\epsilon)}{\nu}\,\partial_t F^T_\epsilon - \nabla_{\mathbf{r}}\big[D^T(\epsilon)\nabla_{\mathbf{r}} F^T_\epsilon\big] + M^T(\epsilon)F^T_\epsilon = -2I^T_{\mathrm{coll}}, \tag{46}$$

where local DOS is given by Eq. (36) and other parameters are defined as [22,27,28]:

$$D^L(\epsilon) = \frac{D}{4}\,\mathrm{tr}\{\hat{\tau}^0 - \hat{Q}^R\hat{Q}^A\} = D\cosh^2\left(\frac{\vartheta - \bar{\vartheta}}{2}\right), \quad (47)$$

$$D^T(\epsilon) = \frac{D}{4}\,\mathrm{tr}\{\hat{\tau}^0 - \hat{\tau}^3\hat{Q}^R\hat{\tau}^3\hat{Q}^A\} = D\cosh^2\left(\frac{\vartheta + \bar{\vartheta}}{2}\right); \quad (48)$$

$$M^T(\epsilon) = \frac{1}{2}\,\mathrm{tr}\{\hat{Q}^R\hat{\Delta} + \hat{\Delta}\hat{Q}^A\} = i\Delta(\sinh\vartheta - \sinh\bar{\vartheta}) = 2\gamma\cosh^2\left(\frac{\vartheta+\bar{\vartheta}}{2}\right)|\sinh\vartheta|^2. \quad (49)$$

The mass, $M^T(\epsilon)$, exists only in the absence of TRS. It goes to zero at large energy as $M^T \to 2\gamma\Delta^2/\epsilon^2$, but acquires a large value $M^T(\epsilon_g) = 2\Delta^{4/3}\gamma^{-1/3}$ near the gap. Such a mass provides a rapid decay of the transversal component of the distribution function to zero. We thus focus here only on the slow longitudinal relaxation.

The corresponding collision integral is given by Eq. (19) (with factor 1/2 to compensate for Nambu doubling of the degrees of freedom), where one should use the Nambu-rotated Λ_ϵ matrices, Eq. (31). This yields, e.g.:

$$\mathrm{tr_N}\{\hat{\tau}^0[\hat{\gamma}^{cl}\hat{\Lambda}_{\epsilon'}\hat{\gamma}^{cl}\hat{\Lambda}_\epsilon - \hat{\Lambda}_\epsilon\hat{\gamma}^{cl}\hat{\Lambda}_{\epsilon'}\hat{\gamma}^{cl}]_K^{(1,2)}\} = 2(F^L_\epsilon - F^L_{\epsilon'})\,\mathrm{Re}[\cosh(\vartheta - \vartheta') + \cosh(\vartheta - \bar{\vartheta}')], \quad (50)$$

where $\vartheta = \vartheta(\epsilon)$ and $\vartheta' = \vartheta(\epsilon')$. As a result, the kinetic equation for the quasiparticles occupation number $f_\epsilon = (1 - F^L_\epsilon)/2$ acquires a form:

$$\frac{\nu(\epsilon)}{\nu}\,\partial_t f_\epsilon - \nabla_{\mathbf{r}}[D^L(\epsilon)\nabla_{\mathbf{r}} f_\epsilon] = I^L_{\mathrm{coll}}[f_\epsilon],$$
$$I^L_{\mathrm{coll}}[f_\epsilon] = \int\frac{d\epsilon'}{2\pi}M^S_{\epsilon,\epsilon'}[N_\omega f_{\epsilon'}(1-f_\epsilon) - (1+N_\omega)f_\epsilon(1-f_{\epsilon'})], \quad (51)$$

where the superconducting phonon matrix element is:

$$M^S_{\epsilon,\epsilon'} = \frac{1}{2}\,\mathrm{Re}[\cosh(\vartheta-\vartheta') + \cosh(\vartheta - \bar{\vartheta}')]M^N_{\epsilon-\epsilon'} = \frac{\nu(\epsilon)}{\nu}\frac{\nu(\epsilon')}{\nu}[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}]M^N_{\epsilon-\epsilon'}, \quad (52)$$

where $\omega = \epsilon - \epsilon'$, the normal state matrix element M^N_ω is given by Eqs. (20), (22) and DOS $\nu(\epsilon)$ is given by Eqs. (36), (42). Motivated by standard TRS notations, we introduced

$$2u_\epsilon v_\epsilon \equiv \frac{\mathrm{Re}[\sinh\vartheta]}{\mathrm{Re}[\cosh\vartheta]}, \quad (53)$$

which is only defined for $|\epsilon| > \epsilon_g$, see Fig. 4. Employing Eqs. (34)–(42), one may show that

$$2u_\epsilon v_\epsilon \approx \begin{cases}\Delta/|\epsilon|; & \epsilon - \Delta \gg \Delta^{1/3}\gamma^{2/3};\\ \sqrt{1-(\gamma/\Delta)^{2/3}}; & |\epsilon - \Delta| \lesssim \Delta^{1/3}\gamma^{2/3}\end{cases}. \quad (54)$$

Since $\nu(\epsilon') = 0$ for $|\epsilon'| < \epsilon_g$, while $[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}]$ tends to a constant, one finds $M^S_{\epsilon,\epsilon'<\epsilon_g} = 0$: as expected the final energy ϵ' has to be outside the spectral gap. For TRS superconductors these results appeared in Ref. [13].

