Superbosonization in disorder and chaos: Role of anomalies

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The superbosonization formula aims at rigorously calculating fermionic integrals via employing supersymmetry. We derive such a supermatrix representation of superfield integrals and specify integration contours for the supermatrices. The derivation is essentially based on the supersymmetric generalization of the Itzykson-Zuber integral in the presence of anomalies in the Berezinian and shows how an integral over supervectors is eventually reduced to an integral over commuting variables. The approach is tested by calculating both one and two point correlation functions in a class of random matrix models. It is argued that the approach is capable of producing nonperturbative results in various systems with disorder, including physics of many-body localization, and other situations hosting localization phenomena.

I. INTRODUCTION

Supersymmetry [1,2] deals with Grassmann numbers that were originally invented in mathematics and later used in quantum field theory as the classical analogues of anticommuting operators. This mathematical construction is proven to be a very useful tool for studies in various fields of physics and in particular in models of quantum chaos, involving random matrix theory and various models of disorder [3–5].

One of the prominent methods employing supersymmetry is the nonlinear supersymmetric σ model [3,4] description of disordered metallic conductors. According to this standard formalism, effective field theory is described by an action with coordinate dependent supermatrix field, $Q(\mathbf{r})$, obeying the constraint,

$$Q^2(\mathbf{r}) = 1. \tag{1}$$

This method has a broad range of applications including study of Anderson localization, mesoscopic fluctuations, levels statistics in a limited volume, quantum chaos. A general form of the free energy functional F is rather simple

$$F[Q] = \frac{\pi \nu}{8} \int \text{Str}[D(\nabla Q)^2 + 2i(\omega + i\delta)Q(\mathbf{r})]d\mathbf{r} \tag{2}$$

containing the classical diffusion coefficient D, the one particle density of states ν, and frequency ω. Although the free energy $F[Q]$, Eq. (2), is written in the limit of a weak disorder, it can be used for strongly disordered samples replacing the gradient by finite differences. At very low energies the effective free energy functional is dominated in a finite volume by the zero spatial mode, $Q(\mathbf{r}) = Q_0$, which is independent of $\mathbf{r}$. In this limit the model is especially simple containing only the second term in Eq. (2). The spectral properties of the theory are *universal* and coincide with those of Wigner-Dyson random matrix ensembles with corresponding symmetries [3–7].

The derivation of the σ model, Eq. (2), from microscopic models is not exact and is based on a saddle point method or invariance arguments applicable at weak disorder or the large size of the matrices in the Wigner-Dyson ensembles. At the same time, the "diffusive" σ model, Eq. (2) is not applicable for, e.g., description of electron motion in ballistic regime,

where characteristic spatial scales are much smaller than the mean free path. Another important problem, that is known to be out of the reach of the nonlinear σ model, is random matrix models with finite range correlations between the matrix elements that do not belong to the Wigner-Dyson ensembles. One of the examples is models of weakly nondiagonal matrices [8,9]. Of course, there are many other models that cannot be reduced to the σ model, Eq. (2).

In many of those models correlation functions of interest can be expressed from the beginning in terms of integrals over supervectors, and the problem arises due to absence of a possibility of using the saddle-point approximation leading to Eqs. (1) and (2). Therefore, it is natural to try to generalize the σ model, Eq. (2), to a model containing supermatrices but without the constraint, Eq. (1). In such a model, the generating functional $Z(J)$ would be expressed in terms of an integral over unconstrained supermatrices, and having calculated this integral, one would be able to compute correlation functions of interest. It should be noticed here that usually many-level or many point correlation functions are really interesting. One-level or one-point averages (average density of states) usually do not bring interesting information about the systems (in the problem of Anderson localization, the average density of states cannot help to distinguish between the metal and insulator).

In principle, some dual representations of a generating functional, initially given by an integral over $N \times N$ Hermitian matrices (color space), are known as color-flavor transformations [10]. They transform the original integral in "color space" to an integral over certain supermanifolds, which are acting in the dual space (flavor space). However, being interesting on its own, this transformation has not yet evolved into a new computational tool.

Trying to find a new method of studying nonstandard problems of the supersymmetry method, a kind of bosonization procedure to the original fermionic functional $Z(J)$ has been suggested some time ago [11,12]. As a result, the partition function has been represented in a supermatrix action formulation without any constraint whatsoever; this approach was claimed to be applicable to the physics of electron motion at all scales. It seemed that the limitations due to the

nonlinearity of the conventional σ-model representation were overcame. Nevertheless the formula of superbosonization was not well understood from the practical point of view, namely the integration method was not specified.

More precisely, in Ref. [12] a new superbosonization formula that allowed the field integral over supervectors be expressed through a supermatrix integral has been derived,

$$\int D\psi\, D\bar{\psi}\; F(\psi \otimes \bar{\psi}) = \int_{\mathbb{H}_n} D\mathcal{A}\; \mathrm{Sdet}\mathcal{A}^{-1}\, F(\mathcal{A}), \quad (3)$$

where $\mathbb{H}_n$ is the linear space of Hermitian n-dimensional complex supermatrices, $\psi \in U(n,1|n,1)$ and $\bar{\psi} \in U(n,1|n,1)$ are supervectors, and $F : \mathbb{H}_n \to \mathbb{G}$ is a formal map with $\mathbb{G}$ representing a superspace [13]. Importantly, the right-hand side of Eq. (3) could be evaluated under general conditions, without reducing it to any mean-field manifold. For this reason, it was suggested that Eq. (3) could be capable of producing nonperturbative results in various models of disorder. One can imagine that Eq. (3) can represent a promising approach for nonperturbative studies in physics of many-body localization [14,15] and other situations where disorder plays an important role [16].

Originally [12], Eq. (3) has been derived rather schematically without discussing contours of integration over the commuting elements of the supermatrix $\mathcal{A}$. An attempt to specify contours of integration has been undertaken in Ref. [17]. Roughly, speaking it was suggested to integrate over the eigenvalues of the boson block from $-\infty$ to ∞, while the integration over the eigenvalues of the fermion-fermion block has to be performed over a compact domain (a circle in the simplest case). Surprisingly, it turned out that such an integration was well defined only in rather uninteresting cases. In particular, it worked perfectly well for correlation functions that required a sufficiently small number $q \leqslant n$ of the bosonic components, where n was a number of artificial "orbitals". In other words, one could use Eqs. (3) for calculation of the density of states in case of the unitary ensemble, while one encountered a singularity of the type $\infty \times 0$, when trying to calculate a two-level correlation function. The situation for, e.g. orthogonal ensemble was even worse and one could not calculate even the density of states in this case. The situation was better when using a sufficiently large number of the "orbitals" n but this could be efficiently closer to results obtained using the standard saddle-point method and therefore less interesting.

These findings have been confirmed rigorously in Ref. [18] but the case $q > n$ was not resolved and it was even concluded that the superbosonization formula, Eq. (3), was not correct for this case. This was a serious obstacle in using the superbosonization for applications to interesting unsolved problems.

In this paper we resolve this long standing problem of the integration in Eq. (3) for the case of hermitian matrices with an arbitrary correlation between the matrix elements. Of course, the suggested approach can be used for disordered systems with a broken time-reversal invariance. We do it integrating over the eigenvalues of the fermion-fermion block along the imaginary axis from $-i\infty$ to $i\infty$ instead of the integration along the circle adopted in Refs. [17,18]. This does not make a difference in the results for $q \leqslant n$ but it makes the integral,

Eq. (3), well defined for $q > n$ and computation of many point correlation functions feasible, thus establishing a new method of calculations for interesting problems.

The paper is organized as follows. In Sec. II we set the basis for the subsequent analysis of the bosonization procedure of Ref. [12] by calculation of a supersymmetric generalization of Itzykson-Zuber (IZ) integral. In Sec. III we show how the formulated supermatrix representation of integrals over supervectors (the so-called bosonized representation) can be evaluated. In particular, we derive the domains of integration, for which the bosonization formula is exact. It is remarkable that this regularized scheme leads to an effective reduction of dimensionality of the domain of integration, which is *noncompact*.

The proof is essentially based on the results discussed in Sec. II: the supersymmetric generalization of the Itzykson-Zuber (IZ) integral [19–24], in situations when a boundary term is crucial due to the presence of singularities in the Berezinian. Emergence of this boundary term in the IZ integral ensures that both representations of the generating functional coincide.

In Sec. IV we apply the regularized superbosonization formula to calculation of correlation functions in random matrix models. We derive both one and two point correlation functions for Hermitian diagonal random matrices with continuously distributed components and correction to the density of states for weakly nondiagonal random matrices [8,9]. Technical details of some of the derivations are presented in Appendices A–C.

