Bounds on complex polarizabilities and a new perspective on scattering by a lossy inclusion

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Here, we obtain explicit formulas for bounds on the complex electrical polarizability at a given frequency of an inclusion with known volume that follow directly from the quasistatic bounds of Bergman and Milton on the effective complex dielectric constant of a two-phase medium. We also describe how analogous bounds on the orientationally averaged bulk and shear polarizabilities at a given frequency can be obtained from bounds on the effective complex bulk and shear moduli of a two-phase medium obtained by Milton, Gibiansky, and Berryman, using the quasistatic variational principles of Cherkaev and Gibiansky. We also show how the polarizability problem and the acoustic scattering problem can both be reformulated in an abstract setting as "Y problems." In the acoustic scattering context, to avoid explicit introduction of the Sommerfeld radiation condition, we introduce auxiliary fields at infinity and an appropriate "constitutive law" there, which forces the Sommerfeld radiation condition to hold. As a consequence, we obtain minimization variational principles for acoustic scattering that can be used to obtain bounds on the complex backwards scattering amplitude. Some explicit elementary bounds are given.

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I. INTRODUCTION

Here, we consider scattering of waves by lossy inclusions. By lossy we mean that the inclusion absorbs energy. If the wavelength inside and outside the inclusion, and attenuation lengths inside the inclusion, are very long compared to the diameter of the inclusion, then one may use a quasistatic approximation, where one uses the usual static equations but with complex-valued fields and complex-valued material moduli. At fixed frequency ω the physical fields in the neighborhood of the inclusion are obtained by multiplying these complex fields by $e^{-i\omega t}$ and then taking the real part. The leading correction to the field at long distances from the inclusion, long compared to the diameter but short compared to the relevant wavelengths or attenuation lengths, is the dipolar part and the relation between it, and the incident field is governed by the polarizability of the inclusion.

In the context of the dielectric problem, a dilute array of scatterers each with polarizability matrix $\boldsymbol{\alpha}$, but randomly orientated so the average polarizability is $(\mathrm{Tr}\,\boldsymbol{\alpha}/3)\mathbf{I}$, has an effective dielectric constant well known to be

$$\epsilon_* \approx 1 + p\ \mathrm{Tr}(\boldsymbol{\alpha})/(3|\Omega|), \tag{1.1}$$

where $|\Omega|$ is the volume of the inclusion Ω, and p is the volume occupied by the inclusion phase in the array. Thus, the low volume fraction limit of the microstructure-independent Bergman-Milton bounds [1–4] on the complex dielectric constant ϵ_* of an isotropic two-phase composite immediately gives one bounds on the complex average polarizability. In this way, bounds on complex polarizabilities were obtained as long ago as 1979 [5], although it was not until 1981 that the results were published [see Fig. 3 in [6], reproduced here in Fig. 1(b)]. The Bergman-Milton bounds were obtained via the analytic approach, using the analytic properties of the effective dielectric constant as a function of the component dielectric constants. From a wider perspective, the bounds are related to bounds on Stieltjes and Herglotz functions, and to the Nevanlinna-Pick interpolation problem on which there is a huge literature. In the case where the bounds on the complex dielectric constant ϵ_* are sharp, such as in two dimensions [2–4], then the corresponding bounds on the complex polarizabilities are also, at least asymptotically, sharp. We mention that analytic representations, similar to those obtained for the effective moduli of composites [3,7–10], have also been obtained for the polarizability tensor [11–13], and for electromagnetic scattering [14,15].

Recently, there has been a resurgence of interest in such bounds on the complex polarizability, or at least the imaginary part which governs the absorption. This is fed by the realization that such bounds are helpful to determine the absorption of radiation of a cloud of dispersed subwavelength-sized metal particles that may be useful for smoke screens [16]. The authors of [16] apparently did not realize that bounds on the complex quaistatic polarizability are in fact a simple corollary of those on the complex dielectric constant of periodic two-phase composites in the small volume fraction limit.

The bounds on the complex dielectric constant have also been obtained using the variational principles of Cherkaev and Gibiansky [17]. In fact, for viscoelastic problems at fixed frequency where one is interested in bounding the complex effective elasticity tensor, it seems that the variational approach is more suitable than the analytic approach [18–20]. Both the variational approach and the the analytic approach have been extended to viscoelastic problems in the time domain, by Carini and Mattei [21] and Mattei and Milton [22], respectively. In this connection, for obtaining bounds on the viscoelastic response at a given time, it seems that the analytic approach is the most suitable method.

Most interesting has been the recent breakthrough result of Miller *et al.* [23] where through astoundingly simple arguments they obtain inclusion shape independent bounds on the scattered power, absorbed power, and their sum (known as the extinction) in terms of the material moduli, frequency, and amplitude of the incident plane wave. Most significantly,

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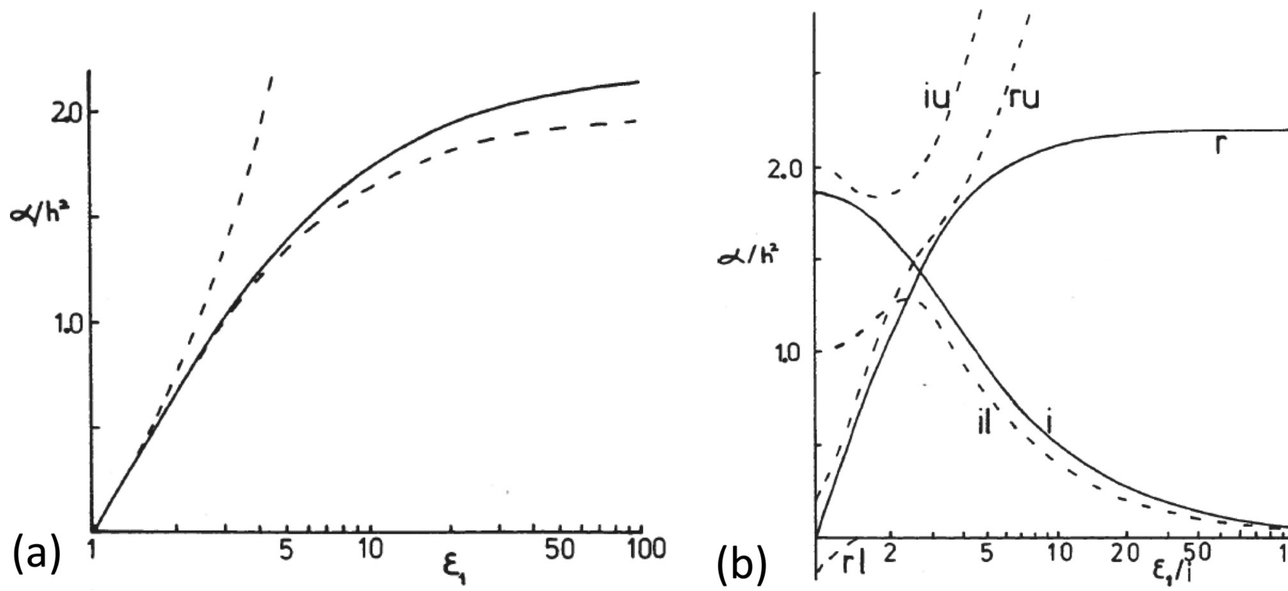

FIG. 1. The dashed lines show bounds on α/h^2 representing the orientationally averaged real and complex polarizability per unit volume of an arbitrarily shaped two-dimensional inclusion. In (a) the dielectric constant ϵ_1 of the inclusion is real, while in (b) it is purely imaginary. The surrounding medium has a dielectric constant of unity. In (b) "ru" and "rl" denote the upper and lower bounds on the real part of α/h^2, while "iu" and "il" denote the upper and lower bounds on the imaginary part of α/h^2. The solid lines are the numerical results for a square-shaped inclusion. The bounds in (a) are asymptotically attained in the cases of the solid circular cylinder and the thin cylindrical shell. The figures are reproductions, with permission of Springer, of Figs. 2 and 3 in [6].

they do not assume that the inclusion is small compared to the wavelength: they use the full time-harmonic Maxwell equations rather than just the quasistatic approximation.

Thus, one wonders if there are some variational minimization principles that apply to scattering by an inclusion. Here, we will see that indeed there are such variational minimization principles. However, with a choice of trial fields, they do not provide a bound on the extinction or, equivalently, the forward scattering amplitude, but rather surprisingly provide a bound on the *backward scattering amplitude*. Thus, it seems that these variational principles do not allow one to recover the extinction bounds of Miller *et al.* [23]. Our approach to obtaining variational principles follows that in Chap. 12 of [10]: since the equations are linear, the variational principles should be quadratic and obtained by expanding a positive-semidefinite quadratic form of the difference between the actual fields and the trial fields, where those terms in the expansion that involve products of the actual field and the trial field need to be integrated by parts (or equivalently evaluated using the orthogonality properties of the relevant subspaces of fields).

We mention that minimization principles have been obtained by Milton, Seppecher, and Bouchitté [24] and Milton and Willis [25] for the full time-harmonic acoustic equations, Maxwell's equations, and elastodynamic equations, in bodies of finite extent containing inhomogeneous lossy media. This advance was made possible by the key realization that these equations can all be suitably manipulated into a form where it is easy to see that one can directly apply the transformation techniques of Cherkaev and Gibiansky [17] to obtain minimization variational principles. While it is not immediately clear how to extend these variational principles to scattering, this is in fact what we will ultimately succeed in doing. For simplicity, we confine our attention to the acoustic problem: electromagnetic and elastodynamic scattering will be considered elsewhere.

We will see that problems of determining polarizabilty tensors and solving scattering by an inclusion can be naturally formulated in an abstract setting as "Y problems." For an introduction to "Y problems" and their significance, see Chaps. 19, 20, and 29 in the book [10], as well as Secs. 23.6, 23.7, and 24.10 therein, and also Chaps. 1, 2, 7, 9, and 10 in the book [26]. Briefly, "Y tensors," and the associated fractional linear transformations linking effective tensors and "Y tensors," first appeared in bounds on the effective moduli of composites, in formulas for effective medium approximations, and in continued fractions for the effective tensor [27–31]. The continued fractions were connected with a hierarchical spitting of the relevant Hilbert space, known as the field equation recursion method, in which "Y problems" make a natural appearance at successive stages of the procedure [30,32]. In the first stage of the procedure, for a two-phase periodic composite, the tensor $\mathbf{Y}_*$ was found to have a direct physical meaning, relating the phase averages of the fluctuating components of the fields [18]. For example, in a dielectric problem with a periodic dielectric constant, a periodic displacement field $\mathbf{d}(\mathbf{x})$ and periodic electric field $\mathbf{e}(\mathbf{x})$, one has

$$\langle \chi_i(\mathbf{d} - \langle \mathbf{d} \rangle) \rangle = -\mathbf{Y}_* \langle \chi_i(\mathbf{e} - \langle \mathbf{e} \rangle) \rangle, \tag{1.2}$$

where $\chi_i(\mathbf{x})$, $i = 1,2$, is the indicator function taking the value 1 in phase i, and 0 in the other phase, and the angular brackets $\langle \ldots \rangle$ denote a volume average over the unit cell of periodicity.

The setting of a Y problem is a Hilbert space, or finite-dimensional vector space, $\mathcal{K}$ that has the decomposition

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{H}, \tag{1.3}$$

where the spaces $\mathcal{E}$ and $\mathcal{J}$ are orthogonal complements, as are the spaces $\mathcal{V}$ and $\mathcal{H}$. Given a linear operator $\mathbf{L}$ mapping $\mathcal{H}$ to $\mathcal{H}$, the Y problem is to find for each given element $\mathbf{E}_1$ of $\mathcal{V}$ the associated fields

$$\mathbf{E}_2,\mathbf{J}_2 \in \mathcal{H}, \quad \mathbf{J}_1 \in \mathcal{V}, \quad \text{such that } \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \in \mathcal{E}, \\ \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \in \mathcal{J}, \quad \mathbf{J}_2 = \mathbf{L}\mathbf{E}_2. \tag{1.4}$$

Note that because $\mathcal{V}$ and $\mathcal{H}$ are orthogonal and span $\mathcal{K}$ any field, or vector, $\mathbf{K} \in \mathcal{K}$ can be split into $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$, where $\mathbf{K}_1 \in \mathcal{V}$ and $\mathbf{K}_2 \in \mathcal{H}$. Assuming that these fields are uniquely determined for each $\mathbf{E}_1 \in \mathcal{V}$, $\mathbf{J}_1$ must be linearly dependent on $\mathbf{E}_1$ and this linear relation

$$\mathbf{J}_1 = -\mathbf{Y}_*\mathbf{E}_1 \tag{1.5}$$

defines the associated operator $\mathbf{Y}_*$, which maps $\mathcal{V}$ to $\mathcal{V}$, or to a subspace of $\mathcal{V}$. The meaning of the spaces $\mathcal{K}$, $\mathcal{E}$, $\mathcal{J}$, $\mathcal{V}$, and $\mathcal{H}$ will of course depend on the problem under consideration and many examples can be given. In the context of a two-phase dielectric periodic composite, as in (1.2), $\mathcal{K}$ is the space of square-integrable periodic fields with zero average over the unit cell, $\mathcal{E}$ is the space of gradients of periodic potentials, $\mathcal{J}$ are those fields in $\mathcal{K}$ that have zero divergence, $\mathcal{V}$ are those fields in $\mathcal{K}$ that are constant in each phase, and $\mathcal{H}$ are those fields whose average over each phase is zero. Another concrete example is an electrical network comprised of a network of m, possibly complex, impedances on one side of the circuit board, and a network of b batteries or oscillating power sources on the other side of the circuit board, with the two networks being connected by terminal nodes drilled through the circuit board. Fields in $\mathcal{K}$ are then $(m + b)$-dimensional vectors whose elements represent the field components in the impedances or batteries. Fields in $\mathcal{E}$ are potential drops, while fields in $\mathcal{J}$ represent currents satisfying the condition that the net flux of current in or out of any node is zero. The subspace $\mathcal{E}$ can also be seen as the column space of the incidence matrix $\mathbf{M}$ of the entire network, and the subspace $\mathcal{J}$ as the null space of $\mathbf{M}^T$, thus accounting for the orthogonality of these subspaces. Fields in $\mathcal{V}$ have elements which are nonzero only in the batteries, while fields in $\mathcal{H}$ have elements which are nonzero only in the impedances. The matrix $\mathbf{L}$ is then diagonal with elements representing the individual impedance values. The tensor $\mathbf{Y}_*$ measures the response of the batteries. The orthogonality of $\mathbf{E}$ and $\mathbf{J}$ coupled with the orthogonality of implies $-\mathbf{J}_1 \cdot \mathbf{E}_1 = \mathbf{J}_2 \cdot \mathbf{E}_2$ or, equivalently, that

$$\mathbf{E}_1\mathbf{Y}_*\mathbf{E}_1 = \mathbf{E}_2 \cdot \mathbf{L}\mathbf{E}_2. \tag{1.6}$$

So, if $\mathbf{L}$ is real and positive semidefinite (as is the case when the impedances are resistors), then (1.6) is a restatement of the fact that the net power provided by the batteries is equal to the net power consumed by the impedances. It also implies $\mathbf{Y}_*$ is positive semidefinite, which is why the minus sign is introduced in the definition (1.5). For more details, see Chap. 19 in [10]. Interestingly, one can perform algebraic operations on Y problems, in the same way that one can perform algebraic operations such as addition, multiplication, and substitution with electrical circuits, and, moreover, if one removes the orthogonality constraints on the subspaces, these operations can be extended to include subtraction and division: one has a complete algebra (see Chap. 7 in [26]).

The advantage of recognizing that determining polarizabilty tensors and solving scattering by an inclusion are both "Y problems" is that one can more or less immediately write variational minimization principles, even when the moduli are complex, and also one can deduce important analytic properties of $\mathbf{Y}_*$ as a function of the component moduli. Both the variational principles and the analytic properties can lead to bounds on $\mathbf{Y}_*$, and thus to bounds on the polarizabilty tensor or on the scattering amplitudes.

II. FORMULATING THE PROBLEM OF DETERMINING THE POLARIZABILITY TENSOR AS A Y PROBLEM

The purpose of this section is twofold: first, to introduce Y problems in a simple setting quite close to that of acoustic scattering, namely, the dielectric problem, in quasistatics, of determining the complex polarizability tensor of a lossy inclusion in a three-dimensional infinite homogeneous dielectric medium; and second to review the accompanying standard analysis as it will have direct parallels in the context of acoustic scattering. In a two-phase periodic composite the simplest associated Y problem is obtained by stripping the constant fields from the underlying equations (see, for example, [18] and Sec. 19.1 in [10]). Similarly for the polarizabilty problem, the associated Y problem is obtained by stripping the constant applied incident fields from the underlying equations.

The permittivity $\boldsymbol{\epsilon}(\mathbf{x})$ is $\boldsymbol{\epsilon}_1$ inside the inclusion and $\boldsymbol{\epsilon}_0$ outside:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}_0 + (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)\chi(\mathbf{x}), \tag{2.1}$$

where $\chi(\mathbf{x})$ is the indicator function taking the value 1 in the inclusion and 0 outside. Let $\mathcal{K}$ denote the Hilbert space of square integrable three-component vector fields. Then, the constitutive law takes the form

$$\underbrace{\mathbf{d}_0 + \mathbf{d}^s(\mathbf{x})}_{\mathbf{d}(\mathbf{x})} = \boldsymbol{\epsilon}(\mathbf{x})[\underbrace{\mathbf{e}_0 + \mathbf{e}^s(\mathbf{x})}_{\mathbf{e}(\mathbf{x})}], \tag{2.2}$$

where $\mathbf{d}_0$ and $\mathbf{e}_0$ are constant fields, with $\mathbf{d}_0 = \boldsymbol{\epsilon}_0\mathbf{e}_0$, while

$$\mathbf{e}^s \in \mathcal{E}, \quad \mathbf{d}^s \in \mathcal{J}, \tag{2.3}$$

in which $\mathcal{E}$ is the space of fields in $\mathcal{K}$ that have zero curl, while $\mathcal{J}$ is the space of fields in $\mathcal{K}$ that have zero divergence. For simplicity, the dielectric tensor outside is assumed to be isotropic, of the form $\boldsymbol{\epsilon}_0 = \epsilon_0\mathbf{I}$, where ϵ_0 is a positive scalar. The electric potential $V(\mathbf{x})$ outside any sphere containing the inclusion has an expansion in spherical harmonics [33], the leading term of which is

$$V^s(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x}/(4\pi\epsilon_0 r^3) + \cdots, \tag{2.4}$$

and the associated electric field $\mathbf{e}^s(\mathbf{x}) = -\nabla V^s(\mathbf{x})$ is

$$\mathbf{e}^s = -\nabla V^s = -\mathbf{b}/(4\pi\epsilon_0 r^3) + 3\mathbf{x}(\mathbf{b} \cdot \mathbf{x})/(4\pi\epsilon_0 r^5) + \cdots. \tag{2.5}$$

So, we see that at large distances the dominant correction to the uniform field comes from terms involving the vector $\mathbf{b}$; this vector is known as the induced dipole moment. The factor of $4\pi\epsilon_0$ has been introduced into the above expansions so that $\mathbf{b}$ has a physical interpretation when inclusion is in free space and ϵ_0 represents the dielectric constant (or, more precisely, the electrical permittivity) of free space. As we will see shortly, $\mathbf{b}$ can then be identified with the first moment of the induced charge density.