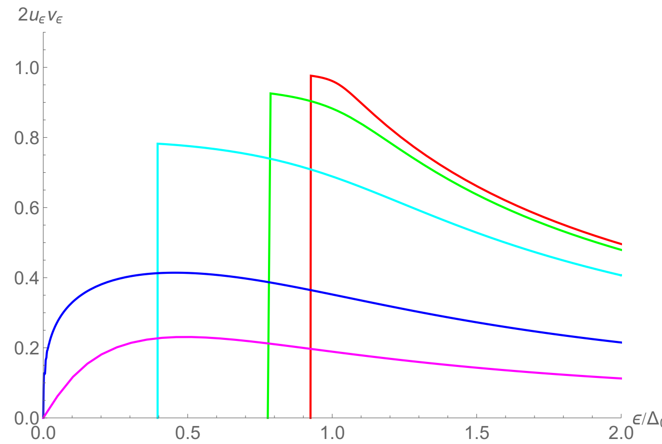

FIG. 4. $2u_\epsilon v_\epsilon$ as a function of energy for the same values of TRS breaking parameter γ as in Fig 1.

The factor $\nu(\epsilon)/\nu$ on the right hand side of Eq. (52) is canceled against the same on the left hand side of Eq. (51). As a result, one finds for the "out" electron-phonon relaxation rate:

$$\frac{1}{\tau_{e:ph}} = \int_{|\epsilon'|>\epsilon_g}\frac{d\epsilon'\nu(\epsilon')}{2\pi\nu}(1+N_\omega)(1-f_{\epsilon'}) \times[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}]M^N_\omega. \quad (55)$$

The energy integral here may be further subdivided onto positive $\epsilon' > \epsilon_g$ and negative $\epsilon' < -\epsilon_g$ regions, representing the inelastic *scattering* and *recombination* processes correspondingly:

$$\frac{1}{\tau_{e:ph}(\epsilon)} = \frac{1}{\tau^{\mathrm{sc}}_{e:ph}(\epsilon)} + \frac{1}{\tau^{\mathrm{rec}}_{e:ph}(\epsilon)}, \quad (56)$$

where

$$\frac{1}{\tau^{\mathrm{sc}}_{e:ph}} = \int_{\epsilon_g}^\infty\frac{d\epsilon'\nu(\epsilon')}{2\pi\nu}(1+N_{\epsilon-\epsilon'})(1-f_{\epsilon'}) \times[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}]M^N_{|\epsilon-\epsilon'|};$$
$$\frac{1}{\tau^{\mathrm{rec}}_{e:ph}} = \int_{\epsilon_g}^\infty\frac{d\epsilon'\nu(\epsilon')}{2\pi\nu}(1+N_{\epsilon+\epsilon'})f_{\epsilon'} \times[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}]M^N_{|\epsilon+\epsilon'|}. \quad (57)$$

To obtain recombination time we changed integration variable $\epsilon' \to -\epsilon'$ and used that $F^L_{-\epsilon'} = -F^L_{\epsilon'}$ and therefore $(1 - f_{-\epsilon'}) = f_{\epsilon'}$. We also employed Eq. (39), which insures that both $\nu(\epsilon')$ and $2u_{\epsilon'}v_{\epsilon'}$ are even functions. For low concentration of nonequilibrium quasiparticles $f_{\epsilon'} \ll 1$ the recombination processes may be disregarded, even though their matrix element tends to be larger.

V. KINETICS OF QUASIPARTICLES TRAPPING

A. Trapping rate

We now focus on trapping of nonequilibrium quasiparticles within the regions with the locally suppressed energy gap. Such suppression is often achieved by breaking TRS, resulting in a spatially dependent TRS breaking parameter $\gamma(\mathbf{r})$. For example, an isolated Abrikosov vortex brings $\gamma(r) = \frac{1}{2}\Delta(\xi/r)^2$.

Quasiparticles with an initial energy $\epsilon > \Delta$ diffuse to the regions with the suppressed gap $\epsilon_g(\mathbf{r}) < \Delta$. There they can inelastically scatter to a final energy ϵ' within the window $\epsilon_g(\mathbf{r}) < \epsilon' < \Delta$ by emitting an acoustic phonon. As a result, they end up being trapped within the spatial region $\epsilon_g(\mathbf{r}) < \epsilon'$, due to Andreev reflections from its boundaries. We evaluate the corresponding trapping rate, assuming very low phonon temperature, $N_\omega \ll 1$, and small concentration of nonequilibrium quasiparticles, i.e., $f_{\epsilon'} \ll 1$. As a result the trapping rate is found as:

$$\frac{1}{\tau_{\rm tr}} = \frac{b_j \pi \Omega_d}{(2\pi)^d} \frac{D p_F^2}{\rho_m v_j^{d+2}} \int_{\epsilon_g}^{\Delta} \frac{d\epsilon' \nu(\epsilon')}{2\pi\nu} [1 - 4u_\epsilon v_\epsilon u_{\epsilon'} v_{\epsilon'}](\epsilon - \epsilon')^d, \tag{58}$$

where the coefficients b_j are defined after Eq. (22).