II. ANOMALY IN SUPERSYMMETRIC ITZYKSON-ZUBER INTEGRAL

A. Supersymmetric Itzykson-Zuber integral

In this section we present useful formulas that will be applied in subsequent sections. Let us note that in all future considerations the integration over the linear space of complex supermatrices, $\int_{\mathbb{H}_n} D\mathcal{A}$, with flat Berezin measure [1,2] is always performed first by diagonalizing the matrix $\mathcal{A}$ and then by integrating over the eigenvalues. We distinguish between "fermion-fermion (FF)" and "boson-boson (BB)" blocks of the matrix $\mathcal{A}$ corresponding respectively to products $\psi_F \otimes \bar{\psi}_F$ and $\psi_B \otimes \bar{\psi}_B$ of anticommuting and commuting components of the supervectors. After the diagonalization of the supermatrix $\mathcal{A}$ one half of the eigenvalues will be in the FF block, and the other part will be in the BB block. We will call these eigenvalues FF and BB eigenvalues, respectively.

We will demonstrate that the integration over the BB eigenvalues should be performed in the infinite interval $\mathbb{R} \equiv \{-\infty, \infty\}$, while the integration over the FF eigenvalues should be performed in the infinite interval $\{-i\infty, i\infty\}$. This contrasts the integration rules of Refs. [17,18], where the integration over the FF eigenvalues was performed along the unit circle.

Let us note that any complex $2n \times 2n$ supermatrix, $\mathcal{A}$, can be diagonalized as $\mathcal{A} = U\mathcal{A}_d \bar{V}$, where $U \in U(n|n)$, $\bar{V} \in U(n|n)/U^{2n}(1)$ are diagonalization matrices restricted correspondingly to the unitary supergroup and its subspace with removed phases. To this end, we are interested in a matrix

integral of the type

$$\Gamma[\{\tilde{b}_j,b_j,\}|\{\tilde{\lambda}_j,\lambda_j\}] = \int DU\, D\bar{V}\ \exp\{\mathrm{Re}\,\mathrm{Str}[UB_d\bar{V}Q_d]\},$$

(4)

where $B_d = \mathrm{diag}\{\tilde{b}_i,b_i\}$ and $Q_d = \mathrm{diag}\{\tilde{\lambda},\lambda,\}$ are the FF and BB eigenvalues of supermatrices B and Q, respectively, and $U \in U(n|n)$, $\bar{V} \in U(n|n)/U^{2n}(1)$. Let us note that this integral is often referred to as supersymmetric Berezin-Karpelevich integral [25] in the literature, and the connection between the latter and the Itzykson-Zuber integral is not straightforward [26]. The result of integration in Eq. (4) reads [20,22,23]

$$\Gamma[\{\tilde{b}_j,b,\}|\{\tilde{\lambda}_j,\lambda_j\}]$$
$$= [1 - \eta(\{\tilde{\lambda}_i,\lambda_i\})]\frac{\prod_i \delta(\tilde{b}_i)\delta(b_i)}{\Delta^2(\{\tilde{b}_j^2,b_j^2\})} + \Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}].$$

(5)

Here $\Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$ is the result of the bulk integration without accounting for the singularity in the Berezinian (if there is such). It has the form

$$\Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$$
$$= \frac{1}{2^{2n^2}(n!)^2}\frac{\det J_0[\tilde{b}_p\tilde{\lambda}_q]_{p,q=1...n}\det J_0[b_l\lambda_m]_{l,m=1...n}}{\Delta(\{\tilde{b}_j^2,b_j^2\})\Delta(\{\tilde{\lambda}_j^2,\lambda_j^2\})},$$

(6)

where

$$\Delta(\{\tilde{b}_j^2,b_j^2\}) = \frac{\prod_{k<r=1}^n(\tilde{b}_k^2 - \tilde{b}_r^2)\prod_{l<m=1}^n(b_i^2 - b_m^2)}{\prod_{p<q=1}^n(\tilde{b}_q^2 - b_p^2)}$$
$$= \det\left[\frac{1}{\tilde{b}_j^2 - b_i^2}\right]_{i,j=1...n}$$

(7)

is the supersymmetric Vandermonde determinant and $J_0[b_p\lambda_q]$ is the zero-order Bessel function. The term $\eta(\{\tilde{\lambda}_i,\lambda_i\})$ is the boundary term arising from the singularities of the Berezinian (This type of the boundary term in the integrals over supermatrices has been found in Refs. [3,27] and is sometimes called Efetov-Wegner boundary term [28,29]). It originates from the regularization of the anomaly in the Berezinian and is given by [23]

$$\eta(\{\tilde{\lambda}_i,\lambda_i\})$$
$$= \frac{1}{\Delta(\{\tilde{\lambda}_j^2,\lambda_j^2\})}\det\left[\frac{1}{\tilde{\lambda}_l^2 - \lambda_k^2}\left(1 - e^{\frac{\tilde{\lambda}_l^2 - \lambda_k^2}{2t}}\right)\right]_{k,l=1,...,n}.$$

(8)

One can easily check that the expression (7) for IZ integral boundary terms, $\Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$, does not fulfill Eq. (9) (see below). The singularity of the Berezinian $\Delta^2(\{\tilde{b}_j^2,b_j^2\})$ in Eq. (9) gives rise to the appearance of the boundary term in $\Gamma[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$.

B. Origin of the boundary term in the Itzykson-Zuber integral

Our aim in this section is to underline the origin of the anomaly of the Berezinian and the implication for the supersymmetric Itzykson-Zuber integral. For this purpose for

any given diagonal complex supermatrix Q_d consider the Gaussian integral,

$$\int DB\ \exp\left\{\frac{1}{2t}\,\mathrm{Str}[(B-Q_d)^2]\right\}$$
$$= \int \prod_i d\tilde{b}_i db_i\ \Delta^2(\{\tilde{b}_j^2,b_j^2\})$$
$$\times \exp\left\{-\frac{1}{2t}\,\mathrm{Str}[B_d^2 + Q_d^2]\right\}$$
$$\times \int DU\ D\bar{V}\ \exp\left\{\frac{1}{2t}\,\mathrm{Re}\ \mathrm{Str}\{UB_d\bar{V}Q_d\}\right\}$$
$$= \int \prod_i d\tilde{b}_i db_i \Delta^2(\{\tilde{b}_j^2,b_j^2\})\ \Gamma[\{\tilde{b}_j,b_j,\}|\{\tilde{\lambda}_j,\lambda_j\}],$$

(9)

where we have the diagonalized complex supermatrix $B \to B_d = \mathrm{diag}\{b_i,\bar{b}_i\}$. In Eq. (9) $\Delta^2(\{\tilde{b}_j^2,b_j^2\})$ is the Berezinian of the transformation, $B = UB_d\bar{V}$, where $U \in U(n|n)$ and $\bar{V} \in U(n|n)/U^{2n}(1)$.

The integral, Eq. (9), is originally gaussian, and integrating separately over all matrix elements of the supermatrix B gives unity. It is clear that changing the variables of the integration cannot modify this result and one must obtain unity also integrating over the eigenvalues. However, one can easily check that the "naive" expression for Itzykson-Zuber integral $\Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$, Eqs. (6) and (7), is not equal to unity. It is the singularity of the Berezinian $\Delta^2(\{\tilde{b}_j^2,b_j^2\})$ in Eq. (9) that gives rise to the appearance of boundary term $\eta(\{\tilde{\lambda}_i,\lambda_i\})$ Eq. (8), in $\Gamma[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$, that was found in Ref. [23]. Existence of this boundary term ensures the condition that the integral Eq. (9) is unity. Hence, the correct answer for supersymmetric Itzykson-Zuber integral Γ has the form Eq. (5).

The following remark is in order. The result, $\Gamma_0[\{\tilde{b}_j,b_j\}|\{\tilde{\lambda}_j,\lambda_j\}]$, of the evaluation of the supersymmetric IZ integral in the absence of singularities was derived by solving the supersymmetric heat equation [20,22], a technique that was developed in Ref. [19] for conventional matrices. It is straightforward to check that the boundary term $\propto(1-\eta)$ in Eq. (5) also satisfies the heat equation.

III. SUPERBOSONIZATION: PROOF AND INTEGRATION CONTOURS

In this section we present a derivation of the superbosonization formula and, in particular, of the bosonized σ model for random matrices. The derivation is similar to the procedure developed in Refs. [3–5,20,30], but here instead of the Hubbard-Stratonovich transformation [31], which in the standard scheme follows the averaging over random matrices, we use the identities from the above section. Actually, the scheme of the derivation is very close to that of Ref. [12] but is more rigorous. It is useful to recall that formal sums of formal products $\Psi \otimes \bar{\Psi}$, where $\Psi \in U(n,1|n,1)$ and $\bar{\Psi} \in U(n,1|n,1)$ are supervectors, constitute a vector space. This vector space is defined, up to isomorphism, by the condition that every antisymmetric, bilinear map $f : U(n,1|n,1) \times \bar{U}(n,1|n,1) \to \mathbb{G}$ determines a unique linear map $g : U(n,1|n,1) \otimes \bar{U}(n,1|n,1) \to \mathbb{G}$ with

$f(\Psi, \bar{\Psi}) = g(\Psi \otimes \bar{\Psi})$. This implies that if we consider a map, $F : \mathbb{H}_n \to \mathbb{G}$, then the integral

$$I_F = \int D\psi \, D\bar{\psi} \, F(\psi \otimes \bar{\psi}) \tag{10}$$

is now well defined. From now on we will restrict ourselves to the case of maps, F, such that the integral I_F in Eq. (10) is convergent.