Since the equations for the fields are linear, there must be a linear relation between the induced dipole moment $\mathbf{b}$ and the applied field $\mathbf{e}_0$. This linear relation

$$\mathbf{b} = \boldsymbol{\alpha}\mathbf{e}_0 \tag{2.6}$$

defines the polarizability tensor $\boldsymbol{\alpha}$ of the inclusion. This tensor has also been called the Pólya-Szegő matrix (see [34,35]).

For a fixed applied field $\mathbf{e}_0$ the vector $\mathbf{b}$ is determined by the integral of the polarization field

$$\mathbf{p}(\mathbf{x}) = [\epsilon(\mathbf{x}) - \epsilon_0]\mathbf{e}(\mathbf{x}) = \mathbf{d}(\mathbf{x}) - \epsilon_0\mathbf{e}(\mathbf{x}) = \mathbf{d}_0 + \mathbf{d}^s(\mathbf{x}) - \epsilon_0[\mathbf{e}_0 + \mathbf{e}^s(\mathbf{x})] = \mathbf{d}^s(\mathbf{x}) - \epsilon_0\mathbf{e}^s(\mathbf{x}) \tag{2.7}$$

over the volume of the inclusion. To see this we follow, for example, the argument given in Sec. 10.1 of [10]. Consider a ball B_{r_0} of very large radius r containing the inclusion. Since the polarization field is zero outside the inclusion, we can equate the integral of the polarization field over the inclusion with the integral of the polarization field over the ball B_{r_0}. Since the displacement field $\mathbf{d}(\mathbf{x})$ has zero divergence, and since $-\mathbf{e}^s(\mathbf{x})$ is the gradient of the electrical potential $V^s(\mathbf{x})$, it follows that for any vector $\mathbf{m}$,

$$\begin{aligned}\int_{B_{r_0}} \mathbf{m}\cdot\mathbf{p}(\mathbf{x})\,d\mathbf{x} &= \int_{B_{r_0}} \mathbf{d}^s(\mathbf{x})\cdot\nabla(\mathbf{m}\cdot\mathbf{x})\,d\mathbf{x} + \epsilon_0\mathbf{m}\cdot\int_{B_{r_0}} \nabla V^s(\mathbf{x})\,d\mathbf{x}(\mathbf{x}) \\ &= \int_{\partial B_{r_0}} (\mathbf{m}\cdot\mathbf{x})\mathbf{d}^s(\mathbf{x})\cdot\mathbf{n} + \epsilon_0 V^s(\mathbf{x})\mathbf{m}\cdot\mathbf{n}\,dS \\ &= \epsilon_0\mathbf{m}\cdot\int_{\partial B_{r_0}} V^s(\mathbf{x})\mathbf{n} - \mathbf{x}[\nabla V^s(\mathbf{x})\cdot\mathbf{n}]\,dS,\end{aligned} \tag{2.8}$$

where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ is the outward normal to the surface ∂B of the ball B. When the radius r of the ball B is sufficiently large, we can use the asymptotic formulas (2.4) and (2.5) to estimate these integrals:

$$\begin{aligned}\int_{\partial B_{r_0}} \mathbf{x}[\nabla V^s(\mathbf{x})\cdot\mathbf{n}]\,dS &\approx -\int_{\partial B_{r_0}} 2\mathbf{x}(\mathbf{b}\cdot\mathbf{x})/\left(4\pi\epsilon_0 r_0^4\right)dS = \frac{2}{3\epsilon_0}\mathbf{b}, \\ \int_{\partial B_{r_0}} V(\mathbf{x})\mathbf{n}\,dS &\approx \int_{\partial B_{r_0}} \mathbf{x}(\mathbf{b}\cdot\mathbf{x})/\left(4\pi\epsilon_0 r_0^4\right)dS = \frac{1}{3\epsilon_0}\mathbf{b},\end{aligned} \tag{2.9}$$

with these approximations becoming increasingly accurate as the radius r of the ball B_{r_0} approaches infinity. By subtracting these expressions and taking the limit as r approaches infinity, we see that

$$\int_{B_{r_0}} \mathbf{p}(\mathbf{x})\,d\mathbf{x} = \mathbf{b}. \tag{2.10}$$

Now we define $\mathcal{V}$ to consist of all fields of the form $\chi(\mathbf{x})\mathbf{v}$ where $\mathbf{v}$ is a constant vector, i.e., which are constant in the inclusion and zero outside, and we define $\mathcal{H}$ as the orthogonal complement of $\mathcal{V}$ in the subspace $\mathcal{K}$, i.e., those fields in $\mathcal{K}$ that have zero average value over the inclusion. Then, we rewrite (2.2) as

$$\mathbf{d}^s(\mathbf{x}) = \boldsymbol{\epsilon}(\mathbf{x})\mathbf{e}^s(\mathbf{x}) + (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)\chi(\mathbf{x})\mathbf{e}_0, \tag{2.11}$$

and express the fields in the form

$$\mathbf{e}^s(\mathbf{x}) = \mathbf{e}_1(\mathbf{x}) + \mathbf{e}_2(\mathbf{x}), \quad \mathbf{d}^s(\mathbf{x}) = \mathbf{d}_1(\mathbf{x}) + \mathbf{d}_2(\mathbf{x}) \quad \text{with } \mathbf{e}_1,\mathbf{d}_1 \in \mathcal{V}, \quad \mathbf{e}_2,\mathbf{d}_2 \in \mathcal{H}. \tag{2.12}$$

The projections onto $\mathcal{V}$ and $\mathcal{H}$ are $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ whose actions on a field $\mathbf{p}(\mathbf{x}) \in \mathcal{K}$ are given by

$$\boldsymbol{\Pi}_1\mathbf{p} = \chi\langle\mathbf{p}\rangle, \quad \boldsymbol{\Pi}_2\mathbf{p} = \mathbf{p} - \chi\langle\mathbf{p}\rangle. \tag{2.13}$$

Applying $\boldsymbol{\Pi}_2$ to both sides of (2.12) gives

$$\mathbf{d}_2 = \boldsymbol{\Pi}_2\boldsymbol{\epsilon}(\mathbf{x})\mathbf{e}_2 = \boldsymbol{\epsilon}(\mathbf{x})\mathbf{e}_2, \tag{2.14}$$

while applying $\boldsymbol{\Pi}_1$ to both sides of (2.12) or, equivalently, subtracting (2.14) from it, gives

$$\mathbf{d}_1 = \boldsymbol{\epsilon}_1\mathbf{e}_1 + (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)\chi(\mathbf{x})\mathbf{e}_0. \tag{2.15}$$

Equations (2.3), (2.12), and (2.14) are the defining equations for a Y problem: given $\mathbf{e}_1 \in \mathcal{V}$, find $\mathbf{d}_1 \in \mathcal{V}$ and $\mathbf{e}_2,\mathbf{d}_2 \in \mathcal{H}$, with $\mathbf{d}_2 = \boldsymbol{\epsilon}\mathbf{e}_2$ such that $\mathbf{e}_1 + \mathbf{e}_2 \in \mathcal{E}$ and $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{J}$. Since $\mathbf{d}_1$ depends linearly on $\mathbf{e}_1$ we may write

$$\mathbf{d}_1 = -\mathbf{Y}_*\mathbf{e}_1, \tag{2.16}$$

which defines the effective Y tensor $\mathbf{Y}_*$. Substituting this in (2.15) gives

$$\mathbf{e}_1 = -(\mathbf{Y}_* + \boldsymbol{\epsilon}_1)^{-1}(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)\chi(\mathbf{x})\mathbf{e}_0. \tag{2.17}$$

Also, by definition of the polarizability tensor $\boldsymbol{\alpha}$,

$$\boldsymbol{\alpha}\mathbf{e}_0 = |\Omega|\langle(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)[\mathbf{e}_0 + \mathbf{e}^s(\mathbf{x})]\rangle = |\Omega|\langle(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)(\mathbf{e}_0 + \mathbf{e}_1)\rangle, \tag{2.18}$$

and so we see that

$$\boldsymbol{\alpha} = |\Omega|[(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0) - (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)(\mathbf{Y}_* + \boldsymbol{\epsilon}_1)^{-1}(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_0)]. \tag{2.19}$$

III. BOUNDS ON THE ORIENTATIONALLY AVERAGED COMPLEX POLARIZABILITY TENSOR

Bounds on the polarizabilty tensor are an obvious consequence of bounds on the effective dielectric constant of composite materials. Consider an inclusion Ω of volume $|\Omega|$ having isotropic dielectic constant ϵ_1 which is surrounded by material with dielectric constant $\epsilon_2 = 1$. We let $\chi_1 = \epsilon_1 - 1$ denote the susceptibility of phase 1 [that is not to be confused with the characteristic function $\chi(\mathbf{x})$]. Let $\boldsymbol{\alpha}$ be its (possibly anisotropic) polarizability tensor. We then consider a dilute suspension of copies of this inclusion, with equally distributed random orientations. Then insert material (or void) with dielectic constant $\epsilon_2 = 1$ outside the inclusions. By symmetry this material has an isotropic effective dielectric constant ϵ_*, which remains isotropic no matter what value the volume fraction $p = |\Omega|/\ell^3$ occupied by the inclusions happens to be. In the limit $p \to 0$, one has the asymptotic formula

$$\epsilon_* \approx 1 + p\ \mathrm{Tr}(\boldsymbol{\alpha})/(d|\Omega|), \tag{3.1}$$

where $\mathrm{Tr}(\boldsymbol{\alpha})$ represents the average polarizability of the inclusions [in which $\mathrm{Tr}(\boldsymbol{\alpha})$ denotes the sum of the diagonal elements of the polarizability tensor], and $d = 2$ or 3 is the dimensionality of the space.

If ϵ_1 is real, then Hashin-Shtrikman [36] established that the effective dielectric constant ϵ_* lies between the formulas

$$1 + \frac{dp(\epsilon_1 - 1)}{d + (1-p)(\epsilon_1 - 1)}, \quad \epsilon_1 + \frac{3(1-p)\epsilon_1(1-\epsilon_1)}{d\epsilon_1 + p(1-\epsilon_1)}, \tag{3.2}$$

where $d = 2$ or 3 is the dimensionality of the composite. Taking the limit $p \to 0$ of each expression and using (3.1) establishes that $\mathrm{Tr}(\boldsymbol{\alpha})/(d|\Omega|)$ must lie between the bounds

$$\chi_1 - \chi_1^2/[d(1+\chi_1)], \quad \chi_1 - \chi_1^2/(\chi_1 + d). \tag{3.3}$$

If ϵ_1 is complex, then the Bergman-Milton [1–3,5,37] bounds imply that ϵ_* lies inside the region of the complex plane bounded by the circular arcs inscribed by the points

$$\begin{aligned}\epsilon_1^{\mathrm{BM}}(v) &= 1 + p(\epsilon_1 - 1)\\ &\quad - \frac{p(1-p)(\epsilon_1-1)^2}{(1-p)\epsilon_1 + p + (d-1)[v/\epsilon_1 + (1-v)]^{-1}},\\ \epsilon_2^{\mathrm{BM}}(w) &= 1 + p(\epsilon_1 - 1)\\ &\quad - \frac{p(1-p)(\epsilon_1-1)^2}{(1-p)\epsilon_1 + p + (d-1)[w\epsilon_1 + (1-w)]},\end{aligned} \tag{3.4}$$

as the real parameters v and w vary along the real axis between 0 and 1. Taking the limit $p \to 0$ of each expression and using (3.1) establishes that $\mathrm{Tr}(\boldsymbol{\alpha})/(d|\Omega|)$ must lie inside the region of the complex plane bounded by the circular arcs inscribed by the points

$$\alpha_1^{\mathrm{BM}}(v) = \chi_1 - \frac{\chi_1^2}{1 + \chi_1 + (d-1)[v/(1+\chi_1) + (1-v)]^{-1}}, \tag{3.5}$$

$$\alpha_2^{\mathrm{BM}}(w) = \chi_1 - \frac{\chi_1^2}{1 + \chi_1 + (d-1)(w\chi_1 + 1)},$$

as the real parameters v and w vary along the real axis between 0 and 1. The bounds (3.5) imply the bounds of Miller, Hsu, Homer Reid, DeLacy, Joannopoulos, Soljačić, and Johnson [16] on the quasistatic absorption of small particles (Miller, private communication). They point out the relevance of these bounds to determining limits on the absorption of light by smoke screens of small metal particles.

In two dimensions, improved bounds were obtained by Milton [2,3] who found that ϵ_* lies inside the region of the complex plane bounded by the circular arcs inscribed by the points

$$\begin{aligned}\epsilon_1^{\mathrm{M}}(v) &= \frac{(p\epsilon_1 + 1 - p + \epsilon_1)(\epsilon_1 + 1) - (1-p)v(\epsilon_1 - 1)^2}{[(1-p)\epsilon_1 + p + 1](\epsilon_1 + 1) - (1-p)v(\epsilon_1 - 1)^2},\\ \epsilon_2^{\mathrm{M}}(w) &= \epsilon_1\frac{(p\epsilon_1 + 2 - p)(\epsilon_1 + 1) - pw(\epsilon_1 - 1)^2}{[(1-p)\epsilon_1 + p + \epsilon_1](\epsilon_1 + 1) - pw(\epsilon_1 - 1)^2}.\end{aligned} \tag{3.6}$$

Taking the limit $p \to 0$ of each expression and using (3.1) establishes that $\mathrm{Tr}(\boldsymbol{\alpha})/(2|\Omega|)$ must lie inside the region of the complex plane bounded by the circular arc and straight line inscribed by the points

$$\begin{aligned}\alpha_1^{\mathrm{M}}(v) &= \frac{2\chi_1(2+\chi_1)}{(2+\chi_1)^2 - v\chi_1^2}, \quad \alpha_2^{\mathrm{M}}(w) = \frac{\chi_1(2+\chi_1)}{2(1+\chi_1)}\\ &\quad - \frac{w\chi_1^3}{(\chi_1+1)(\chi_1+2)},\end{aligned} \tag{3.7}$$

as the real parameters v and w vary along the real axis between 0 and 1.

This extremely simple approach to deriving bounds on the polarizabilty tensor is entirely rigorous once the asymptotic formula (3.1) is established. By this method, rigorous bounds on the real and complex polarizabilty α of two-dimensional inclusions having an isotropic polarizability tensor were established in Figs. 2 and 3 of Milton, McPhedran, and McKenzie [6], reproduced here in Fig. 1, and it was noted that when ϵ_1 is real the bounds are sharp for a disk, and for a very thin annulus. For two-dimensional inclusions that are perfectly conducting (effectively with ϵ_1 being infinite) Pólya and Szegő [34] had shown that the circular disk has the lowest average polarizabilty of any inclusion shape of the same area, where the average is taken over all orientations, and they conjectured that a perfectly conducting sphere had the lowest average polarizabilty of any inclusion shape of the same area. The conjecture is proved by the Hashin-Shtrikman bounds (3.3). A stronger form of the conjecture states that the sphere is the only shape that attains the bounds: this and the related weak Eshelby conjecture were proved in [38] (see also [39] for an independent proof of the weak Eshelby conjecture that states an ellipsoid is the only shape inside which the field is uniform for all uniform applied fields).

When ϵ_1 is real tighter bounds on the anisotropic polarizability tensor $\boldsymbol{\alpha}$ (without averaging over orientations) were obtained by Lipton [40] by considering a dilute array of inclusions all with the same orientation, having an effective dielectric tensor $\boldsymbol{\epsilon}_* \approx \mathbf{I} + p\boldsymbol{\alpha}/|\Omega|$ in the limit $p \to 0$. Lipton obtained the polarizabilty bounds by substituting this expression in the Tartar-Murat-Lurie-Cherkaev bounds [41–44], and taking the limit $p \to 0$.

Lipton [40] similarly derived bounds on the average elastic polarizability tensor, averaged over an ensemble of grain orientations not necessarily distributed randomly with the inclusion and matrix having real moduli, from the low volume fraction limit of the bounds of Avellaneda [45] and noted they were sharp for suitable distributions of platelike inclusions with at most 15 orientations in three dimensions (more recent work of [46] implies that six orientations suffice). Shape-independent bounds on the average elastic polarizability tensor also follow by taking the low volume fraction limit of the "trace bounds" of [47] and Zhikov [48,49]. Capdeboscq and Kang [50] show these can be tightened for inclusions which have some local thickness.

When the bulk modulus κ_1 and shear modulus μ_1 of the given inclusion phase are complex, while the bulk modulus κ_0 and shear modulus μ_0 of the surrounding material are real, then one can again consider the complex effective bulk modulus κ_* and the complex effective shear modulus μ_* of a dilute suspension of copies of the inclusion, randomly orientated, and occupying a volume fraction p tending to zero. The available bounds on κ_* and μ_* are naturally expressed in terms of their Y transforms,

$$\begin{aligned} y_\kappa &= -(1-p)\kappa_1 - p\kappa_0 + \frac{p(1-p)(\kappa_1-\kappa_0)^2}{p\kappa_1+(1-p)\kappa_0-\kappa_*}, \\ y_\mu &= -(1-p)\mu_1 - p\mu_0 + \frac{p(1-p)(\mu_1-\mu_0)^2}{p\mu_1+(1-p)\mu_0-\mu_*}. \end{aligned} \tag{3.8}$$

When the volume fraction p is small, we have

$$\kappa_* \approx (1+p\alpha_\kappa/|\Omega|)\kappa_0, \quad \mu_* \approx (1+p\alpha_\mu/|\Omega|)\mu_0, \tag{3.9}$$

in which α_κ and α_μ are the average bulk and shear polarizabilities, where these are obtained by averaging the possibly anisotropic fourth-order elastic polarizability tensor of the given inclusion over all orientations. Substituting these expressions in (3.8), we see that in the limit $p \to 0$ the formulas for y_κ and y_μ reduce to

$$\begin{aligned} y_\kappa &= -\kappa_1 + \frac{(\kappa_1-\kappa_0)^2}{\kappa_1-\kappa_0(1+\alpha_\kappa/|\Omega|)}, \\ y_\mu &= -\mu_1 + \frac{(\mu_1-\mu_0)^2}{\mu_1-\mu_0(1+\alpha_\mu/|\Omega|)}. \end{aligned} \tag{3.10}$$

Then, the Berryman-Gibiansky-Milton bounds on y_κ and y_μ for viscoelastic media [18–20], that were derived using the Cherkaev-Gibiansky variational principles [17], with y_κ and y_μ replaced by the expressions (3.10), directly give bounds on the possible complex values of the average bulk and shear polarizabilities α_κ and α_μ.

