Fisher information approach to nonequilibrium phase transitions in a quantum XXZ spin chain with boundary noise

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We investigated quantum critical behaviors in the nonequilibrium steady state of a XXZ spin chain with boundary Markovian noise using Fisher information. The latter represents the distance between two infinitesimally close states, and its superextensive size scaling witnesses a critical behavior due to a phase transition since all the interaction terms are extensive. Perturbatively, in the noise strength, we found superextensive Fisher information at anisotropy $|\Delta| \leq 1$ and irrational $\frac{\arccos\Delta}{\pi}$ irrespective of the order of two noncommuting limits, i.e., the thermodynamic limit and the limit of sending $\frac{\arccos\Delta}{\pi}$ to an irrational number via a sequence of rational approximants. From this result we argue the existence of a nonequilibrium quantum phase transition with a critical phase $|\Delta| \leq 1$. From the nonsuperextensivity of the Fisher information of reduced states, we infer that this nonequilibrium quantum phase transition does not have local order parameters but has nonlocal ones, at least at $|\Delta| = 1$. In the nonperturbative regime for the noise strength, we numerically computed the reduced Fisher information which lower bounds the full-state Fisher information and is superextensive only at $|\Delta| = 1$. From the latter result, we derived local order parameters at $|\Delta| = 1$ in the nonperturbative case. The existence of critical behavior witnessed by the Fisher information in the phase $|\Delta| < 1$ is still an open problem. The Fisher information also represents the best sensitivity for any estimation of the control parameter, in our case the anisotropy Δ, and its superextensivity implies enhanced estimation precision which is also highly robust in the presence of a critical phase.

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I. INTRODUCTION

One of the paradigms for nonequilibrium statistical physics consists of the study of nonthermalizing noisy dynamics [1,2]: nonequilibrium phase transitions are nonanalytic changes of nonequilibrium steady states (NESSs). This kind of transition has a much richer phenomenology than equilibrium phase transitions because NESSs lack a universal description in terms of thermodynamic potentials. From a methodological point of view, this situation results in a large variety of universality classes without general tools for their characterization [3,4]. For instance, algebraically decaying correlation functions are not particular to critical phenomena [5]. Also the spectral gap of the Liouvillian, an open-system generalization of the Hamiltonian gap, may vanish in the thermodynamic limit for all parameters, with critical points resulting only in a faster convergence [6,7].

The broad interest in nonequilibrium phase transitions and in the search, pursued in our approach, for universal tools to characterize them stems also from their emergence in a large variety of settings, from complex systems, both physical [8–13] and biological [14–19], to social sciences and economics [20–23]. Furthermore, quantumlike models have been developed to fit phenomena in social sciences and economics [24].

For quantum systems, dynamics with Markovian noise are represented by Lindblad master equations [25,26]. Recently, many investigations shed light on the complex and critical behaviors of quantum NESSs [5–7,27–38]. An interesting paradigmatic master equation consisting of an anisotropic Heisenberg (XXZ) spin chain driven by an unequal noise at its boundaries has a nontrivial steady state with transitions manifesting in transport properties [39].

Exactly solvable models form one of the main pillars of classical statistical mechanics, both in and out of equilibrium. Among important general concepts which are amenable to exact solutions are the NESSs, important nontrivial examples of which are the simple exclusion processes with boundary driving [40]. Similar models do not yet hold for quantum statistical mechanics, as the number of exact solutions for interacting models, particularly out of equilibrium, is very limited. The example of a boundary-driven XXZ model is one of the very few solutions. Nevertheless, the behavior of the nonequilibrium partition function for a few other models that can be exactly solved using a similar boundary noise protocol (e.g., a boundary-driven Fermi-Hubbard model and an integrable $SU(3)$ chain [39]) is qualitatively identical to the one for the isotropic Heisenberg model. This leads us to believe that the boundary-driven XXZ model discussed here may represent an important out-of-equilibrium universality class, and the same type of phase transition may later be seen in other models. This may not be related to integrability, but in nonintegrable systems the numerical analysis required to apply our approach to the NESS will be much harder.

Many equilibrium phase transitions are detected by the Bures metric, also known as fidelity susceptibility [41]. It is proportional to the Fisher information [42–45], except in the presence of pathological behaviors consisting of only removable singularities [46]. While this quantity reduces to standard susceptibilities for thermal phase transitions [47–50], it represents a more sophisticated tool for quantum phase transitions (QPTs), both symmetry-breaking [51,52] and topological [53] ones. It is worth mentioning another nonstandard approach to phase transitions in equilibrium statistical mechanics and in chaotic dynamics which is based on topological changes in isoenergetic manifolds in the phase space [54,55]. For nonequilibrium steady-state quantum phase transitions (NESS-QPTs), which are discussed here, the study of Fisher information is in the very early stages [56,57].

The rationale of our approach relies upon the geometric interpretation of the Fisher information as the distance between two infinitesimally close states with respect to a varying control parameter. Indeed, when all interaction terms are extensive, a superextensive metric implies instability with respect to small changes, e.g., due to critical points separating different phases. In this paper, we exemplify this approach with deep characterizations of the NESS-QPT in the XXZ chain with boundary noise. The above geometric interpretation provides a universal and unifying approach for both equilibrium and nonequilibrium, and possibly unknown, phase transitions, with a clear advantage over the aforementioned nonuniversal tools.

We also investigate relations between nonlocal or local order parameters and the Fisher information of the full state or of reduced states, respectively. This relation is general and does not depend on the model and, as such, can be applied to any phase transition detected by the Fisher information. Moreover, it relies upon the Cramér-Rao bound, i.e., a result from estimation theory [42–44], while previous studies of the fidelity susceptibility focused only on the intuition behind the geometric interpretation, thus missing the connection to order parameters. Connection is intuitive for symmetry-breaking phase transitions where local order parameters are known, so signatures of the phase transition can be found in reduced states. The most interesting application is in phase transitions without known order parameters, like in our case. This application reverses the usefulness argument for the fidelity susceptibility in topological phase transitions: while size scaling of the Fisher information was used to detect transitions without local order parameters, in our case we infer the local or nonlocal nature of order parameters from the size scaling of the Fisher information.

Using the above arguments, we endorse our approach as a powerful tool to characterize general nonequilibrium quantum phase transitions in other systems far beyond previously considered cases, according to the following recipe: superextensivity of the Fisher information in systems with extensive interactions which scale linearly with the volume detects general critical behaviors with at least nonlocal order parameters, and superextensivity of the reduced-state Fisher information further proves local order parameters. Our study opens a different avenue of research in NESS-QPTs, illustrating that complex structures and relevant features can be extracted by the Fisher information in highly nontrivial systems.