As the first step we make use of the identity derived in Appendix A to rewrite the field integral in the left-hand side of Eq. (3) as

$$I_F = \int_{\mathbb{H}_n} D\mathcal{A} F(\mathcal{A}) \int D\psi \, D\bar{\psi} \int_{\mathbb{H}_n} DB$$

$$\times \exp\left\{ i\,\mathrm{Str}[\mathcal{A}B] - i \sum_j \mathrm{Str}[\psi_j \otimes \bar{\psi}_j B] - \delta\,\mathrm{Str}[B^2] \right\}, \tag{11}$$

where δ is an infinitely small variable that ensures the convergence of the integral over the variable B in Eq. (11); it can be dropped once the integral over B is convergent. Now, due to the convergence of the integral in Eq. (11) and the presence of δ, we are free to change the order of the integration over the supermatrix B and the supermatrices $\psi_i \otimes \bar{\psi}_i$. The integration over the supervectors $\psi, \bar{\psi}$ leads to

$$\int D\psi \, D\bar{\psi} \exp\left\{ -i \sum_j \mathrm{Str}[\psi_j \otimes \bar{\psi}_j B] \right\}$$

$$= \int D\psi \, D\bar{\psi} \, e^{i \sum_j \bar{\psi}_j B \psi_j} = \mathrm{Sdet}[iB]. \tag{12}$$

Then, the integral over B acquires the form

$$I_B = \int_{\mathbb{H}_n} DB \, \mathrm{Sdet}[iB] \, e^{i\,\mathrm{Str}[\mathcal{A}B]}, \tag{13}$$

where we dropped δ due to the convergence of the integral Eq. (13). The integral Eq. (13) can be calculated by changing the integration variable B to $B' = \mathcal{A}B$. Supermatrix B' is not necessarily Hermitian, however it obeys the constraint $\mathrm{Str}[B'] = \mathrm{Str}[(B')^\dagger]$. By definition B' is an element of the vector space $\Lambda^2(\mathbb{H}_n)$ (for definition see Appendix B). Taking into account the fact that due to the supersymmetry the Berezinian of the transformation $B' = \mathcal{A}B$ is unity and

$$\mathrm{Sdet}B = \frac{\mathrm{Sdet}B'}{\mathrm{Sdet}\mathcal{A}}, \tag{14}$$

we obtain

$$I_B = \mathrm{Sdet}\mathcal{A}^{-1} \int_{\Lambda^2(\mathbb{H}_n)} DB' \, \mathrm{Sdet}B' \, e^{i\,\mathrm{Str}B'} = C_n \, \mathrm{Sdet}\mathcal{A}^{-1}. \tag{15}$$

The coefficient C_n is calculated in Appendix B, yielding $C_n = 1$. As a result, one arrives at the bosonized representation for the integral Eq. (11)

$$I_F = \int_{\mathbb{H}_n} D\mathcal{A} \, \mathrm{Sdet}\mathcal{A}^{-1} \, F(\mathcal{A}). \tag{16}$$

To finalize this section we remind the reader that in Eq. (16) the integration over the linear space of Hermitian supermatrices, $\int_{\mathbb{H}_n} D\mathcal{A}$, with Berezin measure is understood here as follows: (i) First we diagonalize the matrix $\mathcal{A}$ and then integrate over the eigenvalues. (ii) Integration over "boson-boson" eigenvalues is performed in the infinite interval $\{-\infty, \infty\}$, whereas the integration over the "fermion-fermion" eigenvalues is performed (in contrast to Refs. [17,18]) in the noncompact interval $\{-i\infty, i\infty\}$. In this way, the integral in Eq. (3) over supervectors is reduced to an integral over commuting variables. It is worth mentioning that the presence of $\mathrm{Sdet}\mathcal{A}^{-1}$ in Eq. (3) leads to a singular product $\prod_i \tilde{\lambda}_i^{-1}$, which make the integral very sensitive to the contour of the integration over the FF eigenvalues $\tilde{\lambda}_i$.

Representing the integral over supervectors in terms of an integral over the supermatrices is more than just changing the variables of the integration. Usually, the term bosonization is used for the procedure of a replacement of an electron model by a model describing collective bosonic excitations. For example, the traditional σ model describes so-called diffusion modes instead of electrons in a random potential. As our transformation is exact and is based on the supersymmetry, we find it proper to use the word "superbosonization" for the transformation, Eq. (3), complemented by the rules of integration over the eigenvalues of the supermatrices.

IV. SUPERBOSONIZATION OF RANDOM MATRICES: CORRELATION FUNCTIONS

In this section we develop a technique for calculation of various correlation functions in random matrix theory (RMT), such as the averaged density of states, level-level correlations, eigenfunction correlations, and higher order correlation functions. For this purpose, without loss of generality, we consider an ensemble of N-dimensional Hermitian matrices, $H = \{H_{ij}\}$, with continuously distributed components. For simplicity let us concentrate on the Gaussian probability density function,

$$\mathcal{P}(H) = \prod_{i,j=1}^{N} P_{ij}(H_{ij}), \tag{17}$$

with the distribution functions P_{ij} $(i, j = 1 \ldots N)$ equal to

$$P(H_{ij}) = \frac{1}{2\pi A_{ij}} \exp\left\{ -\frac{H_{ij} H_{ij}^*}{2A_{ij}} \right\}. \tag{18}$$

Then Eqs. (17) and (18) unambiguously define statistical properties of the matrix entries as $\langle H_{ij} \rangle = 0$, $\langle H_{ii}^2 \rangle = A_0$, and $\langle H_{ij}^2 \rangle = A_{ij}$ for $i \neq j$. The Wigner-Dyson unitary ensemble is obtained putting $A_{ij} = \mathrm{const}$ independent on i, j.

A. Correlation functions in the superbosonized representation: General framework

We begin with the generating functional for n-point correlation functions

$$Z(J_1 \ldots J_n) = \int D\psi \, D\psi^+ \, \exp\left\{ i \sum_{i,j=1}^{N} \psi_i^+ \mathcal{M}_{i,j}^{J_j} \psi_j \right\}, \tag{19}$$

with the matrix $\mathcal{M}_{i,j}^{J_j}$ defined as

$$\mathcal{M}_{i,j}^{J_j} = L\delta_{ij}\mathcal{H}_j^{J_j} + LH_{ij},$$

$$\mathcal{H}_j^{J_j} = \left(-E\hat{\mathbb{1}} + \frac{\omega + i0}{2}\Lambda - J_j\,\hat{s}\right). \qquad (20)$$

In Eq. (19) ψ_i are supervectors with n bosonic and n fermionic components, the source terms, J_i ($i = 1 \ldots n$), in Eq. (20) are real parameters multiplied by diagonal $2n \times 2n$ matrices, $\hat{s}$, which break the fermion-boson (FB) symmetry. Parameter E stands for the energy and ω is the frequency. The $2n$-dimensional *supermatrices* L, Λ, and $\hat{s}$ are defined as

$$L = \begin{pmatrix} id_n & 0 \\ 0 & \hat{k} \end{pmatrix}_{FB},$$

$$\Lambda = \begin{pmatrix} \hat{k} & 0 \\ 0 & \hat{k} \end{pmatrix}_{FB}, \qquad (21)$$

$$\hat{s} = \begin{pmatrix} id_n & 0 \\ 0 & -id_n \end{pmatrix}_{FB},$$

with n-dimensional unity matrix, id_n, and n-dimensional diagonal matrix, $\hat{k} = \mathrm{diag}(1, -1)$. The purpose of introducing the matrix, $\hat{k}$, is that it distinguishes between the advanced and the retarded (A/R) Green functions. Finally, $\hat{\mathbb{1}}$ is an identity matrix in both F/B and A/R spaces (from now on we will discontinue explicitly writing $\hat{\mathbb{1}}$ for shortening the presentation).