IV. ACOUSTIC SCATTERING

The polarizability problem is of course a limiting case of the scattering problem when the frequency of the incident field is very low, so that the wavelength of the incident radiation is much larger than the inclusion size. The success in Sec. II in reposing this as a Y problem suggests that we might be able to repose acoustic scattering at any frequency as a Y problem by eliminating the incident fields from the equations.

Let $P^a(\mathbf{x})$ and $\mathbf{v}^a(\mathbf{x})$ be the plane wave pressure and velocity fields that solve the acoustic equations in a homogeneous medium with density ρ_0 and bulk modulus κ_0, i.e.,

$$\underbrace{\begin{pmatrix} -i\mathbf{v}^a \\ -i\nabla\cdot\mathbf{v}^a \end{pmatrix}}_{\mathcal{G}^a} = \underbrace{\begin{pmatrix} -(\omega\rho_0)^{-1}\mathbf{I}_d & 0 \\ 0 & \omega/\kappa_0 \end{pmatrix}}_{\mathbf{Z}_0} \underbrace{\begin{pmatrix} \nabla P^a \\ P^a \end{pmatrix}}_{\mathcal{F}^a}, \tag{4.1}$$

where $\mathbf{I}_d$ is the $d\times d$ identity matrix. Specifically, if $P^a(\mathbf{x}) = p^a e^{i\mathbf{k}_0\cdot\mathbf{x}}$, then these have the solution

$$\begin{aligned} \mathcal{F}^a &= \begin{pmatrix} \nabla P^a \\ P^a \end{pmatrix} = \begin{pmatrix} i\mathbf{k}_0 p^a \\ p^a \end{pmatrix} e^{i\mathbf{k}_0\cdot\mathbf{x}}, \\ \mathcal{G}^a &= \begin{pmatrix} -i\mathbf{v}^a \\ -i\nabla\cdot\mathbf{v}^a \end{pmatrix} = \begin{pmatrix} -ip^a\mathbf{k}_0/(\omega\rho_0) \\ p^a\omega/\kappa_0 \end{pmatrix} e^{i\mathbf{k}_0\cdot\mathbf{x}}, \end{aligned} \tag{4.2}$$

implying that $v^a = p^a\mathbf{k}_0/(\omega\rho_0)e^{i\mathbf{k}_0\cdot\mathbf{x}}$ and that $\mathbf{k}_0$ must have magnitude $k_0 = |\mathbf{k}_0|$ given by

$$k_0 = \sqrt{\omega^2\rho_0/\kappa_0}. \tag{4.3}$$

We define $\mathcal{V}^0$ as the space spanned by all fields of the form

$$\chi(\mathbf{x})\begin{pmatrix} a_1\mathbf{k}_0 \\ a_2 \end{pmatrix} e^{i\mathbf{k}_0\cdot\mathbf{x}}, \tag{4.4}$$

as the complex constants a_1 and a_2 vary and $\mathbf{k}_0$ varies, with $k_0 = |\mathbf{k}_0|$ fixed and given by (4.3). We emphasize that fields in $\mathcal{V}^0$ do not necessarily have the form (4.4) but rather are a linear sum of fields of this form. The space $\mathcal{V}^0$ is the space of fields that exist inside the inclusion when it has the same properties as the matrix, and therefore is the analog of the space $\mathcal{V}$ in the polarizability problem.

Given fields

$$\mathcal{P}(\mathbf{x}) = \begin{pmatrix} \mathbf{p}(\mathbf{x}) \\ p(\mathbf{x}) \end{pmatrix}, \quad \mathcal{P}'(\mathbf{x}) = \begin{pmatrix} \mathbf{p}'(\mathbf{x}) \\ p'(\mathbf{x}) \end{pmatrix}, \tag{4.5}$$

where $\mathbf{p}(\mathbf{x})$ and $\mathbf{p}'(\mathbf{x})$ are d-dimensional vector fields, and $p(\mathbf{x})$ and $p'(\mathbf{x})$ are scalar fields, we define the inner product

$$(\mathcal{P}',\mathcal{P}) = \lim_{r_0\to\infty}\int_t w(t)(\mathcal{P}',\mathcal{P})_{r_0 t}\,dt, \tag{4.6}$$

in which $w(t)$ is some smooth non-negative weighting function, with say the properties that

$$w(t) = 0 \text{ when } t \leqslant 1/2 \text{ or } t \geqslant 2, \text{ and } 1 = \int_{1/2}^{2} w(t)dt \tag{4.7}$$

and

$$\begin{aligned} (\mathcal{P}',\mathcal{P})_r &= \int_{B_r} \overline{\mathcal{P}'(\mathbf{x})}\cdot\mathcal{P}(\mathbf{x}), \\ \text{where } \overline{\mathcal{P}'(\mathbf{x})}\cdot\mathcal{P}(\mathbf{x}) &\equiv \overline{\mathbf{p}'(\mathbf{x})}\cdot\mathbf{p}(\mathbf{x}) + \overline{p'(\mathbf{x})}p(\mathbf{x}), \end{aligned} \tag{4.8}$$

and $\overline{a}$ denotes the complex conjugate of a for any quantity a. We define $\mathcal{K}^0$ as the space of fields $\mathcal{P}^0$ such that the norm $|h\mathcal{P}^0| = (h\mathcal{P}^0,h\mathcal{P}^0)^{1/2}$, with inner product given by (4.6), is finite for all scalar functions $h(\mathbf{x}) \in C_0^\infty(\mathbb{R}^d)$ [where $C_0^\infty(\mathbb{R}^d)$ is the set of all infinitely differentiable functions with compact support] and which additionally have the asymptotic behavior

$$\begin{aligned} \mathcal{P}^0(\mathbf{x}) = {} & \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\begin{pmatrix} \widehat{\mathbf{x}}R_\infty^s(\widehat{\mathbf{x}}) \\ S_\infty^s(\widehat{\mathbf{x}}) \end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \\ & + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\begin{pmatrix} \widehat{\mathbf{x}}R_\infty^i(\widehat{\mathbf{x}}) \\ S_\infty^i(\widehat{\mathbf{x}}) \end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \end{aligned} \tag{4.9}$$

for some complex scalar functions $R_\infty^s(\mathbf{n})$, $S_\infty^s(\widehat{\mathbf{x}})$, $R_\infty^i(\mathbf{n})$, and $S_\infty^i(\mathbf{n})$ defined on the unit sphere $|\mathbf{n}| = 1$, where $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. Here, the superscript s is used because these field components will later be associated with the scattered field. The superscript i is used because these field components will later be associated with incoming fields, though not the incoming fields associated with the incident fields P^a and $\mathbf{v}^a$ as these will be treated separately. The subspace $\mathcal{K}^0$ has been defined in this way to ensure that if $\mathcal{P} \in \mathcal{K}^0$, then so are its real and imaginary parts, as taking real and imaginary parts is crucial to developing variational principles along the lines first suggested by Cherkaev and Gibiansky [17]. We define $\mathcal{K}^s$ as the space of fields $\mathcal{P}^0 \in \mathcal{K}^0$ satisfying the condition that $R_\infty^i(\mathbf{n}) = S_\infty^i(\mathbf{n}) = 0$ for all $\mathbf{n}$. Note that the norm $|\mathcal{P}^0| = (\mathcal{P}^0,\mathcal{P}^0)^{1/2}$ is not finite for fields in $\mathcal{K}^0$ if $R_\infty^s(\mathbf{n})$, $S_\infty^s(\widehat{\mathbf{x}})$, $R_\infty^i(\mathbf{n})$, or $S_\infty^i(\mathbf{n})$ is nonzero. We define $\mathcal{H}^0$ as the orthogonal complement of $\mathcal{V}^0$ in the space $\mathcal{K}^0$.

We are interested in solving

$$\underbrace{\begin{pmatrix} -i(\mathbf{v}^a+\mathbf{v}^s) \\ -i\nabla\cdot(\mathbf{v}^a+\mathbf{v}^s) \end{pmatrix}}_{\mathcal{G}^a+\mathcal{G}^s} = \underbrace{\begin{pmatrix} -(\omega\boldsymbol{\rho})^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla(P^a+P^s) \\ (P^a+P^s) \end{pmatrix}}_{\mathcal{F}^a+\mathcal{F}^s}, \tag{4.10}$$

where $P^s(\mathbf{x})$ is the scattered pressure, $\mathbf{v}^s(\mathbf{x})$ the associated scattered velocity, and $\mathcal{G}^s,\mathcal{F}^s \in \mathcal{K}^s$. Here, the density matrix $\boldsymbol{\rho}$ takes the value $\rho_0\mathbf{I}_d$ outside the inclusion and the value $\boldsymbol{\rho}_1$ inside the inclusion, while the bulk modulus scalar κ takes the value κ_0 outside the inclusion and the value κ_1 inside the inclusion. Due to viscoelasticity (energy loss under oscillatory compression), it is quite natural to have a bulk modulus that is complex with a negative imaginary part. We also allow for the density $\boldsymbol{\rho}_1$ to depend on the frequency ω and be anisotropic and possibly complex valued with a positive imaginary part, even with a negative real part, since this can be the case in metamaterials [51–63]. Alternatively, one can consider electromagnetic scattering off a cylindrical shaped inclusion (not necessarily with a circular cross section) and then the transverse electric and transverse magnetic equations are directly analogous to the two-dimensional acoustic equations. In that context, it is well known that both the electric permittivity tensor and magnetic permeability tensor can be anisotropic and complex valued, with positive-semidefinite imaginary parts.

Now, using the relation (4.1), that $\mathcal{G}^a = \mathbf{Z}_0\mathcal{F}^a$, we rewrite (4.10) as

$$\mathcal{G}^s(\mathbf{x}) = \mathbf{Z}(\mathbf{x})\mathcal{F}^s(\mathbf{x}) + (\mathbf{Z}_1-\mathbf{Z}_0)\chi(\mathbf{x})\mathcal{F}^a, \tag{4.11}$$

in which $\chi(\mathbf{x})\mathcal{F}^a \in \mathcal{V}^0$. We define $\mathcal{E}^0$ as the space of all fields $\mathcal{F}^0$ in $\mathcal{K}^0$ of the form

$$\mathcal{F}^0 = \begin{pmatrix} \nabla P^0(\mathbf{x}) \\ P^0(\mathbf{x}) \end{pmatrix} \tag{4.12}$$

for some scalar field $P^0(\mathbf{x})$, and we define $\mathcal{J}^0$ as the space of all fields $\mathcal{G}^0$ in $\mathcal{K}^0$ of the form

$$\mathcal{G}^0 = \begin{pmatrix} -i\mathbf{v}^0 \\ -i\nabla\cdot\mathbf{v}^0 \end{pmatrix} \tag{4.13}$$

for some vector field $\mathbf{v}^0(\mathbf{x})$. The fields $\mathcal{F}^0$ and $\mathcal{G}^0$, being in $\mathcal{K}^0$, have the asymptotic forms

$$\begin{aligned} \mathcal{F}^0(\mathbf{x}) &= \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{P^s_\infty(\widehat{\mathbf{x}})\begin{pmatrix} ik_0\widehat{\mathbf{x}} \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \\ &\quad + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{P^i_\infty(\widehat{\mathbf{x}})\begin{pmatrix} -ik_0\widehat{\mathbf{x}} \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\}, \\ \mathcal{G}^0(\mathbf{x}) &= \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{V^s_\infty(\widehat{\mathbf{x}})\begin{pmatrix} -i\widehat{\mathbf{x}}/k_0 \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \\ &\quad + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{V^i_\infty(\widehat{\mathbf{x}})\begin{pmatrix} i\widehat{\mathbf{x}}/k_0 \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\}, \end{aligned} \tag{4.14}$$

implying, through (4.12) and (4.13), that at large $|\mathbf{x}|$,

$$\begin{aligned} P^0(\mathbf{x}) &\approx \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}P^s_\infty(\widehat{\mathbf{x}}) + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}P^i_\infty(\widehat{\mathbf{x}}), \\ \mathbf{v}^0(\mathbf{x}) &\approx \widehat{\mathbf{x}}\frac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}V^s_\infty(\widehat{\mathbf{x}}) - \widehat{\mathbf{x}}\frac{e^{-ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}V^i_\infty(\widehat{\mathbf{x}}). \end{aligned} \tag{4.15}$$

The Sommerfeld radiation condition in fact implies that the $P^i_\infty(\widehat{\mathbf{x}})$ and $V^i_\infty(\widehat{\mathbf{x}})$ associated with the actual scattered pressure $P^s(\mathbf{x})$ and scattered velocity $\mathbf{v}^s(\mathbf{x})$ are zero, but we keep these terms as we want to impose a "constitutive law at infinity" that forces $P^i_\infty(\widehat{\mathbf{x}})$ and $V^i_\infty(\widehat{\mathbf{x}})$ to be zero and thus replaces the Sommerfeld radiation condition. Also, we want to define the spaces $\mathcal{E}^0$ and $\mathcal{J}^0$ so that if $\mathcal{F}^0 \in \mathcal{E}^0$ and $\mathcal{G}^0 \in \mathcal{J}^0$, then so are their real and imaginary parts. We extend the definition of $P^s_\infty(\widehat{\mathbf{x}})$ and $V^s_\infty(\widehat{\mathbf{x}})$ to all of $\mathbb{R}^3$ except the origin in the natural way by letting

$$P^s_\infty(\mathbf{x}) = P^s_\infty(\mathbf{x}/|\mathbf{x}|), \quad V^s_\infty(\mathbf{x}) = V^s_\infty(\mathbf{x}/|\mathbf{x}|). \tag{4.16}$$

Then, using the fact that $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ and $\widehat{\mathbf{x}} = \mathbf{x}/\sqrt{\mathbf{x}\cdot\mathbf{x}}$, we obtain

$$\begin{aligned} \nabla P^s(\mathbf{x}) &= \frac{ik_0\mathbf{x}e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|^2}P^s_\infty(\widehat{\mathbf{x}}) - \frac{\mathbf{x}e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|^3}P^s_\infty(\widehat{\mathbf{x}}) \\ &\quad + \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|^2}\mathbf{p}^s(\mathbf{x}/|\mathbf{x}|), \\ \nabla\cdot\mathbf{v}^s(\mathbf{x}) &= \frac{ie^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}V^s_\infty(\widehat{\mathbf{x}}) + (d-2)\frac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|^2}V^s_\infty(\widehat{\mathbf{x}}) \\ &\quad + \frac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|^2}v^s(\mathbf{x}/|\mathbf{x}|), \end{aligned} \tag{4.17}$$

where

$$\mathbf{p}^s(\mathbf{x}/|\mathbf{x}|) = |\mathbf{x}|\nabla P^s_\infty(\mathbf{x}), \quad v^s(\mathbf{x}/|\mathbf{x}|) = \mathbf{x}\cdot\nabla V^s_\infty(\mathbf{x}) \tag{4.18}$$

only depend on $\mathbf{x}/|\mathbf{x}|$ since $\nabla P^s_\infty(\lambda\mathbf{x}) = (1/\lambda)\nabla P^s_\infty(\lambda\mathbf{x})$ and $\nabla V^s_\infty(\lambda\mathbf{x}) = (1/\lambda)\nabla V^s_\infty(\lambda\mathbf{x})$ for all real $\lambda > 0$. The dominant terms in the expressions in (4.17) are the first terms, which justifies those terms in (4.14) that involve $P^s_\infty(\widehat{\mathbf{x}})$ and $V^s_\infty(\widehat{\mathbf{x}})$. The terms that involve $P^i_\infty(\widehat{\mathbf{x}})$ and $V^i_\infty(\widehat{\mathbf{x}})$ are justified in a similar way by extending those functions to all of $\mathbb{R}^3$ except the origin. Using integration by parts, we have the key identity that

$$(\mathcal{F}^0,\mathcal{G}^0)_r = \int_{B_r}\overline{\mathcal{F}^0}\cdot\mathcal{G}^0\,d\mathbf{x} = \int_{\partial B_r} -i\,\overline{P^0(\mathbf{x})}\mathbf{n}\cdot\mathbf{v}^0(\mathbf{x})\,dS. \tag{4.19}$$

From (4.15) we see that when $|\mathbf{x}|$ is large,

$$\begin{aligned} \overline{P^0(\mathbf{x})}\mathbf{n}\cdot\mathbf{v}^0(\mathbf{x}) &\approx \frac{1}{k_0|\mathbf{x}|^2}\big[\overline{P^s_\infty(\widehat{\mathbf{x}})}V^s_\infty(\widehat{\mathbf{x}}) - \overline{P^i_\infty(\widehat{\mathbf{x}})}V^i_\infty(\widehat{\mathbf{x}}) \\ &\quad + e^{2ik_0r}\,\overline{P^i_\infty(\widehat{\mathbf{x}})}V^s_\infty(\widehat{\mathbf{x}}) \\ &\quad - e^{-2ik_0r}\,\overline{P^s_\infty(\widehat{\mathbf{x}})}V^i_\infty(\widehat{\mathbf{x}})\big]. \end{aligned} \tag{4.20}$$

The last two cross terms that involve e^{2ik_0r} and e^{-2ik_0r} obviously oscillate very rapidly with r and will average to zero in the integral (4.6) involving the smooth weight function

$w(t)$. Thus, we get

$$(\mathcal{F}^0,\mathcal{G}^0) = \lim_{r_0\to\infty}\int_t dt\frac{-iw(t)}{k_0 r_0^2 t^2}\int_{B_{r_0 t}} \times\left[\overline{P^s_\infty(\widehat{\mathbf{x}})}V^s_\infty(\widehat{\mathbf{x}}) - \overline{P^i_\infty(\widehat{\mathbf{x}})}V^i_\infty(\widehat{\mathbf{x}})\right]dS = \frac{-i}{k_0}\int_{B_1}\left[\overline{P^s_\infty(\mathbf{n})}V^s_\infty(\mathbf{n}) - \overline{P^i_\infty(\mathbf{n})}V^i_\infty(\mathbf{n})\right]dS. \tag{4.21}$$