Fisher information is also intimately connected to metrology, being the inverse of the smallest variance in the estimation of the varying parameters [42–44]. Superextensivity implies extraordinary enhanced metrological performances. Thus, beyond the aim of NESS-QPTs, our study deepens the connection between quantum noisy dynamics and metrology [57–63], as well as general relations between NESS and quantum information [64,65].

This paper is organized as follows. We define the spin-chain model with boundary Markovian dissipation in Sec. II and the Fisher information with properties relevant for our analysis in Sec. III. In Sec. IV, we discuss the size scaling of Fisher information for perturbatively small dissipation strength and implications of the nonequilibrium phase transition, including the existence of a critical phase and of (non)local order parameters. In Sec. V, we report on the Fisher information and properties of the nonequilibrium phase transition nonperturbatively in the dissipation strength, and in Sec. VI we conclude.

II. SPIN MODEL

We discuss an n-spin chain with the XXZ Hamiltonian

$$H_{\mathrm{XXZ}} = \sum_{j=1}^{n-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right), \qquad (1)$$

which is an archetypical nearest-neighbor interaction in condensed matter [66,67], with σ_j^α being the Pauli matrices of the jth spin. In addition, we consider a uniform magnetic field along the z direction and Markovian dissipation at the boundary of the chain, arriving thus at the following dynamical equation for the density matrix, called the master equation:

$$\frac{d}{dt}\rho(t) = -i\left[\frac{\Omega}{2}M_z + J H_{\mathrm{XXZ}}, \rho(t)\right] + \lambda \sum_{k=1}^{4}\left[L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\}\right], \qquad (2)$$

where

$$L_{1,2} = \sqrt{\frac{1 \pm \mu}{2}}\,\sigma_1^\pm, \quad L_{3,4} = \sqrt{\frac{1 \mp \mu}{2}}\,\sigma_n^\pm \qquad (3)$$

are the so-called Lindblad operators and $M_z = \sum_{j=1}^n \sigma_j^z$ is the total magnetization along the z direction [68].

While the first line of (2) reproduces the standard Schrödinger equation, the second line is the prototypical form of quantum Markovian dissipation under the minimal physical assumption that the resulting time evolution γ_t is a semigroup, i.e., $\gamma_t \gamma_s = \gamma_{s+t}\ \forall\, t,s \geq 0$, trace preserving, and completely positive, i.e., preserves positivity of any initial density matrix even when arbitrarily correlated with ancillary systems.

Markovian master equations can be derived from microscopic models with system-environment interaction that is linear in the Lindblad operators [25,26]. In particular, Markovian master equations with local Lindblad operators, i.e., each environment interacting with a single particle as in Eq. (2), derive from the so-called singular coupling approximation [25,26] or from the weak system-environment coupling if the system Hamiltonian is dominated by the interaction-free part [69], in our case $\Omega \gg J$.

Our model has an exactly solvable steady-state density operator, i.e., a fixed point $\rho_\infty = \lim_{t\to\infty} \rho_t$ of (2), which can be represented in terms of a matrix product ansatz (see

Ref. [39] for a review). This structure will be essential to make our computations efficient.

III. FISHER INFORMATION

Given the aforementioned analytic solution, we compute the Fisher information for variations of the anisotropy Δ,

$$F_\Delta = 8 \lim_{\delta\to 0} \frac{1-\sqrt{\mathcal{F}(\rho_\infty(\Delta),\rho_\infty(\Delta+\delta))}}{\delta^2} = 2\int_0^\infty ds\, \mathrm{Tr}\left[\left(\frac{\partial \rho_\infty}{\partial \Delta} e^{-s\rho_\infty}\right)^2\right], \quad (4)$$

with the Uhlmann fidelity $\mathcal{F}(\rho,\sigma) = (\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}})^2$ [45]. Defining the eigenvalues $\{p_j\}_j$ and the corresponding eigenvectors $\{|j\rangle\}_j$ of the state ρ_∞, the definition (4) of the Fisher information reads [42–44]

$$F_\Delta = 2\sum_{j,l} \frac{|\langle j|\partial_\Delta \rho_\infty |l\rangle|^2}{p_j + p_l}. \quad (5)$$

The connection between the Fisher information F_Δ and estimation theory is summarized in the Cramér-Rao bound, which bounds any estimation variance of Δ [42–44]. If Δ is estimated by the measurement of the observable O, the Cramér-Rao bound reads

$$\mathrm{Var}(\Delta) = \frac{\Delta^2 O}{\left(\frac{\partial}{\partial\Delta}\langle O\rangle\right)^2} \geq \frac{1}{F_\Delta}, \quad (6)$$

where $\Delta^2 O$ is the variance of the observable O and $\mathrm{Var}(\Delta)$ follows from error propagation.

A property of the Uhlmann fidelity that is useful in the following is that it is nondecreasing under the action of trace-preserving and completely positive maps on both the arguments [70]. The partial trace, namely, the average over the degrees of freedom of subsystems, is a trace-preserving and completely positive map. Therefore, the Fisher information computed from (4) but using reduced states, i.e., resulting from partial traces of the full state ρ_∞, is a lower bound to the Fisher information of ρ_∞. In the next sections, we use the relation between local order parameters and the Fisher information computed with reduced states instead of full states, which we explain here.

Good order parameters for phase transitions are nonanalytic quantities at critical points. Consider local expectations $\langle O\rangle$ with

$$O = \sum_{\mathcal{R}} O_{\mathcal{R}}, \quad \langle O_{\mathcal{R}}\rangle = \mathrm{Tr}\left(\rho_\infty^{\mathcal{R}} O_{\mathcal{R}}\right), \quad (7)$$

where $\mathcal{R}$ are subsystems with finite, n-independent size,

$$\rho_\infty^{\mathcal{R}} = \mathrm{Tr}_{\bar{\mathcal{R}}}\rho_\infty \quad (8)$$

is the reduced state resulting from the partial trace over the complement $\bar{\mathcal{R}}$ of the subsystem $\mathcal{R}$, and $O_{\mathcal{R}}$ is an observable of the subsystem $\mathcal{R}$. Divergences of the derivatives of $\langle O\rangle$ are related to the Fisher information $F_\Delta^{\mathcal{R}}$ computed from Eq. (4) using the state $\rho_\infty^{\mathcal{R}}$. Suppose that the anisotropy Δ has to be estimated via measurements of local expectations $\langle O_{\mathcal{R}}\rangle$. The Cramér-Rao bound is a bound for any estimation variance [42–44]:

$$\mathrm{Var}(\Delta) = \frac{\Delta^2 O_{\mathcal{R}}}{\left(\frac{\partial}{\partial\Delta}\langle O_{\mathcal{R}}\rangle\right)^2} \geq \frac{1}{F_\Delta^{\mathcal{R}}}, \quad (9)$$

