Analytical description of the mode hybridization in a restricted two-dimensional model for an electromagnetic cavity containing a thin magnetized slab

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Analytical solutions to the problem of mode splitting caused by a thin plain magnetized slab placed in a rectangular electromagnetic cavity are found in two special cases: an infinite slab between two ideal infinite plain mirrors (1D model) and a 2D model of a cavity confined in the direction of permanent magnetization. Under the resonance conditions, the time-dependent magnetization vectors inside the slab are strongly enhanced, with opposite directions in the split modes. For realistic parameters, the frequency splitting does not depend on the dielectric constant of the slab and the Landau-Lifshitz-Gilbert damping coefficient, whereas the loaded cavity quality factor is not sensitive to the slab position.

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I. INTRODUCTION

Recently, hybridized modes in electromagnetic cavities containing ferromagnetic samples attracted the attention of many researchers [1–19]. This interest is explained, in particular, by possible applications to the problems of quantum information, due to strong coupling of qubits (represented by spin systems) to electromagnetic fields [20]. Reviews of similar systems, such as exciton polaritons in semiconductor microcavities or other hybrid devices, can be found in Refs. [21–24].

Earlier, the effects of strong coupling between electromagnetic and magnetostatic modes were studied, e.g., in Refs. [25,26]. Various geometries were considered up to now. Spherical cavities with spherical samples displaced from the cavity center were investigated theoretically in Ref. [1]. The authors of Refs. [2,6,17] performed experiments with thin ferromagnetic films deposited on a superconducting coplanar waveguide microwave resonator or similar structures. Rectangular 3D cavities containing small spherical samples were used in Refs. [3,4,13,18]. Re-entrant type cavities with two cylindrical posts and a small spherical sample between them were studied in Ref. [5], and multipost cavities were used in Ref. [14]. Coaxial cables with small spherical samples were used in Refs. [7,11]. Numerical simulations of multilayered magnetic structures were performed in Ref. [8]. Experiments with flat samples inside a rectangular cavity were reported in Ref. [9]. The case of a thin film placed between two plane semitransparent mirrors was studied theoretically in Ref. [10] in the framework of the scattering approach. Experiments with this geometry were reported in Ref. [12]. Cylindrical cavities were used in Refs. [15] (with flat samples) and [16] (with a spherical sample near the wall).

The aim of our study is to find analytical solutions for the electromagnetic field in simplified models of rectangular cavities containing thin flat magnetized slabs, in order to see how the mode hybridization happens and what is essentially different in the two split modes. The plan is as follows. Section II contains basic equations. Section III is devoted to a simplified version of the problem studied in Ref. [10], namely, a thin infinite plain slab between two ideal infinite plain mirrors. A more realistic 2D model is studied in Sec. IV. A brief discussion of results is given in Sec. V.

II. BASIC EQUATIONS

We consider a cavity containing a slab with a constant dielectric permeability ε and a constant uniform saturated magnetization vector $\mathbf{M}_0 = (0, M_0, 0)$, whose direction is chosen as the y axis. In addition, a constant uniform magnetic field $\mathbf{H}_0 = (0, H_0, 0)$ is directed along the same axis. The geometry is shown in Fig. 1.

Our goal is to find the structure of possible weak monochromatic time-dependent electric and magnetic fields $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{-i\omega t}$ and $\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$ in the whole volume of the cavity (both inside and outside the slab). The Maxwell equations inside the slab have the form (we use the SI units)

$$\mathrm{rot}\mathbf{H} = \varepsilon_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathrm{rot}\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}(\mathbf{H} + \mathbf{M}), \tag{1}$$

where $\mathbf{M}(\mathbf{r},t) = \mathbf{M}(\mathbf{r})e^{-i\omega t}$ is the time-dependent part of the magnetization vector inside the slab. Outside the slab one should put $\varepsilon = 1$ and $\mathbf{M} = 0$. The evolution of vector $\mathbf{M}$ is governed by the Landau-Lifshitz-Gilbert equation [27–29]

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma [\mathbf{H}_{ef} \times \mathcal{M}] + \frac{\alpha}{M_0}\left[\mathcal{M} \times \frac{\partial \mathbf{M}}{\partial t}\right], \tag{2}$$

where $\mathcal{M} = \mathbf{M}_0 + \mathbf{M}$ is the total magnetization vector, γ is the gyromagnetic ratio, and α is a small damping coefficient. For the materials like yttrium iron garnet $Y_3Fe_5O_{12}$ (YIG), frequently used in experiments, $\gamma/(2\pi\mu_0) \approx 28$ GHz/T and the saturation magnetization $\mu_0 M_0 \approx 0.17$ T [30]. The values of the damping factor α for YIG, given in the available literature, were $\alpha \sim 10^{-3}$ [8,12] and $\alpha \approx 3 \times 10^{-4}$ [15], while a much lower value $\alpha \approx 7 \times 10^{-5}$ was reported in Ref. [31]. The limiting relaxation mechanism is spin-lattice coupling [32]. The effective magnetic field acting on the magnetization vector has the form (if we neglect the anisotropy effects)

$$\mathbf{H}_{ef} = \mathbf{H}_0 + \mathbf{H} + J\nabla^2 \mathbf{M}, \tag{3}$$

where $J > 0$ is the exchange parameter.

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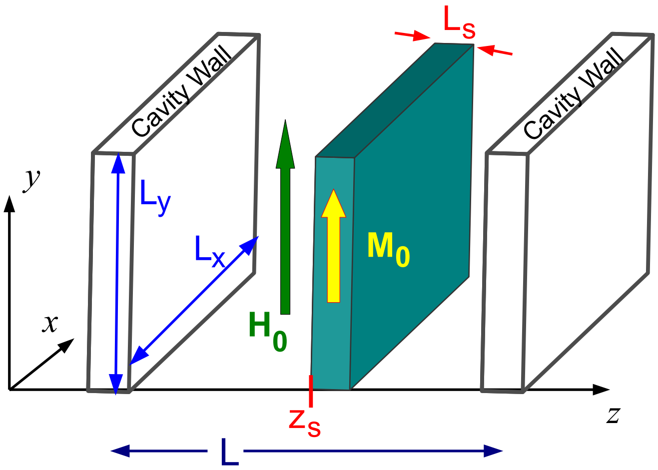

FIG. 1. Geometry of the problem.

