Static and dynamic properties of Josephson weak links with singlet and triplet coupling

Andreas Moor[*] and Anatoly F. Volkov

Theoretische Physik III, Ruhr-Universität Bochum, D-44780 Bochum, Germany

(Received 17 May 2017; published 26 July 2017)

We theoretically study static and dynamic properties of short Josephson junctions (JJs) with singlet and triplet Josephson coupling. In singlet Josephson weak links, two singlet superconductors S are connected with each other by a normal film (N) or wire. Triplet JJs, which we denote S_m-N(F)-S_m, are formed by two singlet BCS superconductors covered by a thin layer of a weak ferromagnet F_w. These superconductors S_m are separated from the N (or F) layer by spin filters, which pass electrons with only one spin orientation. The triplet Cooper pairs propagating from the left (right) superconductors S_m differ from each other not only in polarizations, but also in chiralities. The latter is determined by the magnetization orientation in weak ferromagnets F_w. We obtain analytical formulas for the critical Josephson current in both types of JJs. If the chiralities of the triplet Cooper pairs penetrating into the N film in S_m-N(F)-S_m JJs from the left and right S_m are different, the Josephson current is not 0 in the absence of the phase difference (spontaneous Josephson current). We also calculate the admittance $Y(\Omega)$ for arbitrary frequencies Ω in the case of singlet JJs and for low frequencies in the case of triplet JJs. At low temperatures T, the real part of the admittance $Y'(\Omega)$ in singlet JJs starts to increase from 0 at $\hbar\Omega \geqslant \Delta_{sg}$, but at $T \geqslant \Delta_{sg}$, it has a peak at low frequencies the magnitude of which is determined by inelastic processes. The subgap Δ_{sg} depends on the transparencies of the S/N interfaces and on the phase difference $2\chi_0$. The low-frequency peak in $Y'(\Omega)$ in triplet JJs disappears.

I. INTRODUCTION

The frequency dependence of the admittance $Y(\Omega)$ for uniform superconductors has been calculated long ago in the well-known papers by Mattis and Bardeen and by Abrikosov, Gor'kov, and Khalatnikov [1–3]. It has been shown that the real part of the admittance $Y'(\Omega)$ at zero temperature is 0 unless the frequency Ω does not exceed $2\Delta/\hbar$. This natural result has been confirmed experimentally [4–6]. The admittance of weakly inhomogeneous superconductors has been determined by Larkin and Ovchinnikov [7]. Those authors have also calculated the admittance of a strongly inhomogeneous superconductor with a sufficiently strong Zeeman interaction in the so-called Fulde-Ferrel-Larkin-Ovchinnikov state [8].

Strongly inhomogeneous superconductivity is realized also in S/N structures or in Josephson weak links of different kinds, S/N/S, S/c/S, etc., where N represents a normal film and c indicates a constriction. Superconducting correlations are induced in the N region due to the proximity effect so that a subgap Δ_{sg} ($\Delta_{sg} < \Delta$) may arise in the N film [9,10]. The admittance $Y(\Omega)$ for the S/N/N_{res} structure was calculated in Ref. [11], where N_{res} represents a bulk normal metal (reservoir) or a thick normal film attached to a thinner N film or wire. A peak in $Y'(\Omega)$ was shown to exist at a frequency corresponding to a subgap Δ_{sg} in the N film.

More attention was paid to the study of ac properties of superconducting weak links (see reviews in Refs. [12–14]). The interest in this study increased in recent years due to the rapid progress in experimental techniques and possible applications of Josephson junctions (JJs) [15–18]. The formula for the admittance of a short S/c/S (or S/N/S) weak link was derived in Ref. [19] and analyzed in more detail in our recent paper, [20] (see also Ref. [21], where another approach to

calculations is used). The admittance of long S/N/S junctions has been calculated in Refs. [22] and [23]. In all these papers, an anomalous enhancement of the real part of the admittance $Y'(\Omega)$ at low frequencies Ω has been obtained [19–23]. The enhancement is caused by quasiparticles with energies in the interval $\Delta_{sg} < \epsilon < \Delta$, where the subgap Δ_{sg} depends on the dc phase difference $2\chi_0$.

Interestingly, collective modes may lead to peculiarities in the admittance $Y'(\Omega)$ of a uniform superconductor or weak links under certain conditions. For example, the amplitude mode results in a peak at $\Omega \simeq 2\Delta$ in the admittance of a current-carrying superconductor [24], and the phase mode, or the so-called Carlson-Goldman mode [25–30], leads to some features in the $I-V$ characteristics and in the dependence of $Y(\Omega)$ [29,30].

On the other hand, in the last decade a great deal of attention has been paid to the study of triplet odd-frequency superconductivity [31], which arises in singlet superconductor (S)/ferromagnet (F) structures (see reviews, Refs. [32–38], and references therein). It was shown [39] that the proximity effect induces triplet Cooper pairs in S/F structures with an inhomogeneous magnetization $M(x)$ in the ferromagnet F [40]. These pairs penetrate into the ferromagnet over a relatively large distance l_T, which may be of the order $l_T \sim \sqrt{D/T}$, that is, much larger than the length $l_h \sim \sqrt{D/h}$ of the condensate penetration into a homogeneous ferromagnet with an exchange field h (D is the diffusion coefficient and T the temperature). The long penetration of the condensate is caused by the triplet Cooper pairs with the total spin orientation parallel to the magnetization vector **M** in the ferromagnet. In the case of a magnetically homogeneous F triplet, Cooper pairs penetrating into the F from a singlet superconductor have the total spin oriented perpendicular to the vector **M** and penetrate the ferromagnet over a rather short distance, $\sim l_h$.

The long-range triplet superconducting correlations may provide, in particular, the Josephson coupling in S/F/S

---

[*]amoor@tp3.rub.de

junctions with a relatively thick ferromagnetic film. The appearance of these long-range triplet components has been proved mainly by observing the dc Josephson effect in JJs of different kinds with a ferromagnetic layer(s) [41–53]. Different aspects of the stationary Josephson effect in Josephson junctions of various types have been analyzed in many theoretical papers (see Refs. [54–74] as well as the references in recent review articles, Refs. [36–38]. On the other hand, the dynamics of the triplet component was studied only in uniform superconductors [75,76], and not in Josephson junctions.

In this paper, we calculate and analyze the admittance $Y(\Omega)$ of JJs of two types, S/N/S and S_m/N/S_m (or S_m/F/S_m), where S_m represents a magnetic superconductor. In the latter case it does not matter whether the superconductors S_m are connected by a normal or a ferromagnetic wire. It is merely important that the magnetization vector **M** is parallel to the filter's axis, i.e., **M**||**e**$_z$. The Josephson coupling is provided by the singlet component in JJs of the first type and by the triplet odd-frequency component in JJs of the second type. We consider the case of short JJs, with the distance between the superconducting reservoirs shorter than the coherence length $\xi_S \sim \{\sqrt{D/T}, \sqrt{D/\Delta}\}$. Unlike the case studied in Refs. [19] and [20], where the S/N interface resistance R_{if} was assumed to be negligible compared with the resistance R_N in the N region, we consider here the opposite case, i.e., $R_{if} \gg R_N$. As in the case considered in Refs. [19] and [20], the real part of the admittance $Y'(\Omega)$ in singlet JJs has a maximum at low Ω if the temperature is not too low, $T \gtrsim \Delta_{sg}$. The low-frequency behavior of $Y'(\Omega)$ is described approximately by the expression $Y'(\Omega) \sim [2\gamma_N/(4\gamma_N^2 + \Omega^2)] \exp(-\Delta_{sg}/T)$. Although the general formulas for the current obtained in this paper for the singlet weak links, Eqs. (31), (32), and (34), look similar to those in our previous publication, Ref. [20], they differ essentially because the integrands in the corresponding equations are different [77]. In the case of triplet coupling, a slight enhancement of $Y'(\Omega)$ remains only if the characteristic exchange field h is small compared with the subgap Δ_{sg}. Otherwise no enhancement of $Y'(\Omega)$ appears.

The paper is organized as follows. In Sec. II, we present the main equations. In Sec. III, we calculate the critical Josephson current I_c for an S/N/S Josephson weak link with a coupling via the singlet component. We find also a response of this JJ to a small ac phase variation and present an expression for the admittance of the junction $Y(\Omega)$. In Sec. IV, we obtain and analyze the current I_c and the admittance $Y(\Omega)$ in JJs of the S_m/N/S_m or S_m/F/S_m type, where S_m is a magnetic superconductor serving as a source of fully polarized triplet Cooper pairs [see Fig. 1(b)]. In Sec. V, the results obtained are discussed.

II. BASIC EQUATIONS

We consider JJs of the two types shown in Fig. 1. The JJ or weak links in Fig. 1(a) consist of two bulk BCS superconductors S connected by a normal wire (or film). The JJ depicted in Fig. 1(b) consists of two bulk agnetic superconductors S_m connected by a normal (or ferromagnetic) wire. The superconductors S_m are formed by a BCS superconductor covered with a thin film of a weak ferromagnet F_w with an exchange field **h**. We assume that at the S_m/N interfaces
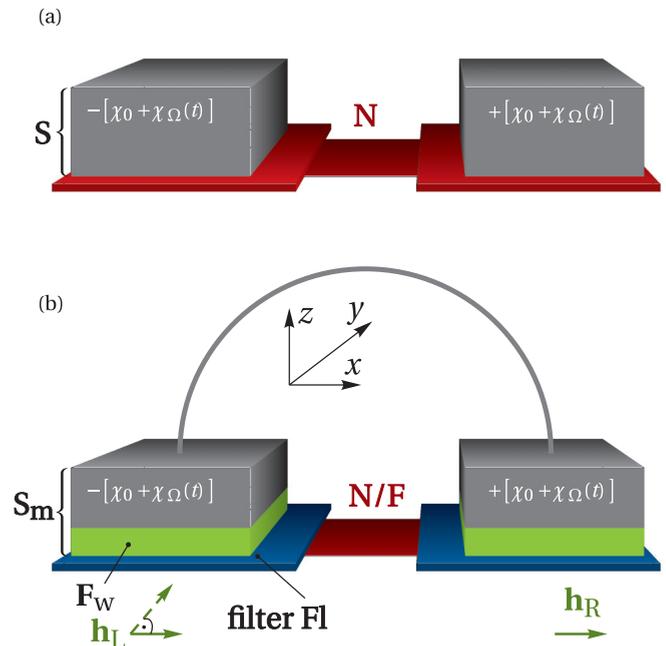
FIG. 1. Schematic of the Josephson junctions (JJs) under consideration. (a) JJ with singlet coupling; (b) JJ with triplet coupling. The magnetic superconductors S_m consist of conventional singlet superconductors covered with thin layers of weak ferromagnets F_w so that triplet Cooper pairs with spin polarizations in the x−z or y−z plane penetrate into the F_w layer due to the proximity effect. The filters Fl let pass Cooper pairs with spins parallel or antiparallel to the z axis. If the exchange fields $h_{R,L}$ at the right and at the left have different directions, a spontaneous current arises which can flow through the loop shown.

there are spin filters which let pass electrons with only one spin direction (**s**||**z**). The exchange field **h** is supposed to be perpendicular to the z axis so that triplet Cooper pairs, which appear in the F film, have a component along the z axis and therefore penetrate through the filters. The singlet Cooper pairs do not pass through the filters. We consider the diffusive case assuming that the mean free path is shorter than the coherence length $\xi_S \simeq \sqrt{D/T_c}$.