where $\Delta^2 O_{\mathcal{R}}$ is the variance of the observable $O_{\mathcal{R}}$ and $\mathrm{Var}(\Delta)$ follows from error propagation. Suppose, instead, we estimate Δ via experimental measurements of the kth derivative $\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle$. The Cramér-Rao bound reads

$$\mathrm{Var}(\Delta) = \frac{\mathrm{Var}\left(\frac{\partial^{k-1}}{\partial\Delta^{k-1}}\langle O_{\mathcal{R}}\rangle\right)}{\left(\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle\right)^2} \geq \frac{1}{F_\Delta^{\mathcal{R}}}, \quad (10)$$

where $\mathrm{Var}(\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle)$ is the variance of the experimental measurements of $\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle$. Such a quantity depends on the measured observables and on instrumental parameters, e.g., the increment of Δ if derivatives are estimated via difference quotients. Therefore, the size scaling of the reduced Fisher information bounds from above the degree of divergence of the derivatives of local expectations (7):

$$\left|\frac{\partial}{\partial\Delta}\langle O\rangle\right| \leq \sum_{\mathcal{R}}\left|\frac{\partial}{\partial\Delta}\langle O_{\mathcal{R}}\rangle\right| \leq \sum_{\mathcal{R}}\sqrt{F_\Delta^{\mathcal{R}}\,\Delta^2 O_{\mathcal{R}}}, \quad (11)$$

$$\left|\frac{\partial^k}{\partial\Delta^k}\langle O\rangle\right| \leq \sum_{\mathcal{R}}\left|\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle\right| \leq \sum_{\mathcal{R}}\sqrt{F_\Delta^{\mathcal{R}}\,\mathrm{Var}\left(\frac{\partial^{k-1}}{\partial\Delta^{k-1}}\langle O_{\mathcal{R}}\rangle\right)}. \quad (12)$$

We use these bounds to infer the existence of local order parameters.

IV. PERTURBATIVE ANALYSIS IN THE DISSIPATION STRENGTH

We start our investigation with the perturbative analysis for small noise strength $\frac{\lambda}{J}$. It is worth stressing that this analysis does not correspond to a perturbation around equilibrium. The zeroth order of the NESS is the completely mixed state which does not depend on any parameter. Therefore, there is no notion of temperature or of other equilibrium properties or traces of phase transitions in the zeroth order. As a consequence, our perturbative analysis already captures genuine nonequilibrium phase transitions. In this case, the NESS [39,71] is

$$\rho_\infty = \frac{1}{2^n}\left(\mathbb{1} + i\frac{\lambda\mu}{2J}(Z - Z^\dagger) + \frac{\lambda^2\mu}{8J^2}\{[Z,Z^\dagger] - \mu(Z-Z^\dagger)^2 - 2\mathrm{Tr}(ZZ^\dagger)\mathbb{1}\}\right) + O\left(\frac{\lambda}{J}\right)^3, \quad (13)$$

where Z is a matrix product operator,

$$Z = \sum_{\{s_1,\ldots,s_N\}\in\{0,+,-\}^N} \langle L|\prod_{j=1}^n A_{s_j}|R\rangle \bigotimes_{j=1}^n \sigma_j^{s_j}. \quad (14)$$

A_{s_j} are tridiagonal matrices on the auxiliary Hilbert space spanned by the orthonormal basis $\{|L\rangle, |R\rangle, |1\rangle, |2\rangle, \ldots, |\lfloor \frac{n}{2} \rfloor\rangle\}$:

$$\begin{aligned} A_0 &= |L\rangle\langle L| + |R\rangle\langle R| + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos(\eta k)|k\rangle\langle k|, \\ A_+ &= |1\rangle\langle R| - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sin(\eta k)|k+1\rangle\langle k|, \\ A_- &= |L\rangle\langle 1| + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sin(\eta(k+1))|k\rangle\langle k+1|, \end{aligned} \tag{15}$$

and $\eta = \arccos \Delta \in \mathbb{R} \cup i\mathbb{R}$. The expansion (13) holds as soon as the zeroth order is larger than the first order in $\frac{\lambda}{J}$. Estimating the magnitude of each order with its Hilbert-Schmidt norm ($||O||_{\mathrm{HS}} = \sqrt{\mathrm{Tr}(OO^\dagger)}$), the validity condition for (13) reads

$$\frac{\lambda}{J} < \frac{\sqrt{2^{n+1}}}{\mu}||Z||_{\mathrm{HS}}^{-1} = \begin{cases} O\left(\frac{1}{\sqrt{n}}\right) & \text{if } |\Delta| < 1, \\ O\left(\frac{1}{n}\right) & \text{if } |\Delta| = 1. \end{cases} \tag{16}$$

For $\Delta > 1$ the upper bound in (16) is the inverse of a superexponential function; thus, the perturbative expansion (13) is not useful.

A. Noncommuting limits for the Fisher information

At the lowest order in $\frac{\lambda}{J}$, the Fisher information is

$$\begin{aligned} F_\Delta &= \frac{\lambda^2\mu^2}{2^{n+1}J^2}\mathrm{Tr}(\partial_\Delta Z\, \partial_\Delta Z^\dagger) + O\left(\frac{\lambda}{J}\right)^4 \\ &= \frac{\lambda^2\mu^2}{J^2}(\widetilde{F}_\Delta + \widehat{F}_\Delta) + O\left(\frac{\lambda}{J}\right)^4, \end{aligned} \tag{17}$$

with the two non-negative contributions

$$\widetilde{F}_\Delta = \frac{1}{2(1-\Delta^2)}\sum_{j=1}^{n}\langle L|\mathbb{A}_0^{j-1}\mathbb{D}\mathbb{A}_0^{n-j}|R\rangle, \tag{18}$$

$$\widehat{F}_\Delta = \frac{1}{8(1-\Delta^2)}\frac{d^2}{d\eta^2}\langle L|\mathbb{A}_0^n|R\rangle \tag{19}$$

and the matrices $\mathbb{A}_0$ and $\mathbb{D}$ on the auxiliary space of the matrix product structure,

$$\mathbb{A}_0 = \sum_{\substack{k,k'=L,R,\\ 1,\ldots,\lfloor \frac{n}{2} \rfloor}} \left[(A_0)_{k,k'}^2 + \frac{1}{2}(A_+)_{k,k'}^2 + \frac{1}{2}(A_-)_{k,k'}^2\right]|k\rangle\langle k'|, \tag{20}$$

$$\begin{aligned} \mathbb{D} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\Bigg[&\mathrm{sgn}(1-\Delta^2)\frac{k^2}{2}|k\rangle\langle k| + \frac{k^2}{4}|k+1\rangle\langle k| \\ &+ \frac{(k+1)^2}{4}|k\rangle\langle k+1|\Bigg]. \end{aligned} \tag{21}$$

The trace in (17) equals a transition amplitude in the doubled auxiliary space, spanned by $\{|k\rangle \otimes |k'\rangle\}_{k,k'=L,R,1,2,\ldots,\lfloor \frac{n}{2} \rfloor}$; for example, the leftmost (rightmost) state on the right-hand sides of Eqs. (18) and (19) is actually $\langle L| \otimes \langle L|$ ($|R\rangle \otimes |R\rangle$). Nevertheless, only the subspace spanned by $\{|k\rangle \otimes |k\rangle\}_{k=L,R,1,2,\ldots,\lfloor \frac{n}{2} \rfloor}$ contributes, and then we apply the mapping $|k\rangle \otimes |k\rangle \to |k\rangle$ to reduce the dimension of the auxiliary space.