The time-independent amplitudes obey the equations

$$\mathrm{rot}\mathbf{H} = -i\omega\varepsilon_0\varepsilon\mathbf{E}, \qquad \mathrm{rot}\mathbf{E} = i\omega\mu_0(\mathbf{H}+\mathbf{M}). \tag{4}$$

The first of these two equations can be written in the following form, which will be used later:

$$\mathbf{E} = i(\omega\varepsilon_0\varepsilon)^{-1}\begin{pmatrix}\partial H_z/\partial y - \partial H_y/\partial z\\ \partial H_x/\partial z - \partial H_z/\partial x\\ \partial H_y/\partial x - \partial H_x/\partial y\end{pmatrix}. \tag{5}$$

The second order equation for the magnetic field vector inside the slab reads

$$\Delta\mathbf{H} + \omega^2\varepsilon\varepsilon_0\mu_0\mathbf{H} = \mathrm{grad}\,\mathrm{div}\mathbf{H} - \omega^2\varepsilon\varepsilon_0\mu_0\mathbf{M}. \tag{6}$$

Outside the slab the right-hand side of this equation should be replaced by zero.

We assume that $|\mathbf{M}| \ll |M_0|$, since $\mathbf{M}$ is induced by the weak harmonic magnetic field with $|\mathbf{H}| \ll |H_0|$. Then we can linearize Eq. (2), neglecting the term $[\mathbf{H}\times\mathbf{M}]$. Under these assumptions, the amplitude vectors of the harmonic magnetization and magnetic field obey the equation (remember that vectors $\mathbf{H}_0$ and $\mathbf{M}_0$ are parallel)

$$-i\omega\mathbf{M} = \gamma[\mathbf{H}_0\times\mathbf{M}] - \gamma[\mathbf{M}_0\times(\mathbf{H}+J\nabla^2\mathbf{M})] - (i\omega\alpha/M_0)[\mathbf{M}_0\times\mathbf{M}]. \tag{7}$$

It is reasonable to suppose that $\nabla^2\mathbf{M} = -q^2\mathbf{M}$ for the harmonic fields in a homogeneous slab, where the constant factor q^2 will be determined later. Then we have $M_y = 0$ and the following equations for two other components of vector $\mathbf{M} = (M_x, M_y, M_z)$:

$$i\omega M_x = \omega_M H_z - \tilde\omega_H M_z, \quad i\omega M_z = \tilde\omega_H M_x - \omega_M H_x, \tag{8}$$

where

$$\omega_H \equiv \gamma H_0, \quad \omega_M \equiv \gamma M_0, \quad \tilde\omega_H \equiv \omega_H - i\omega\alpha + \omega_M J q^2.$$

The solutions to Eqs. (8) are as follows,

$$M_x = \frac{\omega_M(\tilde\omega_H H_x + i\omega H_z)}{\tilde\omega_H^2 - \omega^2}, \tag{9}$$

$$M_z = \frac{\omega_M(\tilde\omega_H H_z - i\omega H_x)}{\tilde\omega_H^2 - \omega^2}. \tag{10}$$

Then Eq. (6) results in the following equations for the magnetic field components inside the slab:

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} - \frac{\partial^2 H_y}{\partial x\partial y} - \frac{\partial^2 H_z}{\partial x\partial z} = -k^2\varepsilon[uH_x + ivH_z], \tag{11}$$

$$\frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial x^2} - \frac{\partial^2 H_y}{\partial z\partial y} - \frac{\partial^2 H_x}{\partial x\partial z} = -k^2\varepsilon[uH_z - ivH_x], \tag{12}$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} - \frac{\partial^2 H_x}{\partial x\partial y} - \frac{\partial^2 H_z}{\partial y\partial z} = -k^2\varepsilon H_y, \tag{13}$$

where $k^2 = \omega^2\varepsilon_0\mu_0$,

$$u = \frac{\tilde\omega_H(\tilde\omega_H + \omega_M) - \omega^2}{\tilde\omega_H^2 - \omega^2}, \qquad v = \frac{\omega\omega_M}{\tilde\omega_H^2 - \omega^2}. \tag{14}$$

III. 1D MODEL

Let us consider first the idealized special case of very big transverse spatial dimensions: $L_x = L_y = \infty$. Then we may look for the solutions depending on the single space variable z perpendicular to the slab surface. (There exist also solutions describing the waveguide propagation along the infinite slab [8,33,34], but we do not consider such propagating modes here.) For these solutions, Eqs. (11)–(13) can be simplified as follows,

$$\frac{d^2 H_x}{dz^2} + k^2\varepsilon[uH_x + ivH_z] = 0, \tag{15}$$

$$uH_z - ivH_x = 0, \tag{16}$$

$$\frac{d^2 H_y}{dz^2} + k^2\varepsilon H_y = 0. \tag{17}$$

The consequence of (10) and (16) is $B_z = H_z + M_z = 0$, as it must be due to the equation $\mathrm{div}\mathbf{B} = dB_z/dz = 0$ in the one-dimensional case under consideration. Using Eq. (16), we can write Eq. (15) as

$$\frac{d^2 H_x}{dz^2} + k^2\varepsilon g^2 H_x = 0, \tag{18}$$

where

$$g^2(\omega) = \frac{(\tilde\omega_H + \omega_M)^2 - \omega^2}{\tilde\omega_H(\tilde\omega_H + \omega_M) - \omega^2}. \tag{19}$$

We see that H_x and H_y components are totally independent. Moreover, the H_y component (directed along the constant magnetic field and magnetization vectors) does not depend on the magnetic parameters H_0 and M_0.