To derive the supersymmetric action for RMT, one has to perform averaging in the generating functional, $Z(J_1 \ldots J_n)$, over realizations of the entries of the random matrix, H. Carrying out such an averaging with the probability distribution defined in Eqs. (17) and (18), one obtains

$$\langle Z(J_1 \ldots J_n)\rangle = \int D\psi\, D\bar{\psi}\, \exp\left\{i\sum_i \bar{\psi}_i \mathcal{H}_i^J \psi_i \right.$$
$$\left. - \frac{1}{2}\sum_{i,j} A_{i,j}(\bar{\psi}_i\psi_j)(\bar{\psi}_j\psi_i)\right\}, \qquad (22)$$

where we have defined $\bar{\psi}_i = \psi_i^+ L$. At this point we note that for the constituent terms of the action (expressions in exponent), Eq. (22), the following identities hold

$$\bar{\psi}_i\mathcal{H}_i^J\psi_i = \mathrm{Str}\left[\psi_i\otimes\bar{\psi}_i\left(-E + \frac{\omega + i0}{2}\Lambda - J_i\right)\right],$$

$$(\bar{\psi}_j\psi_i)(\bar{\psi}_i\psi_j) = -\mathrm{Str}(\psi_i\otimes\bar{\psi}_i\psi_j\otimes\bar{\psi}_j), \qquad (23)$$

The crucial step towards calculation of the correlation functions in Gaussian random matrix theory under consideration is the evaluation of the superintegrals in Eq. (23) from their superbosonized representation Eq. (3). With the help of the superbosonization formula Eq. (3) we can represent the generating functional, $\langle Z(J)\rangle$, in the form

$$\langle Z(J_1 \ldots J_n)\rangle$$
$$= \int_{\mathbb{H}_n} \prod_i DQ_i\, \mathrm{Sdet}[Q_i]^{-1}$$
$$\times \exp\left\{i\sum_i \mathrm{Str}[Q_i\mathcal{H}_i^J] - \frac{1}{2}\sum_{i,j} A_{i,j}\,\mathrm{Str}[Q_iQ_j]\right\},$$
$$(24)$$

where each of the integrals over the linear space of complex Hermitian supermatrices, $\mathbb{H}_n$, should be performed first diagonalizing matrices Q_i and then integrating over their eigenvalues. As was mentioned in the Introduction, integration over BB eigenvalues is performed along the real axis, $(-\infty, \infty)$, whereas integration over FF eigenvalues is performed along the imaginary axis, $(-i\infty, i\infty)$.

In conclusion of this subsection we note that the derivatives of the averaged generating functional, $\langle Z(J_1 \ldots J_n)\rangle$, taken at zero source, $J = 0$, define the advanced and retarded Green functions in RMT [5,20]. The n-point Green functions can be expressed via the derivatives of $\langle Z(J_1 \ldots J_n)\rangle$ functional in a standard way,

$$G^{R/A}(E_1 \ldots E_n) = \frac{1}{\pi^n}\left\langle\prod_{i=1}^n \mathrm{Tr}\left[\frac{1}{E_i - H \pm i0}\right]\right\rangle$$
$$= \frac{1}{(2\pi)^n}\frac{\partial^n}{\partial J_1 \cdots \partial J_n}\langle Z(J_1 \ldots J_n)\rangle|_{J_i=0},$$
$$(25)$$

which define the universal characteristics of RMT. As usual, the sign "+" in the denominator corresponds to the retarded Green function G^R, while the sign "−" to the advanced one G^A.

B. Correlation functions for diagonal random matrices

Let us first show how the method developed here works for diagonal random matrices. Although this case is not the most interesting one, it allows one to understand how the method works. We remind the reader that the conventional nonlinear σ model [3–5] is not applicable in this case.

For diagonal random matrices we have $A_{ij} = 0$ for $i \neq j$, and thus the averaged generating functional Eq. (24) acquires the form

$$\langle Z_0(J_1 \ldots J_n)\rangle = \int_{\mathbb{H}_n}\prod_i DQ_i\,\mathrm{Sdet}[Q_i]^{-1}\exp\left\{i\sum_i \mathrm{Str}[Q_i\mathcal{H}_i^J] - \frac{1}{2}\sum_i A_0\mathrm{Str}[Q_i^2]\right\}, \qquad (26)$$

where supermatrices $\mathcal{H}_i^J$ are given by Eq. (20), $\hat{s}$ is given by Eq. (21), and Q_i are Hermitian supermatrices with n bosonic and n fermionic entries.

Calculation of $\langle Z_0(J_1 \ldots J_n)\rangle$ can be performed in a similar way, as the calculation of C_n in Appendix B. Namely, first we diagonalize the supermatrices Q_i and afterwards perform IZ-type integration. Since the supermatrices Q_i in Eq. (26) are Hermitian, they can be diagonalized upon the rotation by the elements of the unitary supergroup, SU($n|n$). Substituting

transformation $Q = U Q_d U^+$, where $U \in SU(n|n)$, into Eq. (26) we arrive to the following form of the generating functional

$$\langle Z_0(J_1 \ldots J_n) \rangle = \int \prod_i DU_i \int \prod_\alpha D\lambda_{i,\alpha} \, D\tilde{\lambda}_{i,\alpha} \Delta^2(\{\tilde{\lambda}_{i,\alpha}, \lambda_{i,\alpha}\}) \prod_{j,\alpha} \frac{\lambda_{j,\alpha} \pm i0}{\tilde{\lambda}_{j,\alpha} \pm i0}$$

$$\times \exp\left\{ i \sum_i \text{Str}\big[Q_{i,d} U_i \mathcal{H}_{i,d}^J U_i^+\big] - \frac{1}{2} \sum_{i,\alpha} A_0\big(\lambda_{i,\alpha}^2 - \tilde{\lambda}_{i,\alpha}^2\big) \right\}, \tag{27}$$

where the integration over bosonic eigenvalues of Q_i, $\lambda_{i,\alpha}$ ($\alpha = 1, \ldots n$), should be carried out along the real axis, $(-\infty, \infty)$, and the integration over fermionic eigenvalues, $\tilde{\lambda}_{i,\alpha}$ ($\alpha = 1, \ldots n$), should be carried out along the imaginary axis, $(-i\infty, i\infty)$. The infinitesimally small terms $\pm i0$ in Eq. (27) arise after removing $\pm i0$ from H_i^J in Eq. (26) by shifting the variable of the integration Q. Berezinian, $\Delta^2(\{\tilde{\lambda}_{j,\alpha}, \lambda_{j,\alpha}\})$, is the Jacobian of the diagonalization given by

$$\Delta(\{\tilde{\lambda}_{j,\alpha}, \lambda_{j,\alpha}\}) = \prod_{\alpha,\beta=1}^n \frac{(\lambda_{j,\alpha} - \lambda_{j,\beta})(\tilde{\lambda}_{j,\alpha} - \tilde{\lambda}_{j,\beta})}{(\lambda_{j,\alpha} - \tilde{\lambda}_{j,\beta})} = \det_{\alpha,\beta}\left[\frac{1}{\lambda_{j,\alpha} - \tilde{\lambda}_{j,\beta}}\right]. \tag{28}$$

It is transparent that the zero order generating functional Eq. (27) has a factorized form and can be represented as

$$\langle Z_0(J_1 \ldots J_n) \rangle = [\mathcal{Z}_0(J)]^N, \tag{29}$$

where

$$\mathcal{Z}_0(J) = \int \prod_i DU \int \prod_\alpha D\lambda_\alpha \, D\tilde{\lambda}_\alpha \Delta^2(\{\tilde{\lambda}_\alpha, \lambda_\alpha\}) \times \prod_{\alpha=1}^n \frac{\lambda_\alpha \pm i0}{\tilde{\lambda}_\alpha \pm i0} \exp\left\{ i\text{Str}\big[Q_d U \mathcal{H}_d^J U^+\big] - \frac{1}{2} \sum_\alpha A_0\big(\lambda_\alpha^2 - \tilde{\lambda}_\alpha^2\big) \right\}. \tag{30}$$

We see that the calculation of the generating functional for diagonal random matrices reduces to the calculation of the IZ integral. This integral can be calculated employing the result of Sec. II for the unitary supergroup [20,23], $U \in SU(n,n)$:

$$I = \int DU \, \exp\big\{i\text{Str}\big[Q_d U \mathcal{H}_d^J U^+\big]\big\}$$

$$= [1 - \eta(\{\tilde{h}_\alpha, h_\alpha\})] \frac{\prod_\alpha \delta(\lambda_\alpha) \, \delta(\tilde{\lambda}_\alpha)}{\Delta^2(\{\tilde{\lambda}_\alpha, \lambda_\alpha\})} + \frac{1}{2^{n(n-1)} \pi^n n!^2} \frac{\det_{\alpha,\beta}[e^{ih_\alpha \lambda_\beta}] \, \det_{\alpha,\beta}[e^{ih_\alpha \lambda_\beta}]}{\Delta(\{\tilde{h}_\beta, h_\alpha\})\Delta(\{\tilde{\lambda}_\beta, \lambda_\alpha\})}, \tag{31}$$

where the components h_α and $\tilde{h}_\alpha$, ($\alpha = 1 \cdots n$) are BB and FF eigenvalues of $\mathcal{H}_d^J$, respectively, and the Δ functions are defined by Eq. (28). The boundary term η is given by Eq. (8) and reads

$$\eta(\{\tilde{h}_\beta, h_\alpha\}) = \frac{\det[\mu(\tilde{h}_\beta, h_\alpha)]_{\alpha,\beta=1,\ldots,n}}{2^{n(n-1)} \pi^n \, \Delta(\{\tilde{h}_\beta, h_\alpha,\})}, \tag{32}$$