Now, we briefly reread the results of Ref. [57], originally focused on metrology but not on NESS-QPT, and then we report our results in order to end up with a full description of the NESS-QPT. The system undergoes a NESS-QPT at $|\Delta| = 1$, detected by superextensive Fisher information in the leading order,

$$F_\Delta \simeq \frac{\lambda^2\mu^2}{32J^2}n^4 \quad \text{for } \frac{\lambda\mu}{J} < \frac{1}{n} \text{ and large } n. \tag{22}$$

When the rescaled anisotropy parameter $\frac{\eta}{\pi} = \frac{\arccos\Delta}{\pi}$ is rational and $|\Delta| < 1$, the Fisher information in the leading order is

$$F_\Delta \simeq \frac{\lambda^2\mu^2}{J^2}(\widetilde{\xi}\, n^2 + \xi\, n) \quad \text{for } \frac{\lambda\mu}{J} < \frac{1}{\sqrt{n}} \text{ and large } n, \tag{23}$$

with size-independent coefficients $\widetilde{\xi}$ and ξ. Thus, F_Δ cannot be superextensive. Keeping only the leading contribution of the Fisher information in the thermodynamic limit and only afterwards setting $\frac{\eta}{\pi}$ to an irrational number result in $F_\Delta = \frac{\lambda^2\mu^2}{J^2}O(n^5)$, with some oscillations in n damped for more irrational $\frac{\eta}{\pi}$. The latter approach catches the superextensive size scaling of the Fisher information, i.e., the divergent degree of the Fisher information density, when the limit of $\frac{\eta}{\pi}$ approaching irrationals is taken after the thermodynamic limit.

Keeping in mind the just mentioned results of [57], we now present results aiming to complete the characterization of the NESS-QPT. We show that the limit of $\frac{\eta}{\pi}$ approaching an irrational number does not commute with the thermodynamic limit $n \to \infty$ for $|\Delta| < 1$. Consider first the thermodynamic limit and then the limit of $\frac{\eta}{\pi}$ approaching an irrational number via a sequence of rational approximants, say, $\frac{\eta_m}{\pi} = \frac{f_{m+1}}{f_m}$, with $\{f_m\}_m$ being the Fibonacci sequence for $m \geq 3$, which approaches the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ as $m \to \infty$. The coefficient ξ, plotted in Fig. 1, shows the divergence for $m \to \infty$ fitted by

$$\xi = (0.0107 \pm 0.0004) f_m^{3.993\pm0.007}. \tag{24}$$

When $|f_m| \geq \lfloor \frac{n}{2} \rfloor + 1, \xi(|f_m|) = \xi(\lfloor \frac{n}{2} \rfloor + 1)$; thus, $\xi = O(n^4)$ in the limit $m \to \infty$, in agreement with the above results.

We now show the numerical computation of the Fisher information with the opposite order of limits, namely, at irrational $\frac{\eta}{\pi}$ without any assumption about the particle number n. The log-log plot of the rescaled contribution $\frac{J^2}{\lambda^2\mu^2}\widetilde{F}_\Delta$ to the Fisher information, with $\widetilde{F}_\Delta < F_\Delta$, is shown in Fig. 2. We are particularly interested in the superextensivity of F_Δ as a signature of a critical phase, and the remaining contribution to F_Δ, i.e., $\widehat{F}_\Delta$, can scale only linearly with n. This plot shows slower overall growth compared to the Fisher information with the limit order exchanged, with fits given in Table I.

We have thus shown the noncommutativity of the two limits and also the superextensivity of the Fisher information for both limit orders for irrational $\frac{\eta}{\pi}$ which are all critical points. This

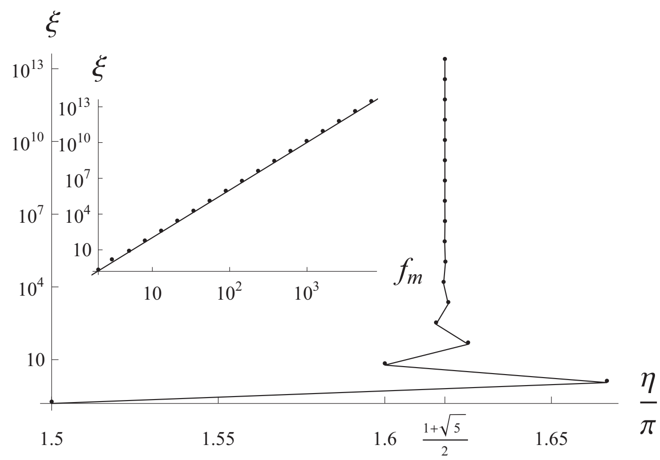

FIG. 1. Semilog plot of the coefficient ξ for $\frac{\eta}{\pi} = \frac{f_{m+1}}{f_m}$, with $\{f_m\}_m$ being the Fibonacci sequence. The inset shows the log-log plot of ξ as a function of f_m, which is perfectly fitted by $(0.0107 \pm 0.0004) f_m^{3.993\pm0.007}$.

indicates that the model has a critical phase for $|\Delta| \leq 1$ with a highly singular behavior.