Let us suppose that the magnetized slab of thickness L_s occupies the region $0 \leqslant z_s < z < z_s + L_s \leqslant L$, where z_s is the position of the left boundary of the slab. Then the function $H_x(z)$ can be written as follows (its form in the empty part of the cavity is determined by the boundary condition $dH_x/dz = E_y = 0$ at the ideal cavity walls):

$$H_x(z) = \begin{cases} W\cos(kz), & 0 < z < z_s\\ U\cos(kg\sqrt{\varepsilon}\,z + \varphi), & z_s < z < z_s + L_s.\\ V\cos[k(z-L)], & z_s + L_s < z < L\end{cases}$$

The continuity conditions for H_x and $E_y \propto \varepsilon^{-1} dH_x/dz$ at the slab surfaces $z = z_s$ and $z = z_s + L_s$ result in two equations:

$$g \tan(kg\sqrt{\varepsilon}\, z_s + \varphi) = \sqrt{\varepsilon} \tan(kz_s), \tag{20}$$

$$g \tan[kg\sqrt{\varepsilon}\,(z_s + L_s) + \varphi] = \sqrt{\varepsilon} \tan[k(z_s + L_s - L)]. \tag{21}$$

Excluding the phase φ we arrive at the equation

$$\tan[k(z_s+L_s-L)] = \frac{g \tan(kg\sqrt{\varepsilon}\, L_s)/\sqrt{\varepsilon} + \tan(kz_s)}{1 - \sqrt{\varepsilon} \tan(kz_s) \tan(kg\sqrt{\varepsilon}\, L_s)/g}. \tag{22}$$

For a thin slab, satisfying the condition $kL_s \ll 1$, the denominator in Eq. (22) can be replaced by unity (unless $|\tan(kz_s)| \gg 1$, but these regions, close to the nodes of the magnetic field in the empty cavity, are not interesting from the point of view of the hybridization phenomena), and we have a simplified equation

$$\frac{g}{\sqrt{\varepsilon}} \tan(kg\sqrt{\varepsilon}\, L_s) = \tan[k(z_s + L_s - L)] - \tan(kz_s). \tag{23}$$

For the totally empty cavity ($L_s = z_s = 0$) we have the well known solutions

$$k_n = \omega_n/c = n\pi/L, \qquad n = 1,2,3,\dots, \tag{24}$$

where $c \equiv (\varepsilon_0\mu_0)^{-1/2}$ is the light velocity in vacuum.

To find solutions to Eq. (23) for nonzero (but small enough) values of the slab thickness $L_s \ll L$ (the criterion of smallness will be given below), we replace the tangent function in the left-hand side of Eq. (23) with its argument and neglect the term kL_s in the right-hand side of this equation. Then, writing $k = k_n + \delta k$ (with $|\delta k|L = |\delta\omega|L/c \ll 1$) and neglecting the second order corrections, we use the chain of relations

$$\tan[k(z_s - L)] - \tan(kz_s) = \tan(kz_s - L\delta k) - \tan(kz_s)$$
$$\approx -(L\delta\omega/c)/\cos^2(kz_s) \approx -(L\delta\omega/c)/\cos^2(n\pi z_s/L).$$

In this way, we transform Eq. (23) to the following one:

$$\delta\omega = -R\omega_n g^2(\omega_n + \delta\omega), \quad R \equiv \frac{L_s}{L}\cos^2\left(n\pi\frac{z_s}{L}\right). \tag{25}$$

Consequently, the maximal influence of the slab on the frequency shift can be observed at those points z_s where the magnetic field amplitude takes maximal values in the empty cavity: $\cos^2(n\pi z_s/L) = 1$.

A. Neglecting damping and exchange effects

Let us begin the analysis of Eq. (25) with the simplest case of $\alpha = J = 0$. For a small ratio $|\delta\omega|/\omega_n \ll 1$ we can write $\omega^2 \approx \omega_n^2 + 2\omega_n\delta\omega$ in the denominator of function (19), neglecting the term $\delta\omega^2$. At the same time, we put $\omega = \omega_n$ in the numerator of this function, taking into account that it is multiplied by the small parameter $R \ll 1$ in the right-hand side of Eq. (25). Thus we arrive at the quadratic equation

$$2(\delta\omega)^2 - \chi\delta\omega - R\eta = 0, \tag{26}$$

where

$$\chi \equiv [\omega_H(\omega_H + \omega_M) - \omega_n^2]/\omega_n, \quad \eta \equiv (\omega_H + \omega_M)^2 - \omega_n^2.$$

Two solutions of Eq. (26)

$$\delta\omega_\pm = \chi/4 \pm \sqrt{\chi^2/16 + R\eta/2} \tag{27}$$

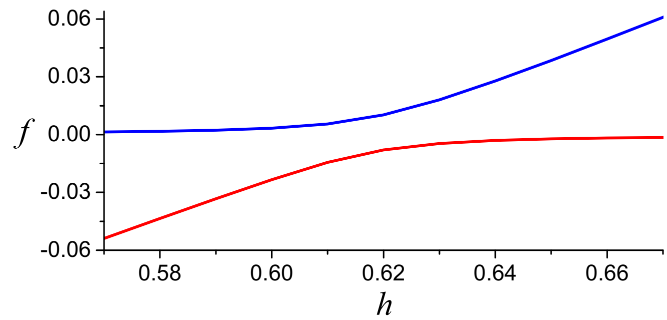

FIG. 2. The dimensionless frequency shift $f \equiv \delta\omega/\omega_n$ versus the dimensionless magnetic field $h \equiv \omega_H/\omega_M = H/M_0$ in the case of $\omega_n = \omega_M$ and $R = 10^{-4}$.

provide the analytical description of the hybridization effect, illustrated in Fig. 2. This effect is mostly pronounced under the resonance condition $\chi = 0$, i.e.,

$$\omega_H(\omega_H + \omega_M) = \omega_n^2, \quad \omega_H^{\rm res} = \sqrt{\omega_n^2 + \omega_M^2/4} - \omega_M/2. \tag{28}$$

The frequency splitting $\Delta = \delta\omega_+ - \delta\omega_-$ in the resonance case (it can be treated also as the avoided-crossing frequency or the Rabi frequency) equals

$$\Delta = \left(R\omega_M\left[\sqrt{\omega_M^2 + 4\omega_n^2} + \omega_M\right]\right)^{1/2}. \tag{29}$$

This dependence is illustrated in Fig. 3.