There are two distinct limits for the trapping rate distinguished by the comparison of the relative excess energy of nonequilibrium quasiparticles, $\delta_\epsilon \equiv (\epsilon - \Delta)/\Delta$, and the relative energy range affected by breaking of TRS, $(\gamma/\Delta)^{2/3}$. Carrying out the integration in Eq. (58) with the help of Eqs. (42), (54), one finds

$$\frac{1}{\tau_{\rm tr}} \propto \frac{D p_F^2 \Delta^{d+1}}{\rho_m v_j^{d+2}} \left(\frac{\gamma}{\Delta}\right)^{\frac{1}{3}} \begin{cases} (\gamma/\Delta)^{\frac{2}{3}(d+1)}; & \delta_\epsilon < (\gamma/\Delta)^{\frac{2}{3}}, \\ \delta_\epsilon^{d+1}; & (\gamma/\Delta)^{\frac{2}{3}} < \delta_\epsilon \lesssim 1. \end{cases} \tag{59}$$

In most metals the longitudinal sound velocity is about twice that of the transversal one. As a result, the *transversal* phonons are about an order of magnitude more efficient in trapping the nonequilibrium quasiparticles than the longitudinal ones. Hereafter we thus restrict ourselves exclusively to the transversal waves. Notice that the transversal phonons are coupled to electrons due to impurity displacement mechanism, which is only present in the disordered limit $ql = \omega l/v_t \lesssim 1$. The characteristic length scale v_t/ω is typically in the range 10–100 nm. We shall assume that the characteristic thickness of superconducting films is larger than that and put $d = 3$ in the subsequent estimates.

B. Trapping power of a single vortex

We now evaluate the total trapping power of a vortex in a superconducting film, defined as a spatial integral of the local trapping rate (58), $P = \int d^2\mathbf{r}/\tau_{\rm tr}(\mathbf{r})$. This quantity may then be used as a sink term in the macroscopic 2D diffusion equation for the density of nonequilibrium quasiparticles, $n(\mathbf{r},t)$,

$$\partial_t n - \nabla[D\nabla n] = -P\,\delta^{(2)}(\mathbf{r})\,n, \tag{60}$$

where the vortex is placed at $r = 0$. The form of the right-hand side of Eq. (60) assumes the vortex core being a perfect sink. This is adequate, if (i) the density of nonequilibrium quasiparticles exceeds substantially the density of equilibrium excitations with energy $\epsilon > \Delta$ in the core region, and (ii) relaxation of nonequilibrium quasiparticles occurs due to "deeply inelastic" processes. The former condition is met in low-temperature experiments, see, e.g., Ref. [3]. Condition (ii) is satisfied for the relaxation by phonon emission (we note in passing that tunneling processes in S-I-N structures [9] provide a counterexample to (ii), resulting in a measurable backflow from a trap).

For an isolated vortex the TRS breaking parameter is a function of the distance r from the core $\gamma = \frac{1}{2}\Delta(\xi/r)^2$. The trapping rate at small γ scales as $\gamma^{1/3} \sim r^{-2/3}$, and thus the integral in the definition of the trapping power is dominated by large distances from the vortex core. Employing Eq. (58), one finds:

$$P = \frac{D(p_F\xi)^2\Delta^4}{10\pi\rho_m v_t^5}\left[\left(\frac{r_c}{2\xi}\right)^2\left((1+\delta_\epsilon)^4 - \delta_\epsilon^4\right) + \left(\frac{R}{\xi}\right)^{4/3}\delta_\epsilon^4\right]. \tag{61}$$

The first term in the square brackets here is the contribution of the vortex core, which we model as a normal cylinder with the radius r_c. The second term is coming from the outer periphery of the vortex core with R being its effective outer radius. It is determined by either a distance between vortices, a penetration depth, or the condition that $\Delta - \epsilon_g(R) \approx T$, where T is the phonon temperature. Indeed, beyond such a radius the trap is too shallow and trapping is not effective because of the activation escape. This leads to $(R/\xi)^{4/3} \approx \Delta/T$ and allows us to rewrite the last term in the brackets of Eq. (61) as $(\delta_\epsilon \Delta^{1/4}/T^{1/4})^4$. The peripheral trapping may dominate, if the typical quasiparticles excess energy grossly exceeds the phonon temperature.