Here, the matrix $\mu_{\alpha\beta} = \mu(\tilde{h}_\beta, h_\alpha)$ is given by

$$\mu(\tilde{h}_\beta, h_\alpha) = \int_{-\infty}^\infty D\lambda \int_{-i\infty}^{i\infty} D\tilde{\lambda} \frac{1}{(\lambda - \tilde{\lambda})} \exp\left\{ -\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih_\alpha \lambda - i\tilde{h}_\beta \tilde{\lambda} \right\}$$

$$= \begin{cases} -\frac{2\pi}{h_\alpha - \tilde{h}_\beta} \exp\left\{ -\frac{h_\alpha^2 - \tilde{h}_\beta^2}{2A_0} \right\} & \text{for} \quad h_\alpha \neq \tilde{h}_\beta, \\ 0 & \text{for} \quad h_\alpha = \tilde{h}_\beta. \end{cases} \tag{33}$$

Now, with the help of the IZ integral, Eq. (31), we can perform integration over U and U^+, namely the parameter space of the unitary supergroup, in the expression for $\mathcal{Z}_0(J)$, Eq. (30). Then, taking into account the determinant form of the super-Vandermonde determinant Eq. (28), we obtain

$$\mathcal{Z}_0(J) = [1 - \eta(\{h_\alpha, \tilde{h}_\alpha\})] + \frac{1}{2^{n(n-1)} \pi^n \, \Delta(\{h_\alpha, \tilde{h}_\beta\})}$$

$$\times \det\left[\int_{-\infty}^\infty D\lambda \int_{-i\infty}^{i\infty} D\tilde{\lambda} \frac{\lambda}{\tilde{\lambda}(\lambda - \tilde{\lambda})} \exp\left\{ -\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih_\alpha \lambda - i\tilde{h}_\beta \tilde{\lambda} \right\}\right]_{\alpha,\beta=1\ldots n}. \tag{34}$$

With this expression for $\mathcal{Z}_0(J)$ we are ready to calculate one and two point (both level-level and eigenfunction-eigenfunction) correlation functions for Gaussian ensemble of unitary diagonal random matrices. These calculations are presented in the next two subsections.

1. Density of states for diagonal random matrices

The averaged density of states is expressed in terms of the imaginary part of the one-point Green function, $G^A(E)$, as follows

$$\rho(E) = \frac{1}{\pi} \, \mathrm{Im} G^A(E). \qquad (35)$$

The function $G^A(E)$ is related to the averaged generating functional via Eq. (25). Employing the factorization property Eq. (29) for one point Green function, one is led to evaluate the integral in Eq. (34) for $n = 1$, which means that all the supermatrices are two dimensional and thus have one bosonic and one fermionic eigenvalue. Then the Bosonic eigenvalue of the supermatrix $\mathcal{H}_d^J$ will have the form $h = E + \omega + J$, while the fermionic eigenvalue will have the form $\tilde{h} = E + \omega - J$. Without lost of generality we can set $\omega = 0$.

For one point Green function one has to take a derivative of the generating functional, $G_0^A(E) = (1/2\pi)\partial\langle Z_0(J)\rangle/\partial J|_{J=0}$, which, as follows from Eqs. (25) and (26), can be equivalently represented as $G_0^A(E) = i\langle \mathrm{Str}[\hat{s}Q]\rangle$. For $n = 1$ the supersymmetric Vandermonde determinant simplifies and acquires the form $\Delta(h,\tilde{h}) = 1/(h - \tilde{h}) = 1/(2J)$. From here one can easily realize that only $1/\Delta(\tilde{h},h)$ term in Eq. (34) contributes to the derivative in the $J = 0$ limit. Therefore, the expression for one point Green function takes the form

$$G_0^A(E) = -\frac{2E}{A_0} + \frac{1}{\pi} \int_{-\infty}^{\infty} D\lambda \int_{-i\infty}^{i\infty} D\tilde{\lambda} \, \frac{\lambda}{(\tilde{\lambda} - i0)(\lambda - \tilde{\lambda})}$$
$$\times \exp\left\{-\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + iE(\lambda - \tilde{\lambda})\right\}. \qquad (36)$$

Evaluation of the integral in Eq. (36), presented in Appendix C, leads to

$$G_0^A(E) = \sqrt{\frac{\pi}{2A_0}}\left(i + \mathrm{erfi}\left[\frac{E}{\sqrt{2A_0}}\right]\right)e^{-\frac{E^2}{2A_0}}, \qquad (37)$$

where $\mathrm{erfi}(x)$ is the imaginary error function

$$\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)n!}. \qquad (38)$$

Equation (37) exactly reproduces the averaged advanced Green function of the Gaussian unitary ensemble of diagonal random matrices (see for example Refs. [8,9]). Substituting Eq. (37) into Eq. (35) we find the density of states $\rho_0(E)$ for diagonal random matrices,

$$\rho_0(E) = \frac{N^2}{\sqrt{2\pi A_0}} \, e^{-\frac{E^2}{2A_0}}. \qquad (39)$$

2. Two point correlation function for diagonal random matrices

In this subsection we show how the superbosonization formula with the flat integration measure, as defined above, works for four-dimensional supermatrices. Namely, we employ the developed technique of superbosonized generating functional Eqs. (24) and (25) for calculation of a two-level correlation function of diagonal random matrices. For simplicity we will concentrate on a level-level correlation function having the

following form

$$K_0^A(E_1, E_2) = \frac{1}{\pi^2}\left\langle \mathrm{Tr}\left[\frac{1}{E_1 - H - i0}\right]\mathrm{Tr}\left[\frac{1}{E_2 - H - i0}\right]\right\rangle$$
$$- N(N-1)G_0^A(E_1)G_0^A(E_2), \qquad (40)$$

where **N** is the size of the matrices. We are aware of the fact that the correlation function $K_0^A(E_1, E_2)$ containing the product of two advanced Green functions is not the most interesting function characterizing the level correlations. However, the computation of this function presented here serves merely as a demonstration of how the method works. We emphasize that the method of integration adopted in Refs. [17,18] does not work when applied to this problem.

For the case of diagonal random matrices the two-point function Eq. (40) can be derived upon evaluating Eqs. (24) and (25). This can be done making use of the factorization property Eq. (29) with $\mathcal{Z}_0(J)$ given by (34). The calculation is straightforward. Since the Vandermonde determinant, $\Delta(\{\tilde{h}_\beta, h_\alpha\})$, in Eqs. (32) and (34) is always inverse proportional to the source terms, J_1 and J_2, it is easy to see that only the $[\Delta(\{h_\alpha, \tilde{h}_\beta\})]^{-1}$ term will contribute to double derivative in Eq. (25) taken at $J_1 = J_2 = 0$. The double derivative of the Vandermonde determinant is equal to

$$\partial_{J_1}\partial_{J_2}[\Delta(\{\tilde{h}_\beta, h_\alpha,\})]^{-1}|_{J_1, J_2 = 0} = 4. \qquad (41)$$

Therefore, we have for the function $K_0^A(E_1, E_2)$

$$\frac{1}{N}K_0^A(E_1, E_2)$$
$$= -\frac{1}{\pi^2}\det[\mu(\{h_\alpha, \tilde{h}_\beta\})]_{\alpha,\beta=1,2}$$
$$+ \frac{1}{\pi^2}\det\left[\int_{-\infty}^{\infty} D\lambda \int_{-i\infty}^{i\infty} D\tilde{\lambda}\frac{\lambda}{(\tilde{\lambda} + i0)(\lambda - \tilde{\lambda})}\right.$$
$$\times \left.\exp\left\{-\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih_\alpha\lambda - i\tilde{h}_\beta\tilde{\lambda}\right\}\right]_{\alpha,\beta=1,2}, \qquad (42)$$

where $\mu(\{\tilde{h}_\beta, h_\alpha\})$ is defined by Eq. (33) and $h_{1,2} = \tilde{h}_{1,2} = E_{1,2}$.

Analysis of the integral over λ and $\tilde{\lambda}$ under the second determinant in Eq. (42) is presented in Appendix C. Result of the integration can be represented as the sum, $\tilde{\eta}(\{h_\alpha, \tilde{h}_\beta\}) + G_0(\{h_\alpha, \tilde{h}_\beta\})$, where

$$G_0(\{\tilde{h}_\beta, h_\alpha\}) = \sqrt{\frac{\pi}{2A_0}}\left(i + \mathrm{erfi}\left[\frac{\tilde{h}_\beta}{\sqrt{2A_0}}\right]\right)e^{-\frac{h_\alpha^2}{2A_0}}. \qquad (43)$$

Substituting now Eqs. (32) and (43) into the determinants in Eq. (42), we come to the result

$$K_0^A(E_1, E_2) = \frac{N}{\pi}\left[\frac{G_0^A(E_2) - G_0^A(E_1)}{E_1 - E_2}\right], \qquad (44)$$

where the function $G_0^A(E)$ is given by Eq. (37). The result for the level-level correlation function for diagonal random matrices, Eq. (44), coinciding with Eq. (40), together with Eq. (37) agrees with the one found in Ref. [32].