In order to find the current I through the system, we need to determine the quasiclassical Green's functions. We employ the same basic equations for the quasiclassical matrix Green's functions $\check{g}$ as in Ref. [20]. In the considered one-dimensional geometry and in the diffusive limit they obey a generalized Usadel equation of the form [30,78–81]

$$-iD\partial_x(\check{g}\partial_x\check{g}) + i(\check{\tau}_3\partial_t\check{g} + \partial_{t'}\check{g}\check{\tau}_3) + [\check{\Sigma}, \check{g}]$$
$$= V(t)\check{g} - \check{g}V(t'), \quad (1)$$

where D is the diffusion coefficient, the matrix $\check{\Sigma}$ describes damping, and V is the electric potential. Equation (1) describes the Green's function in the N film and is complemented by the boundary condition [82–84]

$$\check{g}\partial_x\check{g} = \pm\varkappa_{L,R}[\check{g}, \check{\Gamma} \cdot \check{G}\check{\Gamma}]|_{\pm L}, \quad (2)$$

where $\varkappa_{L,R} = 1/R_{\square L,R}\sigma$, $R_{\square L,R}$ is the interface resistance of the left (right) interface per unit area, and σ is the conductivity of the N metal. Note that this boundary condition ensures the

continuity of the current across the S_m/N or S_m/F interfaces. The matrix $\check{\Gamma} = \text{diag}\{\hat{\Gamma}, \hat{\Gamma}\}$ describes the action of the filters. If the filters allow the passage of only electrons with spins aligned parallel to the z axis, then $\hat{\Gamma} = (\mathcal{T}\hat{1} + \mathcal{U}\hat{X}_{33})/\sqrt{2}$ with $\mathcal{T} = \pm\mathcal{U}$ and $\hat{X}_{ij} = \hat{\tau}_i \cdot \hat{\sigma}_j$ for $i,j = 0,1,2,3$, where τ_i and σ_j are Pauli matrices operating in the Gor'kov-Nambu, respectively, spin spaces (τ_0 and σ_0 are 2×2 unity matrices). The probability that an electron with spin up (down) will pass into the N wire is $\mathcal{T}_{\uparrow,\downarrow} \propto \mathcal{T} \pm \mathcal{U}$. We set $\mathcal{U} = s\mathcal{T}$ with $s = \pm 1$ and the coefficients $\mathcal{T}$ and $\mathcal{U}$ are normalized so that $\mathcal{T} = |\mathcal{U}| = 1$. The sign of the factor s determines the orientation of the spins of electrons (parallel or antiparallel to the vector $\mathbf{e}_z$) passing through the filters. In the case of spin-inactive interfaces, $\hat{\Gamma} = \hat{1}$. We have to solve the Usadel equation, Eq. (1), for the Green's function $\check{g}$ in the N film with the boundary condition, Eq. (2), that connects the Green's function $\check{g}$ in the N film with the known Green's functions $\check{G}$ in the reservoirs (please note the use of lowercase and capital letters to denote the Green's functions in different systems). We neglect the inverse proximity effect on the superconducting reservoirs (see Appendix C).

The matrix Green's function $\check{g}$ consists of the retarded (advanced) Green's functions $\hat{g}^{R(A)}$ (diagonal elements) and the Keldysh function $\hat{g}$ (the off-diagonal $\check{g}_{12}$ element) and obeys the normalization condition

$$\check{g} \cdot \check{g} = \check{1}. \tag{3}$$

The Thouless energy $E_{\text{Th}} = D/L^2$ is assumed to be much larger than Δ and V. This assumption means that the function $\check{g}(x)$ is almost constant in space. Integrating Eq. (1) over x and taking into account the boundary condition, Eq. (2), we obtain for the Fourier component $\check{g}(\epsilon, \epsilon')$

$$\epsilon \check{X}_{30} \cdot \check{g} - \check{g} \cdot \check{X}_{30} \epsilon' = iE_0[\check{g}, \check{G}], \tag{4}$$

where $\check{G} \equiv (\check{G}_R + \check{G}_L)/2$, $\check{X}_{30} = \text{diag}\{\hat{X}_{30}, \hat{X}_{30}\}$, and $E_0 = D\varkappa/L$. For simplicity we assume that the interface resistances are equal, $\varkappa_R = \varkappa_L \equiv \varkappa$. The matrices $\check{G}$ describe electrons passing through the filter,

$$\check{G} = \check{\Gamma} \cdot \check{G} \cdot \check{\Gamma}. \tag{5}$$

Matrices $\check{G}$ and $\check{G}$ consist of the retarded (advanced) ($\hat{G}^{R(A)}$) and Keldysh ($\hat{G}$) Green's functions in the reservoirs,

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G} \\ 0 & \hat{G}^A \end{pmatrix}. \tag{6}$$

The retarded (advanced) Green's functions $\hat{G}^{R(A)}$ are given by

$$\check{G}^{R(A)}(t,t') = \hat{S}(t)[G(t-t')\hat{X}_{30} + F(t-t')\hat{X}_{10}]^{R(A)}\hat{S}^\dagger(t'), \tag{7}$$

where $\hat{S}(t) = \exp(\hat{X}_{30}i\chi(t)/2)$ with the phase of the order parameter in the right or left reservoir, respectively, $\chi_{R,L}(t) = \pm[\chi_0 + \chi_\Omega(t)]_{R,L}$. We set $\chi_R(t) = -\chi_L(t) \equiv \chi(t)$. If the reservoirs are BCS singlet superconductors, the Fourier components of the functions $G^{R(A)}(t-t')$ and $F^{R(A)}(t-t')$ are

$$G_0^{R(A)}(\epsilon) = (\epsilon \pm i\gamma)/\zeta^{R(A)}, \tag{8}$$

$$F_0^{R(A)}(\epsilon) = \Delta/\zeta^{R(A)}, \tag{9}$$

with $\zeta^{R(A)} = \sqrt{(\epsilon \pm i\gamma)^2 - \Delta^2}$ and the damping rate in the superconducting reservoirs γ.

In the case of S_m superconductors, the Green's functions $\hat{G}(\omega)$ have a more complicated structure [85]. In the static case ($\chi = \chi_0 = \text{const}$), in the Matsubara representation [$\epsilon = i\omega \equiv i\pi T(2n+1)$] they have the form

$$\hat{G}(\omega) = \hat{S}_0[G_{\omega h}(\hat{X}_{30} + s\hat{X}_{03}) + F_{\omega h}\hat{X}_\perp]\hat{S}_0^\dagger. \tag{10}$$

Here, $\hat{S}_0 = \exp(\hat{X}_{30}i\chi_0/2)$ and

$$G_{\omega h} = \frac{1}{2}\left(\frac{\omega + ih}{\zeta_{\omega+}} + \frac{\omega - ih}{\zeta_{\omega-}}\right) \equiv \text{Re}\left(\frac{\omega + ih}{\zeta_{\omega+}}\right), \tag{11}$$

$$F_{\omega h} = \frac{\Delta}{2}\left(\frac{1}{\zeta_{\omega+}} - \frac{1}{\zeta_{\omega-}}\right) \equiv i\Delta\,\text{Im}\left(\frac{1}{\zeta_{\omega+}}\right), \tag{12}$$

with $\zeta_{\omega\pm} = \sqrt{(\omega \pm ih)^2 + \Delta^2}$. The factor $s = \pm 1$ characterizes the direction of the triplet Cooper pairs with respect to the z axis. The form of the matrix $\hat{X}_\perp$ depends on the direction of the exchange field $\mathbf{h}$ ($\mathbf{h}||\mathbf{e}_x$ or $\mathbf{h}||\mathbf{e}_y$):

$$\hat{X}_\perp = \frac{1}{\sqrt{2}}\begin{cases} \hat{X}_{11} - s\hat{X}_{22}, & \mathbf{h}||\mathbf{e}_x; \\ \hat{X}_{12} + s_L\hat{X}_{21}, & \mathbf{h}||\mathbf{e}_y. \end{cases} \tag{13}$$

We clarify the origin of matrices in this equation. If the exchange field h is parallel to the z axis ($h||\hat{e}_z$), then the part of the triplet condensate Green's function is $\hat{F}_{\omega h} = F_{\omega h}\hat{X}_{13}$, where the function $F_{\omega h}$ is defined in Eq. (12). Rotation around the x or y axis, respectively, with the aid of the rotation matrix $\hat{U} = \cos\alpha/2 + i\hat{X}_{01(2)}\sin\alpha/2$, with $\alpha = \pi/2$, transforms the function $\hat{F}_{\omega h} \to F_{\omega h}\hat{X}_{12}$ or $\hat{F}_{\omega h} \to F_{\omega h}\hat{X}_{11}$. The action of the spin filter transforms these functions into $\hat{\Gamma}\hat{F}_{\omega h}\hat{\Gamma} \to F_{\omega h}(\hat{X}_{12} + s\hat{X}_{21})$ or $\hat{\Gamma}\hat{F}_{\omega h}\hat{\Gamma} \to F_{\omega h}(\hat{X}_{11} - s\hat{X}_{22})$.