This lack of orthogonality of the subspaces $\mathcal{E}^0$ and $\mathcal{J}^0$ can be remedied by introducing an auxiliary space $\mathcal{A}$ of two-component vector fields $\mathbf{q}(\mathbf{n}) = [q_1(\mathbf{n}),q_2(\mathbf{n})]$ defined, and square integrable, on the unit sphere $|\mathbf{n}| = 1$. We then consider the Hilbert space $\mathcal{K}$ composed of fields $[\mathcal{P},q_1,q_2]$, where $\mathcal{P}\in\mathcal{K}^0$ and $\mathbf{q}(\mathbf{n}) = [q_1(\mathbf{n}),q_2(\mathbf{n})]\in\mathcal{A}$. In general, the field components $q_1(\mathbf{n})$ and $q_2(\mathbf{n})$ need not be related to the functions $R^s_\infty(\mathbf{n})$, $S^s_\infty(\widehat{\mathbf{x}})$, $R^i_\infty(\mathbf{n})$, and $S^i_\infty(\mathbf{n})$ appearing in the asymptotic expansion (4.9). The inner product between two fields $\mathcal{Q} = [\mathcal{P},q_1,q_2]$ and $\mathcal{Q}' = [\mathcal{P}',q_1',q_2']$ in $\mathcal{K}$ is defined as

$$\langle\mathcal{Q}',\mathcal{Q}\rangle = (\mathcal{P}',\mathcal{P}) + \frac{1}{2k_0}\int_{|\mathbf{n}|=1}\overline{q_1'(\mathbf{n})}q_1(\mathbf{n}) + \overline{q_2'(\mathbf{n})}q_2(\mathbf{n})\,dS. \tag{4.22}$$

We define $\mathcal{E}$ to consist of fields $\mathcal{F} = [\mathcal{F}^0,-iP^s_\infty + iP^i_\infty,P^s_\infty + P^i_\infty]\in\mathcal{K}$, where $\mathcal{F}^0\in\mathcal{E}^0$ while $P^s_\infty(\mathbf{n})$ and $P^i_\infty(\mathbf{n})$ are those functions that enter its asymptotic form (4.14). We define $\mathcal{J}$ to consist of fields $\mathcal{G} = [\mathcal{G}^0,V^s_\infty + V^i_\infty,iV^s_\infty - iV^i_\infty]\in\mathcal{K}$, where $\mathcal{G}^0\in\mathcal{J}^0$ while $V^s_\infty(\mathbf{n})$ and $V^i_\infty(\mathbf{n})$ are those functions that enter its asymptotic form (4.14). In each case, the accompanying auxiliary fields are, respectively,

$$\mathbf{q}_\mathcal{F}(\mathbf{n}) = \left[-iP^s_\infty(\mathbf{n}) + iP^i_\infty(\mathbf{n}),P^s_\infty(\mathbf{n}) + P^i_\infty(\mathbf{n})\right],\quad\text{and}$$
$$\mathbf{q}_\mathcal{G}(\mathbf{n}) = \left[V^s_\infty(\mathbf{n}) + V^i_\infty(\mathbf{n}),iV^s_\infty(\mathbf{n}) - iV^i_\infty(\mathbf{n})\right]. \tag{4.23}$$

The auxiliary fields have been defined in this way, in part, to ensure that if $\mathcal{F}\in\mathcal{J}$ and $\mathcal{G}\in\mathcal{J}$ then so too do their real and imaginary parts lie in these subspaces.

Now, the inner product of $\mathcal{F}$ and $\mathcal{G}$ is

$$\langle\mathcal{F},\mathcal{G}\rangle = (\mathcal{F}^0,\mathcal{G}^0) + \frac{1}{k_0}\int_{\partial B_1} i\overline{P^s_\infty(\mathbf{n})}V^s_\infty(\mathbf{n})\,dS + \frac{1}{k_0}\int_{\partial B_1} -i\overline{P^i_\infty(\mathbf{n})}V^i_\infty(\mathbf{n})\,dS = 0, \tag{4.24}$$

which implies the orthogonality of the spaces $\mathcal{E}$ and $\mathcal{J}$. Similarly, we extend the definition of $\mathcal{V}^0$: $\mathcal{V}$ consists of pairs $\mathcal{P}_1 = [\mathcal{P}^0_1,0,0]$ where $\mathcal{P}^0_1\in\mathcal{V}^0$. We define $\mathcal{H}^0$ as the orthogonal complement of $\mathcal{V}^0$ in the space $\mathcal{K}^0$, and $\mathcal{H}$ as the orthogonal complement of $\mathcal{V}$ in the space $\mathcal{K}$: it consists of fields $\mathcal{P}_2 = [\mathcal{P}^0_2,q_1,q_2]$, where $\mathbf{q} = [q_1,q_2]\in\mathcal{A}$ and $\mathcal{P}^0_2\in\mathcal{H}^0$, which implies

$$\int_\Omega\overline{\mathcal{P}^0_2(\mathbf{x})}\cdot\begin{pmatrix}a_1\mathbf{k}_0\\ a_2\end{pmatrix}e^{i\mathbf{k}_0\cdot\mathbf{x}}\,d\mathbf{x} = 0\quad\text{for all } a_1,a_2. \tag{4.25}$$

The fields $\mathcal{F}^s$ and $\mathcal{G}^s$ that solve (4.10) are, respectively, in $\mathcal{E}^0$ and $\mathcal{J}^0$

$$\mathcal{F}^s\in\mathcal{E}^0,\qquad\mathcal{G}^s\in\mathcal{J}^0, \tag{4.26}$$

and we express them in the form

$$\mathcal{F}^s(\mathbf{x}) = \mathcal{F}^s_1(\mathbf{x}) + \mathcal{F}^s_2(\mathbf{x}),\quad\mathcal{G}^s(\mathbf{x}) = \mathcal{G}^s_1(\mathbf{x}) + \mathcal{G}^s_2(\mathbf{x}),$$
$$\text{with }\mathcal{F}^s_1,\mathcal{G}^s_1\in\mathcal{V}^0,\quad\mathcal{F}^s_2,\mathcal{G}^s_2\in\mathcal{H}^0. \tag{4.27}$$

Clearly, we have

$$\mathcal{G}^s_1 = \mathcal{G}^a + \mathcal{G}^s_1 - \mathbf{Z}_0\mathcal{F}^a = \mathbf{Z}_1(\mathcal{F}^a + \mathcal{F}^s_1) - \mathbf{Z}_0\mathcal{F}^a, \tag{4.28}$$

and substracting this formula for $\mathcal{G}^s_1(\mathbf{x})$,

$$\mathcal{G}^s_1(\mathbf{x}) = \mathbf{Z}(\mathbf{x})\mathcal{F}^s_1(\mathbf{x}) + (\mathbf{Z}_1 - \mathbf{Z}_0)\chi(\mathbf{x})\mathcal{F}^a, \tag{4.29}$$

from (4.11) we see that

$$\mathcal{G}^s_2(\mathbf{x}) = \mathbf{Z}(\mathbf{x})\mathcal{F}^s_2(\mathbf{x}). \tag{4.30}$$

Of course, since $\mathcal{F}^s$ and $\mathcal{G}^s$ lie in $\mathcal{K}^s$, rather than just $\mathcal{K}^0$, the asymptotic components $P^i_\infty(\mathbf{n})$ and $V^i_\infty(\mathbf{n})$ are zero. However, let us remove this restriction and allow nonzero values of $P^i_\infty(\mathbf{n})$ and $V^i_\infty(\mathbf{n})$, that we will then show must be zero. The associated fields $\mathcal{F}_2 = [\mathcal{F}^s_2,-iP^s_\infty + iP^i_\infty,P^s_\infty + P^i_\infty]\in\mathcal{H}$ and $\mathcal{G}_2 = [\mathcal{G}^s_2,V^s_\infty + V^i_\infty,iV^s_\infty - iV^i_\infty]\in\mathcal{H}$ have auxiliary components $\mathbf{q}_\mathcal{F}$ and $\mathbf{q}_\mathcal{G}$ given by (4.23). We require that these auxiliary components satisfy the constitutive law

$$\mathbf{q}_\mathcal{G} = \frac{i\omega}{\kappa_0}\mathbf{q}_\mathcal{F} \tag{4.31}$$

vskip-2ptor, equivalently, we have

$$V^s_\infty + V^i_\infty = \omega(P^s_\infty - P^i_\infty)/\kappa_0,$$
$$iV^s_\infty - iV^i_\infty = i\omega(P^s_\infty + P^i_\infty)/\kappa_0. \tag{4.32}$$

Additionally, the constitutive law (4.30) allows us to relate the asymptotic terms of $\mathcal{G}^s_2(\mathbf{x})$ and $\mathcal{F}^s_2(\mathbf{x})$ giving

$$V^s_\infty = \omega P^s_\infty/\kappa_0,\quad V^i_\infty = \omega P^i_\infty/\kappa_0. \tag{4.33}$$

In conjunction with (4.32), this forces

$$V^i_\infty(\mathbf{n}) = P^i_\infty(\mathbf{n}) = 0, \tag{4.34}$$

as desired. Thus, we have replaced the Sommerfeld radiation condition with the constitutive law (4.31).

There is a natural division of the Hilbert space $\mathcal{H}$ into three orthonormal subpaces: the space $\mathcal{S}_1$ of fields $\mathcal{P}_2 = [\mathcal{P}^s_2,0,0]$ where $\mathcal{P}^s_2(\mathbf{x})\in\mathcal{H}^s$ is nonzero only in the inclusion phase; the space $\mathcal{S}_2$ of fields $\mathcal{P}_2 = [\mathcal{P}^s_2,0,0]$ where $\mathcal{P}^s_2(\mathbf{x})\in\mathcal{H}^s$ is nonzero only in the matrix phase; and the space $\mathcal{S}_3$ of fields $\mathcal{P}_2 = [0,q_1,q_2]$ where $\mathbf{q} = [q_1,q_2]\in\mathcal{A}$. In the first two cases, we define the action of an operator $\mathbf{L}$ on these fields to be $\mathbf{L}\mathcal{P}_2 = [\mathbf{Z}_1\mathcal{P}^s_2,0,0]$ and $\mathbf{L}\mathcal{P}_2 = [\mathbf{Z}_0\mathcal{P}^s_2,0,0]$, respectively, and in the third case to be $\mathbf{L}\mathcal{P}_2 = [0,(i\omega/\kappa_0)q_1,(i\omega/\kappa_0)q_2]$ to agree with (4.31). More generally, the action of $\mathbf{L}$ on any field $\mathcal{P}_2$ in $\mathcal{H}$ is obtained by resolving $\mathcal{P}_2$ into its components in these three subspaces, and summing the action of $\mathbf{L}$ on the component fields. With this definition, (4.30) and (4.31) imply the constitutive law

$$\mathcal{G}_2 = \mathbf{L}\mathcal{F}_2. \tag{4.35}$$

Equations (4.26), (4.27), and (4.35) are the defining equations for a Y problem: given $\mathcal{F}_1\in\mathcal{V}$, find $\mathcal{G}_1\in\mathcal{V}$ and $\mathcal{F}_2,\mathcal{G}_2\in\mathcal{H}$, with $\mathcal{G}_2 = \mathbf{L}\mathcal{F}_2$ such that $\mathcal{F}_1 + \mathcal{F}_2\in\mathcal{E}$ and $\mathcal{G}_1 + \mathcal{G}_2\in\mathcal{J}$. Since $\mathcal{G}_1$ depends linearly on $\mathcal{F}_1$ we may write

$$\mathcal{G}_1 = -\mathbf{Y}_*\mathcal{F}_1, \tag{4.36}$$

which defines the effective Y operator $\mathbf{Y}_*$. Since $\mathcal{V}$ consists of fields $\mathcal{P}_1 = [\mathcal{P}_1^s,0,0]$ where $\mathcal{P}_1^s \in \mathcal{V}^0$ we can equivalently write the relation (4.36) as

$$\mathcal{G}_1^s = -\mathbf{Y}_*^0\mathcal{F}_1^s, \tag{4.37}$$

which defines the effective Y operator $\mathbf{Y}_*^0$. Substituting this in (4.29) gives

$$\mathcal{F}_1^s = -(\mathbf{Y}_*^0 + \mathbf{Z}_1)^{-1}(\mathbf{Z}_1 - \mathbf{Z}_0)\chi(\mathbf{x})\mathcal{F}^a. \tag{4.38}$$

We emphasize that $\mathbf{Y}_*^0$ is a linear operator which maps fields in $\mathcal{V}^0$ to fields in $\mathcal{V}^0$: it is not a matrix that acts locally on the fields. Notice that the orthogonality of the spaces $\mathcal{E}$ and $\mathcal{J}$ and the orthogonality of the spaces $\mathcal{V}$ and $\mathcal{H}$ imply

$$\begin{aligned}0 &= \langle \mathcal{F}_1 + \mathcal{F}_2, \mathcal{G}_1 + \mathcal{G}_2\rangle = \langle \mathcal{F}_1, \mathcal{G}_1\rangle + \langle \mathcal{F}_2, \mathcal{G}_2\rangle \\ &= -\left(\mathcal{F}_1^s, \mathbf{Y}_*^0\mathcal{F}_1^s\right) + \langle \mathcal{F}_2, \mathbf{L}\mathcal{F}_2\rangle.\end{aligned} \tag{4.39}$$

Now, let

$$P^{a'}(\mathbf{x}) = e^{-i\mathbf{k}_0'\cdot\mathbf{x}} \quad \text{and}$$
$$\mathbf{v}^{a'}(\mathbf{x}) = -i(\omega\rho_0)^{-1}\nabla e^{-i\mathbf{k}_0'\cdot\mathbf{x}} = -\mathbf{k}_0'(\omega\rho)^{-1}e^{-i\mathbf{k}_0'\cdot\mathbf{x}} \tag{4.40}$$

be another plane wave pressure and associated velocity field that solve the acoustic equations in the homogeneous medium with density ρ_0 and bulk modulus κ_0, i.e.,

$$\underbrace{\begin{pmatrix} -i\mathbf{v}^{a'} \\ -i\nabla\cdot\mathbf{v}^{a'}\end{pmatrix}}_{\mathcal{G}^{a'}} = \underbrace{\begin{pmatrix} -(\omega\rho_0)^{-1}\mathbf{I}_d & 0 \\ 0 & \omega/\kappa_0\end{pmatrix}}_{\mathbf{Z}_0}\underbrace{\begin{pmatrix}\nabla P^{a'} \\ P^{a'}\end{pmatrix}}_{\mathcal{F}^{a'}}. \tag{4.41}$$

Using the key identity we have that

$$\begin{aligned} I_1 &\equiv \int_{B_{r_0}} \mathcal{F}^{a'}\cdot(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)\,d\mathbf{x} \\ &= \int_{B_{r_0}} [\mathcal{F}^{a'}\cdot\mathcal{G}^s - (\mathbf{Z}_0\mathcal{F}^{a'})\cdot\mathcal{F}^s]\,d\mathbf{x} \\ &= \int_{B_{r_0}} (\mathcal{F}^{a'}\cdot\mathcal{G}^s - \mathcal{G}^{a'}\cdot\mathcal{F}^s)\,d\mathbf{x} \\ &= \int_{\partial B_{r_0}} -iP^{a'}\mathbf{n}\cdot\mathbf{v}^s + iP^s\mathbf{n}\cdot\mathbf{v}^{a'}\,dS. \end{aligned} \tag{4.42}$$

Clearly, the integrand on the left-hand side vanishes outside Ω and so the integral must be independent of the radius r of the ball B_{r_0} (so long as it contains the inclusion). So, one can evaluate this integral by taking the limit $r_0 \to \infty$, which will be done in the next section. The identity (4.42) is the analog of the identity (2.8) that for the polarization problem expresses an integral over the inclusion in terms of the far field.

Alternatively, using (4.29), we can write the left-hand side of the equation as

$$\begin{aligned} I_1 &= \int_{B_{r_0}} \mathcal{F}^{a'}\cdot(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)\,d\mathbf{x} \\ &= \int_{\mathbb{R}^3} \chi\mathcal{F}^{a'}\cdot\left(\mathcal{G}_1^s + \mathcal{G}_2^s\right) - \chi\mathcal{F}^{a'}\cdot(\mathbf{Z}_0\mathcal{F}_1^s) - (\mathbf{Z}_0\chi\mathcal{F}^{a'})\cdot\mathcal{F}_2^s\,d\mathbf{x} \\ &= \int_\Omega \mathcal{F}^{a'}\cdot(\mathbf{Z}_1 - \mathbf{Z}_0)\mathcal{F}_1 + \mathcal{F}^{a'}\cdot(\mathbf{Z}_1 - \mathbf{Z}_0)\mathcal{F}^a\,d\mathbf{x} \\ &= \int_\Omega \mathcal{F}^{a'}\cdot[(\mathbf{Z}_1 - \mathbf{Z}_0) - (\mathbf{Z}_1 - \mathbf{Z}_0)(\mathbf{Y}_* + \mathbf{Z}_1)^{-1} \\ &\quad\times(\mathbf{Z}_1 - \mathbf{Z}_0)]\mathcal{F}^a\,d\mathbf{x}, \end{aligned} \tag{4.43}$$

where we have used the fact that $\chi\mathcal{F}^{a'}$ and $\mathbf{Z}_0\chi\mathcal{F}^{a'}$ are in $\mathcal{V}^0$, and hence orthogonal to $\mathcal{G}_2^s$ and $\mathcal{F}_2^s$. Let $\widetilde{\mathcal{V}}$ be that subspace of $\mathcal{V}$ comprised of all linear combinations of fields of the form

$$\chi(\mathbf{x})\mathcal{F}^a = \chi(\mathbf{x})\begin{pmatrix} ip^a\mathbf{k}_0 \\ p^a\end{pmatrix}e^{i\mathbf{k}_0\cdot\mathbf{x}}, \tag{4.44}$$

as p^a and $\mathbf{k}_0$ vary, with $k_0 = |\mathbf{k}_0|$ fixed and given by (4.3). Clearly, both $\chi(\mathbf{x})\mathcal{F}^a$ and $\chi(\mathbf{x})\mathcal{F}^{a'}$ lie in $\widetilde{\mathcal{V}}$ so if $\widetilde{\Pi}$ denotes the projection operator onto $\widetilde{\mathcal{V}}$, we have the identity

$$\begin{aligned} I_1 &= \lim_{r_0\to\infty}\int_{\partial B_{r_0}} -iP^{a'}\mathbf{n}\cdot\mathbf{v}^s + iP^s\mathbf{n}\cdot\mathbf{v}^{a'}\,dS \\ &= \int_\Omega \mathcal{F}^{a'}\cdot\mathbf{\Lambda}\mathcal{F}^a\,d\mathbf{x}, \end{aligned} \tag{4.45}$$

where $\mathbf{\Lambda}$ is the scattering operator

$$\mathbf{\Lambda} = \widetilde{\Pi}[(\mathbf{Z}_1 - \mathbf{Z}_0) - (\mathbf{Z}_1 - \mathbf{Z}_0)(\mathbf{Y}_*^0 + \mathbf{Z}_1)^{-1}(\mathbf{Z}_1 - \mathbf{Z}_0)]\widetilde{\Pi}, \tag{4.46}$$

with this expression being analogous to the expression (2.19) for the polarizability tensor. Thus, the bilinear form $I_1(\mathcal{F}^{a'},\mathcal{F}^a)$ defines $\mathbf{\Lambda}$ and we will see in the next section that I_1 can be determined from the far-field scattering amplitudes $P_\infty^s(\mathbf{n})$.