We now use the parameters of devices investigated in experiment [3] to estimate P with the help of Eq. (61) and compare it with the experimental findings. In Ref. [3] the trapping power of individual vortices in an aluminum film was measured to be $P = 6.7 \times 10^{-2}$ cm^2/s at the base temperature of $T = 20$ mK. The relevant parameters of the film were [3] $D = 18$ cm^2/s; $v_t = 3.0 \times 10^5$ cm/s; $\rho_m = 2.7$ g/cm^3; $E_F = 11.7$ eV; $\Delta = 1.8 \times 10^{-4}$ eV. With these parameters one finds $D(p_F\xi)^2\Delta^4/(\rho_m v_t^5) = 6.8 \times 10^{-3}$ cm^2/s. As a result, the core contribution to the trapping power in Eq. (61) is about two orders of magnitude smaller than the observed value for a reasonable estimate of r_c and δ_ϵ. At the base temperature, $(R/\xi)^{4/3} \approx \Delta/T = 10^2$, the peripheral contribution may provide trapping of the right order of magnitude only if $\delta_\epsilon \sim 1$. For the geometry of devices in Ref. [3], there are no reasons to expect that the quasiparticles are so "hot" in the vicinity of the vortices. Therefore, albeit the peripheral contribution adds to the trapping power, it is not sufficient to explain the observed value of P.

C. Trapping by a current-carrying constriction

Trapping rate was also measured [15] in a nanobridge closed by a flux-biased superconducting loop. The flux bias was creating a supercurrent flowing through the constriction. The supercurrent breaks TRS and thus suppresses the energy gap in the nanobridge itself as well as in the adjacent leads, carrying the stray currents. Assuming $3D$ leads, the stray current density may be estimated as $j_s(r) = I_s/(2\pi r^2)$, where r is distance from the constriction and I_s is the total supercurrent through the constriction.

To apply our theory we need to find a relation between the local supercurrent density $j_s(r)$ and the local TRS breaking parameter $\gamma(r)$. Assuming an applied vector potential $\mathbf{A}(r)$, the supercurrent density is obtained by variation of the action (24) over the (quantum component of the) vector potential and is given by

$$-j_s = Ae\nu D\mathrm{Im}\int_0^\infty d\epsilon\, F^L_\epsilon \sinh^2\vartheta$$
$$= Ae\nu D\mathrm{Im}\int_{\vartheta(0)}^0 d\vartheta \frac{d\epsilon}{d\vartheta}\sinh^2\vartheta, \qquad (62)$$

where in the second equation we put $T=0$ and changed the integration variable to ϑ. We will also assume that the vector potential A, which creates the supercurrent is the sole source of the breaking TRS symmetry (i.e., no additional magnetic field is present) and thus $\gamma = 2DA^2$. Performing the integration one finds (we traded here A for γ):

$$j_s = e\nu\gamma^{3/2}\sqrt{\frac{D}{2}}\,\mathrm{Re}\,h\left(\frac{\Delta}{\gamma}\right);$$
$$h(x) = x\sin^{-1}x - \frac{2}{3}\left(1-\sqrt{1-x^2}\left(1+\frac{1}{2}x^2\right)\right). \qquad (63)$$

Notice that $\Delta = \Delta(\gamma)$ according to Eq. (44). The resulting critical current $j_s(\gamma)$ is plotted in Fig. 3. For small j_s the effective TRS breaking parameter γ is found from Eq. (63) as

$$\gamma(r) = \frac{8}{\pi^2}\Delta\left(\frac{j_s(r)}{e\nu\xi\Delta^2}\right)^2 = \frac{2\Delta}{\pi^4}\left(\frac{I_s}{e\nu\xi^3\Delta^2}\right)^2\left(\frac{\xi}{r}\right)^4. \qquad (64)$$

According to Eq. (59) the trapping rate far from the constriction (i.e., for small γ) scales as $\tau_{\rm tr}^{-1}(r) \sim \gamma^{1/3} \sim I_s^{2/3}r^{-4/3}$. To calculate the total trapping rate τ_T^{-1} of the constriction, measured in Ref. [15], one integrates this expression over the volume of the leads and multiplies by n_{qp}—concentration of nonequilibrium quasiparticles (in notations of Ref. [15] $n_{qp} = x_{qp}\nu\Delta$, where x_{qp} is the dimensionless fraction of broken Cooper pairs). The aforementioned volume integral is coming from large distances R and thus $\tau_T^{-1} \sim I_s^{2/3}R^{5/3}$. As explained above, the outer radius R is limited by thermally activated escape and is estimated from $(\gamma(R)/\Delta)^{2/3} = T/\Delta$. This leads to $R \sim I_s^{1/2}T^{-3/8}$. As a result, the trapping rate of the constriction, coming from its outer periphery, is given by:

$$\frac{1}{\tau_T} \propto \frac{Dp_F^2(\Delta\delta_\epsilon)^4}{\rho_m v_t^5}n_{qp}\xi^3\left(\frac{I_s}{e\nu\xi^3\Delta^2}\right)^{2/3}\left(\frac{\Delta}{T}\right)^{5/8}. \qquad (65)$$

(For a 2D pattern of stray currents one finds $\tau_T^{-1} \sim I_s^2T^{-1}$.) Both dependencies on current and temperature are in a qualitative agreement with the data of Ref. [15]. The "core" contribution, due to trapping on localized Andreev bound states, was calculated in Ref. [15] and found to be about two orders of magnitude less than the observed value. The peripheral trapping, discussed here, may well account for this discrepancy, though quantitative comparison is impeded by the uncertainty in δ_ϵ and n_{qp}.