B. Reduced Fisher information and the absence of local order parameters

A critical phase detected by the Fisher information was also observed in the XY model with boundary noise which is mapped to a free Fermion model [56]. Our model has Fermion interactions, i.e., the anisotropy term, and the above singular behavior. A critical phase with several peaks of the Fisher information was also found in the topological phase transition of the Kitaev honeycomb model [53] without local order parameters. This analogy demands a deeper understanding of the NESS-QPT in terms of order parameters. We undertake

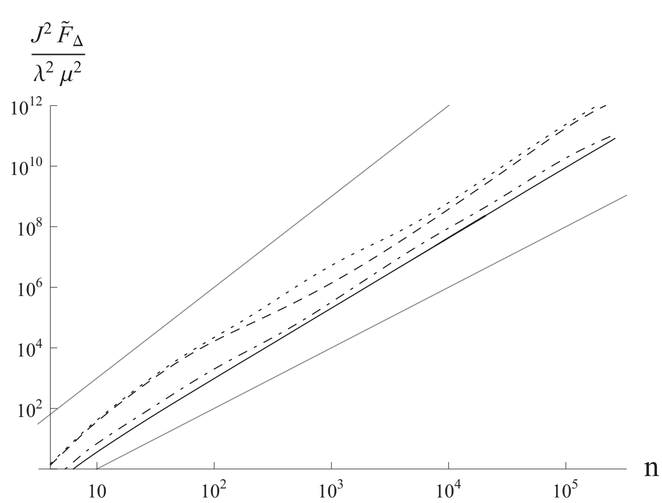

FIG. 2. Log-log plots of the contribution $\frac{J^2}{\lambda^2\mu^2}\widetilde{F}_\Delta$ to the Fisher information as a function of n for irrational $\frac{\eta}{\pi}$: $\eta = \pi\varphi$, with $\varphi = \frac{1+\sqrt{5}}{2}$ being the golden ratio (solid black line), $\eta = \pi\frac{\sqrt{3}}{2}$ (dotted line), $\eta = \pi^2$ (dashed line), and $\eta = \pi e$ (dot-dashed line). For comparison we also plot the slopes of power laws $10^{-2}n^2$ and n^3 (solid gray lines).

TABLE I. Fits of the size scalings plotted in Fig. 2. The fits are more precise when the oscillations are smaller.

$\frac{\eta}{\pi}$	Fit: $\frac{J^2}{\lambda^2\mu^2}\widetilde{F}_\Delta$
$\varphi = \frac{1+\sqrt{5}}{2}$	$(2.112 \pm 0.002)10^{-2}\, n^{2.32788\pm0.00009}$
$\frac{\sqrt{3}}{2}$	$(3.0 \pm 0.3)10^{-1}\, n^{2.37\pm0.01}$
π	$(1.9 \pm 0.2)10^{-1}\, n^{2.35\pm0.02}$
e	$(3.5 \pm 0.2)10^{-2}\, n^{2.341\pm0.006}$

this investigation based on the Fisher information of reduced states.

Defining the set $\mathcal{R} = \{\mathcal{R}_j\}_{j=1,\ldots,|\mathcal{R}|}$ made of $|\mathcal{R}|$ spins at increasing positions $\mathcal{R}_j$, the reduced NESS of this chain portion is

$$\rho_\infty^{\mathcal{R}} = \frac{1}{2^{|\mathcal{R}|}}\left[\mathbb{1}_{2^{|\mathcal{R}|}} + i\frac{\lambda\mu}{2J}(Z_{\mathcal{R}} - Z_{\mathcal{R}}^\dagger)\right] + O\left(\frac{\lambda}{J}\right)^2, \quad (25)$$

where $\mathbb{1}_{2^{\mathcal{R}}}$ is the $2^{\mathcal{R}} \times 2^{\mathcal{R}}$ identity matrix and

$$Z_{\mathcal{R}} = \sum_{\substack{\{s_{\mathcal{R}_j}\}_{j=1,\ldots,|\mathcal{R}|}\\ \in\{0,+,-\}^{|\mathcal{R}|}}} \langle L|A_{s_{\mathcal{R}_1}} \prod_{j=2}^{|\mathcal{R}|} A_0^{\mathcal{R}_j-\mathcal{R}_{j-1}-1} A_{s_{\mathcal{R}_j}}|R\rangle \bigotimes_{j=1}^{|\mathcal{R}|} \sigma_{\mathcal{R}_j}^{s_{\mathcal{R}_j}}. \quad (26)$$

In Eq. (26), we have used the fact that $A_0|R\rangle = |R\rangle$ and $\langle L|A_0 = \langle L|$.

We show upper bounds for the reduced Fisher information and the nonincreasing n dependence of local expectations, thus of $\Delta^2 O_{\mathcal{R}}$ and $\mathrm{Var}(\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle)$. These results, together with Eqs. (11) and (12), imply that there are no local order parameters, i.e., nonanalytic expectations (7). For instance, extensive expectations $\langle O\rangle$, e.g., with a number $O(n)$ of subsystems such as $\mathcal{R}$ labeling single spins or neighboring couples, cannot have superextensive derivatives.

We start this analysis by bounding the reduced Fisher information of arbitrary subsystems $\mathcal{R}$ with an n-independent size $|\mathcal{R}|$ at order $\frac{\lambda^2}{J^2}$:

$$F_\Delta^{\mathcal{R}} \leq O\left(\frac{1}{n}\right), \quad (27)$$

which follows from the matrix operator structure of ρ_∞ and $\rho_\infty^{\mathcal{R}}$ through the following logical steps:

(i) Coefficients of ρ_∞ expanded in the tensor basis made of Pauli matrices are generated by products of sequences of n tridiagonal matrices on an auxiliary space [65], as shown in Eq. (14).

(ii) The dependence of these coefficients on n enters through the number of matrices in the sequence generating a $\lfloor\frac{n}{2}\rfloor$-dimensional auxiliary subspace.

(iii) The coefficients of $\rho_\infty^{\mathcal{R}}$ in the Pauli tensor basis have an analogous structure, as shown in Eq. (26), but with the diagonal matrix A_0 in the matrix product at positions corresponding to traced-out spins.

(iv) This diagonal matrix A_0 does not have raising and lowering operators, and thus, the dimension of the generated auxiliary subspace equals $\frac{|\mathcal{R}|}{2}$. As a consequence, the dependence of $\rho_\infty^{\mathcal{R}}$ on n is manifest only from the exponents $\mathcal{R}_j - \mathcal{R}_{j-1} + 1 < n$.

(v) The modulus of A_0 is strictly upper bounded by the identity matrix at $|\Delta| < 1$. Therefore, the coefficients of $\rho_\infty^{\mathcal{R}}$ in the Pauli tensor basis are upper bounded in the modulus by an exponentially decaying function due to $A_0^{\mathcal{R}_j-\mathcal{R}_{j-1}-1}$ if some $\mathcal{R}_j - \mathcal{R}_{j-1} - 1$ grows with n or by an n-independent contribution if $\mathcal{R}_j - \mathcal{R}_{j-1} - 1 = O(n^0)$ for $j \in [2,|\mathcal{R}|]$. This already proves that local expectations, and thus $\Delta^2 O_{\mathcal{R}}$ and $\mathrm{Var}(\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle)$, do not increase with n. The above n dependence, together with the definition (4) and the range $\frac{\lambda}{J} < \frac{1}{\sqrt{n}}$ of the perturbation expansion, implies the bound (27) for $|\Delta| < 1$.