The hybridization region corresponds to $\chi^2 < 8R\eta$:

$$\left|\omega_H - \omega_H^{\rm res}\right| < \frac{\omega_n\Delta}{\sqrt{\omega_n^2 + \omega_M^2/4}}. \tag{30}$$

Outside this region we have separate photon and magnon modes. The photon mode (horizontal line in Fig. 2) has the frequency close to ω_n (it almost does not depend on the static magnetic field H_0), whereas the frequency of magnon mode is close to $\sqrt{\omega_H(\omega_H + \omega_M)}$, in accordance with Ref. [35] for the plain slab.

Now we can evaluate the realm of validity of the approximations made above. The manipulations with the right-hand side of Eq. (23) are justified provided $\omega_n L_s \ll L\delta\omega$. In view of (29) we obtain the condition

$$\frac{L_s}{L} \ll \frac{M_0}{H_0}\cos^2(n\pi z_s/L). \tag{31}$$

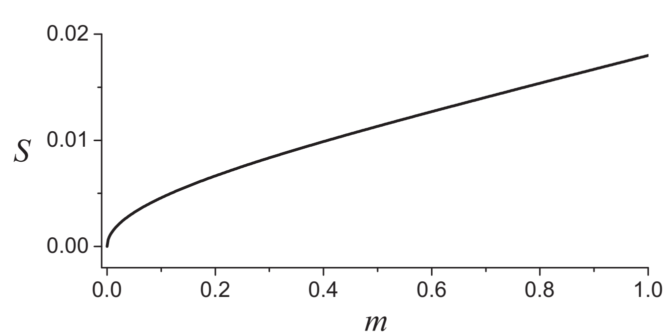

FIG. 3. The dimensionless Rabi frequency $S \equiv \Delta/\omega_n$ versus the ratio $m \equiv \omega_M/\omega_n$ for $R = 10^{-4}$.

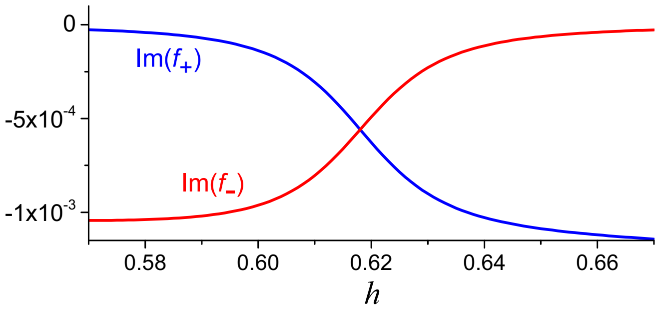

FIG. 4. The imaginary parts of dimensionless frequency shifts $f \equiv \delta\omega_\pm/\omega_n$ versus the dimensionless magnetic field $h \equiv H/M_0$ for $\alpha = 10^{-3}$, $\omega_n = \omega_M$, and $R = 10^{-4}$.

The simplification of the left-hand side of Eq. (23) is justified provided

$$\frac{L_s}{L} \ll \left[\frac{H_0 \cos^2(n\pi z_s/L)}{M_0 \varepsilon^2}\right]^{1/3}. \tag{32}$$

Consequently, simple approximations do not work in the regions with small amplitudes of the magnetic field, where $|\cos(n\pi z_s/L)| \ll 1$. But these regions are not interesting, if the goal is to obtain a maximal frequency splitting. Outside these regions, the assumption $L_s \ll L$ is sufficient, if $M_0 \sim H_0$ and ε is not too big.

B. Account of damping

If $\alpha > 0$ (but $J = 0$), then the denominator of function $g^2(\omega_n + \delta\omega)$ can be approximated as

$$\tilde{\omega}_H(\tilde{\omega}_H + \omega_M) - \omega^2 \approx \omega_n \chi - 2\omega_n \delta\omega - i\omega_n \alpha(2\omega_H + \omega_M),$$

where the higher-order terms $\delta\omega^2$, $\alpha\delta\omega$, and α^2 are neglected. Instead of (26) we have now the equation (neglecting the higher-order terms $R\alpha$, $R\delta\omega$, and so on)

$$2(\delta\omega)^2 + (i\alpha\xi - \chi)\delta\omega - R\eta = 0, \qquad \xi \equiv 2\omega_H + \omega_M. \tag{33}$$

Its two solutions are as follows,

$$\delta\omega_\pm = (\chi - i\alpha\xi)/4 \pm \sqrt{(\chi - i\alpha\xi)^2/16 + R\eta/2}. \tag{34}$$

For small values of α (remember that $\alpha < 10^{-3}$ for the materials like YIG), real parts of $\delta\omega_\pm$ are practically the same as that discussed in the preceding subsection. A typical behavior of the imaginary parts of functions (34) is shown in Fig. 4.