VI. DISCUSSION OF THE RESULTS

We have developed a unified theory for treating electron-phonon kinetics in disordered normal metals and superconductors, including superconductors with broken TRS. The latter case is particularly important for evaluation of the trapping rate of nonequilibrium quasiparticles in the regions, where the energy gap is suppressed by a local magnetic field or a supercurrent. Quasiparticle traps are proven to be useful for increasing coherence time of superconducting qubits.

Our theory shows that the trapping rate $\tau_{\rm tr}^{-1}$ is a very sensitive function of the TRS breaking parameter γ, which at low temperature scales as $\tau_{\rm tr}^{-1} \propto \gamma^{1/3}$. As a result, even the regions with a weak breaking of TRS, such as a far periphery of a vortex or a constriction, may provide a significant contribution to the overall trapping power of "hot" quasiparticles. The quantitative comparison with the experiment [15] requires detailed knowledge of nonequilibrium quasiparticles energy distribution, which is not available at the moment. Our estimates show that in order to account for the observed trapping rates, the nonequilibrium quasiparticles excess energy should be $\epsilon - \Delta \sim \Delta \gg T$, where T is the phonon bath temperature.

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APPENDIX A: ALTERNATIVE DERIVATION OF THE ELECTRON-PHONON ACTION

Here we provide an alternative derivation of Eqs. (11), which is based on first principles Coulomb interactions between the electrons and the lattice as well as impurity drag by the lattice displacements. The first of these effects leads to the standard Coulomb action

$$S_C = \int dt\left[\frac{1}{2}\sum_{\mathbf{q}}\varphi_{\mathbf{q},t}U_C^{-1}\varphi_{-\mathbf{q},t} + \sum_{\mathbf{r}}\varphi_{\mathbf{r},t}(\rho_0\mathrm{div}\,\mathbf{u} - \rho_e)_{\mathbf{r},t}\right], \qquad (A1)$$

where $\varphi_{\mathbf{r},t}$ is the fluctuating scalar potential, $U_C = 4\pi e^2/q^2$ is the bare Coulomb interaction, $\rho_e(\mathbf{r},t) = \bar\psi(\mathbf{r},t)\psi(\mathbf{r},t) - \rho_0$ is the excess electron density, while $\rho_0\mathrm{div}\,\mathbf{u}$ is the excess lattice density.

The second effect is more subtle and pertains to the disordered limit $ql \ll 1$, where $l = v_F\tau$ is the elastic mean free path and τ is the elastic mean free time. It originates from the fact that the impurities are frozen into the crystal lattice and therefore are also subject to the displacement $\mathbf{u}(\mathbf{r},t)$ [11–14,17,19]. Therefore hitherto static random disorder potential becomes a dynamic object $V_{\rm dis}(\mathbf{r}) \to V_{\rm dis}(\mathbf{r}+\mathbf{u}(\mathbf{r},t))$. It is convenient to shift $\mathbf{r}$ to write the interaction of the electron density with the disorder potential as

$$H_{\rm dis} = \sum_{\mathbf{r}} V_{\rm dis}(\mathbf{r})\rho_e(\mathbf{r}-\mathbf{u}(\mathbf{r},t),t). \qquad (A2)$$

Performing averaging over the Gaussian distribution of short-ranged disorder, one finds the following action

$$iS_{\rm dis} = -\frac{1}{4\pi\nu\tau}\iint dt dt'\sum_{\mathbf{r}}\bar\psi_{\mathbf{r}-\mathbf{u},t}\psi_{\mathbf{r}-\mathbf{u},t}\bar\psi_{\mathbf{r}-\mathbf{u}',t'}\psi_{\mathbf{r}-\mathbf{u}',t'}, \qquad (A3)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$ and $\mathbf{u}' = \mathbf{u}(\mathbf{r},t')$. One can now rearrange the fermionic fields and decouple the four-fermion action with the help of the nonlocal in time field $Q_{t,t'}(\mathbf{r})$. This leads to the following term:

$$\bar{\psi}_{\mathbf{r}-\mathbf{u},t} Q_{t,t'}(\mathbf{r})\psi_{\mathbf{r}-\mathbf{u}',t'} \approx \bar{\psi}_{\mathbf{r},t}\big[Q_{t,t'}(\mathbf{r}) - \hat{\mathcal{L}}_1 + \tfrac{1}{2}\hat{\mathcal{L}}_2\big]\psi_{\mathbf{r},t'}. \tag{A4}$$