V. EXPRESSING THE SCATTERED FIELD IN TERMS OF INTEGRALS OVER THE INCLUSION

Here, our goal is to evaluate the integral on the right-hand side of (4.42) using the asymptotic formula

$$P^s(\mathbf{x}) = \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}P_\infty^s(\widehat{\mathbf{x}}), \text{ with } \widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \tag{5.1}$$

for the scattered pressure field, and the associated asymptotic formula for the scattered velocity field $\mathbf{v}^s = -i(\omega\rho_0)^{-1}\nabla P^s(\mathbf{x})$. The calculation is the analog of the calculation (2.9), that expresses a far-field integral in terms of the dipole moment.

Suppose we take a ball B of radius r. Then, the outwards unit normal to the ball surface is $\mathbf{n} = \mathbf{x}/r$ and consequently $\mathbf{n}\cdot\mathbf{x} = r$. Using the fact that $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ and $\widehat{\mathbf{x}} = \mathbf{x}/\sqrt{\mathbf{x}\cdot\mathbf{x}}$, this gives

$$\mathbf{n}\cdot\nabla P^s(\mathbf{x}) = \frac{\partial P^s(\mathbf{x})}{\partial r} \approx \frac{ik_0e^{ik_0r}}{r}P_\infty^s(\widehat{\mathbf{x}}) - \frac{e^{ik_0r}}{r^2}P_\infty^s(\widehat{\mathbf{x}}). \tag{5.2}$$

Hence, at large distances, keeping $\widehat{\mathbf{x}}$ fixed, the dominant term in the above expression for $\mathbf{n}\cdot\nabla P^s(\mathbf{x})$ is the first term. So just keeping this, we obtain

$$\mathbf{n}\cdot\mathbf{v}^s = -i(\omega\rho_0)^{-1}\mathbf{n}\cdot\nabla P^s(\mathbf{x}) \approx (\omega\rho_0)^{-1}\frac{k_0e^{ik_0r}}{r}P_\infty^s(\widehat{\mathbf{x}}). \tag{5.3}$$

Recall the pressure field $P^{a'}(\mathbf{x})$ and associated velocity field $\mathbf{v}^{a'}(\mathbf{x})$ are given by (4.40). So, we need to evaluate

$$\begin{aligned} I_1 &= \int_{\partial B_r} -iP^{a'}\mathbf{n}\cdot\mathbf{v}^s + iP^s\mathbf{n}\cdot\mathbf{v}^{a'}\,dS \\ &\approx -i(\omega\rho_0)^{-1}\int_{\partial B_r} e^{\mathbf{k}_0'\cdot\mathbf{x}}e^{ik_0r}P_\infty^s(\widehat{\mathbf{x}})(k_0 + \mathbf{n}\cdot\mathbf{k}_0')/r\,dS. \end{aligned} \tag{5.4}$$

Without loss of generality, let us suppose that the x_1 axis has been chosen in the direction of $\mathbf{k}_0'$, so $e^{-i\mathbf{k}_0'\cdot\mathbf{x}} = e^{-ik_0x_1}$ and $\mathbf{n}\cdot\mathbf{k}_0' = k_0n_1 = k_0x_1/r$. Let us use cylindrical coordinates (x_1,ϱ,θ) where $\varrho = \sqrt{x_2^2+x_3^2}$ and $\tan\theta = x_3/x_2$, so that $x_2 = \varrho\cos\theta$ and $x_3 = \varrho\sin\theta$. We then introduce the ratio $t = x_1/r$ and express

$$P_\infty^s(\widehat{\mathbf{x}}) = P_\infty^s(\theta,t). \tag{5.5}$$

Thus, in cylindrical coordinates the far-field expression for the scattered pressure field at ∂B becomes

$$P^s(\mathbf{x}) \approx \frac{e^{ik_0r}}{r}P_\infty^s(\theta,x_1/r). \tag{5.6}$$

We choose as our variables of integration the parameters $t = x_1/r$ and θ. In terms of t and θ, we have

$$x_1 = rt,\quad \varrho = r\sqrt{1-t^2},\quad e^{-i\mathbf{k}_0'\cdot\mathbf{x}} = e^{-ik_0rt},$$
$$(k_0+\mathbf{n}\cdot\mathbf{k}_0')/r = k_0(1+t)/r,$$
$$dS = \varrho\,d\theta\,dx_1/\sqrt{n_2^2+n_3^2} = r^2\,d\theta\,dt, \tag{5.7}$$

where n_2 and n_3 are the components of the vector $\mathbf{n} = \mathbf{x}/r$. The only term in the integration that involves θ is $P_\infty^s(\theta,h)$, so integrating this over θ defines

$$p_\infty(t) \equiv \int_0^{2\pi} P_\infty^s(\theta,t)\,d\theta. \tag{5.8}$$

We obtain

$$I_1 \approx -i(\omega\rho_0)^{-1}I_2,\quad I_2 = \int_{-1}^{1} rf(t)e^{irg(t)}\,dt, \tag{5.9}$$

where

$$g(t) = -k_0t + k_0,\quad f(t) = k_0(1+t)p_\infty(t). \tag{5.10}$$

Asymptotic expressions in the limit $r\to\infty$ for integrals taking the form of I_2 in (5.9) are available when $g(t)$ has a nonzero derivative $g'(t) = -k_0$ for $1 \geqslant t \geqslant -1$, which is clearly the case, and one has [64]

$$\lim_{r\to\infty} I_2 = \frac{e^{irg(1)}f(1)}{ig'(1)} - \frac{e^{irg(-1)}f(-1)}{ig'(-1)}. \tag{5.11}$$

We have $g'(1) = g'(-1) = -k_0$, while (5.10) and (5.6) imply

$$g(1) = e0,\quad g(-1) = 2k_0,$$
$$f(1) = 2k_0p_\infty(1) = 4k_0\pi\,P_\infty^s(\mathbf{k}_0'/k_0),\quad f(-1) = 0, \tag{5.12}$$

that when substituted in (5.11) gives

$$\lim_{r\to\infty} I_2 = 4i\pi\,P_\infty^s(\mathbf{k}_0'/k_0), \tag{5.13}$$

which is independent of α as expected. Hence, we obtain an exact expression for I_1:

$$I_1 = 4k_0\pi\,P_\infty^s(\mathbf{k}_0'/k_0)/(\omega\rho_0). \tag{5.14}$$

VI. A MINIMIZATION VARIATIONAL PRINCIPLE FOR ACOUSTIC SCATTERING

The fact that acoustic scattering can be regarded as a "Y problem" naturally leads to minimization variational principles. Here, we follow the more or less standard approach for deriving these variational principles (see [32] and Sec. 19.6 of [10]), using the transformation techniques of Gibiansky and Cherkaev [17]. Some adaptation is needed to allow for the fact that the matrix phase is lossless. This requires one to choose trial fields that solve the wave equation exactly in the matrix phase.

For $\mathbf{x}$ in the inclusion phase we can take real and imaginary parts of the constitutive law (4.30) to give

$$\begin{pmatrix}\mathrm{Re}\,[\mathcal{G}_2^s(\mathbf{x})]\\ \mathrm{Im}\,[\mathcal{G}_2^s(\mathbf{x})]\end{pmatrix} = \begin{pmatrix}-\mathbf{Z}_1'' & \mathbf{Z}_1'\\ \mathbf{Z}_1' & \mathbf{Z}_1''\end{pmatrix}\begin{pmatrix}\mathrm{Im}\,[\mathcal{F}_2^s(\mathbf{x})]\\ \mathrm{Re}\,[\mathcal{F}_2^s(\mathbf{x})]\end{pmatrix}, \tag{6.1}$$

where $\mathbf{Z}_1'$ and $\mathbf{Z}_2''$ denote the real and imaginary parts of $\mathbf{Z}_1 = \mathbf{Z}_1' + i\mathbf{Z}_1''$. Let us begin by supposing that ω is real and that $1/\kappa_1$ and ρ have strictly positive imaginary parts so that $\mathbf{Z}_1''$ is a positive-definite matrix. Then, following the ideas of Cherkaev and Gibiansky [17], that were extended to wave equations by Milton, Seppecher, and Bouchitté [24] and Milton and Willis [25], we can rewrite this constitutive law in the inclusion phase as

$$\underbrace{\begin{pmatrix}-\mathrm{Im}\,[\mathcal{F}_2^s(\mathbf{x})]\\ \mathrm{Im}\,[\mathcal{G}_2^s(\mathbf{x})]\end{pmatrix}}_{\mathbf{J}_2^0(\mathbf{x})} = \underbrace{\begin{pmatrix}[\mathbf{Z}_1'']^{-1} & -[\mathbf{Z}_1'']^{-1}\mathbf{Z}_1'\\ -\mathbf{Z}_1'[\mathbf{Z}_1'']^{-1} & \mathbf{Z}_1''+\mathbf{Z}_1'[\mathbf{Z}_1'']^{-1}\mathbf{Z}_1'\end{pmatrix}}_{\mathcal{L}_1} \times \underbrace{\begin{pmatrix}\mathrm{Re}\,[\mathcal{G}_2^s(\mathbf{x})]\\ \mathrm{Re}\,[\mathcal{F}_2^s(\mathbf{x})]\end{pmatrix}}_{\mathbf{E}_2^0(\mathbf{x})}, \tag{6.2}$$

where the matrix $\mathcal{L}_1$ is now positive definite. In the matrix phase a relation like (6.2) does not hold as $\mathbf{Z}_0$ is real. However, what enters the variational principle is $\mathbf{E}_2^0(\mathbf{x})\cdot\mathbf{J}_2^0(\mathbf{x}) = \mathbf{E}_2^0(\mathbf{x})\cdot\mathcal{L}\mathbf{E}_2^0(\mathbf{x})$. This will remain finite (and in fact approaches zero) as the imaginary part of $\mathbf{Z}$ tends to zero if the fields $\mathbf{J}_2^0(\mathbf{x})$ and $\mathbf{E}_2^0(\mathbf{x})$ defined in (6.2) are required to have components satisfying

$$\mathrm{Im}\,[\mathcal{G}_2^s(\mathbf{x})] = \mathbf{Z}_0\,\mathrm{Im}\,[\mathcal{F}_2^s(\mathbf{x})],\quad \mathrm{Re}\,[\mathcal{G}_2^s(\mathbf{x})] = \mathbf{Z}_0\,\mathrm{Re}\,[\mathcal{F}_2^s(\mathbf{x})], \tag{6.3}$$

as implied by (4.30), where $\mathbf{Z}_0$ is real. Thus, we have

$$\mathbf{E}_2^0(\mathbf{x})\cdot\mathbf{J}_2^0(\mathbf{x}) = -\mathrm{Re}\,[\mathcal{G}_2^s(\mathbf{x})]\cdot\mathrm{Im}\,[\mathcal{F}_2^s(\mathbf{x})] + \mathrm{Re}\,[\mathcal{F}_2^s(\mathbf{x})]\cdot\mathrm{Im}\,[\mathcal{G}_2^s(\mathbf{x})] = 0 \tag{6.4}$$

for all $\mathbf{x}$ in the matrix. Similarly, on the subspace $\mathcal{S}_3$ where $\mathbf{L}$ maps a field $[0,\{\mathbf{q}_\mathcal{F}\}_1,\{\mathbf{q}_\mathcal{F}\}_2]$ to $[0,\{\mathbf{q}_\mathcal{G}\}_1,\{\mathbf{q}_\mathcal{G}\}_2]$ the constitutive law (4.31), $\mathbf{q}_\mathcal{G} = \frac{i\omega}{\kappa_0}\mathbf{q}_\mathcal{F}$, can be rewritten, analogously to (6.2), as

$$\underbrace{\begin{pmatrix}-\mathrm{Im}[\mathbf{q}_\mathcal{F}(\mathbf{n})]\\ \mathrm{Im}[\mathbf{q}_\mathcal{G}(\mathbf{n})]\end{pmatrix}}_{\mathbf{t}(\mathbf{n})} = \underbrace{\begin{pmatrix}\kappa_0/\omega & 0\\ 0 & \omega/\kappa_0\end{pmatrix}}_{\mathcal{L}_3}\underbrace{\begin{pmatrix}\mathrm{Re}[\mathbf{q}_\mathcal{G}(\mathbf{n})]\\ \mathrm{Re}[\mathbf{q}_\mathcal{F}(\mathbf{n})]\end{pmatrix}}_{\mathbf{s}(\mathbf{n})}, \tag{6.5}$$

where $\mathcal{L}_3$ is clearly positive definite. Now, suppose we have a real trial pressure field $\underline{P}(\mathbf{x})$ and a purely imaginary trial velocity field $\underline{\mathbf{v}}(\mathbf{x})$, such that the associated real fields

$$\underline{\mathcal{F}}^0(\mathbf{x}) = \begin{pmatrix}\nabla\underline{P}(\mathbf{x})\\ \underline{P}(\mathbf{x})\end{pmatrix},\qquad \underline{\mathcal{G}}^0(\mathbf{x}) = \begin{pmatrix}-i\underline{\mathbf{v}}\\ -i\nabla\cdot\underline{\mathbf{v}}\end{pmatrix} \tag{6.6}$$

have the asymptotic forms

$$\underline{\mathcal{F}}^0(\mathbf{x}) = \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\frac{\underline{P}_\infty(\widehat{\mathbf{x}})}{2}\begin{pmatrix} ik_0\widehat{\mathbf{x}} \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\frac{\overline{\underline{P}_\infty(\widehat{\mathbf{x}})}}{2}\begin{pmatrix} -ik_0\widehat{\mathbf{x}} \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\},$$
$$\underline{\mathcal{G}}^0(\mathbf{x}) = \frac{e^{ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\frac{\underline{V}_\infty(\widehat{\mathbf{x}})}{2}\begin{pmatrix} -i\widehat{\mathbf{x}}/k_0 \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} + \frac{e^{-ik_0|\mathbf{x}|}}{|\mathbf{x}|}\left\{\frac{\overline{\underline{V}_\infty(\widehat{\mathbf{x}})}}{2}\begin{pmatrix} i\widehat{\mathbf{x}}/k_0 \\ 1\end{pmatrix} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\}, \quad (6.7)$$

for some choice of complex-valued functions $\underline{P}_\infty(\mathbf{n})$ and $\underline{V}_\infty(\mathbf{n})$, and are such that in the matrix (outside the inclusion)

$$\underline{\mathcal{G}}^0(\mathbf{x}) = \mathbf{Z}_0\underline{\mathcal{F}}^0(\mathbf{x}), \quad (6.8)$$

so that the trial fields satisfy (6.3). Thus, in the matrix the trial fields are required to be solutions to the acoustic wave equation. From the asymptotic forms (6.7) and (4.23) we see that the accompanying auxiliary fields are

$$\mathbf{q}_{\underline{\mathcal{F}}}(\mathbf{n}) = [\mathrm{Im}[\underline{P}_\infty(\mathbf{n})], \mathrm{Re}[\underline{P}_\infty(\mathbf{n})]],$$
$$\text{and } \mathbf{q}_{\underline{\mathcal{G}}}(\mathbf{n}) = [\mathrm{Re}[\underline{V}_\infty(\mathbf{n})], -\mathrm{Im}[\underline{V}_\infty(\mathbf{n})]]. \quad (6.9)$$

The fields $\underline{\mathcal{F}}^0(\mathbf{x})$ and $\underline{\mathcal{G}}^0(\mathbf{x})$ can be expressed as

$$\underline{\mathcal{F}}^0 = \underline{\mathcal{F}}_1^0 + \underline{\mathcal{F}}_2^0, \quad \underline{\mathcal{G}}^0 = \underline{\mathcal{G}}_1^0 + \underline{\mathcal{G}}_2^0,$$
$$\text{with} \quad \underline{\mathcal{F}}_1^0,\underline{\mathcal{G}}_1^0 \in \mathcal{V}^0, \quad \underline{\mathcal{F}}_2^0,\underline{\mathcal{G}}_2^0 \in \mathcal{H}^0. \quad (6.10)$$

So, if we define

$$\underline{\mathbf{E}}_2^0(\mathbf{x}) = \begin{pmatrix}\underline{\mathcal{G}}_2^0(\mathbf{x}) \\ \underline{\mathcal{F}}_2^0(\mathbf{x})\end{pmatrix}, \quad (6.11)$$

it follows from (6.8) that $\underline{\mathcal{G}}_2^0(\mathbf{x}) = \mathbf{Z}_0\underline{\mathcal{F}}_2^0(\mathbf{x})$ and then (6.2) and (6.3) imply

$$\underline{\mathbf{E}}_2^0(\mathbf{x})\cdot\underline{\mathbf{J}}_2^0(\mathbf{x}) = 0 \quad (6.12)$$

for all $\mathbf{x}$ in the matrix.