C. Nondiagonal contributions to the density of states for almost diagonal matrices

In order to show how the superbosonization technique works for less trivial random matrix theories, we calculate in this section a correction to the density of states in the model of almost diagonal matrices [8,9] up to the second order in the bandwidth b. By definition, statistical properties of nondiagonal matrices are described by a single, always positive function, $\mathcal{F}(r)$, as $A_{ij} = b^2 \mathcal{F}(|i-j|)$, $i \neq j$. Function $\mathcal{F}(r)$ can adopt any form provided that it has a maximum at the center of the band, $r = 0$, and decays with the bandwidth b as r becomes large. For small b we have the ensemble of almost diagonal random matrices, while for large b we approach the Wigner-Dyson Gaussian unitary ensemble (GUE).

We consider the case of $b \ll 1$ when the standard nonlinear σ model is not applicable. Then, expanding the exponent in Eq. (24) in b, we have

$$\langle Z(J) \rangle = \langle Z_0(J) \rangle + b^2 \langle Z_1(J) \rangle, \tag{45}$$

where zero order in b^2 contribution, $\langle Z_0(J) \rangle$, corresponds to diagonal random matrices considered in the previous subsection. Technically, calculation of the correction, $b^2 \langle Z_1(J) \rangle$, is similar to that of $\langle Z_0(J) \rangle$. It is determined by the form of A_{ij} for almost diagonal matrices as follows

$$
\begin{aligned}
\langle Z_1(J) \rangle &= \frac{1}{2} \sum_{i,j} \mathcal{F}(|i-j|) \langle \mathrm{Str}[Q_i Q_j] \rangle \\
&= -\frac{1}{2} \int_{\mathbb{H}_n} \prod_i DQ_i \ \mathrm{Sdet}[Q_i]^{-1} \\
&\quad \times \frac{1}{2} \sum_{i,j} \mathcal{F}(|i-j|) \ \mathrm{Str}[Q_i Q_j] \\
&\quad \times \exp\left\{ i \sum_i \mathrm{Str}[Q_i \mathcal{H}_i^J] - \frac{1}{2} \sum_i A_0 \ \mathrm{Str}[Q_i^2] \right\},
\end{aligned}
\tag{46}
$$

where, as usual, integration goes over the linear space $\mathbb{H}_n$ with the flat measure. Then, the correction to the advanced Green function, $b^2 G_1^A(E)$, is expressed in terms of the correction, $b^2 \langle Z_1(J) \rangle$, to the averaged generating functional,

$$
\begin{aligned}
G_1^A(E) &= \frac{\partial \langle Z_1(J) \rangle}{\partial J} \\
&= \frac{i}{2} \sum_{k, i \neq j} \mathcal{F}(|i-j|) \ \langle \mathrm{Str}[\hat{s} Q_k] \ \mathrm{Str}[Q_i Q_j] \rangle. \tag{47}
\end{aligned}
$$

In Eq. (47) the averaging, $\langle \ldots \rangle$, is defined as

$$
\begin{aligned}
\langle F[Q] \rangle &= \int DQ \ \mathrm{Sdet}[Q]^{-1} \ F[Q] \\
&\quad \times \exp\left\{ i \sum_i \mathrm{Str}[Q \mathcal{H}^J] - \frac{1}{2} A_0 \ \mathrm{Str}[Q^2] \right\}. \tag{48}
\end{aligned}
$$

Averaging in the right hand side of Eq. (47) can be performed using the identity, $\langle Q \rangle = \langle \frac{1}{2} \mathrm{Str}[\hat{s} Q] \rangle i d_{2n}$. Next, we make use

of this identity to represent the average in Eq. (47) as

$$
\begin{aligned}
&\langle \mathrm{Str}[\hat{s} Q_k] \rangle_{Q_k} \ \langle \mathrm{Str}[Q_i Q_j] \rangle_{Q_{i,j}} \\
&= \langle \mathrm{Str}[\hat{s} Q] \rangle_Q \left\langle \frac{1}{2} \mathrm{Str}[\hat{s} Q] \mathrm{Str}[Q] \right\rangle_Q \delta_{ki} \quad (\text{or } \delta_{kj}) \\
&= -\frac{i}{2} \ G_0(E) \ \frac{\partial \langle \mathrm{Str}[Q] \rangle^J}{\partial J} \delta_{ki} \quad (\text{or, alternatively } \delta_{kj}),
\end{aligned}
\tag{49}
$$

where, according to Eq. (48), we have

$$
\begin{aligned}
\frac{\partial \langle \mathrm{Str}[Q] \rangle^J}{\partial J} &= \frac{\partial}{\partial J} \int DQ \ \mathrm{Sdet}[Q]^{-1} \ \mathrm{Str}[Q] \\
&\quad \times \exp\left\{ i \sum_i \mathrm{Str}[Q H^J] - \frac{1}{2} A_0 \ \mathrm{Str}[Q^2] \right\}. \tag{50}
\end{aligned}
$$

As described above, now again, one has to diagonalize the supermatrix Q and reduce the expression Eq. (46) to IZ integral. For that purpose, we first notice that the only difference between the expressions for $\langle \mathrm{Str}[Q] \rangle^J$ and $Z_0(J)$ is the presence of the term $\mathrm{Str}[Q]$ under integral, which, after diagonalization for the one-point Green functions ($n = 1$ case), produces an additional $\lambda - \tilde{\lambda}$ term under the integral in Eq. (34). Secondly, the boundary $1 - \eta$ term does not contribute here, because the presence of δ functions in Eq. (31) together with $\lambda - \tilde{\lambda}$ in the integral makes it zero. Repeating now the calculation for $G_0(E)$ and keeping in mind the two observations above, one finds

$$
\begin{aligned}
\frac{\partial \langle \mathrm{Str}[Q] \rangle^J}{\partial J} &= \frac{1}{\pi} \int_{-\infty}^{\infty} D\lambda \int_{-i\infty}^{i\infty} D\tilde{\lambda} \ \frac{\lambda}{(\tilde{\lambda} + i0)} \\
&\quad \times \exp\left\{ -\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + i E(\lambda - \tilde{\lambda}) \right\} \\
&= \frac{2i}{A_0} \ N \ \mathcal{R}_N[\mathcal{F}] \ G_0^A(E). \tag{51}
\end{aligned}
$$

Substituting now Eqs. (49) and (51) into the expression for the first order correction to the Green function $G_1(E)$, Eq. (47), we obtain

$$
G_1(E) = N \ \mathcal{R}_N[\mathcal{F}] \ G_0^A(E) \ \frac{1}{A_0} \left[G_0^A(E) - 1 \right],
$$

where $\mathcal{R}_N[\mathcal{F}] \equiv 2 \sum_{l=1}^{N} \mathcal{F}(l)$. Then, for the first order in b^2 correction to density of states, $\rho_1(E) = \pi^{-1} \mathrm{Im} G_1^A(E)$, one easily finds

$$
\rho_1(E) = \mathcal{R}_N[\mathcal{F}] \ \rho_0(E) \ \frac{1}{A_0} \left(E \ \sqrt{\frac{2\pi}{A_0}} \ \mathrm{erfi}\left[\frac{E}{\sqrt{2 A_0}} \right] - 1 \right), \tag{52}
$$

which exactly reproduces the results first obtained with the help of the virial expansion [8,9].

V. OUTLOOK

We have presented a new scheme of computations using the superbosonization formula, Eq. (3), first proposed in Ref. [12].

We have proven that this formula is exact and have given a precise recipe for the performing integration for many point correlation functions for the unitary ensemble. In contrast to a previous study [17,18] the integration over the eigenvalues in the fermion-fermion block of the supermatrices is performed from $-i\infty$ to $i\infty$ and not along a circle. This way of the integration has allowed us to obtain regular integrals and calculate them in several cases.

The proof of our approach and proposed method of computation of the integrals is heavily based on the supersymmetric extension of the Itzykson-Zuber integral. This integral in known only for systems with broken time-reversal symmetry (unitary ensemble) and this is why we consider here only such systems. At the same time, the proposed method of the integration over the eigenvalues of the supermatrices when one integrates over the eigenvalues in the boson-boson block from $-\infty$ to ∞ and over the eigenvalues in the fermion-fermion block from $-i\infty$ to $i\infty$ looks very general. This encourages us to make a guess that this way of the integration can also be used for time reversal invariant ensembles. Of course, such a guess must be checked and proven in the future.

We have demonstrated that the application of the bosonization formula to random band matrix (RBM) [33–38] models with small bandwidth b reproduces the perturbative expansion of DOS obtained by virial expansion [8]. We have also computed the simplest two-point correlation function containing a product of two advanced Green functions for the ensemble of diagonal matrices. Of course, calculating an average product of both retarded and advanced Green functions would be a more interesting task but we leave it for future study. It is important at the moment that our method allows us to calculate many-point correlations functions for cases where the way of the integration developed in Refs. [17,18] does not work. We have made comparison with the known results only for checking our approach and demonstration of details of the computation.

Equation (3) complemented by our recipe of the integration is exact and most general representation of the integrals over supervectors in terms of integrals over supermatrices. The traditional nonlinear σ model, Eqs. (1) and (2), can be obtained using the saddle-point approximation for calculation of the integral over the supermatrix Q is less general. Taking into account a success of the latter in solving numerous problems (see, e.g. Ref. [3]) we believe that its generalization can also bring new interesting results.