The two curves intersect at the point of resonance with $\chi = 0$, where $\mathrm{Im}(\delta\omega_\pm^{\mathrm{res}}) = -(i\alpha/4)\sqrt{4\omega_n^2 + \omega_M^2}$. Note that this value (which is much smaller than the frequency splitting Δ) does not depend on the slab thickness L_s and its position z_s. The quality factor of the cavity with ideal walls at the resonance equals

$$Q_{\mathrm{res}} = \frac{\omega}{2|\mathrm{Im}\,\delta\omega^{\mathrm{res}}|} \approx \frac{\omega_n}{\alpha\sqrt{\omega_n^2 + \omega_M^2/4}}. \tag{35}$$

For a nonideal cavity with high unloaded quality factor Q_0 one can use the formula $Q_{\mathrm{tot}}^{-1} = Q_0^{-1} + Q_{\mathrm{res}}^{-1}$.

C. Account of exchange effects

To evaluate the contribution of exchange effects, let us suppose that $J > 0$ but $\alpha = 0$. Since the magnetization vector components are proportional to the magnetic field components inside the slab, the following consistency condition must be satisfied: $q^2 = k^2\varepsilon g^2$. Comparing this equality with Eq. (25) and taking $k \approx k_n$, we arrive at the relation $Jq^2 \approx -\delta\omega\varepsilon J k_n^2/(R\omega_n)$. Therefore the contribution of terms proportional to J in the denominator of function $g^2(\omega)$ has the following order of magnitude for $\omega_M \sim \omega_H \sim \omega_n$ (neglecting the second order corrections):

$$(2\omega_H + \omega_M)\omega_M J q^2 \sim -\omega_n \delta\omega\varepsilon J k_n^2/R.$$

Since $J \approx 3 \times 10^{-16}$ m^2 and $\varepsilon \approx 15$ for typical magnetic materials used in experiments (YIG) [30], the dimensionless parameter $\varepsilon J k_n^2$ is very small for the low frequency modes in cavities with $L > 1$ cm (when $k_n \sim \pi/L$): $\varepsilon J k_n^2 < 10^{-9}$. On the other hand, $R > 10^{-4}$ for slabs of thickness $L_s > 1$ μm (in the regions of maximal magnetic field of the field mode). Consequently, the exchange effects are negligible in the case concerned.

D. What is different for the two hybridized modes?

For small frequency changes, $|\delta\omega_\pm| \ll \omega_n$, the changes of the wave number k are also small. Therefore the H_x component of the magnetic field (parallel to the slab) is close to its values in the empty cavity outside the slab. Moreover, due to the continuity of H_x at the slab surfaces, this field component almost does not change inside the slab as well. Then, what is significantly different for the hybridized modes with the frequencies $\omega_\pm = \omega_n + \delta\omega_\pm$?

The answer is given by Eq. (16), relating the magnetic field components inside the slab. For $|\delta\omega_\pm| \ll \omega_n$ it can be written as (we neglect here the damping corrections)

$$H_z = \frac{i\omega_M}{\chi - 2\delta\omega} H_x. \tag{36}$$

Consequently, at the resonance $\chi = 0$ the magnetic field component H_z inside the slab (it is perpendicular to the slab surface) becomes much stronger than H_x, and it has opposite signs for the ω_+ and ω_- modes:

$$H_z^{\mathrm{res}} = -\frac{i\omega_M}{2\delta\omega} H_x = -M_z^{\mathrm{res}}. \tag{37}$$

Moreover, the x component of the magnetization vector $\mathbf{M}$ also becomes big, and the directions of the time-dependent parts of the magnetization vector are opposite for the split modes:

$$M_x^{\mathrm{res}} = -\frac{\omega_n \omega_M}{2\omega_H \delta\omega} H_x. \tag{38}$$

IV. 2D MODELS

Real cavities have finite extensions in all three dimensions. The main difficulty of solving equations in this case is the necessity to take into account the boundary conditions not only at the surfaces with $z = const$, but also at the boundaries described by the equations $x = const$ and $y = const$. In the following subsections we show that for some restricted 2D models analytical solutions still can be obtained, although this seems to be impossible in the generic case.

A. Confinement in the direction of external magnetic field

Let us consider a cavity having a finite extension in the direction of external constant magnetic field, so that $-L_y/2 < y < L_y/2$ (but $-\infty < x < \infty$). Then the magnetic field vector must depend on two variables, z and y. In this case Eqs. (11)–(13) become

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + k^2\varepsilon[uH_x + ivH_z] = 0, \tag{39}$$

$$\frac{\partial^2 H_z}{\partial y^2} - \frac{\partial^2 H_y}{\partial z \partial y} + k^2\varepsilon[uH_z - ivH_x] = 0, \tag{40}$$

$$\frac{\partial^2 H_y}{\partial z^2} - \frac{\partial^2 H_z}{\partial y \partial z} + k^2\varepsilon H_y = 0. \tag{41}$$

One can check that the split solutions of the preceding section with $H_y = \partial H_x/\partial y = \partial H_z/\partial y = 0$ hold in the new case, as well (as soon as the only nonzero component of the electric field is E_y). On the other hand, we have no more the solution with $H_x = H_z = 0$ but $H_y \neq 0$ (which did not depend on the slab magnetization), because the boundary condition $E_x = 0$ at $y = \pm L_y/2$ cannot be satisfied in view of Eq. (40).

We look for the factorized solutions with $H_y \neq 0$:

$$H_x = \psi(z)\beta(y), \;\; H_z = \gamma(z)\delta(y), \;\; H_y = \mu(z)\nu(y). \tag{42}$$

Such a factorization is possible for Eq. (39) provided

$$\beta'' = -r^2\beta(y), \qquad \delta(y) = \xi\beta(y), \qquad r,\xi = const. \tag{43}$$

(Hereafter the primes mean derivatives of functions with respect to their arguments.) Without any loss of generality we may take $\xi = 1$ [otherwise we can simply redefine function $\gamma(z)$]. Then we arrive at the equation

$$\psi'' + (k^2\varepsilon u - r^2)\psi(z) + ik^2\varepsilon v\gamma(z) = 0. \tag{44}$$

Equation (40) with functions (42) assumes the form

$$\beta(y)[(k^2\varepsilon u - r^2)\gamma(z) - ik^2\varepsilon v\psi(z)] = \mu'(z)\nu'(y). \tag{45}$$