We need to find the current I given by the expression (this equation is written in more detailed form in Appendix D)

$$I = (16\kappa R_\square e)^{-1} \int d\bar{\epsilon}\,\{(\check{g}\partial_x\check{g})^K\}_{30}$$

$$= (16R_\square e)^{-1} \int d\bar{\epsilon}\,\{[\check{g}, \check{\Gamma}\check{G}\check{\Gamma}]^K\}_{30} \tag{14}$$

as a response to a small periodic variation of the phase $\chi_\Omega(t) = \chi_\Omega\cos(\Omega t)$ in the presence of a constant phase difference χ_0. Here, $\bar{\epsilon} = (\epsilon + \epsilon')/2$ and we have introduced the notation $\{(\check{g}\partial_x\check{g})^K\}_{ij} \equiv \text{Tr}\{\hat{X}_{ij}(\check{g}\partial_x\check{g})^K\}/4$. Therefore, we have to solve Eq. (4) and to find the function $\check{g}$. First, we consider the case of singlet superconductors.

Unlike a similar system with perfectly penetrable interfaces considered in Refs. [19] and [20], in our case there are barriers at the S/N interfaces so that the interfaces have a finite resistance $R_{\text{if}} \equiv R_\square$. Moreover, we assume that the interface resistance is much higher than the resistance of the N layer, $R_L = 2L/\sigma$, i.e., the inequality

$$R_\square \gg R_L \tag{15}$$

is fulfilled.

III. SINGLET COUPLING

In this section, we consider an S/N/S junction in the absence of spin filters at the interfaces, i.e., we set $\check{\Gamma} = \check{1}$. This means

that the matrices $\hat{G}$ and $\check{G}$ coincide. Therefore, the Josephson coupling is realized through singlet Cooper pairs penetrating the N region due to the proximity effect. First, we analyze the stationary case.

A. Stationary case

In this case, it is convenient to use the Matsubara representation of the Green's functions. The matrices $[\hat{G}_\omega]_{R,L}$ read

$$[\hat{G}_\omega]_{R,L} = [G_\omega \hat{X}_{30} + F_\omega \exp(\pm i\chi_0 \hat{X}_{30}) \cdot \hat{X}_{10}]_{R,L}. \quad (16)$$

The function G_ω is equal to $G_{\omega h}$ at $h = 0$ [see Eq. (11)] and $F_\omega = (\Delta/\omega)G_\omega$.

From Eq. (4) we obtain an equation for the stationary Green's function $\hat{g}_\omega$. One can write this equation in the form

$$[\hat{M}_\omega, \hat{g}_\omega] = 0, \quad (17)$$

where $\hat{M}_\omega = \tilde{\omega}\hat{X}_{30} + \tilde{\Delta}_\omega \hat{X}_{10}$ and $\tilde{\omega} = \omega(1 + E_0/\zeta_\omega)$, $\tilde{\Delta}_\omega = [E_0\Delta/\zeta_\omega]\cos\chi_0$, $\zeta_\omega = \sqrt{\omega^2 + \Delta^2}$, $\tilde{\zeta}_\omega = \sqrt{\tilde{\omega}^2 + \tilde{\Delta}_\omega^2}$. A solution satisfying the normalization condition, Eq. (3), is

$$\hat{g}_\omega = \frac{\hat{M}_\omega}{\tilde{\zeta}_\omega}. \quad (18)$$

The retarded function $\hat{g}^R(\epsilon) = g_\epsilon^R \hat{X}_{30} + f_\epsilon^R \hat{X}_{10}$, which is equal to $\hat{g}_\omega$ at $\omega = -i\epsilon$, determines the density of states (DOS) $\nu(\epsilon) = \mathrm{Re}(g_\epsilon)$. One can easily obtain a transparent formula for $\nu(\epsilon)$ in the limiting cases of large and small E_0.

For large E_0 ($E_0 \gg \Delta$) we have $\tilde{\epsilon} \simeq \epsilon_0/\zeta_0^R$ and $\tilde{\Delta} \simeq (\zeta_0^R)^{-1}\Delta_0\cos\chi_0$ with $\zeta_0^R = \sqrt{(\epsilon + i\gamma_N)^2 - \Delta_0^2\cos^2\chi_0}$. We obtain the standard formula for a BCS superconductor $\nu(\epsilon) = \mathrm{Re}(\epsilon/\sqrt{\epsilon^2 - (\Delta\cos\chi_0)^2})$ with a gap in the quasiparticle spectrum $\Delta|\cos\chi_0|$. In the opposite limit of small E_0 ($E_0 \ll \Delta$), we obtain $\nu(\epsilon) = \mathrm{Re}(\epsilon/\sqrt{\epsilon^2 - (E_0\cos\chi_0)^2})$, that is, the energy gap or subgap $\Delta_{sg} = E_0\cos\chi_0$ is much smaller than Δ [9].

One can easily calculate the dc Josephson current I_J using Eqs. (14) and (18). In the considered static case, the expression for the current can be written in the form

$$I_J = \frac{i\pi T}{2eR_\square} \sum_{\omega \geqslant 0}\{[\hat{g}_\omega, \hat{G}_{\omega R}]\}_{30}. \quad (19)$$

Here, we have used the form of the Keldysh function $\hat{g} = (\hat{g}^R - \hat{g}^A)\tanh(\epsilon\beta)$ [in this case, $\epsilon = \epsilon' = \bar{\epsilon} \equiv (\epsilon + \epsilon')/2$] and transformed the integral in Eq. (14) into the sum over poles of $\tanh(\epsilon\beta)$. Substituting Eqs. (16) and (18) into this equation, we find a standard relation for the Josephson current [12–14],

$$I_J = I_c \sin(2\chi_0), \quad (20)$$

where the critical current of the considered JJ with the Josephson coupling through the singlet condensate equals

$$I_{cS} = \frac{2\pi T}{eR_\square} E_0\Delta^2 \sum_{\omega \geqslant 0} \frac{1}{\tilde{\zeta}_\omega \zeta_\omega^2}. \quad (21)$$

Now we turn to the response of the singlet JJ to an ac voltage V_Ω.

B. Nonstationary case

We have to find the response $\delta\check{g}$ to a phase variation $\chi_\Omega(t) = \chi_\Omega(t)_R = -\chi_\Omega(t)_L$ or to an ac voltage $V_\Omega(t)$ applied to the considered JJ and coupled to $\chi_\Omega(t)$ via the Josephson relation

$$2eV_\Omega(t) = \hbar\partial\chi_\Omega(t)/\partial t. \quad (22)$$

In order to determine a variation of the Green's function in the N region $\delta\hat{g}^{R(A)}$, we linearize Eq. (4) and take into account that in the case of harmonic variation of the phase, $\chi_\Omega(t) = \chi_\Omega \exp(-i\Omega t)$, the functions $\delta\hat{g}^{R(A)}(\epsilon, \epsilon')$ can be represented in the form $\delta\hat{g}^{R(A)}(\epsilon, \epsilon') = \delta\hat{g}^{R(A)} 2\pi\delta(\epsilon - \epsilon' - \Omega)$. For the functions $\delta\hat{g}^{R(A)}$ we obtain

$$[\hat{M}(\epsilon_+) \cdot \delta\hat{g} - \delta\hat{g} \cdot \hat{M}(\epsilon_-)]^{R(A)}$$
$$= iE_0\{\hat{g}_{0+} \cdot \delta\hat{G} - \delta\hat{G} \cdot \hat{g}_{0-}\}^{R(A)}, \quad (23)$$

where $\hat{M}^{R(A)}(\epsilon) = [\epsilon\hat{X}_{30} + iE_B\hat{G}_0]^{R(A)}$ and $\epsilon_\pm^R = \bar{\epsilon} \pm \Omega/2 + i\gamma_N$ (we take into account a damping rate γ_N in the normal metal N). One can represent the left-hand side in the form

$$[\tilde{\zeta}_+\hat{g}_{0+} \cdot \delta\hat{g} - \delta\hat{g} \cdot \hat{g}_{0-}\tilde{\zeta}_-]^{R(A)}$$
$$= iE_0\{\hat{g}_{0+} \cdot \delta\hat{G} - \delta\hat{G} \cdot \hat{g}_{0-}\}^{R(A)}, \quad (24)$$

where the functions $\hat{g}_{0\pm} = \hat{g}_0(\bar{\epsilon} \pm \Omega/2)$ and $\delta\hat{G}^{R(A)}$ are defined in Eqs. (A4)–(A6) (see Appendix A). The normalization condition, Eq. (3), yields

$$(\hat{g}_{0\pm} \cdot \hat{g}_{0\pm})^{R(A)} = \hat{1}, \quad (25)$$

$$[\hat{g}_{0+} \cdot \delta\hat{g} + \delta\hat{g} \cdot \hat{g}_{0-}]^{R(A)} = 0. \quad (26)$$

Using these equations, we obtain a solution of Eq. (24):

$$\delta\hat{g}^{R(A)} = \left[\frac{iE_0}{\tilde{\zeta}_+ + \tilde{\zeta}_-}[\delta\hat{G} - \hat{g}_{0+} \cdot \delta\hat{G} \cdot \hat{g}_{0-}]\right]^{R(A)}. \quad (27)$$

The Keldysh matrix $\delta\hat{g}$ is represented as the sum of regular and anomalous parts, $\delta\hat{g}^{reg}$ and $\delta\hat{g}^{an}$ [30,86],

$$\delta\hat{g} = \delta\hat{g}^{reg} + n(\bar{\epsilon})\hat{g}^{an}, \quad (28)$$

where $n(\bar{\epsilon}, \Omega) = [\tanh(\epsilon_+\beta) - \tanh(\epsilon_-\beta)]$ and

$$\delta\hat{g}^{reg} = \delta\hat{g}^R\tanh(\epsilon_-\beta) - \tanh(\epsilon_+\beta)\delta\hat{g}^A. \quad (29)$$

The anomalous function can be found by using the same simple procedure as in Refs. [20] and [30]. We find

$$\hat{g}^{an} = \frac{iE_0}{\tilde{\zeta}_+^R + \tilde{\zeta}_-^A}[\delta\hat{G}^{an} - \hat{g}_{0+}^R \cdot \delta\hat{G}^{an} \cdot \hat{g}_{0-}^A]. \quad (30)$$

The expression for $\hat{G}^{an}$ is provided in Appendix A, Eq. (A10).