(vi) At $|\Delta| = 1$, the eigenvalues of A_0 are ± 1, and local expectations remain nonincreasing with n. Furthermore, the derivative in the definition (4) gives an additional multiplicative factor upper bounded by n when deriving the term $A_0^{\mathcal{R}_j-\mathcal{R}_{j-1}-1}$, but with the exponential damping suppressed in the limit $|\Delta| \to 1$. This multiplicative factor is, however, insufficient to compensate the smallness of $\frac{\lambda}{J} < \frac{1}{n}$, which is the validity range of the perturbation expansion at $|\Delta| = 1$. This again implies the bound (27).

Remarkable examples are reduced states $\rho_\infty^{\mathcal{R}}$ of contiguous blocks of spins, i.e., $\mathcal{R} = [\mathcal{R}_{\min},\mathcal{R}_{\max}]$, with all traced-out spins near the boundaries. These reduced states, i.e., Eq. (25) with $\mathcal{R}_j - \mathcal{R}_{j-1} - 1 = 0$, have exactly the same analytic form of the full steady state ρ_∞ with n replaced by the number of spins $|\mathcal{R}| = \mathcal{R}_{\max} - \mathcal{R}_{\min}$ in the subsystem. The system is thus self-similar.

Summing up, the kth derivatives of any observable with $k = O(n^0)$ lack divergent behaviors. If both k and $\mathcal{R}_j - \mathcal{R}_{j-1} - 1$ grow with n, the kth derivatives can diverge because of repeated derivations of the terms $A_0^{\mathcal{R}_j-\mathcal{R}_{j-1}-1}$. Since the lower bound in (10) does not depend on k, this divergence must be compensated by the numerator on the left-hand side. Intuitively, to measure derivatives at increasing orders, e.g., via different quotients, we need to distinguish many measurements all at values of Δ that lie within a very, ideally vanishingly, small interval. Thus, the measurements of such derivatives become very hard, witnessed by large $\mathrm{Var}(\frac{\partial^k}{\partial\Delta^k}\langle O_{\mathcal{R}}\rangle)$, and so is the consequent determination of Δ as imposed by the Cramér-Rao bound (10).

The above discussion is insufficient for the reduced states of a single spin, i.e., $\mathcal{R} = \{j\}$, because it is completely mixed at order $\frac{\lambda}{J}$, i.e., $\rho_\infty^{\mathcal{R}=\{j\}} = \frac{\mathbb{1}_2}{2} + O(\frac{\lambda}{J})^2$, and the next order is

$$\rho_\infty^{\mathcal{R}=\{j\}} = \frac{1}{2}\left(\mathbb{1}_2 + \frac{\lambda^2\mu}{4J^2}\gamma_j\sigma_j^z\right) + O\left(\frac{\lambda}{J}\right)^3, \tag{28}$$

with

$$\gamma_j = \langle L|\mathbb{A}_0^{j-1}\mathbb{A}_z\mathbb{A}_0^{n-j}|R\rangle, \tag{29}$$

$$\mathbb{A}_z = \frac{1}{2}\sum_{\substack{k,k'=L,R,\\ 1,\dots,\lfloor\frac{n}{2}\rfloor}}\left[(A_+)^2_{k,k'} - (A_-)^2_{k,k'}\right]|k\rangle\langle k'|, \tag{30}$$

and $\mathbb{A}_0$ defined in Eq. (20). As before, we have applied the mapping $|k\rangle\otimes|k\rangle \to |k\rangle$ to reduce the dimension of the auxiliary space because only the subspace spanned by $\{|k\rangle\otimes|k\rangle\}_{k=L,R,1,2,\dots,\lfloor\frac{n}{2}\rfloor}$ contributes.

The corresponding single-spin reduced Fisher information is

$$F_\Delta^{\mathcal{R}=\{j\}} = \frac{\left(\frac{\partial}{\partial\Delta}\langle\sigma_j^z\rangle\right)^2}{\Delta^2\sigma_j^z} = \frac{\lambda^4\mu^2}{256J^4}\left(\frac{\partial\gamma_j}{\partial\Delta}\right)^2 + O\left(\frac{\lambda}{J}\right)^5. \tag{31}$$

From numerical computations, we found intensive and even very small $F_\Delta^{\mathcal{R}=\{j\}}$ at $|\Delta| < 1$. This, together with the validity condition $\frac{\lambda}{J} < \frac{1}{\sqrt{n}}$ of the perturbative expansion at $|\Delta| < 1$, implies a bound tighter than (27), namely,

$$F_\Delta^{\mathcal{R}=\{j\}} \le O\left(\frac{\lambda^4}{J^4}n^0\right) < O\left(\frac{1}{n^2}\right). \tag{32}$$

The zeroth and first orders of γ_j around $|\Delta| = 1$ can be analytically computed truncating the auxiliary space to the subsystem spanned by $\{|L\rangle,|R\rangle,|1\rangle,|2\rangle\}$:

$$\gamma_j|_{\Delta=\pm1} = \tfrac{1}{4}(n-2j+1)[1 \pm (n-2)(\Delta \mp 1)] + O(\Delta \mp 1)^2. \tag{33}$$

Therefore, the Fisher information of the reduced state (28) at $|\Delta| = 1$ is

$$F_{\Delta=\pm1}^{\mathcal{R}=\{j\}} = \frac{\lambda^4\mu^2}{256J^4}(n-2)^2(n-2j+1)^2 + O\left(\frac{\lambda}{J}\right)^5. \tag{34}$$

The Fisher information of the single-spin reduced state (34) exhibits an apparent superextensive behavior, i.e., a power-law size scaling with an exponent between 2 and 4, depending on the spin position j. Nevertheless, this power law is reduced by the validity range of the perturbative expansion, namely, $\frac{\lambda}{J} < \frac{1}{n}$ at $|\Delta| = 1$. Therefore, the bound on the reduced Fisher information is

$$F_\Delta^{\mathcal{R}=\{j\}} \le O\left(\frac{\lambda^4}{J^4}n^4\right) < O(n^0). \tag{35}$$

Although the above bound does not allow for local order parameters, its increase with respect to (27) suggests that the superextensivity of $F_{\Delta=\pm1}^{\mathcal{R}=\{j\}}$ might gradually emerge when increasing the order of $\frac{\lambda}{J}$ and in the nonperturbative regime, as we discuss in Sec. V.

These results imply that there are no local order parameters detecting the NESS-QPT, like for topological phase transitions.