It is factorized if $\beta(y) = \zeta\nu'(y)$ with $\zeta = const$. Choosing $\zeta = 1$ [redefining if necessary function $\mu(z)$] we arrive at the equation

$$\mu'(z) = (k^2\varepsilon u - r^2)\gamma(z) - ik^2\varepsilon v\psi(z). \tag{46}$$

Equation (41) becomes

$$\mu'' + k^2\varepsilon\mu(z) = -\rho^2\gamma'(z) \tag{47}$$

if

$$\nu'' + \rho^2\nu(y) = 0, \qquad \rho = const. \tag{48}$$

The equation $\text{div}\mathbf{B} = \text{div}(\mathbf{H} + \mathbf{M}) = 0$ can be factorized provided

$$\nu'(y)[\mu(z) + u\gamma'(z) - iv\psi'(z)] = 0. \tag{49}$$

This condition is satisfied automatically if $\nu'(y) \equiv 0$. But for the finite cavity length in the y direction, we must satisfy the boundary conditions $E_x = E_z = 0$ at the surfaces $y = \pm L_y/2$.

The electric field vector (5) assumes the following form in the factorized case:

$$\mathbf{E} = i(\omega\varepsilon_0\varepsilon)^{-1}\begin{pmatrix} -\nu(y)[\rho^2\gamma(z) + \mu'(z)] \\ \nu'(y)\psi'(z) \\ \rho^2\nu(y)\psi(z) \end{pmatrix}. \tag{50}$$

Consequently, the condition $\nu(y) = 0$ at $y = \pm L_y/2$ determines the solutions to Eq. (48) in the form

$$\nu(y) = \cos(\rho_m y), \quad \rho_m = (1 + 2m)\pi/L_y, \;\; m = 0,1,\ldots.$$

Then we have $r = \rho_m$ and

$$\beta(y) = \delta(y) = \nu'(y) = -\rho_m \sin(\rho_m y).$$

Consequently, in the limit $L_y \to \infty$ we have $\nu(y) = 1$ and $H_x = H_z = 0$. But for $L_y < \infty$ the nonzero component H_y is accompanied with nonzero components H_x and H_z.

It is easy to verify with the aid of Eq. (49), that outside the slab (where $\varepsilon = u = 1$ and $v = 0$) equations for the functions $\psi(z)$ and $\mu(z)$ become identical:

$$\psi'' + (k^2 - r^2)\psi(z) = 0, \qquad \mu'' + (k^2 - r^2)\mu(z) = 0.$$

Inside the slab we use the standard procedure, looking for solutions in the form

$$\psi(z) = \psi_0 e^{iqz}, \qquad \gamma(z) = \gamma_0 e^{iqz}, \qquad \mu(z) = \mu_0 e^{iqz}.$$

Thus we arrive at the matrix equation

$$\begin{pmatrix} k^2\varepsilon u - r^2 - q^2 & ik^2\varepsilon v & 0 \\ -ik^2\varepsilon v & k^2\varepsilon u - r^2 & -iq \\ 0 & iqr^2 & k^2\varepsilon - q^2 \end{pmatrix}\begin{pmatrix}\psi_0 \\ \gamma_0 \\ \mu_0\end{pmatrix} = 0.$$

The corresponding characteristic equation can be reduced to the form

$$uq^4 - [(k^2\varepsilon u - r^2)(u+1) - k^2\varepsilon v^2]q^2 + (k^2\varepsilon u - r^2)^2 - (k^2\varepsilon v)^2 = 0. \tag{51}$$

Its solutions are

$$q^2 = (2u)^{-1}\{(k^2\varepsilon u - r^2)(u+1) - k^2\varepsilon v^2 \pm\sqrt{[(k^2\varepsilon u - r^2)(u-1) - k^2\varepsilon v^2]^2 + 4r^2k^2\varepsilon v^2}\}. \tag{52}$$

Hereafter the solutions corresponding to the negative sign before the square root will be marked as q_b^2, whereas solutions with the positive sign will be marked as q_s^2. This is explained by the observation that in the resonance case, when parameter u is small,

$$u \approx \frac{\omega_n}{\omega_H\omega_M}(2\delta\omega - \chi + i\alpha\xi), \qquad v \approx -\omega_n/\omega_H, \tag{53}$$

q_b^2 is "big", whereas q_s^2 is relatively "small":

$$q_b^2 \approx -\frac{r^2 + k^2\varepsilon v^2}{u}, \qquad q_s^2 \approx \frac{(k^2\varepsilon v)^2 - r^4}{k^2\varepsilon v^2 + r^2}. \tag{54}$$

Let us suppose, for the sake of simplicity, that the slab occupies the region $0 < z < L_s \ll L$. The solutions outside the slab, satisfying the conditions $E_x = E_y = 0$ at $z = L$, have the following form, according to Eq. (50):

$$\psi(z) = \Psi_0\cos[a(z-L)], \qquad \mu(z) = M_0\cos[a(z-L)],$$
$$\gamma(z) = -M_0\sin[a(z-L)]/a, \qquad a^2 \equiv k^2 - r^2.$$

The solutions inside the slab can be written as follows:

$$\psi(z) = \sum_{\lambda=b,s} (\Psi_{\lambda+}e^{iq_\lambda z} + \Psi_{\lambda-}e^{-iq_\lambda z}), \tag{55}$$

$$\gamma(z) = \sum_{\lambda=b,s} \Gamma_\lambda(\Psi_{\lambda+}e^{iq_\lambda z} + \Psi_{\lambda-}e^{-iq_\lambda z}), \tag{56}$$

$$\mu(z) = \sum_{\lambda=b,s} M_\lambda(\Psi_{\lambda+}e^{iq_\lambda z} - \Psi_{\lambda-}e^{-iq_\lambda z}), \tag{57}$$

where

$$\Gamma_\lambda = \frac{q_\lambda^2 + r^2 - k^2\varepsilon u}{ik^2\varepsilon v}, \qquad M_\lambda = \frac{iq_\lambda r^2}{q_\lambda^2 - k^2\varepsilon}\Gamma_\lambda. \tag{58}$$

Note that Eq. (49) is satisfied automatically for these solutions, since it appears coinciding with Eq. (51).