Knowing the variations of the Green's functions, $\delta\hat{g}^{R(A)}$ and $\hat{g}^{an}$, we determine the variation of the current $\delta I_\Omega = I_\Omega \exp(-i\Omega t)$, the Fourier component of which we write as the sum of the regular, I_Ω^{reg}, and anomalous, I_Ω^{an}, parts. The amplitude I_Ω can be written as the sum of the regular, I_Ω^{reg}, and anomalous, I_Ω^{an}, parts,

$$I_\Omega = I_\Omega^{reg} + I_\Omega^{an}. \quad (31)$$

Here,

$$I_\Omega^{\text{reg}} = (16 R_\square e)^{-1} \int d\bar\epsilon \, \{j^R \tanh(\epsilon_- \beta) - j^A \tanh(\epsilon_+ \beta)\}_{30},$$

(32)

with

$$
\begin{aligned}
j^{R(A)} = \{&\hat{X}_{30}[(\hat{g}_{0+}\delta\hat{G}_R - \delta\hat{G}_R\hat{g}_{0-}) \\
&- (\hat{G}_{R0+}\delta\hat{g} - \delta\hat{g}\hat{G}_{R0-})]\}_{30}^{R(A)},
\end{aligned}
$$

(33)

and

$$I_\Omega^{\text{an}} = (16 R_\square e)^{-1} \int d\bar\epsilon \, n(\bar\epsilon) j^{\text{an}},$$

(34)

where the "anomalous current" j^{an} coincides with j^R if the functions $\hat{g}_{0-}^R$, $\delta\hat{G}_R^R$, and G_{0-}^R in Eq. (33) are replaced with $\hat{g}_{0-}^A$, $\delta\hat{G}_R^{\text{an}}$, and G_{0-}^A, respectively.

Using Eqs. (29) and (30) for the matrices in Eqs. (33), we can write $j^{R(A)}$ and j^{an} in terms of the known Green's functions in the reservoirs. We write j^{an} as the sum of two currents,

$$j^{\text{an}} \equiv j_1^{\text{an}} + j_2^{\text{an}},$$

(35)

where $j_{1,2}^{\text{an}}$ correspond to the first (second) term in Eq. (33), respectively. They can be written as follows (see Appendix A):

$$
\begin{aligned}
j_1^{\text{an}} = -i\frac{\chi_\Omega}{2}\big[&(\tilde{g}_{0+}^R - \tilde{g}_{0-}^A)(G_+^R - G_-^A) \\
&- (F_+^R + F_-^A)(\tilde{f}_+^R + \tilde{f}_-^A)\cos\chi_0\big],
\end{aligned}
$$

(36)

$$
\begin{aligned}
j_2^{\text{an}} = \chi_\Omega \frac{E_0}{\tilde{\zeta}_+^R + \tilde{\zeta}_-^A}\big[1 + \tilde{g}_{0+}^R\tilde{g}_{0-}^A + \tilde{f}_+^R\tilde{f}_-^A\big] \\
\times (F_+^R + F_-^A)^2 \sin^2\chi_0.
\end{aligned}
$$

(37)

The currents $j^{R(A)}$ are given by the same formulas if all the functions of the form F_-^A (F_+^R) are replaced with F_-^R (F_+^A).

One can show that the anomalous current leads to an enhancement in the conductance Y_Ω at low frequencies, as it occurs in S/c/S junctions without barriers [19] and in long S/N/S junctions [22,23]. Consider the case of low frequencies ($\Omega \ll \Delta_{\text{sg}}$). The main contribution to the current is due to the first term in Eq. (36) for j_1^{an} and due to the "current" j_2^{an}. We obtain

$$I_\Omega^{\text{an}} = \frac{V_\Omega}{2R_\square}C_\square + I_2^{\text{an}},$$

(38)

where the temperature-dependent function $C_\square$ is

$$C_\square = \int_\Delta^\infty d(\epsilon\beta)\frac{\epsilon\bar\epsilon}{\zeta(\epsilon)\tilde{\zeta}(\epsilon)\cosh^2(\epsilon\beta)}.$$

(39)

The first term determines the admittance of the system, $1/(2R_\square)$, in the normal state. The second term in Eq. (38) is given by

$$
\begin{aligned}
I_2^{\text{an}} = -\frac{\hbar\Omega E_0}{16 R_\square e}\chi_\Omega \int d\bar\epsilon \, \beta \frac{1 + \tilde{g}_{0+}^R\tilde{g}_{0-}^A + \tilde{f}_+^R\tilde{f}_-^A}{\tilde{\zeta}_+^R + \tilde{\zeta}_-^A} \\
\times \left(\frac{F_+^R + F_-^A}{\cosh(\bar\epsilon\beta)}\right)^2 \sin^2\chi_0.
\end{aligned}
$$

(40)

We take into account that in the interval $\Delta_{\text{sg}} \equiv E_0|\cos\chi_0| \leqslant \bar\epsilon \ll \Delta$ the functions $F_+^R = F_-^A \simeq -i$ and
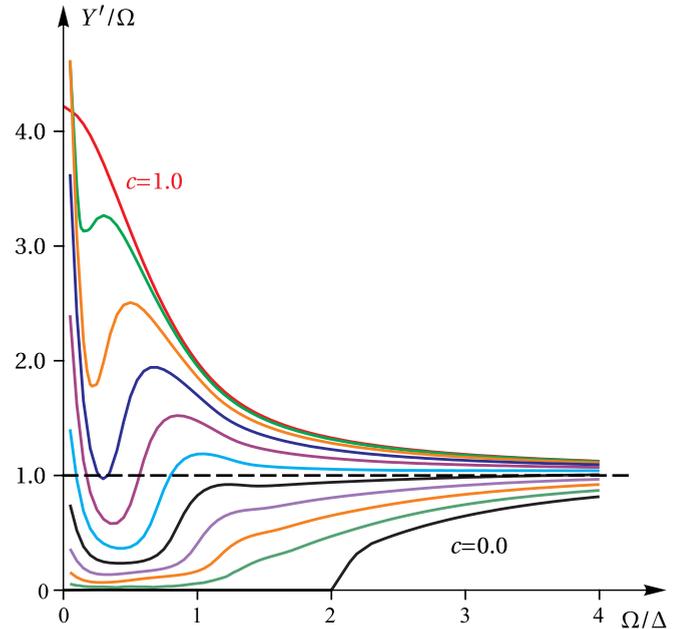
FIG. 2. Frequency dependence of the real part of the admittance in the case of an S/N/S Josephson junction (scaled to its value in the normal state). Here, $c = \cos\chi_0$ and the values of c are incremented by 0.1 starting with 0.0, at the bottom right, and ending with 1.0, at the top left. Note that at small values of c the admittance is a monotonous function, whereas for larger c a maximum emerges.

$\tilde{f}_+^R \simeq -\tilde{f}_-^A \simeq \tilde{f}^R$ so that $1 + \tilde{g}_{0+}^R\tilde{g}_{0-}^A + \tilde{f}_+^R\tilde{f}_-^A \simeq -2(\tilde{f}^R)^2 = -2\tilde{\Delta}^2/\tilde{\zeta}^2$, and $\tilde{\zeta}_+^R + \tilde{\zeta}_-^A \simeq (\Omega + 2i\gamma_N)\bar\epsilon/\tilde{\zeta}$ with $\tilde{\zeta} = \sqrt{\bar\epsilon^2 - \tilde{\Delta}^2}$. Thus, we obtain

$$eR_\square I_2^{\text{an}} = \frac{E_0^2 \sin^2(2\chi_0)}{8T|\cos\chi_0|}\frac{2\gamma_N + i\Omega}{(2\gamma_N)^2 + \Omega^2}J(\chi_0),$$

(41)

where the integral $J(\chi_0)$ is

$$
\begin{aligned}
J(\chi_0) &= \int_1^\infty \frac{dx}{\cosh^2(x\tilde\beta)x\sqrt{x^2 - 1^2}} \\
&= \begin{cases} \pi/2, & \Delta_{\text{sg}} \ll 2T, \\ 2\sqrt{2T/\Delta_{\text{sg}}}\exp(-\Delta_{\text{sg}}/T), & 2T \ll \Delta_{\text{sg}} \ll \Delta, \end{cases}
\end{aligned}
$$

(42)

where $\tilde\beta = \Delta_{\text{sg}}/2T$ and $\Delta_{\text{sg}} \equiv E_0|\cos\chi_0|$.

This means that the real part of the admittance reaches a maximum at $\Omega = 0$ and the magnitude of this maximum is determined by the damping in the spectrum γ_N and temperature T. At temperatures T below the subgap Δ_{sg} the contribution of the anomalous current is exponentially small. In Fig. 2 we depict the frequency dependence of the real part of the admittance $Y_\Omega' = \text{Re}\{I_\Omega/V_\Omega\}$ at different values of χ_0.

IV. TRIPLET COUPLING

In this section, we consider a system with magnetic superconductors S_m as reservoirs. As noted above, these superconductors may be bulk superconductors S covered with a thin layer of a weak ferromagnet F with an exchange field **h** oriented in the x or y direction [see Fig. 1(b)]. The triplet

Cooper pairs that penetrate into the ferromagnetic layers have the total spin lying in the $y-z$ or the $x-z$ plane. The chosen form of the matrix element of tunneling through the filters Fl [see Eq. (2)], $\check{\Gamma}$, implies that only triplet Cooper pairs fully polarized in the z direction can pass through the filters. Triplet Cooper pairs penetrating into the N film from the left and from the right differ not only in their spin polarization, but also in their chiralities. The chiralities are not equal if the **h** vectors in the left and right reservoirs have different orientations, **h**$||$**e**$_x$ or **h**$||$**e**$_y$. Again, we consider first a stationary case.