We have expanded fermionic fields to the second order in the displacement $\mathbf{u}$, which brings the two operators:

$$\hat{\mathcal{L}}_1 = \overleftarrow{\nabla}\cdot\mathbf{u}\, Q_{t,t'}(\mathbf{r}) + Q_{t,t'}(\mathbf{r})\mathbf{u}'\cdot\overrightarrow{\nabla};$$
$$\hat{\mathcal{L}}_2 = \overleftarrow{\nabla}\overleftarrow{\nabla}\cdot\cdot\,\mathbf{u}\mathbf{u}\, Q + \overleftarrow{\nabla}\cdot\mathbf{u}\, 2Q\mathbf{u}'\cdot\overrightarrow{\nabla} + Q\,\mathbf{u}'\mathbf{u}'\cdot\cdot\,\overrightarrow{\nabla}\overrightarrow{\nabla}, \tag{A5}$$

where the arrows above the gradient operators show direction of the differentiation in the context of Eq. (A4). The action is now quadratic in the unshifted fermionic fields which may be integrated out in the standard way, leading to the determinant:

$$\mathrm{Tr}\log\left\{G_0^{-1} - \varphi + \frac{i}{2\tau}\left[Q - \hat{\mathcal{L}}_1 + \frac{1}{2}\hat{\mathcal{L}}_2\right]\right\}, \tag{A6}$$

where $\varphi = \varphi_{\mathbf{r},t}$ is the scalar potential coming from the Coulomb interactions, Eq. (A1).

From this point on, one proceeds along the standard root of deriving the Keldysh nonlinear sigma model [22]. To this end one passes to the Keldysh 2×2 structure, by splitting the contour on forward and backward branches and performing the Keldysh rotation. One then realizes that the soft diffusive modes of the action are described by the manifold $\hat{Q}^2 = 1$ and therefore one can write $\hat{Q} = \hat{\mathcal{R}}^{-1}\hat{\Lambda}\hat{\mathcal{R}}$, where $\hat{\Lambda}$ is the Green function in coinciding spatial points, Eq. (5). This way Eq. (A6) may be rewritten as:

$$\mathrm{Tr}\log\left\{1 + \hat{G}\hat{\mathcal{R}}\big[G_0^{-1},\hat{\mathcal{R}}^{-1}\big] - \hat{G}\hat{\mathcal{R}}\left[\hat{\varphi} + \frac{i}{2\tau}\hat{\mathcal{L}}_1 - \frac{i}{4\tau}\hat{\mathcal{L}}_2\right]\hat{\mathcal{R}}^{-1}\right\}. \tag{A7}$$

Finally, one expands the logarithm here to the lowest nonvanishing orders. This way one obtains the standard nonlinear sigma-model action (first neglecting $\hat{\mathcal{L}}_{1,2}$ terms):

$$i S_0 = \frac{i\nu}{2}\mathrm{Tr}\{\hat{\varphi}\hat{\sigma}^1\hat{\varphi}\} - \frac{\pi\nu}{4}\mathrm{Tr}\{D(\partial_{\mathbf{r}}\hat{Q})^2 - 4\partial_t\hat{Q} - 4i\hat{\varphi}\hat{Q}\}. \tag{A8}$$

The first term on the right hand side here represents static polarizability (i.e., screening) of the electronic band. It comes from the so-called retarded-retarded and advanced-advanced loops. The dynamic screening is encoded in $\pi\nu\mathrm{Tr}\{\hat{\varphi}\hat{Q}\}$ term along with fluctuations of the $\hat{Q}$ field around its stationary point $\hat{\Lambda}$.

We focus now onto the phonon-induced $\hat{\mathcal{L}}_{1,2}$ terms, which originate from the motion of the impurities relative to the electronic liquid. It is easy to see that the first order in $\hat{\mathcal{L}}_1$ vanishes. One is thus left with the three terms: (i) first order in $\hat{\mathcal{L}}_1$ and in $i\hat{\mathcal{R}}\mathbf{v}_F\cdot\overrightarrow{\nabla}\hat{\mathcal{R}}^{-1}$; (ii) first order in $\hat{\mathcal{L}}_2$, and (iii) second order in $\hat{\mathcal{L}}_1$. A straightforward, but somewhat lengthy evaluation of these three terms results in

$$i S_{(\mathrm{i})} = -i\pi\nu\frac{v_F p_F}{d}\mathrm{Tr}\{\hat{\mathbf{u}}\cdot\nabla\hat{Q}\} = i\pi\rho_0\mathrm{Tr}\{\mathrm{div}\hat{\mathbf{u}}\,\hat{Q}\}; \tag{A9}$$

$$i S_{(\mathrm{ii})} = -i\frac{\pi\nu}{2\tau}\frac{p_F^2}{d}\mathrm{Tr}\{\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} - \hat{\mathbf{u}}\,\hat{Q}\hat{\mathbf{u}}\,\hat{Q}\}; \tag{A10}$$