The conditions $E_x = E_y = 0$ at $z = 0$ impose the following restrictions on coefficients $\Psi_{\lambda\pm}$:

$$G_b\Phi_{b+} + G_s\Phi_{s+} = 0, \quad q_b\Phi_{b-} + q_s\Phi_{s-} = 0, \tag{59}$$

where

$$G_\lambda = r^2\Gamma_\lambda + iq_\lambda M_\lambda, \qquad \Phi_{\lambda\pm} = \Psi_{\lambda+} \pm \Psi_{\lambda-}.$$

Four other equations follow from the continuity conditions for functions H_x, E_x, H_y, and E_y at $z = L_s$:

$$\sum_{\lambda=b,s} (\Psi_{\lambda+}e^{iq_\lambda L_s} + \Psi_{\lambda-}e^{-iq_\lambda L_s}) = \Psi_0 \cos\phi,$$

$$\sum_{\lambda=b,s} G_\lambda(\Psi_{\lambda+}e^{iq_\lambda L_s} + \Psi_{\lambda-}e^{-iq_\lambda L_s}) = -M_0 \sin\phi \frac{k^2\varepsilon}{a},$$

$$\sum_{\lambda=b,s} M_\lambda(\Psi_{\lambda+}e^{iq_\lambda L_s} - \Psi_{\lambda-}e^{-iq_\lambda L_s}) = M_0 \cos\phi,$$

$$\sum_{\lambda=b,s} iq_\lambda(\Psi_{\lambda+}e^{iq_\lambda L_s} - \Psi_{\lambda-}e^{-iq_\lambda L_s}) = -\Psi_0 a\varepsilon \sin\phi,$$

where $\phi \equiv a(L_s - L)$.

For $L_s \ll L$, we can approximate the exponential functions in the left-hand sides of these four equations as $\exp(iq_\lambda L_s) \approx 1 + iq_\lambda L_s$. Then, taking into account Eq. (59), we arrive at simplified equations, containing the same coefficients $\Phi_{\lambda\pm}$:

$$\Phi_{b+} + \Phi_{s+} = \Psi_0 \cos\phi, \tag{60}$$

$$iL_s(q_bG_b\Phi_{b-} + q_sG_s\Phi_{s-}) = -M_0 \sin\phi k^2\varepsilon/a, \tag{61}$$

$$M_b\Phi_{b-} + M_s\Phi_{s-} + iL_s(q_bM_b\Phi_{b+} + q_sM_s\Phi_{s+}) = M_0 \cos\phi, \tag{62}$$

$$L_s\left(q_b^2\Phi_{b+} + q_s^2\Phi_{s+}\right) = \Psi_0 a\varepsilon \sin\phi. \tag{63}$$

The exact characteristic equation for the set of equations (59)–(63) is factorized as follows:

$$[iL_sq_bq_s(G_b - G_s)\cos\phi + (k^2\varepsilon/a)(M_bq_s - M_sq_b)\sin\phi] \times \left[L_s\left(G_bq_s^2 - G_sq_b^2\right)\cos\phi - (G_b - G_s)a\varepsilon \sin\phi\right] = 0.$$

Therefore we arrive at two equations, which can be written after some algebra as follows:

$$\tan\phi = \frac{iaL_sq_bq_s(G_s - G_b)}{k^2\varepsilon(M_bq_s - M_sq_b)} = L_sa, \tag{64}$$

$$\tan\phi = L_s\frac{G_bq_s^2 - G_sq_b^2}{a\varepsilon(G_b - G_s)}. \tag{65}$$

Equation (64) does not contain any magnetic parameter. It describes a small frequency shift in the mode with the main polarization of variable magnetic field (H_y) directed along the imposed constant one.

The explicit form of Eq. (65) is rather involved in the general case. But it can be simplified significantly in the most interesting special case of strong hybridization, when $|u| \ll 1$. Then, using Eq. (54) and remembering that $L \gg L_s$, one can arrive at the equation

$$\tan(L\sqrt{(\omega/c)^2 - r^2}) = \frac{L_s(\omega/c)^2v^2}{u\sqrt{(\omega/c)^2 - r^2}}. \tag{66}$$

If $L_s = 0$, then $\omega = \omega_n = c\sqrt{(n\pi/L)^2 + r^2}$. For $0 < L_s \ll L$ we put $\omega = \omega_n + \delta\omega$, neglecting terms proportional to $(\delta\omega)^2$ on both sides of Eq. (66). Following the same procedures as in Secs. III A and III B, we can transform Eq. (66) in the vicinity of resonance to the same quadratic equation (33), with $R = L_s/L$ and $\eta = \eta_{\text{res}} = \omega_n^2\omega_M/\omega_H$. Consequently, there is no influence of the transverse cavity length L_y (in the direction of applied constant magnetic field) on the strength of frequency splitting. [Although the unperturbed cavity eigenfrequency ω_n depends on L_y through the quantity $r = \rho_m = (1 + 2m)\pi/L_y$.]

B. Difficulties in the case of confinement in the direction perpendicular to the external field

If $L_y = \infty$, but $-L_x/2 < x < L_x/2$, then equations for the magnetic field $\mathbf{H}(x,z)$ take the form

$$\frac{\partial^2 H_x}{\partial z^2} - \frac{\partial^2 H_z}{\partial x \partial z} + k^2\varepsilon[uH_x + ivH_z] = 0, \tag{67}$$

$$\frac{\partial^2 H_z}{\partial x^2} - \frac{\partial^2 H_x}{\partial x \partial z} + k^2\varepsilon[uH_z - ivH_x] = 0, \tag{68}$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + k^2\varepsilon H_y = 0. \tag{69}$$

Then the H_y component is totally independent of H_x and H_z. Moreover, this mode does not feel the presence of the slab magnetization and the constant magnetic field. Therefore there is no frequency splitting in this mode.