A. Stationary case

We take the element $\{11\}$ of Eq. (4), that is, the equation for the retarded Green's function $\hat{g}^R$. First, we write this equation for a stationary case in the Matsubara representation, i.e., we set $\epsilon = i\omega$ and obtain

$$\omega[\hat{X}_{30}, \hat{g}_\omega] = E_0[\hat{g}_\omega, \hat{G}], \tag{43}$$

where $\hat{G} = \hat{G}_{||} + \hat{G}_\perp$ with $\hat{G}_{||} = G_{\omega h}(\hat{X}_{30} + s\hat{X}_{03})$, $s = (s_R + s_L)/2$, and the condensate Green's function

$$\hat{G}_\perp = 2^{-1}F_{\omega h}[(\hat{X}_{\perp R} + \hat{X}_{\perp L}) \cos \chi_0$$
$$+ i\hat{X}_{30}(\hat{X}_{\perp R} - \hat{X}_{\perp L}) \sin \chi_0]. \tag{44}$$

The functions $G_{\omega h}$ and $F_{\omega h}$ are defined as $G_{\omega h} = G_{\omega+}$ and $F_{\omega h} = F_{\omega-}$ in Eqs. (11) and (12).

In the case of equal chiralities we have $\hat{X}_{\perp R} = \hat{X}_{\perp L}$. If the chiralities of the left and right S_m are different, we have $\hat{X}_{\perp R} \neq \hat{X}_{\perp L}$, where the form of the matrices $\hat{X}_{\perp R,L}$ is given by Eq. (13).

First, we consider the case of identical chiralities, i.e., $\hat{X}_{\perp R} = \hat{X}_{\perp L}$, and polarizations ($s_R = s_L$).

1. Identical chiralities

In this case, the second term in Eq. (44) vanishes. If the polarizations on the left and on the right are opposite, the Josephson current is 0 [85]. A solution of Eq. (43) is searched in the form

$$\hat{g}_\omega = a_{30}\hat{X}_{30} + a_{03}\hat{X}_{03} + b_\perp \hat{X}_{\perp R}. \tag{45}$$

From Eq. (43) we find

$$(\omega + 2E_0 G_{\omega h})b_\perp = E_0 F_{\omega h}(a_{30} + sa_{03}) \cos \chi_0. \tag{46}$$

The normalization condition $\hat{g}_\omega \cdot \hat{g}_\omega = \hat{1}$ yields

$$a_{30}^2 + a_{03}^2 + b_\perp^2 = 1, \tag{47}$$

$$2a_{30}a_{03} + sb_\perp = 0. \tag{48}$$

We find from Eqs. (46)–(48)

$$b_\perp = \frac{R}{\sqrt{1 + 2R^2}} \tag{49}$$

and

$$a_{30} = 1 + sa_{03}, \tag{50}$$

$$sa_{03} = -\frac{R^2}{A[1 + A]}, \tag{51}$$

(a)

(b)
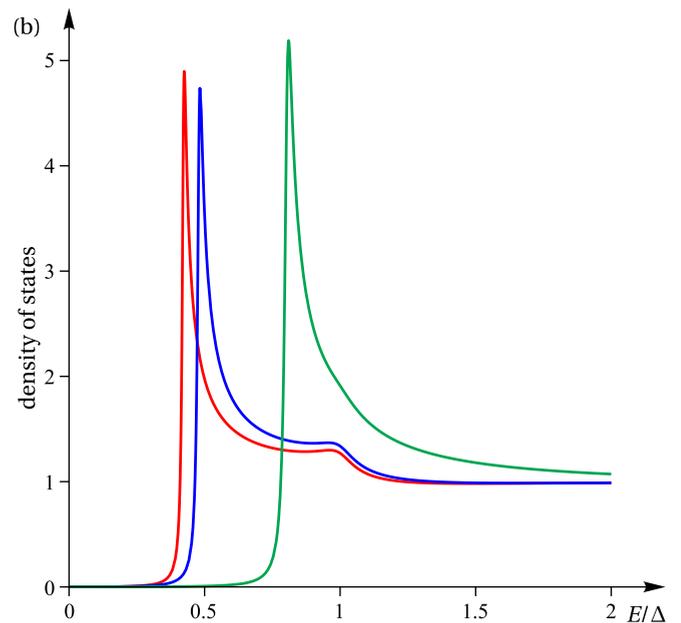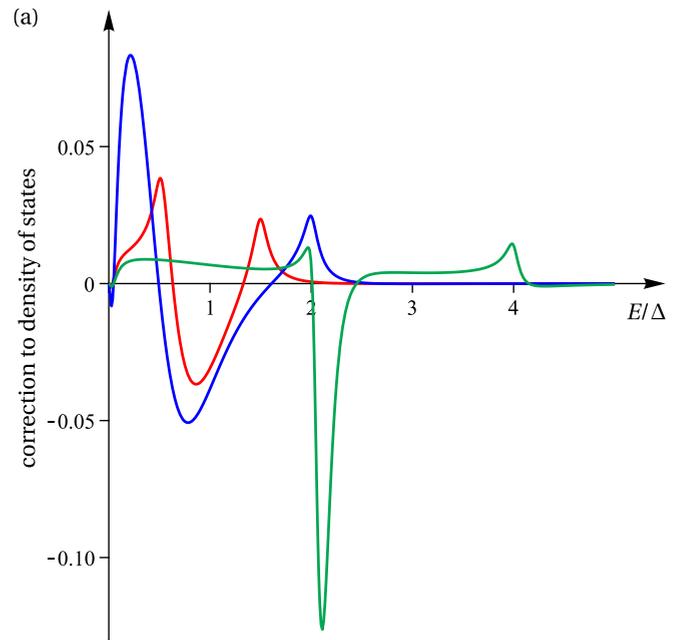
FIG. 3. The energy dependence of the DOS $\nu(\epsilon) = \mathrm{Re}(g_\epsilon)$ for (a) $S_m/N/S_m$ and (b) S/N/S Josephson junctions. One can see that the correction to the DOS of the normal state is small. We set the value of the cosine to $c = 0.8$. (a) The case of an $S_m/N/S_m$ contact; the curves correspond to $h = 0.5\Delta$ (red), $h = 1.0\Delta$ (blue), and $h = 3.0\Delta$ (green). (b) The case of an $S_m/N/S_m$ contact; we vary the parameter E_0 and the curves correspond to $E_0 = 0.8\Delta$ (red), $E_0 = 1.0\Delta$ (blue), and $E_0 = 5.0\Delta$ (green).

where $A = \sqrt{1 + 2R^2}$ and $R = (\omega + 2E_0 G_{\omega h})^{-1}$ $(E_0 F_{\omega h}) \cos \chi_0$.

The parameter sa_{03} determines a correction to the DOS of the N region $\delta\nu(\epsilon)$ [$\delta\nu(\epsilon) = sa_{03}(\omega)$ at $\omega = -i\epsilon$] due to the proximity effect. In Fig. 3 we plot the DOS for $E_0/\Delta = 1$, $\cos \chi_0 = 0.8$, and different h values. We see that the correction to the DOS of the normal metal [$\nu_N(\epsilon) = 1$] is small, and

consequently, there is no gap in the spectrum [33,87]. For comparison, we plot also the DOS of the considered singlet JJ where there is a gap Δ_{sg} in the excitation spectrum.

Expression (49) allows us to find the Josephson current I_J,

$$I_J = i \frac{\pi T}{e R_\square} \sum_{\omega \geqslant 0} \{[\hat{g}_\omega, \hat{G}_{\omega R}]\}_{30}, \tag{52}$$

where

$$\hat{G}_{\omega R} = G_{\omega h}(\hat{X}_{30} + s \hat{X}_{03}) + F_{\omega h}(\cos \chi_0 + i \hat{X}_{30} \sin \chi_0)\hat{X}_{\perp R}. \tag{53}$$

We find

$$I_{J,a} = I_{T,a} \sin(2\chi_0), \tag{54}$$

$$I_{T,a} = \frac{\pi T}{2e R_\square} E_0 \sum_{n \geqslant 0}^{\infty} \frac{F_{\omega h}^2}{\sqrt{(\omega + 2E_0 G_{\omega h})^2 + 2(E_0 F_{\omega h} \cos \chi_0)^2}}. \tag{55}$$

When the total spins of triplet Cooper pairs stemming from the right (left) reservoirs are oriented in the same direction, Eq. (55) determines the critical currents of a π Josephson junction since $F_{\omega h}$ is an imaginary quantity so that $F_{\omega h}^2 < 0$. In Fig. 4 we plot the temperature and h dependence of the critical current $I_{T,a}$. We see that the temperature dependence of the critical current $I_c(T)$ is not monotonous for some values of the exchange field h [62,63]. As a function of h, the absolute value of the critical current $|I_T|$ increases with h from 0 and reaches a maximum at a finite value of h_m [Fig. 4(b)].

2. Different chiralities

In the case of different chiralities but equal polarizations the solution is found in a similar way. We look for the matrix $\hat{g}_\omega$ in the form

$$\hat{g}_\omega = a_{30}\hat{X}_{30} + a_{03}\hat{X}_{03} + B_{\perp R}\hat{X}_{\perp R} + B_{\perp L}\hat{X}_{\perp L}. \tag{56}$$

We find $B_{\perp R} = B_{\perp L} \equiv B_\perp$ with

$$B_\perp = \frac{R(\cos \chi_0 - s \sin \chi_0)}{\sqrt{1 + 4R^2(\cos \chi_0 - s \sin \chi_0)^2}}. \tag{57}$$

The current is determined by Eq. (52) and we find

$$I_{J,b} = I_{Tb} \cos(2\chi_0), \tag{58}$$

$$I_{T,b} = s \frac{\pi T}{2e R_\square} \sum_{n \geqslant 0}^{\infty}$$
$$\times \frac{F_{\omega h}^2}{\sqrt{(\omega + 2E_0 G_{\omega h})^2 + (E_0 F_{\omega h})^2(\cos 2\chi_0 - s \sin 2\chi_0)}}. \tag{59}$$