Finally, we would like to mention that to this point disordered systems have been actually successfully studied using supersymmetric σ model (including statistical properties of the energy levels in small metallic disordered grains), and we mostly focused here on a field theory for random matrix ensembles and nonperturbative effects therein. Another field of great interest of course is the nonperturbative study of various correlation functions in strongly interacting systems. Examples of such systems that potentially can be studied nonperturbatively using superbosonization include among others (i) the field theory of many-body localization in random spin chains [15]; (ii) quantum phase transitions at the boundary of topological superconductors in two and three dimensions, which have been argued to support supersymmetry at long distances and times [39]; (iii) translation invariant lattices

with an odd number of Majorana fermions per unit cell [40].

APPENDIX A: INTEGRAL OVER $\mathbb{H}_n$

Let $\mathbb{H}_n$ be the linear space of Hermitian $2n \times 2n$ supermatrices. Then for all $\mathcal{A} \in \mathbb{H}_n$ the convergent integral,

$$\hat{\Upsilon}(\mathcal{A}) = \lim_{\eta \to 0} \int_{\mathbb{H}_n} DB \exp\{i\,\mathrm{Str}[\mathcal{A}B] - \tilde{\eta}\,\mathrm{Str}[B^2]\}, \quad \text{(A1)}$$

taken over $\mathbb{H}_n$ with Berezin measure satisfies the condition

$$\int_{\mathbb{H}_n} D\mathcal{A}' \, \hat{\Upsilon}(\mathcal{A}' - \mathcal{A}) \equiv 1. \quad \text{(A2)}$$

Moreover, for any map, $\mathcal{F} : \mathbb{H}_n \to \mathbb{G}$, that converges exponentially (or faster), the identity

$$\mathcal{F}(Q) \equiv \int_{\mathbb{H}_n} D\mathcal{A}\mathcal{F}(\mathcal{A})\hat{\Upsilon}(\mathcal{A} - Q) \quad \text{(A3)}$$

always holds.

To derive identities (A2) and (A3) for $\hat{\Upsilon}(\mathcal{A})$, one can first formally perform integration in the definition Eq. (A1) of $\hat{\Upsilon}(\mathcal{A})$, under the limit. This integration yields

$$\hat{\Upsilon}(\mathcal{A}) = \lim_{\eta \to 0} \left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta}\right\}, \quad \text{(A4)}$$

where the limit is well defined. With the help of Eq. (A4), the right hand side of Eq. (A3) can be represented as follows

$$\int_{\mathbb{H}_n} D\mathcal{A}\mathcal{F}(\mathcal{A})\hat{\Upsilon}(\mathcal{A} - Q)$$

$$= \int_{\mathbb{H}_n} D\mathcal{A}\mathcal{F}(Q + \mathcal{A})\hat{\Upsilon}(\mathcal{A})$$

$$= \lim_{\eta \to 0} \left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \int_{\mathbb{H}_n} D\mathcal{A}\{\mathcal{F}(Q) + \mathrm{Str}[\mathcal{F}'(\mathcal{A})\mathcal{A}] + \cdots$$

$$+ \mathrm{Str}[\mathcal{F}^{(n)}(\mathcal{A})\mathcal{A}^n]/n! + \cdots\} \exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta}\right\}$$

$$= \mathcal{F}(Q) + \lim_{\eta \to 0} \left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \int_{\mathbb{H}_n} D\mathcal{A}\{\mathrm{Str}[\mathcal{F}'(\mathcal{A})\mathcal{A}] + \cdots$$

$$+ \mathrm{Str}[\mathcal{F}^{(n)}(\mathcal{A})\mathcal{A}^n]/n! + \cdots\} \exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta}\right\}, \quad \text{(A5)}$$

where we have Tailor expanded the function $\mathcal{F}(Q + \mathcal{A})$ around $\mathcal{A} = 0$. We note that such an expansion exists due to the specific constraints on the function $\mathcal{F}$, outlined in Sec. III. To finalize our proof, it is left to show that

$$\lim_{\eta \to 0} \left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \int_{\mathbb{H}_n} D\mathcal{A}\mathrm{tr}[\mathcal{A}^n] \exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta}\right\} = 0. \quad \text{(A6)}$$

Equation (A6) is proven by introducing the generating functional,

$$W_\eta(K) = \left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \int_{\mathbb{H}_n} D\mathcal{A} \exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta} + \mathrm{Str}[K\mathcal{A}]\right\}$$
$$= \exp\{\eta\mathrm{Str}[K^2]\}, \qquad (A7)$$

and observing that

$$\lim_{\eta\to0}\left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}} \int_{\mathbb{H}_n} D\mathcal{A}\,\mathrm{Str}[\mathcal{A}^n]\exp\left\{-\frac{\mathrm{Str}\mathcal{A}^2}{4\eta}\right\}$$
$$= \lim_{\eta\to0}\left[\frac{1}{4\pi\eta}\right]^{\frac{N}{2}}\mathrm{Str}\left[\frac{\delta^n W_\eta(K)}{\delta\mathcal{A}_1\cdots\delta\mathcal{A}_n}\right]\Bigg|_{K=0} = 0. \quad (A8)$$

APPENDIX B: CALCULATION OF C_n

Let the formal sums of Hermitian superbivectors (product of two supermatrices, each of them being from $\mathbb{H}_n$) constitute a vector space $\Lambda^2(\mathbb{H}_n)$ called the second exterior power of $\mathbb{H}_n$. Then the integral

$$C_n = \int_{\Lambda^2(\mathbb{H}_n)} DB'\,\mathrm{Sdet}B'\,e^{i\mathrm{Str}B'} \qquad (B1)$$

over the vector space $\Lambda^2(\mathbb{H}_n)$ is unity, $C_n = 1$.

According to the proposed prescription, one evaluates integral Eq. (B1) first by diagonalizing the supermatrix B'. As already stated, any given complex $2n \times 2n$ supermatrix B' can be diagonalized by the transformation $B' = UB'_d\bar{V}$, where $U \in U(n|n)$, $\bar{V} \in U(n|n)/U^{2n}(1)$. Substituting this transformation into Eq. (13) we obtain an integral over the eigenvalues, B'_d, and diagonalization "angles" U and $\bar{V}$; the latter integral is nothing but supersymmetric Itzykson-Zuber [19] integral.

In order to evaluate the integral in Eq. (15), we consider the following generalized integral

$$\int DB\,\mathrm{Sdet}[B]\exp\left\{\frac{i}{t}\,\mathrm{Str}[Q_dB]\right\}$$
$$= \int DB\,\mathrm{Sdet}[B]\exp\left\{\frac{1}{2t}\,\mathrm{Str}[B^2 - Q_d^2]\right.$$
$$\left. - \frac{1}{2t}\,\mathrm{Str}[(B - iQ_d)^2]\right\}$$
$$= \int \prod_i db_i\,d\tilde{b}_i\,\Delta^2(\{b_j^2,\tilde{b}_j^2\})$$
$$\times \left[\prod_i \frac{\tilde{b}_i}{b_i}\exp\left\{\frac{1}{2t}\left[(b_i^2 - \tilde{b}_i^2) - (\lambda_i^2 - \tilde{\lambda}_i^2)\right]\right\}\right]$$
$$\times \Gamma[\{b_j,\tilde{b}_j\}|\{\lambda_j,\tilde{\lambda}_j\}], \qquad (B2)$$

which coincides with Eq. (B2) in the case when Q_d is an identity matrix. Before setting $Q_d = id$, first let us note that for any complex $2n \times 2n$ supermatrix of the following diagonal form:

$$\Lambda = \begin{pmatrix} x\otimes id_n & 0 \\ 0 & y\otimes id_n \end{pmatrix}, \quad x,y\in\mathbb{R}, \qquad (B3)$$

where id_n is the $n \times n$ identity matrix, the η term has the form

$$\eta_\Lambda(x,y) = \left(1 - e^{-\frac{x^2-y^2}{2t}}\right)^n. \qquad (B4)$$

The term, $\Gamma_0[\{b_j,\tilde{b}_j\}|x,y]$, [see Eq. (5)] corresponding to the matrix Λ vanishes. This is because the Vandermonde determinant, $\Delta_\Lambda(x,y)$, in the denominator will cancel one of the determinants involving Bessel function in the nominator. However the next determinant, which is equal to zero, remains. Thus, we see that if our $2n$ dimensional complex supermatrix $Q_d = id$ (which means $x = y$ above), then the corresponding η term $\eta_{id}(1,1) = (1 - e^{-\frac{1-1}{2t}})^n = 0$. Therefore, from Eq. (5), we obtain

$$\Gamma[\{b_j,\tilde{b}_j\}|\{1\ldots1\}] = \frac{\prod_i \delta(b_i)\delta(\tilde{b}_i)}{\Delta^2(\{b_j^2,\tilde{b}_j^2\})}. \qquad (B5)$$

Substituting Eq. (B5) into Eq. (B2), where as Q_d a unity matrix is taken with $t = 1$, one obtains

$$C_n = \int DB'\,\mathrm{Sdet}[B']\,e^{i\mathrm{Str}B'}$$
$$= \int \prod_i db_i\,d\tilde{b}_i\,\Delta^2(\{b_j^2,\tilde{b}_j^2\})$$
$$\times \left\{\prod_i \frac{\tilde{b}_i}{b_i}\,e^{\frac{1}{2}[b_i^2-\tilde{b}_i^2]}\right\}\Gamma[\{b_j,\tilde{b}_j\}|\{1\ldots1\}]$$
$$= \int \prod_i db_i\,d\tilde{b}_i\left\{\prod_i \frac{\tilde{b}_i}{b_i}\,e^{\frac{1}{2}[b_i^2-\tilde{b}_i^2]}\right\}\prod_i \delta(b_i)\delta(\tilde{b}_i) = 1. \qquad (B6)$$

The last equality holds, since our integration contours are shifted by an infinitesimal δ and $i\delta$ with respect to the imaginary and real axis correspondingly. This completes the computation of C_n.