$$i S_{(\mathrm{iii})} = i\frac{\pi\nu}{2\tau}\frac{p_F^2}{d}\mathrm{Tr}\{\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} - \hat{\mathbf{u}}\,\hat{Q}\hat{\mathbf{u}}\,\hat{Q}\} + \frac{\pi\nu D\,p_F^2}{4}\mathrm{Tr}\big\{[\hat{Q},\partial^\mu\hat{\mathbf{u}}^\nu][\hat{Q},\partial^\eta\hat{\mathbf{u}}^\lambda]\big\}\,\Upsilon_{\mu\nu,\eta\lambda}, \tag{A11}$$

where $\Upsilon_{\mu\nu,\eta\lambda}$ is given by Eq. (12). Notice that the leading orders in $S_{(\mathrm{ii})}$ and $S_{(\mathrm{iii})}$ exactly cancel each other. The second subleading term in Eq. (A11) originates from gradient operators in $\hat{\mathcal{L}}_1$ acting on displacements $\hat{\mathbf{u}}$, as opposed to the Green functions G.

The scalar linear coupling $S_{(\mathrm{i})}$ may be combined with the potential term in Eq. (A8) by shifting the potential $\hat{\varphi} \to \hat{\phi} = \hat{\varphi} + \frac{\rho_0}{\nu}\mathrm{div}\hat{\mathbf{u}}$. In the limit of the strong Coulomb interactions, $U_C \to \infty$ in Eq. (A1), this allows us to eliminate Fröhlich deformation potential electron-phonon coupling. Indeed, the static screening $\frac{\nu}{2}\mathrm{Tr}\{\hat{\varphi}\hat{\sigma}^1\hat{\varphi}\}$ in Eq. (A8) along with the interaction term $\hat{\varphi}\hat{\sigma}^1\rho_0\mathrm{div}\,\hat{\mathbf{u}}$ in Eq. (A1) upon the aforementioned shift results in $\frac{\nu}{2}\mathrm{Tr}\{\hat{\phi}\hat{\sigma}^1\hat{\phi}\} - \frac{\rho_0^2}{2\nu}\mathrm{div}\hat{\mathbf{u}}\,\hat{\sigma}^1\mathrm{div}\hat{\mathbf{u}}$. The first term here stays for the screened electron-electron interactions, unaffected by lattice displacement, while the second one serves to renormalize upward the longitudinal sound velocity. This latter effect is already accommodated by using the correct value of v_l and thus no other effects of the scalar electron-phonon coupling $S_{(\mathrm{i})}$ remain.

The only remaining term thus is the second quadrupole term in Eq. (A11), which coincides exactly with Eq. (11). The latter was derived using phenomenological Schmid form, Eq. (2), of the electron-phonon coupling. The present first principles derivation provides thus an independent justification for the Schmid theory [12].

APPENDIX B: ULTRASONIC ATTENUATION

1. Normal metals

For the sake of completeness we outline calculation of the ultrasonic attenuation. It is found by integrating out electronic degrees of freedom, $\hat{Q}$, and focusing on modification of the phonon propagator (15) due to electron-phonon coupling. In the leading approximation it is given by the action (11), where one puts $\hat{Q} = \hat{\Lambda}$, cf. Eq. (5), and integrates over the energy:

$$i S_{\hat{\Lambda},\mathbf{u}} = -\nu p_F^2\sum_{\mathbf{q},\omega}\bar{\mathbf{u}}^{\mu,\alpha}_{\mathbf{q},\omega}\hat{K}^{\alpha\beta}(\omega)\frac{Dq^2}{d+2}\left[\delta_{\mu\nu} + \frac{d-2}{d}\frac{q_\mu q_\nu}{q^2}\right]\mathbf{u}^{\nu,\beta}_{\mathbf{q},\omega}, \tag{B1}$$

where the kernel is

$$\hat{K}^{\alpha\beta}(\omega) = \frac{1}{4}\int d\epsilon\,\mathrm{Tr}\{\hat{\gamma}^\alpha\hat{\gamma}^\beta - \hat{\Lambda}_{\epsilon-\omega}\hat{\gamma}^\alpha\hat{\Lambda}_\epsilon\hat{\gamma}^\beta\}. \tag{B2}$$

For normal metals one finds, cf. Eq. (5), the dissipative Caldeira-Leggett kernel:

$$\hat{K}(\omega) = \begin{pmatrix} 0 & -\omega \\ \omega & 2\omega\mathcal{B}_\omega \end{pmatrix} \tag{B3}$$

and $\mathcal{B}_\omega = \int d\epsilon\,(1 - F_{\epsilon-\omega}F_\epsilon)/(2\omega)$ is the bosonic distribution function. It provides damping to the phonon action, Eq. (13): $(\omega \pm i0)^2 - (\omega_{\mathbf{q}}^{(j)})^2 \to \omega^2 \pm i\gamma_{\mathbf{q}}^{(j)}\omega - (\omega_{\mathbf{q}}^{(j)})^2$ (along with the fluctuation-dissipation related noise), where the damping factors are [19,20,23]:

$$\gamma_{\mathbf{q}}^{(j)} = c_j\,\frac{\nu p_F^2}{\rho_m}\,Dq^2, \tag{B4}$$

where $c_t = 2/(d+2)$ and $c_l = c_t[1 + (d-2)/d]$.