Suppose now we prescribe

$$\mathrm{Re}\left(\mathcal{F}_1^s\right) = \underline{\mathcal{F}}_1^0, \quad \mathrm{Re}\left(\mathcal{G}_1^s\right) = \underline{\mathcal{G}}_1^0, \quad (6.13)$$

and let $\mathcal{F}^s = \mathcal{F}_1^s + \mathcal{F}_2^s$ and $\mathcal{G}^s = \mathcal{G}_1^s + \mathcal{G}_2^s$ be the associated solutions of the Y problem. Then, as shown in Appendix A, we have the variational inequality

$$\int_\Omega \underline{\mathbf{E}}_2^0(\mathbf{x})\cdot\mathcal{L}_1\underline{\mathbf{E}}_2^0(\mathbf{x})\,d\mathbf{x} + \int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathcal{L}_3\underline{\mathbf{s}}(\mathbf{n})\,dS \geqslant \times\left(\mathrm{Im}\left(\mathcal{F}_1^s\right), \mathrm{Re}\left(\mathcal{G}_1^s\right)\right) - \left(\mathrm{Re}\left(\mathcal{F}_1^s\right), \mathrm{Im}\left(\mathcal{G}_1^s\right)\right) = -(\mathbf{J}_1^0,\mathbf{E}_1^0), \quad (6.14)$$

where

$$\underline{\mathbf{s}}(\mathbf{n}) = \begin{pmatrix}\mathbf{q}_{\underline{\mathcal{G}}}(\mathbf{n}) \\ \mathbf{q}_{\underline{\mathcal{F}}}(\mathbf{n})\end{pmatrix}, \quad \mathbf{J}_1^0(\mathbf{x}) = \begin{pmatrix}-\mathrm{Im}\left[\mathcal{F}_1^s(\mathbf{x})\right] \\ \mathrm{Im}\left[\mathcal{G}_1^s(\mathbf{x})\right]\end{pmatrix},$$
$$\mathbf{E}_1^0 = \begin{pmatrix}\mathrm{Re}\left[\mathcal{G}_1^s(\mathbf{x})\right] \\ \mathrm{Re}\left[\mathcal{F}_1^s(\mathbf{x})\right]\end{pmatrix}. \quad (6.15)$$

From the definition (4.37) of the Y operator $\mathbf{Y}_*^0$, we have $\mathcal{G}_1^s = -\mathbf{Y}_*^0\mathcal{F}_1^s$ and this relation can then be manipulated into the form

$$\underbrace{\begin{pmatrix}-\mathrm{Im}\left(\mathcal{F}_1^s\right) \\ \mathrm{Im}\left(\mathcal{G}_1^s\right)\end{pmatrix}}_{\mathbf{J}_1} = -\mathcal{Y}\underbrace{\begin{pmatrix}\mathrm{Re}\left(\mathcal{G}_1^s\right) \\ \mathrm{Re}\left(\mathcal{F}_1^s\right)\end{pmatrix}}_{\mathbf{E}_1}, \quad (6.16)$$

which defines the associated operator $\mathcal{Y}$, and the fields $\mathbf{J}_1$ and $\mathbf{E}_1$. Then, the right-hand sides of (6.15) can then be identified with the quadratic form associated with $\mathcal{Y}$:

$$\left(\mathrm{Im}\left(\mathcal{F}_1^s\right), \mathrm{Re}\left(\mathcal{G}_1^s\right)\right) - \left(\mathrm{Re}\left(\mathcal{F}_1^s\right), \mathrm{Im}\left(\mathcal{G}_1^s\right)\right) = -(\mathbf{E}_1,\mathbf{J}_1) = (\mathbf{E}_1,\mathcal{Y}\mathbf{E}_1). \quad (6.17)$$

Consequently, we have the variational principle

$$(\mathbf{E}_1,\mathcal{Y}\mathbf{E}_1) = \min_{\underline{\mathbf{E}}_2^0}\int_\Omega \underline{\mathbf{E}}_2^0(\mathbf{x})\cdot\mathcal{L}_1\underline{\mathbf{E}}_2^0(\mathbf{x})\,d\mathbf{x} + \int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathcal{L}_3\underline{\mathbf{s}}(\mathbf{n})\,dS, \quad (6.18)$$

where the minimum is over all fields $\underline{\mathbf{E}}_2^0$ such that $\underline{\mathbf{E}}^0 = \mathbf{E}_1 + \underline{\mathbf{E}}_2^0$ is of the form

$$\underline{\mathbf{E}}^0 = \underline{\mathbf{E}}^0(\mathbf{x}) = \begin{pmatrix}\underline{\mathcal{G}}^0(\mathbf{x}) \\ \underline{\mathcal{F}}^0(\mathbf{x})\end{pmatrix}, \quad (6.19)$$

with $\underline{\mathcal{G}}^0(\mathbf{x})$ and $\underline{\mathcal{F}}^0(\mathbf{x})$ being of the form (6.6) for some real $\underline{P}(\mathbf{x})$ and a purely imaginary vector field $\underline{\mathbf{v}}(\mathbf{x})$. Additionally, the constitutive relation (6.8) must hold in the matrix. As the right-hand side of (6.18) is non-negative, we deduce that $\mathcal{Y}$ is a positive-semidefinite operator.

Expressing $\underline{\mathcal{F}}_2^0(\mathbf{x})$ and $\underline{\mathcal{G}}_2^0(\mathbf{x})$ in terms of their component fields,

$$\underline{\mathcal{F}}_2^0(\mathbf{x}) = \begin{pmatrix}\mathbf{F}(\mathbf{x}) \\ f(\mathbf{x})\end{pmatrix}, \quad \underline{\mathcal{G}}_2^0(\mathbf{x}) = \begin{pmatrix}\mathbf{G}(\mathbf{x}) \\ g(\mathbf{x})\end{pmatrix}, \quad (6.20)$$

the inequality (6.14) takes the equivalent form

$$\int_\Omega \begin{pmatrix}\mathbf{G}(\mathbf{x}) \\ \mathbf{F}(\mathbf{x})\end{pmatrix}\cdot\mathbf{R}\begin{pmatrix}\mathbf{G}(\mathbf{x}) \\ \mathbf{F}(\mathbf{x})\end{pmatrix} + \begin{pmatrix}g(\mathbf{x}) \\ f(\mathbf{x})\end{pmatrix}\cdot\mathbf{H}\begin{pmatrix}g(\mathbf{x}) \\ f(\mathbf{x})\end{pmatrix}d\mathbf{x} + \int_{|\mathbf{n}|=1}\kappa_0|\underline{V}_\infty(\mathbf{n})|^2/\omega + \omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0\,dS$$
$$\geqslant \left(\mathrm{Im}\left(\mathcal{F}_1^s\right), \mathrm{Re}\left(\mathcal{G}_1^s\right)\right) - \left(\mathrm{Re}\left(\mathcal{F}_1^s\right), \mathrm{Im}\left(\mathcal{G}_1^s\right)\right) \quad (6.21)$$

in which

$$\mathbf{R} = \begin{pmatrix}\omega(\mathbf{r}'')^{-1} & -(\mathbf{r}'')^{-1}\mathbf{r}' \\ -\mathbf{r}'(\mathbf{r}'')^{-1} & [\mathbf{r}'' + \mathbf{r}'(\mathbf{r}'')^{-1}\mathbf{r}']/\omega\end{pmatrix},$$
$$\mathbf{H} = \begin{pmatrix}(\omega h'')^{-1} & -(h'')^{-1}h' \\ -h'(h'')^{-1} & \omega[h'' + h'(h'')^{-1}h']\end{pmatrix}, \quad (6.22)$$

and $\mathbf{r} = -\boldsymbol{\rho}_1^{-1}$, and $h = 1/\kappa_1$. When $\mathbf{r}''$ is very small we have

$$\begin{pmatrix}\mathbf{G}(\mathbf{x}) \\ \mathbf{F}(\mathbf{x})\end{pmatrix}\cdot\mathbf{R}\begin{pmatrix}\mathbf{G}(\mathbf{x}) \\ \mathbf{F}(\mathbf{x})\end{pmatrix} \approx [\omega\mathbf{G}(\mathbf{x}) - \mathbf{r}'\mathbf{F}(\mathbf{x})]\cdot(\omega\mathbf{r}'')^{-1} \times[\omega\mathbf{G}(\mathbf{x}) - \mathbf{r}'\mathbf{F}(\mathbf{x})], \quad (6.23)$$

so if this is to remain finite in the limit $\mathbf{r}'' \to 0$ (i.e., when $\boldsymbol{\rho}_1$ is real) we need to choose the trial fields so that

$$\mathbf{F}(\mathbf{x}) = -\omega\boldsymbol{\rho}_1\mathbf{G}(\mathbf{x}) \text{ for all } \mathbf{x} \in \Omega. \quad (6.24)$$

Then, taking the limit $\mathbf{r}'' \to 0$, the variational inequality (6.21) reduces to

$$\int_\Omega \begin{pmatrix} g(\mathbf{x}) \\ f(\mathbf{x}) \end{pmatrix} \cdot \mathbf{H}\begin{pmatrix} g(\mathbf{x}) \\ f(\mathbf{x}) \end{pmatrix} d\mathbf{x} + \int_{|\mathbf{n}|=1} \kappa_0 |\underline{V}_\infty(\mathbf{n})|^2/\omega + \omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0\, dS \geqslant \left(\mathrm{Im}\left(\mathcal{F}_1^s\right), \mathrm{Re}\left(\mathcal{G}_1^s\right)\right) - \left(\mathrm{Re}\left(\mathcal{F}_1^s\right), \mathrm{Im}\left(\mathcal{G}_1^s\right)\right). \quad (6.25)$$

VII. LINK BETWEEN THE POWER ABSORBED AND SCATTERED BY THE INCLUSION AND Im($\mathbf{Y}_*^0$)

The imaginary part of the quadratic form associated with $\mathbf{Y}_*^0$ has a physical interpretation in terms of the power absorbed and scattered by the inclusion. In elastodynamics the power absorption by a body Ω, having a possibly complex density $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_1' + i\boldsymbol{\rho}_1''$ (with real and imaginary parts $\boldsymbol{\rho}_1'$ and $\boldsymbol{\rho}_1''$), is given by formula (2.5) in [25] and (taking into account our choice of $e^{-i\omega t}$ for the time dependence, rather than $e^{i\omega t}$) can be written as

$$A = \frac{1}{2}\int_\Omega \omega\overline{\mathbf{v}^0}\cdot\boldsymbol{\rho}_1''\mathbf{v}^0 + \mathrm{Re}(\overline{-i\omega\mathbf{e}^0}\cdot\boldsymbol{\sigma}^0)\, d\mathbf{x}, \quad (7.1)$$

where $\mathbf{v}^0 = -i\omega\mathbf{u}^0$ is the complex velocity field, $\mathbf{u}^0$ is the complex displacement field, $\mathbf{e}^0 = [\nabla\mathbf{u}^0 + (\nabla\mathbf{u}^0)^T]/2$ is the strain, and its time derivative $-i\omega\mathbf{e}^0 = [\nabla\mathbf{v}^0 + (\nabla\mathbf{v}^0)^T]/2$ is the strain rate, and $\boldsymbol{\sigma}^0$ is the stress. In a fluid one has $\boldsymbol{\sigma}^0 = -P^0\mathbf{I}$ where $P^0(\mathbf{x})$ is the pressure, and hence the above expression reduces to

$$\begin{aligned} A &= \frac{1}{2}\int_\Omega \omega\,\mathrm{Im}(\overline{\mathbf{v}^0}\cdot\boldsymbol{\rho}_1\mathbf{v}^0) - \mathrm{Re}(\overline{\nabla\cdot\mathbf{v}^0}P^0)\, d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega \mathrm{Im}(\overline{i\mathbf{v}^0}\cdot\nabla P^0) + \mathrm{Im}(\overline{i\nabla\cdot\mathbf{v}}P^0)\, d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega \mathrm{Im}(-i\mathbf{v}\cdot\overline{\nabla P^0}) + \mathrm{Im}(-i\nabla\cdot\mathbf{v}\overline{P^0})\, d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega \mathrm{Im}(\overline{\mathcal{F}^0}\cdot\mathcal{G}^0)\, d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega \mathrm{Re}(\mathcal{F})\cdot\mathrm{Im}(\mathcal{G}^0) - \mathrm{Im}(\mathcal{F}^0)\cdot\mathrm{Re}(\mathcal{G}), \end{aligned} \quad (7.2)$$

where

$$\mathcal{G}^0 = \begin{pmatrix} -i\mathbf{v}^0 \\ -i\nabla\cdot\mathbf{v}^0 \end{pmatrix}, \quad \mathcal{F}^0 = \begin{pmatrix} \nabla P^0 \\ P^0 \end{pmatrix}. \quad (7.3)$$

Thus, by taking the imaginary part of the key identity (4.19) we see that twice the imaginary part of the left-hand side can be identified with the time-averaged power absorbed in the ball B_r and, consequently, by conservation of energy,

$$\frac{1}{2}\mathrm{Im}\int_{\partial B_r} -i\,\overline{P^0(\mathbf{x})}\mathbf{n}\cdot\mathbf{v}^0(\mathbf{x})\, dS$$

can be identified with the time-averaged power flowing inwards through the boundary ∂B_r. Hence, in the identity

$$\frac{1}{2}\mathrm{Im}\left(\mathcal{F}_1^s, \mathbf{Y}_*^0\mathcal{F}_1^s\right) = \frac{1}{2}\mathrm{Im}\int_\Omega \left(\overline{\mathcal{F}_2^s}\cdot\mathbf{Z}_1\mathcal{F}_2^s\right) d\mathbf{x} + \frac{1}{2}\mathrm{Im}\int_{\partial B_r} i\,\overline{P^s(\mathbf{x})}\mathbf{n}\cdot\mathbf{v}^s(\mathbf{x})\, dS, \quad (7.4)$$

implied by (4.39) and (4.20) (with $P_\infty^i = 0$ and $V_\infty^i = 0$), we see that the first term on the right can be identified with the time-averaged power absorbed by the field $\mathcal{F}_2^s$ in the inclusion, while the second term on the right can be identified with the time-averaged power radiated to infinity by the scattered field.

The total time-averaged power absorbed by the inclusion has contributions both from the field $\mathcal{F}_2^s$ and from the fields $\mathcal{F}^a + \mathcal{F}_1^s$, and is given by

$$\begin{aligned} &\frac{1}{2}\mathrm{Im}\int_\Omega \left[\overline{\left(\mathcal{F}^a + \mathcal{F}_1^s + \mathcal{F}_2^s\right)}\cdot\mathbf{Z}_1\left(\mathcal{F}^a + \mathcal{F}_1^s + \mathcal{F}_2^s\right)\right] d\mathbf{x} \\ &\quad = \frac{1}{2}\mathrm{Im}\int_\Omega \left[\overline{\left(\mathcal{F}^a + \mathcal{F}_1^s\right)}\cdot\mathbf{Z}_1\left(\mathcal{F}^a + \mathcal{F}_1^s\right)\right] d\mathbf{x} \\ &\quad + \frac{1}{2}\mathrm{Im}\int_\Omega \left(\overline{\mathcal{F}_2^s}\cdot\mathbf{Z}_1\mathcal{F}_2^s\right) d\mathbf{x}, \end{aligned} \quad (7.5)$$

where in obtaining this last identity we have used the orthogonality of the spaces $\mathcal{V}^0$ and $\mathcal{H}^0$. The last term in (7.5) is that which enters (7.4).

The time-averaged extinction power, being the sum of the total absorbed power and scattered power, should be W, where from (7.4) and (7.5),

$$\begin{aligned} 2W &= \mathrm{Im}\left(\mathcal{F}^a + \mathcal{F}_1^s, \chi\mathbf{Z}_1\left(\mathcal{F}^a + \mathcal{F}_1^s\right)\right) + \mathrm{Im}\left(\mathcal{F}_1^s, \mathbf{Y}_*^0\mathcal{F}_1^s\right) \\ &= \left(\mathcal{F}^a + \mathcal{F}_1^s, \chi\,\mathrm{Im}(\mathbf{Z}_1)\left(\mathcal{F}^a + \mathcal{F}_1^s\right)\right) + \left(\mathcal{F}_1^s, \mathrm{Im}(\mathbf{Y}_*^0)\mathcal{F}_1^s\right). \end{aligned} \quad (7.6)$$

This provides the desired link between W and Im($\mathbf{Y}_*^0$). Further manipulations, carried out in Appendix B, provide alternative expressions for W, namely,

$$2W = \mathrm{Im}\int_{B_r} \overline{p^a\mathcal{F}^{a\prime}}\cdot(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)\, d\mathbf{x}, \quad (7.7)$$

which is similar to the form of the optical theorem given in [23,65,66], and

$$W = 2k_0\pi\,\mathrm{Im}[\overline{p^a}P_\infty^s(\mathbf{k}_0/k_0)]/(\omega\rho_0), \quad (7.8)$$

which is the well-known form of the optical theorem [67,68] for acoustic scattering.

VIII. VARIATIONAL PRINCIPLES FOR THE BACKWARDS SCATTERING AMPLITUDE

For $\mathbf{x}$ in the inclusion phase, the constitutive law implies $\mathcal{G}^a + \mathcal{G}_1^s = \mathbf{Z}_1(\mathcal{F}^a + \mathcal{F}_1^s)$ which analogously to (6.1) and (6.2) can be manipulated into the form

$$\mathbf{J}^a + \mathbf{J}_1 = \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1), \quad \text{where } \mathbf{E}^a = \begin{pmatrix} \mathrm{Re}[\mathcal{G}^a(\mathbf{x})] \\ \mathrm{Re}[\mathcal{F}^a(\mathbf{x})] \end{pmatrix}, \quad \mathbf{J}^a = \begin{pmatrix} -\mathrm{Im}[\mathcal{F}^a(\mathbf{x})] \\ \mathrm{Im}[\mathcal{G}^a(\mathbf{x})] \end{pmatrix} \quad (8.1)$$

and $\mathbf{J}_1$ and $\mathbf{E}_1$ are defined in (6.16). Thus, the formula (7.6) for the extinction power can be rewritten as

$$\begin{aligned} 2W &= \mathrm{Im}\left(\mathcal{F}^a + \mathcal{F}_1^s, \chi\left(\mathcal{G}^a + \mathcal{G}_1^s\right)\right) - \mathrm{Im}\left(\mathcal{F}_1^s, \mathcal{G}_1^s\right) \\ &= (\mathbf{E}^a + \mathbf{E}_1, \chi(\mathbf{J}^a + \mathbf{J}_1)) - (\mathbf{E}_1, \mathbf{J}_1) \\ &= (\mathbf{E}^a + \mathbf{E}_1, \chi\mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) + (\mathbf{E}_1, \mathcal{Y}\mathbf{E}_1). \end{aligned} \tag{8.2}$$

As $\mathcal{L}_1$ and $\mathcal{Y}$ are positive-semidefinite operators, this formula suggests that a variational principle might be obtained from a consideration of the non-negativity of the quadratic form

$$\begin{aligned} &(\mathbf{E}^a + \underline{\mathbf{E}}_1 - \mathbf{E}^a - \mathbf{E}_1, \mathcal{L}_1(\mathbf{E}^a + \underline{\mathbf{E}}_1 - \mathbf{E}^a - \mathbf{E}_1)) + (\underline{\mathbf{E}}_1 - \mathbf{E}_1, \mathcal{Y}(\underline{\mathbf{E}}_1 - \mathbf{E}_1)) \\ &\quad = (\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathcal{L}_1(\mathbf{E}^a + \underline{\mathbf{E}}_1)) + (\mathbf{E}^a + \mathbf{E}_1, \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) + (\underline{\mathbf{E}}_1 \mathcal{Y}\underline{\mathbf{E}}_1) + (\mathbf{E}_1, \mathcal{Y}\mathbf{E}_1) \\ &\quad\quad -2(\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) - 2(\underline{\mathbf{E}}_1, \mathcal{Y}\mathbf{E}_1), \end{aligned} \tag{8.3}$$