In this case, the Josephson current appears even in the absence of a phase difference (the so-called anomalous current) and on the polarization of triplet Cooper pairs [58,63,72,85,88–93]. One can easily prove that the Josephson current is 0 if the triplet components injected from the right and left S_m have opposite spin orientations ($s_R = -s_L = \pm 1$) [85,91]. It is interesting that at a given phase difference
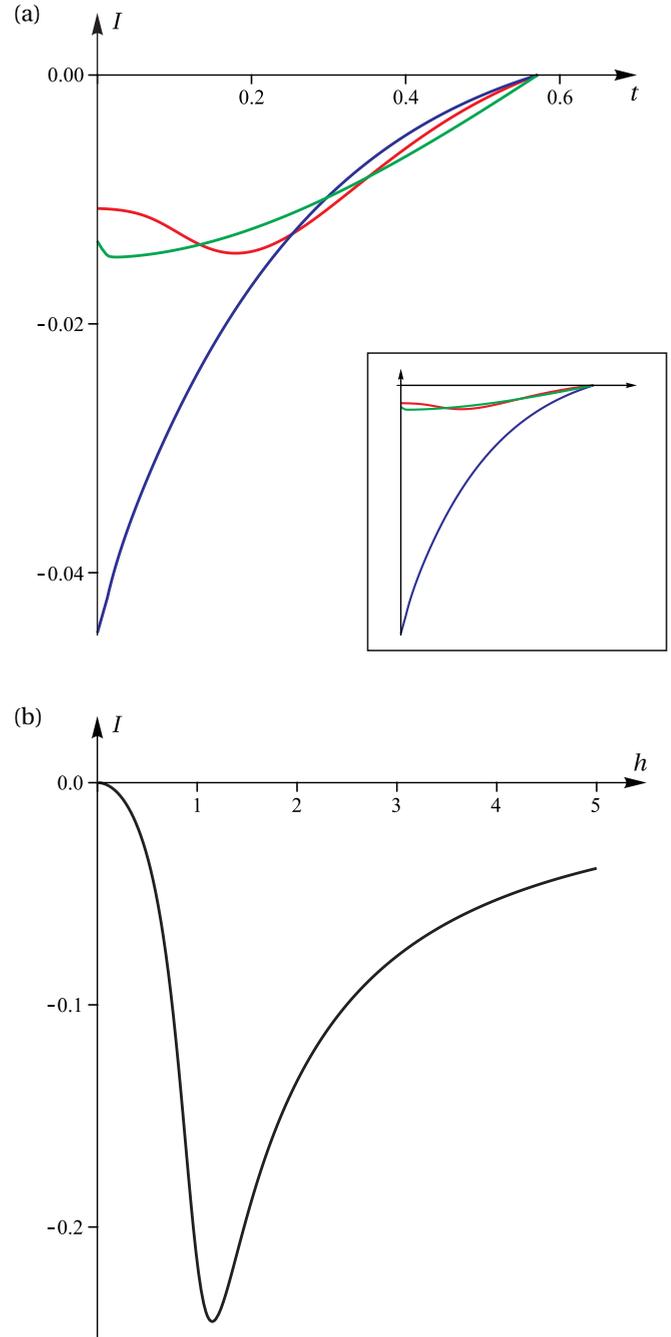
FIG. 4. The Josephson critical current $I_{T,a}$ for JJs with triplet coupling as a function of (a) the temperature $t = T/\Delta$ and (b) the exchange field h. The temperature dependence of $I_{T,a}$ is not monotonous at some values of the exchange field h [62,63]. In plotting the temperature dependence in (a) we set $E_0 = 0.5\Delta$, $c = 0.5$, and the curves correspond to $h = 0.5\Delta$ (red), $h = 1.0\Delta$ (blue), and $h = 3.0\Delta$ (green). Note that the blue curve, corresponding to $h = 1.0\Delta$, is scaled by the factor of 0.3 and the unscaled curve is shown in the inset. Plotting the h dependence we set $E_0 = 0.5\Delta$, $c = 0.5$, and $T = 0.1\Delta$.

$2\chi_0$, the direction of the Josephson current depends on the spin polarization direction of triplet Cooper pairs, i.e., it is determined by the sign of s. This situation is analogous to the case of superconductors with a special type of spin-orbit

interaction where the direction of electron motion depends on the spin polarization [94,95].

As obtained in Ref. [91] (see also Ref. [93]), the quasiclassical approach leads to this correct result if the spin selection is realized through filters. If the filters are replaced by strong ferromagnets, the triplet Cooper pairs described in the quasiclassical approximation penetrate into the N region independently of the spin direction and the current I_T is not 0 for either spin direction.

Consider now a linear response of the system to an applied low ac voltage $\delta V(t)$.

B. Nonstationary case

In the case of triplet coupling considered in this section we are interested in the low-frequency range where only the anomalous Green's function $\hat{g}^{\mathrm{an}}$ are essential. We assume also that the chiralities of the triplet components penetrating the N region are identical, i.e., $\hat{X}_{\perp R} = \hat{X}_{\perp L} = \hat{X}_{11} - s\hat{X}_{22}$.

The anomalous Green's function $\hat{g}^{\mathrm{an}}$ obeys the linearized Eq. (6),

$$
\zeta_+^R \hat{g}_n^R \hat{g}^{\mathrm{an}} - \hat{g}^{\mathrm{an}} \hat{g}_n^A \zeta_-^A
$$
$$
= i E_0 \big[\hat{g}_0^R \hat{G}^{\mathrm{an}} - \hat{G}^{\mathrm{an}} \hat{g}_0^A - \hat{G}_+^R \hat{g}^{\mathrm{an}} + \hat{g}^{\mathrm{an}} \hat{G}_-^A \big], \quad (60)
$$

where the matrices $\hat{g}_n^{R(A)} = \pm \hat{X}_{30}$ are the matrix Green's functions in the N region in the absence of the proximity effect, $\zeta^{R(A)}(\bar{\epsilon}) = \bar{\epsilon} \pm i\gamma_N$, that is, we take into account the damping γ_N, $\epsilon_\pm = \bar{\epsilon} \pm \Omega/2$, and Ω is the frequency of the phase variation $\chi_\Omega(t) = \chi_\Omega \cos(\Omega t)$. The matrices $\hat{g}_0^{R(A)}$ describe the N (F) wire in the stationary state. The obtained results for the N region remain valid also for the F region if the magnetization vector $\mathbf{M}$ is oriented along the z axis.

The method of finding a solution for a nonstationary equation used in Sec. III, is not applicable directly to the case of triplet coupling. We employ here a perturbation method regarding E_0 as a small parameter and representing $\hat{g}^{\mathrm{an}}$ as a series in the powers of E_0, i.e., $\hat{g}^{\mathrm{an}} = \hat{g}_1^{\mathrm{an}} + \hat{g}_2^{\mathrm{an}} + \hat{g}_3^{\mathrm{an}} + \cdots$. The anomalous current I_Ω^{an} is determined by the third-order Green's function $\hat{g}_3^{\mathrm{an}}$ [see Eqs. (B19)–(B21) in Appendix B]. We calculate the "spectral current" j_3^{an} with the aid of $\hat{g}_3^{\mathrm{an}}$. It has the form

$$
j_3^{\mathrm{an}} = -\big\{ \hat{G}_{R+}^R \cdot \hat{g}_{3\perp}^{\mathrm{an}} - \hat{g}_{3\perp}^{\mathrm{an}} \cdot \hat{G}_{R-}^A \big\}_{30}
$$
$$
= \frac{\chi_\Omega}{4} \frac{E_0^3 (\cos \chi_0 \sin \chi_0)^2}{\zeta_{0+}^R + \zeta_{0-}^A} \big(F_{T+}^R + F_{T-}^A \big)^2 B(\epsilon), \quad (61)
$$

where the functions $F_{T\pm}^{R(A)}$ are defined in Eq. (B2) and the function B in Eq. (B21). The anomalous part of the current is

$$
I_\Omega^{\mathrm{an}} = \frac{1}{16\pi e R_\square} \int d\bar{\epsilon} \, n(\bar{\epsilon}) j_{3\perp}^{\mathrm{an}}. \quad (62)
$$

We consider limiting cases of large and small exchange field h.

(a) $h \gg \Delta$. In this case,

$$
F_T^{R(A)} = \frac{\Delta}{2} \bigg[\frac{1}{\zeta(\epsilon+h)} - \frac{1}{\zeta(\epsilon-h)} \bigg]^{R(A)} \simeq \mp \frac{\Delta\epsilon}{h^2}. \quad (63)
$$

Therefore, we have $F_{T+}^R + F_{T-}^A \simeq -\Delta\Omega/h^2$. Calculating the integral in Eq. (62) at $\{T, E_0\} \ll \Delta$, we find for the admittance

$$
Y_\perp^{\mathrm{an}}(\Omega) R_\square = -\frac{E_0 \sin^2 2\chi_0}{32\hbar(-i\Omega + \gamma)} \bigg(\frac{E_0}{\Delta} \bigg)^2 \bigg(\frac{\hbar\Omega}{\Delta} \bigg)^2 \bigg(\frac{\Delta}{h} \bigg)^8. \quad (64)
$$

We see that there is no singularity in the admittance at small Ω. Since the condensate functions $F_T^{R/(A)}$ are small in this case, the anomalous contribution makes a small contribution to the admittance in the normal state $(1/2R_\square)$.

(b) $h \ll \Delta$. In this case, the admittance is

$$
Y(\Omega) R_\square = -\frac{2E_0 \sin^2 2\chi_0}{\hbar(-i\Omega + \gamma)} \bigg(\frac{E_0}{\Delta} \bigg)^2 \bigg(\frac{T}{\Delta} \bigg)^2 \bigg(\frac{h}{\Delta} \bigg)^4 c_1, \quad (65)
$$

where $c_1 = \int_0^\infty dx \, (x/\cosh x)^2 \approx 0.82$.

A singularity exists in this case, but the amplitude of the maximum of $Y'(\Omega)$ at $\Omega \to 0$ is rather small, as the parameters (h/Δ) and (E_0/Δ) are small.

V. CONCLUSIONS

In this paper, we have analyzed the static and dynamic properties of S/N/S and S_m/N(F)/S_m Josephson weak links. In particular, the dc Josephson current I_J was calculated as a function of the temperature and of the parameter $E_0/\Delta(0) = [E_{\mathrm{Th}}/\Delta(0)][R_L/R_\square]$, where $E_{\mathrm{Th}} = D/(2L)^2$ is the Thouless energy and $R_L/R_\square$ is the ratio of the resistance R_L of the N film (or wire) with dimension L to the interface resistance (per unit area) $R_\square$. The factor $R_L/R_\square$ is assumed to be small, but $E_{\mathrm{Th}}/\Delta(0)$ large, so that their product can be arbitrary. The coupling in the S/N/S and S_m/N/S_m junctions is realized through the singlet and fully polarized triplet Cooper pairs, respectively. In this case, the term in the Hamiltonian $\mathbf{h}_{\mathrm{str}}\hat{X}_{33}$, which describes the action of an exchange field $\mathbf{h}_{\mathrm{str}}$ on the spins of electrons, commutes with matrices in Eq. (13) related to the triplet Cooper pairs. Therefore, for these Cooper pairs (long-range component) it doe not matter whether the bridge consists of a normal metal or of a ferromagnet.