2. Superconductors

In the superconducting case the $\hat{K}(\omega)$ kernel, Eq. (B2), depends on the rotation angle in the Nambu space. For its retarded, i.e., $\alpha = q$ and $\beta = cl$, component one finds:

$$K^R(\omega) = -\frac{1}{4}\int d\epsilon\big[\cosh(\vartheta - \vartheta')F_{\epsilon'}^L - \cosh(\bar{\vartheta} - \bar{\vartheta}')F_\epsilon^L + \cosh(\vartheta - \bar{\vartheta}')\big(F_{\epsilon'}^L - F_\epsilon^L\big)\big], \tag{B5}$$

where $\epsilon' = \epsilon - \omega$. Since for $\epsilon < \epsilon_g$, $\mathrm{Im}\vartheta(\epsilon) = -\pi/2$ there is no contribution to the integral from the region $\epsilon,\epsilon' < \epsilon_g$. The imaginary part of this expression, coming from the two energy intervals $\pm\epsilon_g < \epsilon < \pm\epsilon_g + \omega$, is $\sim\omega^2$ at small ω, which gives a small renormalization to the sound velocity. The real part, responsible for the attenuation, takes the following form:

$$\mathrm{Re}K^R(\omega) = \int d\epsilon\,\frac{\nu(\epsilon)}{\nu}\frac{\nu(\epsilon')}{\nu}[1 - 4u_\epsilon v_\epsilon u_{\epsilon'}v_{\epsilon'}](f_{\epsilon'} - f_\epsilon), \tag{B6}$$

where $2u_\epsilon v_\epsilon$ is given by Eq. (53), $f_\epsilon = (1 - F_\epsilon^L)/2$ is the quasiparticle occupation number, and the integral runs over the two intervals $\epsilon < -\epsilon_g$ and $\epsilon > \epsilon_g + \omega$, where both DOS are nonzero.

For TRS superconductors the above expression takes the form:

$$\mathrm{Re}K^R(\omega) = \int d\epsilon\,\frac{\epsilon\epsilon' - \Delta^2}{\sqrt{\epsilon^2 - \Delta^2}\sqrt{\epsilon'^2 - \Delta^2}}(f_{\epsilon'} - f_\epsilon). \tag{B7}$$

In the case $\omega \ll \Delta, T$ one may expand over ω and use the fact that the fraction in Eq. (B7) tends to 1 as $\omega \to 0$. As a result, $\mathrm{Re}K^R(\omega) = -2\omega\int_\Delta^\infty d\epsilon\,(df_\epsilon/d\epsilon) = 2\omega f_\Delta$. This way the ultrasound attenuation coefficient in equilibrium disordered TRS superconductors is found to be [11,23,30]:

$$\gamma_{\mathbf{q}}^S = 2f\left(\frac{\Delta(T)}{T}\right)\gamma_{\mathbf{q}}^N, \tag{B8}$$

where $f(\Delta/T)$ is the Fermi function. (The same relation holds in the clean case as well [31].)

It is worth noticing that the celebrated Eq. (B8) does not hold in the TRS broken case. Indeed, in the limit $\omega \to 0$, the factor $(\frac{\nu(\epsilon)}{\nu})^2[1 - (2u_\epsilon v_\epsilon)^2] = (\mathrm{Re}\cosh\vartheta)^2 - (\mathrm{Re}\sinh\vartheta)^2 \neq 1$ for complex $\vartheta(\epsilon)$. In the small temperature case, $T < \gamma^{2/3}\Delta^{1/3} < \epsilon_g$, Eqs. (42), (54) lead to:

$$\mathrm{Re}K^R(\omega) = \frac{-4\omega}{3\gamma^{2/3}\Delta^{1/3}}\int_{\epsilon_g}^\infty d\epsilon(\epsilon - \epsilon_g)\frac{df_\epsilon}{d\epsilon} = \frac{4\omega}{3\gamma^{2/3}\Delta^{1/3}}\int_{\epsilon_g}^\infty d\epsilon f_\epsilon. \tag{B9}$$

In equilibrium this brings:

$$\gamma_{\mathbf{q}}^S = \frac{4T}{3\gamma^{2/3}\Delta^{1/3}}\,e^{-\epsilon_g/T}\,\gamma_{\mathbf{q}}^N, \tag{B10}$$

i.e., there is an additional small factor $\sim T/(\gamma^{2/3}\Delta^{1/3})$ in comparison with the TRS case (it may be overcompensated, though, by the fact that $\epsilon_g < \Delta$).

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