C. Nonlocal order parameters

Although there are no local order parameters at the lowest order in $\frac{\lambda}{J}$, there are nonlocal order parameters which detect at least the onset of the critical phase $|\Delta| = 1$, for instance, the expectation of

$$O_\Delta = 2^{n+1}\frac{J}{\mu}\frac{\partial}{\partial\lambda}\rho_\infty\bigg|_{\lambda=0} \tag{36}$$

or its limit $O_{\Delta\to\pm1}$ if one prefers a Δ-independent operator. The expectation of (36) satisfies

$$\langle O_\Delta\rangle = \frac{\lambda\mu}{J}\langle L|\mathbb{A}_0^n|R\rangle + O\left(\frac{\lambda}{J}\right)^2 \xrightarrow[\Delta\to\pm1]{} \frac{\lambda\mu}{8J}n(n-1) + O\left(\frac{\lambda}{J}\right)^2 \tag{37}$$

and

$$\frac{\partial}{\partial\Delta}\langle O_\Delta\rangle \xrightarrow[\Delta\to\pm1]{} \mp\frac{\lambda\mu}{12J}n(n-1)(n-2) + O\left(\frac{\lambda}{J}\right)^2. \tag{38}$$

The variance of O_Δ is

$$\Delta^2 O_\Delta = 2\langle L|\mathbb{A}_0^n|R\rangle + O\left(\frac{\lambda}{J}\right) \xrightarrow[\Delta\to\pm1]{} \frac{1}{4}n(n-1) + O\left(\frac{\lambda}{J}\right). \tag{39}$$

The nonlocal order parameter then has a superextensive derivative, i.e., a divergent density of the derivative in the thermodynamic limit. The density derivative is $\frac{1}{n}\frac{\partial}{\partial\Delta}O_\Delta = O(n^{2-\alpha})$ for $\frac{\lambda\mu}{J} = O(\frac{1}{n^\alpha})$, with $\alpha\in(1,2)$, compatible with the range of validity $\frac{\lambda\mu}{J} < \frac{1}{n}$ of the perturbative expansion for $|\Delta| = 1$. The ratio $(\frac{\partial}{\partial\Delta}\langle O_\Delta\rangle)^2/\Delta^2 O_\Delta$ also has the same scaling of the Fisher information $F_\Delta|_{|\Delta|=1}$ (22) almost saturating the Cramér-Rao bound (6) with $O = O_\Delta$.

V. NONPERTURBATIVE ANALYSIS IN THE DISSIPATION STRENGTH

In order to investigate the nonperturbative behavior of the Fisher information, we consider the steady state of the master equation (2) with $\mu = 1$ which is known for any λ [39,72–74]:

$$\rho_\infty = \frac{SS^\dagger}{\mathrm{Tr}(SS^\dagger)}, \quad S = \sum_{\substack{\{s_1,\dots,s_n\}\\ \in\{0,+,-\}^n}} \langle 0|\prod_{j=1}^n B_{s_j}|0\rangle \bigotimes_{j=1}^n \sigma_j^{s_j}, \tag{40}$$

with the matrix product operator S and tridiagonal matrices B_{s_j} on the auxiliary Hilbert space spanned by the orthonormal basis $\{|0\rangle,|1\rangle,|2\rangle,\dots,|\lfloor\frac{n}{2}\rfloor\rangle\}$,

$$B_0 = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\sin[\eta(s-k)]}{\sin(\eta s)}|k\rangle\langle k|,$$
$$B_+ = -\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\sin[\eta(k+1)]}{\sin(\eta s)}|k\rangle\langle k+1|, \tag{41}$$
$$B_- = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\sin[\eta(2s-k)]}{\sin(\eta s)}|k+1\rangle\langle k|,$$

with s given by $8i\sin\eta\cot(s\eta) = \lambda$.

While the numerical or analytical computation of the full-state Fisher information (4) is very hard, the computation of reduced states of small subsystems and their reduced Fisher information is feasible. The reduced Fisher information is a lower bound of the full-state Fisher information because the Uhlmann fidelity is a nondecreasing function under the action of trace-preserving and completely positive maps, like the partial trace, on both the arguments [70]. Therefore, superextensivity of the reduced Fisher information immediately implies superextensivity of the full state's Fisher information, which is the more general signature of the phase transition. As explained in Sec. III, superextensivity of the reduced Fisher information also provides additional knowledge, e.g., proving the existence of and deriving local order parameters.

The jth spin reduced state is diagonal in the σ_j^z basis:

$$\rho_\infty^{(j)} = \frac{1}{2}(\mathbb{1} + \gamma_j\sigma_j^z), \tag{42}$$

$$\gamma_j = \langle\sigma_j^z\rangle = \frac{\langle 0|\mathbb{B}_0^{j-1}\mathbb{B}_z\mathbb{B}_0^{n-j}|0\rangle}{\langle 0|\mathbb{B}_0^n|0\rangle}, \tag{43}$$

and

$$\mathbb{B}_0 = \sum_{k,k'=0}^{\lfloor\frac{n}{2}\rfloor}\left[|(B_0)_{k,k'}|^2 + \frac{1}{2}|(B_+)_{k,k'}|^2 + \frac{1}{2}|(B_-)_{k,k'}|^2\right] \times |k\rangle\langle k'|, \tag{44}$$

$$\mathbb{B}_z = \frac{1}{2}\sum_{k,k'=0}^{\lfloor\frac{n}{2}\rfloor}[|(B_+)_{k,k'}|^2 - |(B_-)_{k,k'}|^2]|k\rangle\langle k'|. \tag{45}$$

We have again applied the mapping $|k\rangle\otimes|k\rangle \to |k\rangle$ to reduce the dimension of the auxiliary space because only the subspace spanned by $\{|k\rangle\otimes|k\rangle\}_{k=0,1,2,\dots,\lfloor\frac{n}{2}\rfloor}$ contributes.

Therefore, the jth spin reduced Fisher information is

$$F_\Delta^{\mathcal{R}=\{j\}} = \frac{\left(\frac{\partial}{\partial\Delta}\langle\sigma_j^z\rangle\right)^2}{\Delta^2\sigma_j^z}, \tag{46}$$

saturating the Cramér-Rao bound (9) with $O_{\mathcal{R}=\{j\}} = \sigma_j^z$. The derivative $\frac{\partial}{\partial\Delta}\langle\sigma_j^z\rangle$ and $F_\Delta^{\{j\}}$ are both superextensive at $|\Delta| = 1$, as shown in Fig. 3 for $\Delta = 1$. The superextensive size scalings of $F_\Delta^{\{j\}}$ are fitted with power laws listed in Table II. The case $\Delta = -1$ gives similar results. The reduced Fisher information $F_\Delta^{\{j\}}$ is also symmetric with respect to reflection of the spin chain around its center.

TABLE II. Fits of the size scalings plotted in Fig. 3.