The superconducting magnetic reservoirs S_m consist of conventional BCS superconductors covered with a thin F layer with exchange field $\mathbf{h}$. Spin filters at S_m/N interfaces let pass only Cooper pairs with the spin oriented along the z axis. The Josephson current I_J is 0 if Cooper pairs penetrating the N (or F) region from the left and right reservoirs have opposite polarizations. The form of the dependence of I_J on the phase difference $2\chi_0$ is determined not only by the relative polarization of triplet Cooper pairs, but also by the so-called chiralities, that is, by the mutual directions of the vectors $\mathbf{h}_{R/L}$ in the right and left S_m reservoirs. In the case of identical chiralities $(\mathbf{h}_R\mathbf{h}_L/h_R h_L = 1)$, the Josephson current has the standard form $I_J = I_{T,a}\sin(2\chi_0)$, whereas in the case of different chiralities $(\mathbf{h}_R\mathbf{h}_L = 0)$, the phase difference dependence, $I_J = I_{T,b}\cos(2\chi_0)$, is rather unusual. This means that in the second case the Josephson current flows in the absence of a phase difference [58,63,72,85,89–91,93].

Interestingly, the current $I_{T,b}$ changes sign by inversion of the polarization direction, i.e., the direction of the Josephson current at a given phase difference $(2\chi_0)$ is determined by the polarization of the triplet Cooper pairs. This means that one can change the direction of the Josephson current by

reversing the polarization of the spin filters. This phenomenon, to some extent, is similar to the spin-orbit interaction when the direction of electron motion depends on its spin polarization [94,95]. On the other hand, by changing the direction of the $\mathbf{h}_{\mathrm{R,L}}$ vectors at $\chi_0 = 0$, one can switch the current I_{J} on or off since the chirality is changed. Therefore, one can control the charge current by switching the polarization of the weak ferromagnet $\mathrm{F_w}$.

We have also calculated the admittance $Y(\Omega)$ for both singlet and triplet JJs in the presence of a dc Josephson current. The dependence of $Y(\Omega)$ for the singlet JJs has been found at all frequencies. At low temperatures T, the real part of the admittance $Y'(\Omega)$ starts to increase at $\Omega \geqslant \Delta_{\mathrm{sg}}$ but shows a sharp peak at low frequencies if T is not too low. The anomalous behavior of $Y'(\Omega)$ is related to the presence of a gap and to the contribution of quasiparticles in the energy interval $\Delta_{\mathrm{sg}} < \epsilon < \Delta$. Such anomalous behavior of $Y'(\Omega)$ was shown to be absent in triplet JJs with fully polarized Cooper pairs because there is no gap in these JJs. Thus, measuring the real part of the admittance $Y'(\Omega)$ at low frequencies, one can extract useful information about the odd-frequency triplet component in $\mathrm{S_m/N/S_m}$ Josephson weak links.

APPENDIX A: SINGLET JOSEPHSON JUNCTION

The Green's functions in the right (left) reservoirs, $\hat{G}_{\mathrm{R,L}}$, in the stationary case read

$$\hat{G}_{\mathrm{R,L}}^{R(A)} = [G_0 \hat{X}_{30} + i F_0 (\cos \chi_0 \pm i \hat{X}_{30} \sin \chi_0)] \hat{X}_{10}^{R(A)}, \quad \text{(A1)}$$

where $2\chi_0$ is the phase difference between superconducting reservoirs. The Green's function in Eq. (23) is defined as follows:

$$\hat{G}^{R(A)} = (1/2)(\hat{G}_{\mathrm{R}} + \hat{G}_{\mathrm{L}})^{R(A)}. \quad \text{(A2)}$$

With the help of Eq. (A1) this equation can be written as

$$\hat{G}^{R(A)} = [G_0 \hat{X}_{30} + i F_0 \cos \chi_0 \hat{X}_{10}]^{R(A)}. \quad \text{(A3)}$$

The variations of the Green's functions $\delta \hat{G}_{\mathrm{R,L}}^{R(A)}$ due to the ac phase χ_Ω are

$$\delta \hat{G}_{\mathrm{R,L}}^{R(A)} = \delta \{ \exp(i \chi_\Omega \hat{X}_{30}/2)[G_0 \hat{X}_{30} + i F_0(\cos \chi_0 \\ \pm i \hat{X}_{30} \sin \chi_0) \hat{X}_{10}]^{R(A)} \exp(-i \chi_\Omega \hat{X}_{30}/2) \} \quad \text{(A4)}$$

or

$$\delta \hat{G}_{\mathrm{R,L}}^{R(A)} = \frac{i \chi_\Omega}{2} [(G_{0-} - G_{0+}) \hat{1} + i(F_{0+} + F_{0-}) \\ \times (\hat{X}_{30} \cos \chi_0 \pm i \hat{1} \sin \chi_0) \hat{X}_{10}]^{R(A)}, \quad \text{(A5)}$$

so that

$$\delta \hat{G}^{R(A)} = -\frac{i \chi_\Omega}{2} [(F_{0+} + F_{0-})^{R(A)} \sin \chi_0] \hat{X}_{10}. \quad \text{(A6)}$$

The anomalous Green's function $\hat{g}^{\mathrm{an}}$ is determined by Eq. (30), where the functions $\hat{g}_0^{R(A)}$ can be easily obtained from Eq. (17),

$$\hat{g}_0^{R(A)}(\epsilon) = [\tilde{g}(\epsilon) \hat{X}_{30} + i \tilde{f}(\epsilon) \hat{X}_{10}]^{R(A)}, \quad \text{(A7)}$$

where $\tilde{g} = \tilde{\epsilon}/\tilde{\zeta}(\tilde{\epsilon})$, $\tilde{f} = \tilde{\Delta}^2(\tilde{\epsilon})/\tilde{\zeta}(\tilde{\epsilon})$, $\tilde{\zeta}(\tilde{\epsilon}) = \sqrt{\tilde{\epsilon}^2 - \tilde{\Delta}^2(\tilde{\epsilon})}$, $\tilde{\epsilon} = \epsilon[1 + i E_0/\zeta(\epsilon)]$, $\tilde{\Delta}(\epsilon) = [i E_0 \Delta/\zeta(\epsilon)] \cos \chi_0$ with

$\zeta(\epsilon) = \sqrt{\epsilon^2 - \Delta^2}$. The anomalous Green's function in reservoirs $\hat{G}^{\mathrm{an}}$ is defined according to

$$(T_+ - T_-)\hat{G}^{\mathrm{an}} = \delta \hat{G} - \delta \hat{G}^R T_- + T_+ \delta \hat{G}^A, \quad \text{(A8)}$$

where $T_\pm \equiv \tanh(\epsilon_\pm \beta)$, and the matrices $\delta \hat{G}$ and $\delta \hat{G}^{R(A)}$ are variations of the Keldysh and retarded (advanced) Green's functions in the presence of the ac phase variation $\chi_\Omega = (\chi_\Omega)_{\mathrm{R}} = -(\chi_\Omega)_{\mathrm{L}}$. The matrix $\hat{G}_{\mathrm{R,L}}^{\mathrm{an}}(\epsilon, \epsilon')$ can be represented in the form $\hat{G}_{\mathrm{R,L}}^{\mathrm{an}}(\epsilon, \epsilon') = \hat{G}_{\mathrm{R,L}}^{an} 2\pi \delta(\epsilon - \epsilon' - \Omega)$. For the function $\hat{G}_{\mathrm{R,L}}^{\mathrm{an}}$ we obtain

$$\hat{G}_{\mathrm{R,L}}^{\mathrm{an}} = \pm \frac{i \chi_\Omega}{2} \{ \hat{X}_{30} \cdot [(\hat{G}_-^R - \hat{G}_-^A)T_- - \hat{G}_-^R T_- + T_+ \hat{G}_-^A] \\ - [(\hat{G}_+^R - \hat{G}_+^A)T_+ - \hat{G}_+^R T_- + T_+ \hat{G}_+^A] \cdot \hat{X}_{30} \}. \quad \text{(A9)}$$

Here, the matrices $\hat{G}_\pm^{R(A)} \equiv \hat{G}_0^{R(A)}(\epsilon_\pm)$ are given in Eqs. (9) and (A2). For the matrix $\hat{G}^{\mathrm{an}} = (\hat{G}_{\mathrm{R}}^{\mathrm{an}} + \hat{G}_{\mathrm{L}}^{\mathrm{an}})/2$ we get from Eq. (A9)

$$\hat{G}^{\mathrm{an}} = -\frac{i \chi_\Omega}{2} (F_+^R + F_-^A) \hat{X}_{10} \sin \chi_0. \quad \text{(A10)}$$

The functions $F_+^R = F_0^R(\epsilon_\pm)$ and $F_0^R(\epsilon)$ are defined in Eq. (9). Using Eqs. (A7) and (A10), we obtain for the anomalous Green's function $\hat{g}^{\mathrm{an}}$

$$\hat{g}^{\mathrm{an}} = \frac{\chi_\Omega E_0}{2(\tilde{\zeta}_+^R + \tilde{\zeta}_-^A)} (F_+^R + F_-^A) \\ \times [1 + \tilde{g}_+ \tilde{g}_- + \tilde{f}_+ \tilde{f}_-] \hat{X}_{10} \sin \chi_0. \quad \text{(A11)}$$

Using Eqs. (A1), (A7), and (A11), we obtain expression (34) for the current, with j_1^{an} and j_2^{an} defined in Eqs. (36) and (37).