APPENDIX C: EVALUATION OF DOUBLE INTEGRALS

Here we will evaluate the following integral

$$\mathcal{I}(h,\tilde{h}) = \int_{-\infty}^{\infty} d\lambda \int_{-i\infty}^{i\infty} d\tilde{\lambda}\,\frac{\lambda}{(\tilde{\lambda}-i0)(\lambda-\tilde{\lambda})}$$
$$\times \exp\left\{-\frac{A_0}{2}(\lambda^2-\tilde{\lambda}^2) + ih\lambda - i\tilde{h}\tilde{\lambda}\right\}. \quad (C1)$$

Making use of the decoupling

$$\frac{\lambda}{(\tilde{\lambda}-i0)(\lambda-\tilde{\lambda})} = \left(\frac{1}{\tilde{\lambda}-i0} + \frac{1}{\lambda-\tilde{\lambda}}\right), \qquad (C2)$$

we represent the double integral, $\mathcal{I}(h,\tilde{h})$, as the sum

$$\mathcal{I}(h,\tilde{h}) = \mathcal{I}_1(h,\tilde{h}) + \mathcal{I}_2(h,\tilde{h}), \qquad (C3)$$

where

$$\mathcal{I}_1(h,\tilde{h}) = \int_{-\infty}^{\infty} d\lambda \int_{-i\infty}^{i\infty} d\tilde{\lambda}\,\frac{1}{(\tilde{\lambda}-i0)}$$
$$\times \exp\left\{-\frac{A_0}{2}(\lambda^2-\tilde{\lambda}^2) + ih\lambda - i\tilde{h}\tilde{\lambda}\right\},$$

$$\mathcal{I}_2(h,\tilde{h}) = \int_{-\infty}^{\infty} d\lambda \int_{-i\infty}^{i\infty} d\tilde{\lambda}\,\frac{1}{(\lambda-\tilde{\lambda})}$$
$$\times \exp\left\{-\frac{A_0}{2}(\lambda^2-\tilde{\lambda}^2) + ih\lambda - i\tilde{h}\tilde{\lambda}\right\}. \quad (C4)$$

In the following two subsections we will evaluate integrals $\mathcal{I}_1(h,\tilde{h})$ and $\mathcal{I}_2(h,\tilde{h})$, respectively.

1. Calculation of $\mathcal{I}_1$

In order to evaluate $\mathcal{I}_1(h,\tilde{h})$ we recall that

$$\frac{1}{(\tilde{\lambda} - i0)} = \mathrm{P}\,\frac{1}{\tilde{\lambda}} + i\pi\delta(\tilde{\lambda}), \tag{C5}$$

where symbol P denotes the principal value of the integral. Then for $\mathcal{I}_1(h,\tilde{h})$ we have

$$\mathcal{I}_1(A_0,h,\tilde{h}) = i\pi \int_{-\infty}^{\infty} \exp\left\{-\frac{A_0}{2}\lambda^2 + ih\lambda\right\} + \tilde{\mathcal{I}}_1(A_0,h,\tilde{h}), \tag{C6}$$

with

$$\tilde{\mathcal{I}}_1(A_0,h,\tilde{h}) = \int_{-\infty}^{\infty} d\lambda\,\mathrm{P}\int_{-i\infty}^{i\infty}\frac{d\tilde{\lambda}}{\tilde{\lambda}} \\ \times \exp\left\{-\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih\lambda - i\tilde{h}\tilde{\lambda}\right\}. \tag{C7}$$

The presence of the principle value in Eq. (C7) insures the possibility of bringing the integral to the Gaussian form first by taking the derivative over $\tilde{h}$:

$$\frac{\partial \tilde{\mathcal{I}}_1(A_0,h,\tilde{h})}{\partial \tilde{h}} = -i\int_{-\infty}^{\infty} d\lambda \int_{-i\infty}^{i\infty} d\tilde{\lambda} \\ \times \exp\left\{-\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih\lambda - i\tilde{h}\tilde{\lambda}\right\} \\ = \sqrt{\frac{2\pi}{A_0}}\,e^{-h^2/(2A_0)}\sqrt{\frac{2\pi}{A_0}}\,e^{\tilde{h}^2/(2A_0)} \\ = \frac{2\pi}{A_0}\exp\left\{-\frac{h^2 - \tilde{h}^2}{2A_0}\right\}. \tag{C8}$$

Then the function $\tilde{\mathcal{I}}_1$ itself will have the form

$$\tilde{\mathcal{I}}_1(A_0,h,\tilde{h}) = \frac{2\pi}{A_0}\,e^{-h^2/(2A_0)}\left(\int_0^{\tilde{h}} d\tilde{h}_1 e^{\tilde{h}_1^2/(2A_0)} + C\right), \tag{C9}$$

with $C = (A_0/2\pi)\exp(h^2/2A_0)\tilde{\mathcal{I}}_1(A_0,h,0)$. On the other hand we have that

$$\tilde{\mathcal{I}}_1(A_0,h,0) = \int_{-\infty}^{\infty} d\lambda\,\mathrm{P}\int_{-i\infty}^{i\infty}\frac{d\tilde{\lambda}}{\tilde{\lambda}} \\ \times \exp\left\{-\frac{A_0}{2}(\lambda^2 - \tilde{\lambda}^2) + ih\lambda\right\} = 0 \tag{C10}$$

suggesting $C = 0$. Substituting Eq. (C9) into Eq. (C6) we obtain

$$\mathcal{I}_1(A_0,h,\tilde{h}) = \pi\sqrt{\frac{2\pi}{A_0}}\exp\left(-\frac{h^2}{2A_0}\right)\left[i + \mathrm{erfi}\left(\frac{\tilde{h}}{\sqrt{2A_0}}\right)\right]. \tag{C11}$$

2. Calculation of $\mathcal{I}_2$

Introducing a new variable, $\lambda' = -i\tilde{\lambda}$, we rewrite integral $\mathcal{I}_2(h,\tilde{h})$ as

$$\mathcal{I}_2(h,\tilde{h}) = i\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda'\,\frac{1}{\lambda - i\lambda'} \\ \times \exp\left\{-\frac{A_0}{2}\left(\lambda - \frac{ih}{A_0}\right)^2 - \frac{A_0}{2}\left(\lambda' - \frac{\tilde{h}}{A_0}\right)^2 \\ - \frac{h^2 - \tilde{h}^2}{2A_0}\right\}. \tag{C12}$$

As the next step we shift variables λ and λ' by ih/A_0 and $\tilde{h}/A_0$, respectively. Then Eq. (C12) will acquire the form

$$\mathcal{I}_2(h,\tilde{h}) = ie^{-\frac{h^2 - \tilde{h}^2}{2A_0}}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \\ \times \frac{e^{-\frac{A_0}{2}(\lambda^2 + \lambda'^2)}}{\lambda + \frac{ih}{A_0} - i\left(\lambda' + \frac{\tilde{h}}{A_0}\right)}. \tag{C13}$$

It is convenient to evaluate integral Eq. (C13) after switching to polar coordinates, $\lambda - i\lambda' = ue^{i\theta}$:

$$\mathcal{I}_2(h,\tilde{h}) = ie^{-\frac{h^2 - \tilde{h}^2}{2A_0}}\int_0^{\infty} u\,du \int_0^{2\pi} d\theta\,\frac{e^{-\frac{A_0}{2}u^2}}{\frac{ih}{A_0} - \frac{i\tilde{h}}{A_0} + ue^{i\theta}}, \tag{C14}$$

and integrate first over θ and only then over u. Result reads

$$\mathcal{I}_2(h,\tilde{h}) = ie^{-\frac{h^2 - \tilde{h}^2}{2A_0}}\frac{2\pi}{i\left(\frac{h}{A_0} - \frac{\tilde{h}}{A_0}\right)}(1 - \delta_{h\tilde{h}})\int_0^{\infty} du\,ue^{-\frac{A_0}{2}u^2} \\ = \frac{2\pi}{h - \tilde{h}}(1 - \delta_{h\tilde{h}})e^{-\frac{h^2 - \tilde{h}^2}{2A_0}}. \tag{C15}$$

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