The sum of the last two terms in (8.3), each of which involves both $\mathbf{E}_1$ and $\underline{\mathbf{E}}_1$, can be replaced by the expression

$$\begin{aligned} -2(\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) - 2(\underline{\mathbf{E}}_1, \mathcal{Y}\mathbf{E}_1) &= -2(\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathbf{J}^a + \mathbf{J}_1) + 2(\underline{\mathbf{E}}_1, \mathbf{J}_1) \\ &= -2(\mathbf{E}^a, \mathbf{J}^a + \mathbf{J}_1) - 2(\underline{\mathbf{E}}_1, \mathbf{J}^a) \\ &= -2(\mathbf{E}^a + \mathbf{E}_1, \mathbf{J}^a + \mathbf{J}_1) + 2(\mathbf{E}_1, \mathbf{J}_1) + 2(\mathbf{E}_1, \mathbf{J}^a) - 2(\underline{\mathbf{E}}_1, \mathbf{J}^a) \\ &= -2(\mathbf{E}^a + \mathbf{E}_1, \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) - 2(\mathbf{E}_1, \mathcal{Y}\mathbf{E}_1) + 2(\mathbf{E}_1, \mathbf{J}^a) - 2(\underline{\mathbf{E}}_1, \mathbf{J}^a). \end{aligned} \tag{8.4}$$

By substituting this back in (8.3) one sees that one has the variational principle

$$(\mathbf{E}^a + \mathbf{E}_1, \mathcal{L}_1(\mathbf{E}^a + \mathbf{E}_1)) + (\mathbf{E}_1, \mathcal{Y}\mathbf{E}_1) - 2(\mathbf{E}_1, \mathbf{J}^a) = \min_{\underline{\mathbf{E}}_1} (\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathcal{L}_1(\mathbf{E}^a + \underline{\mathbf{E}}_1)) + (\underline{\mathbf{E}}_1 \mathcal{Y}\underline{\mathbf{E}}_1) - 2(\underline{\mathbf{E}}_1, \mathbf{J}^a). \tag{8.5}$$

The variational principle derived in Sec. VI can then be substituted into this expression and we obtain

$$\begin{aligned} 2W - 2(\mathbf{E}_1, \mathbf{J}^a) &= \min_{\underline{\mathbf{P}}, \underline{\mathbf{v}}} (\mathbf{E}^a + \underline{\mathbf{E}}_1, \mathcal{L}_1(\mathbf{E}^a + \underline{\mathbf{E}}_1)) + (\underline{\mathbf{E}}_2, \mathcal{L}_1\underline{\mathbf{E}}_2) - 2(\underline{\mathbf{E}}_1, \mathbf{J}^a) + \int_{|\mathbf{n}|=1} \kappa_0 |\underline{V}_\infty(\mathbf{n})|^2/\omega + \omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0 \, dS \\ &= \min_{\underline{\mathbf{P}}, \underline{\mathbf{v}}} (\mathbf{E}^a + \underline{\mathbf{E}}, \mathcal{L}_1(\mathbf{E}^a + \underline{\mathbf{E}})) - 2(\underline{\mathbf{E}}, \chi\mathbf{J}^a) + \int_{|\mathbf{n}|=1} \kappa_0 |\underline{V}_\infty(\mathbf{n})|^2/\omega + \omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0 \, dS, \end{aligned} \tag{8.6}$$

where here $\underline{\mathbf{P}}$ is a real trial pressure field, and $\underline{\mathbf{v}}$ is a purely imaginary trial velocity field, and the real field $\underline{\mathbf{E}}(\mathbf{x})$ is given in terms of them through the equations

$$\begin{aligned} &\underline{\mathbf{E}} = \underline{\mathbf{E}}_1 + \underline{\mathbf{E}}_2 = \begin{pmatrix} \underline{\mathcal{G}}^0(\mathbf{x}) \\ \underline{\mathcal{F}}^0(\mathbf{x}) \end{pmatrix}, \\ &\underline{\mathcal{F}}^0(\mathbf{x}) = \begin{pmatrix} \nabla \underline{P}(\mathbf{x}) \\ \underline{P}(\mathbf{x}) \end{pmatrix} \in \mathcal{E}^0, \quad \underline{\mathcal{G}}^0(\mathbf{x}) = \begin{pmatrix} -i\underline{\mathbf{v}} \\ -i\nabla\cdot\underline{\mathbf{v}} \end{pmatrix} \in \mathcal{J}^0, \end{aligned} \tag{8.7}$$

where $\mathcal{E}^0$ and $\mathcal{J}^0$ consist of all fields of the form (4.12) and (4.13), respectively.

This variational principle has the advantage that the quantity on the right-hand side of (8.6) is easy to numerically compute for a given choice of $\underline{\mathbf{E}}$: it is not necessary to determine the individual component fields $\underline{\mathbf{E}}_1$ and $\underline{\mathbf{E}}_2 = \underline{\mathbf{E}} - \underline{\mathbf{E}}_1$. To obtain a physical interpretation for the quantity $-2(\mathbf{E}_1, \mathbf{J}^a)$ appearing on the left-hand side of (8.6) note that

$$\begin{aligned} -2(\mathbf{E}_1, \mathbf{J}^a) &= 2(\mathrm{Re}\,\mathcal{G}_1, \mathrm{Im}\,\mathcal{F}^a) + 2(\mathrm{Re}\,\mathcal{F}_1, -\mathrm{Im}\,\mathcal{G}^a) \\ &= 2(\mathrm{Re}(\mathcal{G}_1 - \mathbf{Z}_0\mathcal{F}_1), \mathrm{Im}\,\mathcal{F}^a) \\ &= 2(\mathrm{Re}(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s), \mathrm{Im}\,\mathcal{F}^a) \\ &= 2\int_{B_r} \mathrm{Im}(\mathcal{F}^a)\cdot[\mathrm{Re}(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)]\, d\mathbf{x} \\ &= 2\int_{B_r} \mathrm{Im}(\mathcal{F}^a)\cdot\mathrm{Re}\,\mathcal{G}^s - \mathrm{Im}(\mathcal{G}^a)\cdot\mathrm{Re}\,\mathcal{F}^s\, d\mathbf{x} \\ &= 2\int_{\partial B_r} \mathrm{Im}(P^a)\,\mathrm{Im}(\mathbf{n}\cdot\mathbf{v}^s) + \mathrm{Re}(\mathbf{n}\cdot\mathbf{v}^a)\,\mathrm{Re}(P^s)\, dS. \end{aligned} \tag{8.8}$$

Using the asymptotic forms of the fields as $r \to \infty$ we get

$$-2(\mathbf{E}_1, \mathbf{J}^a) = 2\int_{\partial B_r} \frac{k_0\,\mathrm{Im}(p^a e^{i\mathbf{k}_0\cdot\mathbf{x}})\,\mathrm{Im}\left[P_\infty^s(\widehat{\mathbf{x}})e^{ik_0 r}\right]}{r\omega\rho_0} + \frac{(\mathbf{n}\cdot\mathbf{k}_0)\,\mathrm{Re}(p^a e^{i\mathbf{k}_0\cdot\mathbf{x}})\,\mathrm{Re}\left[P_\infty^s(\widehat{\mathbf{x}})e^{ik_0 r}\right]}{r\omega\rho_0}. \tag{8.9}$$

Choosing our coordinates so that the positive x_1 axis points in the direction of $\mathbf{k}_0$, i.e., so that $\mathbf{k}_0\cdot\mathbf{x}=k_0x_1=k_0rt$ and $\mathbf{n}\cdot\mathbf{k}_0=k_0t$ where $t=x_1/r$, and making the substitutions

$$\begin{gathered}\mathrm{Im}(p^ae^{i\mathbf{k}_0\cdot\mathbf{x}})=(-ip^ae^{ik_0rt}+i\overline{p^a}e^{-ik_0rt})/2,\quad \mathrm{Re}(p^ae^{i\mathbf{k}_0\cdot\mathbf{x}})=(p^ae^{ik_0rt}+\overline{p^a}e^{-ik_0rt})/2,\\ \mathrm{Im}\left[P^s_\infty(\widehat{\mathbf{x}})e^{ik_0r}\right]=\left[-iP^s_\infty(\widehat{\mathbf{x}})e^{ik_0r}+i\overline{P^s_\infty(\widehat{\mathbf{x}})}e^{-ik_0r}\right]/2,\quad \mathrm{Re}\left[P^s_\infty(\widehat{\mathbf{x}})e^{ik_0r}\right]=\left[P^s_\infty(\widehat{\mathbf{x}})e^{ik_0r}+\overline{P^s_\infty(\widehat{\mathbf{x}})}e^{-ik_0r}\right]/2,\end{gathered} \tag{8.10}$$

we are left with $-2(\mathbf{E}_1,\mathbf{J}^a)$ being the sum of the two integrals

$$\begin{aligned}\frac{1}{2}\int_{\partial B_r}\frac{k_0(t-1)p^aP^s_\infty(\widehat{\mathbf{x}})e^{ik_0r(1+t)}}{r\omega\rho_0}&=\frac{p^a}{2\omega\rho_0}\int_{-1}^{1}rk_0(t-1)p_\infty(t)e^{ik_0r(1+t)}\,dt,\\ \frac{1}{2}\int_{\partial B_r}\frac{k_0(t+1)\overline{p^a}P^s_\infty(\widehat{\mathbf{x}})e^{ik_0r(1-t)}}{r\omega\rho_0}&=\frac{\overline{p^a}}{2\omega\rho_0}\int_{-1}^{1}rk_0(t+1)p_\infty(t)e^{ik_0r(1-t)}\,dt,\end{aligned} \tag{8.11}$$

and their complex conjugates, in which $p_\infty(t)$ is defined by (5.8). The integrals are of the same form as the integral I_2 in (5.9), with appropriate choices of $f(t)$ and $g(t)$. Using the formula (5.11) we can evaluate them in the limit $r\to\infty$ and they equal, respectively,

$$-ip^a2\pi k_0P^s_\infty(-\mathbf{k}_0/k_0)/(\omega\rho_0)\quad\text{and}\quad i\overline{p^a}2\pi k_0P^s_\infty(\mathbf{k}_0/k_0)/(\omega\rho_0).$$

Adding them and then adding the total to its complex conjugate gives

$$\begin{aligned}-2(\mathbf{E}_1,\mathbf{J}^a)&=4\pi k_0\,\mathrm{Im}\left[p^aP^s_\infty(-\mathbf{k}_0/k_0)\right]/(\omega\rho_0)-4\pi k_0\,\mathrm{Im}\left[\overline{p^a}P^s_\infty(\mathbf{k}_0/k_0)\right]/(\omega\rho_0)\\&=4\pi k_0\,\mathrm{Im}\left[p^aP^s_\infty(-\mathbf{k}_0/k_0)\right]/(\omega\rho_0)-2W,\end{aligned} \tag{8.12}$$

where we have used the expression (7.8) for W given by the optical theorem. Thus, we have the variational principle

$$4\pi k_0\,\mathrm{Im}\left[p^aP^s_\infty(-\mathbf{k}_0/k_0)\right]/(\omega\rho_0)=\min_{\underline{\mathbf{P}},\underline{\mathbf{v}}}(\mathbf{E}^a+\underline{\mathbf{E}},\mathcal{L}_1(\mathbf{E}^a+\underline{\mathbf{E}}))-2(\underline{\mathbf{E}},\chi\mathbf{J}^a)+\int_{|\mathbf{n}|=1}\kappa_0|\underline{V}_\infty(\mathbf{n})|^2/\omega+\omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0\,dS. \tag{8.13}$$

It is interesting that this variational principle, with some choice of trial fields $\underline{\mathbf{P}}$ and $\underline{\mathbf{v}}$, does not give a desired bound on W or, equivalently, on the forward scattering amplitude, but rather bounds the *backwards* scattering amplitude $P^s_\infty(-\mathbf{k}_0/k_0)$.

We note that the physical pressure field associated with the incoming wave is $\mathrm{Re}[P^a(\mathbf{x})e^{-i\omega t}]$ where t is the time. Accordingly, if we shift our origin of time, by replacing t with $t-t_0$, the physical pressure field associated with the incoming wave is $\mathrm{Re}[\widetilde{P}^a(\mathbf{x})e^{-i\omega t}]$ where

$$\widetilde{P}^a(\mathbf{x})=P^a(\mathbf{x})e^{i\omega t_0}=\widetilde{p}^ae^{i\mathbf{k}_0\cdot\mathbf{x}}\ \text{ where }\ \widetilde{p}^a=p^ae^{i\omega t_0}. \tag{8.14}$$

The associated scattered pressure field is then

$$\widetilde{P}^s(\mathbf{x})=P^s(\mathbf{x})e^{i\omega t_0},\quad\text{with }\widetilde{P}^s_\infty(\widehat{\mathbf{x}})=P^s_\infty(\widehat{\mathbf{x}})e^{i\omega t_0}. \tag{8.15}$$

Consequently, with p^a and $P^s_\infty(\widehat{\mathbf{x}})$ replaced by $\widetilde{p}^a$ and $\widetilde{P}^s_\infty(\widehat{\mathbf{x}})$, the variational principle (8.13) becomes

$$4\pi k_0\,\mathrm{Im}[e^{2i\omega t_0}p^aP^s_\infty(-\mathbf{k}_0/k_0)]/(\omega\rho_0)=\min_{\underline{\mathbf{P}},\underline{\mathbf{v}}}(\widetilde{\mathbf{E}}^a+\underline{\mathbf{E}},\mathcal{L}_1(\widetilde{\mathbf{E}}^a+\underline{\mathbf{E}}))-2(\underline{\mathbf{E}},\chi\widetilde{\mathbf{J}}^a)+\int_{|\mathbf{n}|=1}\kappa_0|\underline{V}_\infty(\mathbf{n})|^2/\omega+\omega|\underline{P}_\infty(\mathbf{n})|^2/\kappa_0\,dS, \tag{8.16}$$

where

$$\widetilde{\mathbf{E}}^a(\mathbf{x})=\begin{pmatrix}\mathrm{Re}[e^{i\omega t_0}\mathcal{G}^a(\mathbf{x})]\\ \mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\end{pmatrix},\quad \widetilde{\mathbf{J}}^a(\mathbf{x})=\begin{pmatrix}-\mathrm{Im}\left[e^{i\omega t_0}\mathcal{F}^s_1(\mathbf{x})\right]\\ \mathrm{Im}\left[e^{i\omega t_0}\mathcal{G}^s_1(\mathbf{x})\right]\end{pmatrix}. \tag{8.17}$$

Thus, by varying t_0, and appropriately changing the trial fields, one get bounds that "wrap around" the possible complex values of the backwards scattering amplitude $P^s_\infty(-\mathbf{k}_0/k_0)$. By choosing the origin of time appropriately, we can assume that p^a is real and positive. Then, for example, (8.13) provides an upper bound on $\mathrm{Im}[P^s_\infty(-\mathbf{k}_0/k_0)]$, while (8.16) with t_0 chosen so that $e^{2i\omega t_0}=-1$ provides a lower bound on $\mathrm{Im}[P^s_\infty(-\mathbf{k}_0/k_0)]$. Similarly (8.16), with t_0 chosen so that $e^{2i\omega t_0}=-i$ or $e^{2i\omega t_0}=i$, gives us upper and lower bounds on $\mathrm{Re}[P^s_\infty(-\mathbf{k}_0/k_0)]$.

The simplest choice for the trial field $\underline{\mathbf{E}}$ is of course $\underline{\mathbf{E}}=0$ and (still assuming the origin of time has been chosen so p^a is real and positive) this gives

$$\begin{aligned}&4\pi k_0\,\mathrm{Im}[e^{2i\omega t_0}p^aP^s_\infty(-\mathbf{k}_0/k_0)]/(\omega\rho_0)\leqslant(\mathbf{E}^a,\mathcal{L}_1\mathbf{E}^a)\\&\leqslant\int_\Omega\begin{pmatrix}\mathbf{Z}_0\,\mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\\ \mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\end{pmatrix}\cdot\begin{pmatrix}[\mathbf{Z}_1'']^{-1}&-[\mathbf{Z}_1'']^{-1}\mathbf{Z}_1'\\-\mathbf{Z}_1'[\mathbf{Z}_1'']^{-1}&\mathbf{Z}_1''+\mathbf{Z}_1'[\mathbf{Z}_1'']^{-1}\mathbf{Z}_1'\end{pmatrix}\begin{pmatrix}\mathbf{Z}_0\,\mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\\ \mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\end{pmatrix}d\mathbf{x}\\&\leqslant\int_\Omega\mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\cdot[\mathbf{Z}_1''+(\mathbf{Z}_1'-\mathbf{Z}_0)[\mathbf{Z}_1'']^{-1}(\mathbf{Z}_1'-\mathbf{Z}_0)]\,\mathrm{Re}[e^{i\omega t_0}\mathcal{F}^a(\mathbf{x})]\,d\mathbf{x}\end{aligned}$$

$$\leqslant \int_\Omega \mathrm{Re}(e^{i\omega t_0}\nabla P^a)\cdot[\mathbf{r}''+(\mathbf{r}'-\mathbf{r}_0)(\mathbf{r}'')^{-1}(\mathbf{r}'-\mathbf{r}_0)]\,\mathrm{Re}(e^{i\omega t_0}\nabla P^a)/\omega$$
$$+\omega\,\mathrm{Re}(e^{i\omega t_0}P^a)[h_1''+(h_1'-h_0)(h_1'')^{-1}(h_1'-h_0)]\,\mathrm{Re}(e^{i\omega t_0}P^a)\,d\mathbf{x}$$
$$\leqslant \frac{(p^a)^2\mathbf{k}_0\cdot[\mathbf{r}_1''+(\mathbf{r}_1'-\mathbf{r}_0)(\mathbf{r}_1'')^{-1}(\mathbf{r}_1'-\mathbf{r}_0)]\mathbf{k}_0}{\omega}\int_\Omega\{\sin[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}$$
$$+\,\omega(p^a)^2[h_1''+(h_1'-h_0)^2(h_1'')^{-1}]\int_\Omega\{\cos[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}, \tag{8.18}$$

and $\mathbf{r}_1=-\boldsymbol{\rho}_1^{-1}=\mathbf{r}_1'+i\mathbf{r}_1''$, and $h_1=1/\kappa_1=h_1'+ih_1''$. If both the inclusion phase and the matrix phase are isotropic, so that $\mathbf{r}_1'=r_1'\mathbf{I}$ and $\mathbf{r}_1''=r_1''\mathbf{I}$ then the bound becomes

$$4\pi k_0\,\mathrm{Im}[e^{2i\omega t_0}p^aP_\infty^s(-\mathbf{k}_0/k_0)]/(\omega\rho_0)\leqslant\frac{k_0^2(p^a)^2[r_1''+(r_1'-r_0)^2/r_1'']}{\omega}\int_\Omega\{\sin[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}$$
$$+\,\omega(p^a)^2[h_1''+(h_1'-h_0)^2/h_1'']\int_\Omega\{\cos[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}. \tag{8.19}$$