Spin position	Fit: $F_{\Delta=1}^{\{j\}}$
1	$(4.36\pm0.06)10^{-1}\,n^{1.992.\pm0.002}$
$\lfloor\frac{n}{4}\rfloor$	$(8.5\pm0.9)10^{-4}\,n^{3.97.\pm0.01}$
$\lfloor\frac{n}{2}\rfloor$	$(6.8\pm0.2)10^{-3}\,n^{1.993.\pm0.003}$

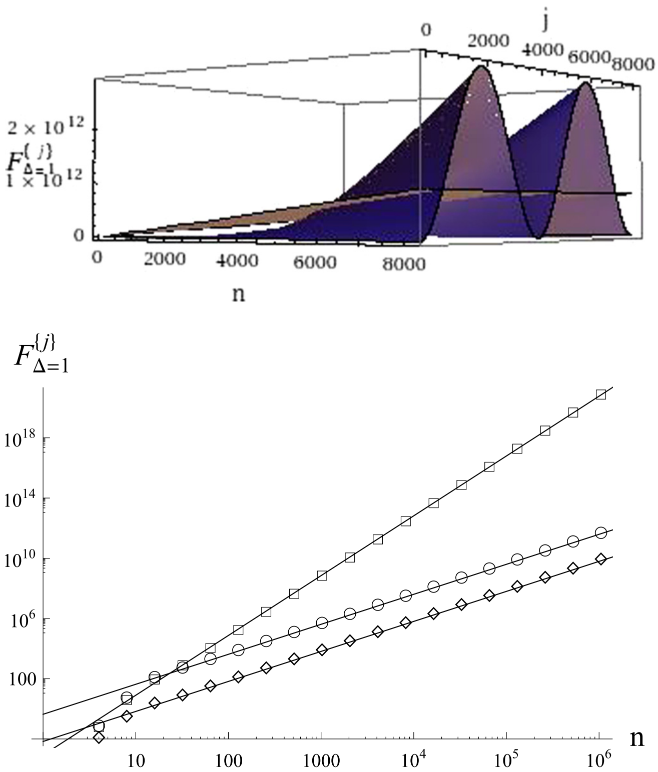

FIG. 3. Top: The Fisher information $F^{\{j\}}_{\Delta=1}$ of the jth spin reduced state as a function of n and j. The superextensivity is manifest from the comparison with the plane $10^8 n$. Bottom: Log-log plots of $F^{\{j\}}_{\Delta=1}$ as a function of n, set to powers of 2, for $\lambda = 1$ and $j = 1$ (circles), $j = \lfloor \frac{n}{4} \rfloor$ (squares), and $j = \lfloor \frac{n}{2} \rfloor$ (diamonds). The solid lines are the corresponding fits (see Table II), excluding the first three points of each line, which clearly deviate from the large-n behavior.

As a consequence of the above superextensivity, the magnetization profile $\langle \sigma^z_j \rangle$ is an intensive, local order parameter for the critical points $|\Delta| = 1$, with a diverging derivative $\frac{\partial}{\partial \Delta} \langle \sigma^z_j \rangle$, plotted in Fig. 4. The finite-size scaling of $\frac{\partial}{\partial \Delta} \langle \sigma^z_j \rangle$ equals the square root of that of the reduced Fisher information $F^{\mathcal{R}=\{j\}}_{\Delta}$ from (46) because the variance in the denominator is $\Delta^2 \sigma^z_j = 1 - \langle \sigma^z_j \rangle^2 = O(n^0)$, in agreement with Fig. 4. Extensive local order parameters are the magnetizations $\sum_{j \in \mathcal{R}} \langle \sigma^z_j \rangle$

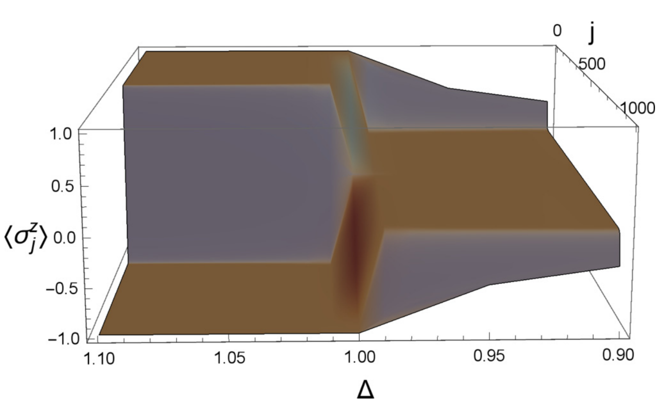

FIG. 4. Plot of the magnetization profile $\langle \sigma^z_j \rangle$ of the jth spin as a function of j and Δ for $n = 1000$ and $\lambda = 1$.

for any macroscopic but not centrosymmetric portion $\mathcal{R}$ of the chain. For centrosymmetric portions, the divergences at spin positions j and $n - j$ cancel each other. Other extensive local order parameters are $\sum_{j \in \mathcal{R}} f(\langle \sigma^z_j \rangle)$ with even functions $f(\cdot)$ and for any set $\mathcal{R}$, even centrosymmetric ones.

The reduced Fisher information does not show superextensive size scaling at $|\Delta| < 1$. Therefore, the superextensivity of the full-state Fisher information at $|\Delta| < 1$, and thus the presence of a critical phase in the nonperturbative regime, is still an open question.

VI. CONCLUSIONS

We derived characterizations of the NESS-QPT of the XXZ model with boundary noise, starting from the Fisher information. We identified a critical phase defined by the anisotropy range $|\Delta| \leq 1$, with irrational $\frac{\eta}{\pi}$ being critical points, for small dissipation. For instance, we observed a clear divergence for $\frac{\eta}{\pi}$ approaching the golden ratio through the Fibonacci sequence and superextensive Fisher information at different irrational $\frac{\eta}{\pi}$. This critical behavior lacks local order parameters but exhibits nonlocal ones. Moreover, it was observed for a small dissipation strength which vanishes for an infinite particle number. This limit might be considered to be similar to reducing the XYZ model to the XY model, which still exhibits a phase transition. Moreover, other topological characterizations of phase transitions already revealed critical points with nonanalytic microcanonical entropy at finite size which becomes smoother and analytic in the thermodynamic limit [54,55].

At nonperturbative dissipation, the reduced Fisher information provides a superextensive lower bound to the full-state Fisher information at $|\Delta| = 1$ together with local order parameters, e.g., the magnetization profile. Since the reduced Fisher information of the nonperturbative NESS is not superextensive for $|\Delta| < 1$, it is still an open question whether the Fisher information is superextensive.

We have proved the power of the Fisher information approach to characterize NESS-QPTs. We suggest that this approach will be useful for many other critical phenomena, such as classical nonequilibrium phase transitions [1–4], in quenched and dynamical systems [30,32], and in chaotic systems [54,75]. Superextensive Fisher information also identifies probes for the estimation of the control parameter with enhanced performance [42–44]. Our system, having a NESS with a very low or vanishing entanglement, is also relevant for enhanced metrological schemes without entanglement [76].

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