The situation is different for the H_x-H_z polarizations. It is easy to verify that for $v = 0$, Eqs. (67) and (68), together with the boundary condition $E_y = 0$ at $x = \pm L_x/2$, permit for solutions in the factorized form

$$H_x = \psi(z)\sin(r_mx + \phi_m), \quad H_z = \gamma(z)\cos(r_mx + \phi_m),$$

where

$$r_m = m\pi/L_x, \quad \phi_m = m\pi/2, \quad m = 1,2,3,\ldots. \tag{70}$$

But such a factorization is impossible for $v \neq 0$, i.e., in the presence of a magnetized slab. In this case we have to look for solutions in the form of superpositions

$$H_x = \sum_{m=1}^{\infty} \psi_m(z) \sin(r_m x + \phi_m), \tag{71}$$

$$H_z = \sum_{m=1}^{\infty} \gamma_m(z) \cos(r_m x + \phi_m). \tag{72}$$

Then Eqs. (67) and (68) assume the form

$$\sum_{m=1}^{\infty} \sin(\alpha_m)[\psi_m''(z) + r_m \gamma_m'(z) + k^2 \varepsilon u \psi_m(z)] + i v k^2 \varepsilon \sum_{m=1}^{\infty} \cos(\alpha_m) \gamma_m(z) = 0, \tag{73}$$

$$\sum_{m=1}^{\infty} \cos(\alpha_m)\left[\left(k^2 \varepsilon u - r_m^2\right)\gamma_m(z) - r_m \psi_m'(z)\right] - i v k^2 \varepsilon \sum_{m=1}^{\infty} \sin(\alpha_m) \psi_m(z) = 0, \tag{74}$$

where $\alpha_m = r_m x + \phi_m$. To obtain the equations for functions $\psi_m(z)$ and $\gamma_m(z)$, one can multiply Eq. (73) by $\sin(\alpha_n)$, Eq. (74) by $\cos(\alpha_n)$, and integrate over dx from $-L_x/2$ to $L_x/2$. Using the formulas (with $l \equiv L_x/2$)

$$\int_{-l}^{l} \sin\alpha_m \sin\alpha_n dx = \int_{-l}^{l} \cos\alpha_m \cos\alpha_n dx = l\delta_{mn},$$

$$\int_{-l}^{l} \cos\alpha_m \sin\alpha_n dx = \frac{1-(-1)^{m+n}}{2(r_n + r_m)} + \frac{1-(-1)^{m-n}}{2(r_n - r_m)},$$

one can arrive at the following set of coupled equations:

$$\psi_n'' + r_n \gamma_n' + k^2 \varepsilon u \psi_n = \frac{4i}{\pi} k^2 \varepsilon v \sum_{m=n+\text{odd}}^{\infty} \frac{n \gamma_m(z)}{m^2 - n^2},$$

$$\left(k^2 \varepsilon u - r_m^2\right)\gamma_m = r_m \psi_m' + \frac{4i}{\pi} k^2 \varepsilon v \sum_{j=m+\text{odd}}^{\infty} \frac{j \psi_j(z)}{j^2 - m^2},$$

where the infinite sums should be taken over all those values of summation indexes, whose difference from n or m is an odd number (note that these sums do not depend on parameter L_x, as soon as $r_m = m\pi/L_x$). But it is unclear, how to solve these equations. Obviously, closing the cavity also in the y direction will make the problem even more complicated.

V. DISCUSSION

We have obtained simple analytical expressions for the frequency splitting and enhanced magnetization in the frameworks of a one-dimensional model. The main consequences of this model are as follows. The frequency splitting happens for the mode, where the time-dependent magnetic vector is perpendicular to the constant uniform magnetization and external magnetic field vectors. The orthogonal component of the time-dependent magnetic field (parallel to the constant magnetization vector) does not feel the slab magnetization. The real part of the frequency splitting does not depend on the magnetic damping coefficient (unless the slab thickness is extremely small). The change of the cavity quality factor does not depend on the position and thickness of the slab (unless it is too thin or too thick). The magnetic field distribution outside the slab is almost the same for the two split modes, even under the resonance condition. At the same time, the time-dependent magnetization vectors inside the slab have opposite directions in the split modes. The amplitudes of these vectors are strongly enhanced at the resonance, both in the parallel and perpendicular directions with respect to the slab surface (but perpendicular to the constant magnetization vector). The enhancement factor is proportional to the ratio $(L/L_s)^{1/2}$. Of course, these results are valid for weakly excited modes, when the linearization of the Landau-Lifshitz-Gilbert equation (2) can be justified.

We have succeeded also in obtaining analytical solutions for the restricted 2D model, when the cavity is confined in the direction of external constant magnetic field but not confined in the perpendicular direction in the slab plane. It is interesting that the final result concerning the frequency splitting is the same as in the 1D case, with the only difference that the 1D empty cavity eigenfrequency $\omega_n = n\pi c/L$ should be replaced by the 2D eigenfrequency $\omega_{nm} = (\omega_n^2 + c^2 r_m^2)^{1/2}$. However, adding the confinement in the third direction leads to complicated equations, which probably do not permit analytical solutions.

Probably, it could be interesting to verify experimentally in real 3D cavities the following qualitative results of the simple 1D and 2D models. (1) That the change of the cavity quality factor does not depend on the position of magnetized sample (far from the nodes of the variable magnetic field) and its thickness. (2) That the frequency splitting varies with the change of the sample position in the same way as the amplitude of cavity magnetic field in the selected mode. (3) That the frequency splitting does not depend on the magnetic damping coefficient.

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