We can express the bound directly in terms of the real and imaginary parts of the complex density $\rho_1=\rho_1'+i\rho_1''$ and complex bulk modulus $\kappa_1=\kappa_1'+i\kappa_1''$ using the identities

$$[r_1''+(r_1'-r_0)^2/r_1'']=[\rho_1''+(\rho_1'-\rho_0)^2/\rho_1'']/\rho_0^2,\qquad [h_1''+(h_1'-h_0)^2/h_1'']=-[\kappa_1''+(\kappa_1'-\kappa_0)^2/\kappa_1'']/\kappa_0^2, \tag{8.20}$$

giving

$$4\pi k_0\,\mathrm{Im}\left[e^{2i\omega t_0}p^aP_\infty^s(-\mathbf{k}_0/k_0)\right]/(\omega\rho_0)\leqslant-\frac{\omega(p^a)^2[\rho_1''+(\rho_1'-\rho_0)^2/\rho_1'']}{\rho_0\kappa_0}\int_\Omega\{\sin[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}$$
$$-\frac{\omega(p^a)^2[\kappa_1''+(\kappa_1'-\kappa_0)^2/\kappa_1'']}{\kappa_0^2}\int_\Omega\{\cos[(\mathbf{k}_0\cdot\mathbf{x})+\omega t_0]\}^2\,d\mathbf{x}, \tag{8.21}$$

where we have replaced k_0^2 with $\omega^2\rho_0/\kappa_0$. This clearly then implies

$$\frac{4\pi|P_\infty^s(-\mathbf{k}_0/k_0)|}{p^ak_0|\Omega|}\leqslant\frac{[\rho_1''+(\rho_1'-\rho_0)^2/\rho_1'']}{\rho_0}-\frac{[\kappa_1''+(\kappa_1'-\kappa_0)^2/\kappa_1'']}{\kappa_0}, \tag{8.22}$$

in which $|\Omega|$ is the volume of Ω and $|P_\infty^s(-\mathbf{k}_0/k_0)|$ is the modulus of the backwards scattering amplitude $P_\infty^s(-\mathbf{k}_0/k_0)$. Note that both terms on the right-hand side of (8.22) are non-negative because $\rho_1''\geqslant 0$ and $\kappa_1''\leqslant 0$. This bound implies that to ensure the backscattering is small when ρ_1'' and κ_1'' are small, one should match ρ_1' and κ_1' to equal ρ_0 and κ_0, respectively.

IX. CONCLUSION

Perhaps the most important contribution of this paper is showing that Sommerfeld's radiation condition can be replaced by an appropriate "constitutive law" at infinity, akin to the perfectly matched layer (PML) technique in numerical analysis. The formulation of scattering as an appropriately defined Y problem puts scattering under the umbrella of a wide class of problems and motives further investigation into the theory of Y problems. It also raises the question as to what other physical or mathematical problems can be reformulated as Y problems. It is interesting that the variational principles only give bounds on the backward scattering amplitude rather than the desired forward scattering amplitude. We have no physical insight into why backscattering is subject to these bounds. As shown in Secs. VII and VIII, some of the quantities first entering the variational principle are related to power dissipation and scattered power, and indeed this was what motivated consideration of the quadratic form (8.3). However, surprisingly, these power terms cancel out of the final variational principle. One wonders if the variational principles can be tweaked in some way to produce bounds on the scattering amplitude in any direction.

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APPENDIX A: DERIVATION OF THE VARIATIONAL INEQUALITY

Here, we derive the variational inequality (6.14). We have

$$\left[\underline{\mathcal{F}}_1^0+\underline{\mathcal{F}}_2^0,\,\mathrm{Im}(\underline{P}_\infty),\,\mathrm{Re}(\underline{P}_\infty)\right]\in\mathcal{E},\quad\left[\underline{\mathcal{G}}_1^0+\underline{\mathcal{G}}_2^0,\,\mathrm{Re}(\underline{V}_\infty),\,-\mathrm{Im}(\underline{V}_\infty)\right]\in\mathcal{J}. \tag{A1}$$

Recall that $\mathrm{Re}\,(\mathcal{F}_1^s)$ and $\mathrm{Re}\,(\mathcal{G}_1^s)$ are prescribed as in (6.13) that $\mathcal{F}^s=\mathcal{F}_1^s+\mathcal{F}_2^s$ and $\mathcal{G}^s=\mathcal{G}_1^s+\mathcal{G}_2^s$ are the associated solutions of the Y problem. Since

$$\big[\mathrm{Re}\,(\mathcal{F}_1^s)+\mathrm{Re}\,(\mathcal{F}_2^s),\mathrm{Im}\,(P_\infty^s),\mathrm{Re}\,(P_\infty^s)\big]\in\mathcal{E},\quad \big[\mathrm{Im}\,(\mathcal{F}_1^s)+\mathrm{Im}\,(\mathcal{F}_2^s),-\mathrm{Re}\,(P_\infty^s),\mathrm{Im}\,(P_\infty^s)\big]\in\mathcal{E} \tag{A2}$$

and

$$\big[\mathrm{Re}\,(\mathcal{G}_1^s)+\mathrm{Re}\,(\mathcal{G}_2^s),\mathrm{Re}\,(V_\infty^s),-\mathrm{Im}\,(V_\infty^s)\big]\in\mathcal{J},\quad \big[\mathrm{Im}\,(\mathcal{G}_1^s)+\mathrm{Im}\,(\mathcal{G}_2^s),\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\in\mathcal{J} \tag{A3}$$

lie in orthogonal spaces, and since $\mathcal{V}$ and $\mathcal{H}$ are orthogonal, we deduce that

$$\begin{aligned}
-\big(\mathrm{Re}\,(\mathcal{F}_1^s),\mathrm{Im}\,(\mathcal{G}_1^s)\big)&=\big\langle\big[\mathrm{Re}\,(\mathcal{F}_2^s),\mathrm{Im}\,(P_\infty^s),\mathrm{Re}\,(P_\infty^s)\big],\big[\mathrm{Im}\,(\mathcal{G}_2^s),\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle\\
&=\big(\mathrm{Re}\,(\mathcal{F}_2^s),\mathrm{Im}\,(\mathcal{G}_2^s)\big)+\big\langle\big[0,\mathrm{Im}\,(P_\infty^s),\mathrm{Re}\,(P_\infty^s)\big],\big[0,\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle,\\
\big(\mathrm{Im}\,(\mathcal{F}_1^s),\mathrm{Re}\,(\mathcal{G}_1^s)\big)&=-\big\langle\big[\mathrm{Im}\,(\mathcal{F}_2^s),-\mathrm{Re}\,(P_\infty^s),\mathrm{Im}\,(P_\infty^s)\big],\big[\mathrm{Re}\,(\mathcal{G}_2^s),\mathrm{Re}\,(V_\infty^s),-\mathrm{Im}\,(V_\infty^s)\big]\big\rangle\\
&=-\big(\mathrm{Im}\,(\mathcal{F}_2^s),\mathrm{Re}\,(\mathcal{G}_2^s)\big)+\big\langle\big[0,\mathrm{Re}\,(P_\infty^s),-\mathrm{Im}\,(P_\infty^s)\big],\big[0,\mathrm{Re}\,(V_\infty^s),-\mathrm{Im}\,(V_\infty^s)\big]\big\rangle.
\end{aligned}\tag{A4}$$

Similarly (A1) with (6.13) imply

$$\begin{aligned}
-\big(\mathrm{Re}\,(\mathcal{F}_1^s),\mathrm{Im}\,(\mathcal{G}_1^s)\big)&=\big\langle\big[\underline{\mathcal{F}}_2^0,\mathrm{Im}(\underline{P}_\infty),\mathrm{Re}(\underline{P}_\infty)\big],\big[\mathrm{Im}\,(\mathcal{G}_2^s),\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle\\
&=\big(\underline{\mathcal{F}}_2^0,\mathrm{Im}\,(\mathcal{G}_2^s)\big)+\big\langle\big[0,\mathrm{Im}(\underline{P}_\infty),\mathrm{Re}(\underline{P}_\infty)\big],\big[0,\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle,\\
\big(\mathrm{Im}\,(\mathcal{F}_1^s),\mathrm{Re}\,(\mathcal{G}_1^s)\big)&=-\big\langle\big[\mathrm{Im}\,(\mathcal{F}_2^s),-\mathrm{Re}\,(P_\infty^s),\mathrm{Im}\,(P_\infty^s)\big],\big[\underline{\mathcal{G}}_2^0,\mathrm{Re}(\underline{V}_\infty),-\mathrm{Im}(\underline{V}_\infty)\big]\big\rangle\\
&=-\big(\mathrm{Im}\,(\mathcal{F}_2^s),\underline{\mathcal{G}}_2^0\big)+\big\langle\big[0,\mathrm{Re}\,(P_\infty^s),-\mathrm{Im}\,(P_\infty^s)\big],\big[0,\mathrm{Re}(\underline{V}_\infty),-\mathrm{Im}(\underline{V}_\infty)\big]\big\rangle.
\end{aligned}\tag{A5}$$

Now, defining $\underline{\mathbf{s}}(\mathbf{n})$ as in (6.15) we clearly have

$$\begin{aligned}
0&\leqslant\int_\Omega\big[\mathbf{E}_2^0(\mathbf{x})-\underline{\mathbf{E}}_2^0(\mathbf{x})\big]\cdot\mathcal{L}_1\big[\mathbf{E}_2^0(\mathbf{x})-\underline{\mathbf{E}}_2^0(\mathbf{x})\big]\,d\mathbf{x}+\int_{|\mathbf{n}|=1}[\mathbf{s}(\mathbf{n})-\underline{\mathbf{s}}(\mathbf{n})]\cdot\mathcal{L}_3[\mathbf{s}(\mathbf{n})-\underline{\mathbf{s}}(\mathbf{n})]\,dS\\
&=\int_\Omega\underline{\mathbf{E}}_2^0(\mathbf{x})\cdot\mathcal{L}_1\underline{\mathbf{E}}_2^0(\mathbf{x})\,d\mathbf{x}+\int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathcal{L}_3\underline{\mathbf{s}}(\mathbf{n})\,dS-2\big(\underline{\mathbf{E}}_2^0,\mathbf{J}_2^0\big)+\big(\mathbf{E}_2^0,\mathbf{J}_2^0\big)\\
&\quad-2\int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS+\int_{|\mathbf{n}|=1}\mathbf{s}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS,
\end{aligned}\tag{A6}$$

where $\mathbf{s}(\mathbf{n})$ and $\mathbf{t}(\mathbf{n})$ are defined in (6.5), and we have used the constitutive laws (6.2) and (6.5), and the identities (6.4) and (6.12). Since

$$\begin{aligned}
\big(\mathbf{E}_2^0,\mathbf{J}_2^0\big)&=-\big(\mathrm{Re}\,(\mathcal{G}_2^s),\mathrm{Im}\,(\mathcal{F}_2^s)\big)+\big(\mathrm{Re}\,(\mathcal{F}_2^s),\mathrm{Im}\,(\mathcal{G}_2^s)\big),\\
\int_{|\mathbf{n}|=1}\mathbf{s}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS&=\big\langle\big[0,\mathrm{Re}\,(V_\infty^s),-\mathrm{Im}\,(V_\infty^s)\big],\big[0,\mathrm{Re}\,(P_\infty^s),-\mathrm{Im}\,(P_\infty^s)\big]\big\rangle\\
&\quad+\big\langle\big[0,\mathrm{Im}\,(P_\infty^s),\mathrm{Re}\,(P_\infty^s)\big],\big[0,\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle,\\
\big(\underline{\mathbf{E}}_2^0,\mathbf{J}_2^0\big)&=-\big(\underline{\mathcal{G}}_2^0,\mathrm{Im}\,(\mathcal{F}_2^s)\big)+\big(\underline{\mathcal{F}}_2^0,\mathrm{Im}\,(\mathcal{G}_2^s)\big),\\
\int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS&=\big\langle\big[0,\mathrm{Re}(\underline{V}_\infty),-\mathrm{Im}(\underline{V}_\infty)\big],\big[0,\mathrm{Re}\,(P_\infty^s),-\mathrm{Im}\,(P_\infty^s)\big]\big\rangle\\
&\quad+\big\langle\big[0,\mathrm{Im}(\underline{P}_\infty),\mathrm{Re}(\underline{P}_\infty)\big],\big[0,\mathrm{Im}\,(V_\infty^s),\mathrm{Re}\,(V_\infty^s)\big]\big\rangle,
\end{aligned}\tag{A7}$$

the identities (A5) imply

$$\begin{aligned}
\big(\mathbf{E}_2^0,\mathbf{J}_2^0\big)+\int_{|\mathbf{n}|=1}\mathbf{s}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS&=\big(\mathrm{Im}\,(\mathcal{F}_1^s),\mathrm{Re}\,(\mathcal{G}_1^s)\big)-\big(\mathrm{Re}\,(\mathcal{F}_1^s),\mathrm{Im}\,(\mathcal{G}_1^s)\big),\\
\big(\underline{\mathbf{E}}_2^0,\mathbf{J}_2^0\big)+\int_{|\mathbf{n}|=1}\underline{\mathbf{s}}(\mathbf{n})\cdot\mathbf{t}(\mathbf{n})\,dS&=\big(\mathrm{Im}\,(\mathcal{F}_1^s),\mathrm{Re}\,(\mathcal{G}_1^s)\big)-\big(\mathrm{Re}\,(\mathcal{F}_1^s),\mathrm{Im}\,(\mathcal{G}_1^s)\big).
\end{aligned}\tag{A8}$$

Substituting these in (A6) gives the variational inequality (6.14).

APPENDIX B: CONNECTION WITH OPTICAL THEOREMS

Here, we show that the expression (7.6) for the extinction power can be connected to other expressions for W, that are generally known as optical theorems. From (7.6) it follows that

$$2W=\mathrm{Im}\,\big(\mathcal{F}^a+\mathcal{F}_1^s,\chi\mathbf{Z}_1(\mathcal{F}^a+\mathcal{F}_1^s)\big)-\mathrm{Im}\,\big(\mathcal{F}_1^s,\mathcal{G}_1^s\big)$$

$$\begin{aligned}
&= \mathrm{Im}(\mathcal{F}^a,\chi\mathbf{Z}_1\mathcal{F}^a) + \mathrm{Im}\left(\mathcal{F}_1^s,\chi\mathbf{Z}_1\mathcal{F}^a\right) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_1\mathcal{F}_1^s\right) + \mathrm{Im}\left(\mathcal{F}_1^s,\mathbf{Z}_1\mathcal{F}_1^s\right)\\
&\quad - \mathrm{Im}\left(\mathcal{F}_1^s,\mathbf{Z}_1\mathcal{F}_1^s + (\mathbf{Z}_1-\mathbf{Z}_0)\chi\mathcal{F}^a\right)\\
&= \mathrm{Im}(\mathcal{F}^a,\chi\mathbf{Z}_1\mathcal{F}^a) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_1\mathcal{F}_1^s\right) + \mathrm{Im}\left(\mathcal{F}_1^s,\mathbf{Z}_0\chi\mathcal{F}^a\right).
\end{aligned} \tag{B1}$$

Since $\mathbf{Z}_0$ is real, we also have

$$\begin{aligned}
0 &= \mathrm{Im}\left(\mathcal{F}^a + \mathcal{F}_1^s,\chi\mathbf{Z}_0\left(\mathcal{F}^a + \mathcal{F}_1^s\right)\right)\\
&= \mathrm{Im}(\mathcal{F}^a,\chi\mathbf{Z}_0\mathcal{F}^a) + \mathrm{Im}\left(\mathcal{F}_1^s,\chi\mathbf{Z}_0\mathcal{F}^a\right) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_0\mathcal{F}_1^s\right) + \mathrm{Im}\left(\mathcal{F}_1^s,\mathbf{Z}_0\mathcal{F}_1^s\right)\\
&= \mathrm{Im}\left(\mathcal{F}_1^s,\chi\mathbf{Z}_0\mathcal{F}^a\right) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_0\mathcal{F}_1^s\right).
\end{aligned} \tag{B2}$$

Substituting this back in (B1), and again using the fact that $\mathrm{Im}(\mathcal{F}^a,\chi\mathbf{Z}_0\mathcal{F}^a) = 0$, gives

$$\begin{aligned}
2W &= \mathrm{Im}(\mathcal{F}^a,\chi(\mathbf{Z}_1-\mathbf{Z}_0)\mathcal{F}^a) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_1\mathcal{F}_1^s\right) + \mathrm{Im}\left(\mathcal{F}^a,\mathbf{Z}_0\mathcal{F}_1^s\right)\\
&= \mathrm{Im}\left(\mathcal{F}^a,\chi(\mathbf{Z}_1-\mathbf{Z}_0)\left(\mathcal{F}^a + \mathcal{F}_1^s\right)\right)\\
&= \mathrm{Im}(\mathcal{F}^a,\chi(\mathbf{Z}_1-\mathbf{Z}_0)(\mathcal{F}^a + \mathcal{F}^s)).
\end{aligned} \tag{B3}$$

This is analogous to the form of the optical theorem given in [23,65,66]. We can further reduce it to

$$\begin{aligned}
2W &= \mathrm{Im}(\mathcal{F}^a,(\mathbf{Z}-\mathbf{Z}_0)(\mathcal{F}^a + \mathcal{F}^s))\\
&= \mathrm{Im}(\mathcal{F}^a,\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)\\
&= \mathrm{Im}\int_{B_r} \overline{p^a}\mathcal{F}^{a\prime}\cdot(\mathcal{G}^s - \mathbf{Z}_0\mathcal{F}^s)\,d\mathbf{x},
\end{aligned} \tag{B4}$$

where $\mathcal{F}^{a\prime} = \overline{\mathcal{F}^a/p^a}$. Thus, using the results of Sec. V, and making the substitution $\mathbf{k}_0' = \mathbf{k}_0$ we get

$$W = 2k_0\pi\,\mathrm{Im}[\overline{p^a}P_\infty^s(\mathbf{k}_0/k_0)]/(\omega\rho_0), \tag{B5}$$

which is the well-known form of the optical theorem [67,68] for acoustic scattering.

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