APPENDIX B: TRIPLET JOSEPHSON JUNCTION. PERTURBATIVE APPROACH

1. Stationary case

We consider the case of a low energy E_0 ($E_0 \ll \Delta$) and obtain corrections $\delta \hat{g}_0^{R(A)}$ to the stationary Green's functions $\hat{g}_0^{R(A)} \equiv \pm \hat{X}_{30}$ as well as the anomalous function $\hat{g}^{\mathrm{an}}$ in a nonstationary case. Consider first the stationary case and represent the matrix $\hat{g}_{\mathrm{st}}^{R(A)}$ in the form of an expansion in powers of E_0, i.e., $\hat{g}_{\mathrm{st}}^{R(A)} = [\hat{g}_0 + \delta_1 \hat{g} + \delta_2 \hat{g} + \cdots]^{R(A)}$.

Equation (43) can be written for $\hat{g}_{\mathrm{st}}^{R(A)}$ in the form [for brevity we drop the indices $R(A)$]

$$\epsilon[\hat{X}_{30}, \hat{g}_{\mathrm{st}}] = i E_0[\hat{g}_{\mathrm{st}}, \hat{G}_0], \quad \text{(B1)}$$

where the matrix $\hat{G}_0$ is

$$\hat{G}_0 = G_{\mathrm{T}} \hat{X}_{30} + i F_{\mathrm{T}}(\cos \chi_0 + i \sin \chi_0 \hat{X}_{30}) \hat{X}_\perp, \quad \text{(B2)}$$

with $G_{\mathrm{T}} = 2^{-1}[G_0(\epsilon + h) + G_0(\epsilon - h)]$, $F_{\mathrm{T}} = 2^{-1}[F_0(\epsilon + h) - F_0(\epsilon - h)]$, and the functions $G_0^{R(A)}$ and $F_0^{R(A)}$ are defined in Eq. (9). The first-order correction obeys the equation

$$\epsilon[\hat{X}_{30}, \delta_1 \hat{g}] = i E_0[\hat{g}_0, \hat{G}_0], \quad \text{(B3)}$$

and the solution is

$$\delta_1 \hat{g} = -E_0 \cos \chi_0 \left(\frac{F_{\mathrm{T}}}{\zeta_0} \right) \hat{X}_\perp, \quad \text{(B4)}$$

with $\zeta_0^{R(A)} = \pm(\epsilon \pm i\gamma)$. The second-order correction $\delta_2\hat{g}$ satisfies the equation

$$\epsilon[\hat{X}_{30}, \delta_2\hat{g}] = i E_0[\delta_1\hat{g}, \hat{G}_0]. \tag{B5}$$

The normalization conditione yields

$$\hat{g}_0 \cdot \delta_2\hat{g} + \delta_2\hat{g} \cdot \hat{g}_0 + (\delta_1\hat{g})^2 = 0. \tag{B6}$$

The solution satisfying Eqs. (B5) and (B6) is represented in the form

$$\delta_2\hat{g} = a\hat{X}_{30} + b\hat{X}_\perp, \tag{B7}$$

with

$$a = -\frac{(E_0 \cos\chi_0)^2}{2}\left(\frac{F}{\zeta_0\zeta_\Delta}\right)^2, \tag{B8}$$

$$b = i E_0^2\left(\frac{F}{\zeta_0}\right)^2 \cos\chi_0, \tag{B9}$$

where $\zeta_\Delta = \sqrt{\epsilon^2 - \Delta^2}$.

2. Nonstationary case

Consider now the nonstationary case. In order to find the anomalous Green's function $\hat{g}^{\mathrm{an}}$, we use Eqs. (B4)–(B7). This function obeys Eq. (60), where the matrix $\hat{G}^{\mathrm{an}}$ looks similar to that provided in Eq. (A9),

$$\hat{G}^{\mathrm{an}} = -\frac{i\chi_\Omega}{2}[F_+^R + F_-^A]\hat{X}_\perp \sin\chi_0. \tag{B10}$$

The first-order correction $\hat{g}_1^{\mathrm{an}}$ obeys the equation

$$\epsilon_+\hat{X}_{30} \cdot \hat{g}_1^{\mathrm{an}} - \hat{g}_1^{\mathrm{an}} \cdot \hat{X}_{30}\epsilon_- = i E_0[\hat{g}_{0+} \cdot \hat{G}^{\mathrm{an}} - \hat{G}^{\mathrm{an}} \cdot \hat{g}_{0-}], \tag{B11}$$

where $\hat{g}_{0+} \equiv \hat{g}_{0+}^R$ and $\hat{g}_{0-} \equiv \hat{g}_{0-}^A$. Taking into account that $\hat{g}_{0+} = -\hat{g}_{0-} = \hat{X}_{30}$ and Eq. (A10), we obtain that

$$\hat{g}_1^{\mathrm{an}} = 0. \tag{B12}$$

The second-order correction $\hat{g}_2^{\mathrm{an}}$ satisfies the equation

$$\zeta_+\hat{g}_{0+} \cdot \hat{g}_2^{\mathrm{an}} - \hat{g}_2^{\mathrm{an}} \cdot \hat{g}_{0-}\zeta_- = i E_0[\delta_1\hat{g}_+ \cdot \hat{G}^{\mathrm{an}} - \hat{G}^{\mathrm{an}} \cdot \delta_1\hat{g}_-]. \tag{B13}$$

We use the normalization condition

$$\hat{g}_{0+} \cdot \hat{g}_2^{\mathrm{an}} + \hat{g}_2^{\mathrm{an}} \cdot \hat{g}_{0-} = 0, \tag{B14}$$

so that the solution of Eq. (B13) is

$$\hat{g}_2^{\mathrm{an}} = g_2^{\mathrm{an}}\hat{X}_{30}, \tag{B15}$$

where g_2^{an} is

$$g_2^{\mathrm{an}} = \frac{i E_0}{\zeta_+ + \zeta_-}G^{\mathrm{an}}(\delta_1 g_+ - \delta_1 g_-). \tag{B16}$$

We need to find the third-order correction $\hat{g}_3^{\mathrm{an}}$ which obeys the equations

$$\zeta_+\hat{g}_{0+} \cdot \hat{g}_3^{\mathrm{an}} - \hat{g}_3^{\mathrm{an}} \cdot \hat{g}_{0-}\zeta_-$$
$$= i E_0\left[\delta_2\hat{g}_+ \cdot \hat{G}^{\mathrm{an}} - \hat{G}^{\mathrm{an}} \cdot \delta_2\hat{g}_- + \hat{g}_2^{\mathrm{an}} \cdot \hat{G}_{0-} - \hat{G}_{0+} \cdot \hat{g}_2^{\mathrm{an}}\right] \tag{B17}$$

$$\hat{g}_{0+} \cdot \hat{g}_3^{\mathrm{an}} + \hat{g}_3^{\mathrm{an}} \cdot \hat{g}_{0-} + \delta_1\hat{g}_+ \cdot \hat{g}_2^{\mathrm{an}} + \hat{g}_2^{\mathrm{an}} \cdot \delta_1\hat{g}_- = 0. \tag{B18}$$

The contribution to the current is given merely by the part $\hat{g}_{3\perp}^{\mathrm{an}}$ of $\hat{g}_3^{\mathrm{an}}$ which is proportional to the matrix $\hat{X}_\perp$. We obtain for $\hat{g}_{3\perp}^{\mathrm{an}}$

$$\hat{g}_{3\perp}^{\mathrm{an}} = g_{3\perp}^{\mathrm{an}}\hat{X}_\perp, \tag{B19}$$

where $g_{3\perp}^{\mathrm{an}}$ is given by

$$g_{3\perp}^{\mathrm{an}} = \frac{\chi_\Omega}{4}\frac{E_0^3(\cos\chi_0)^2 \sin\chi_0}{\zeta_+ + \zeta_-}(F_{\mathrm{T}+} + F_{\mathrm{T}-})B, \tag{B20}$$

and the function B is

$$B = \left(\frac{F_{\mathrm{T}+}}{\zeta_+}\right)^2 - 2\left(\frac{F_{\mathrm{T}+}F_{\mathrm{T}-}}{\zeta_+\zeta_-}\right) - \left(\frac{F_{\mathrm{T}-}}{\zeta_-}\right)^2. \tag{B21}$$

In obtaining Eqs. (B19)–(B21) we used expressions for the corrections $\delta_1\hat{g}_0^{R(A)}$ and $\delta_2\hat{g}_0^{R(A)}$ to the Green's functions $\hat{g}_{\mathrm{n}}^{R(A)}$ in the static case. Note that this formula is applicable both for the triplet and for the singlet JJ, S/N/S, because the only property we used is that the matrix

$$\hat{X}_\perp = \begin{cases} \hat{X}_{11} - s\hat{X}_{22}, & \text{triplet JJ}, \\ \hat{X}_{10}, & \text{singlet JJ}, \end{cases} \tag{B22}$$

anticommutes with the matrix $\hat{X}_{30}$.

Using Eqs. (B19)–(B21), we obtain Eqs. (61)–(62).

APPENDIX C: INVERSE PROXIMITY EFFECT

Consider, for simplicity, the contact between a singlet superconductor S and a normal metal N shown in Fig. 1(a). The Green's functions in the superconductor $\hat{G}_{\omega\mathrm{S}}(z)$ in a stationary case obeys the equation

$$-D_\mathrm{S}\partial_z(\hat{G}_{\omega\mathrm{S}} \cdot \partial_z\hat{G}_{\omega\mathrm{S}}) + \omega[\hat{X}_{30}, \hat{G}_{\omega\mathrm{S}}] + \Delta[\hat{X}_{10}, \hat{G}_{\omega\mathrm{S}}] = 0. \tag{C1}$$

The boundary condition for $\hat{G}_{\omega\mathrm{S}}$ is

$$\hat{G}_{\omega\mathrm{S}} \cdot \partial_z\hat{G}_{\omega\mathrm{S}} = \varkappa_\mathrm{S}[\hat{G}_{\omega\mathrm{S}}, \hat{g}_{\omega\mathrm{N}}], \tag{C2}$$

where $\varkappa_\mathrm{S} = (R_\square\sigma_\mathrm{S})^{-1}$. We integrate Eq. (C1) over the thickness d_S of the S film assuming that the function $\hat{G}_{\omega\mathrm{S}}(z)$ is almost constant (later, we check this assumption) and taking into account the boundary condition, Eq. (C2).

We obtain the equation

$$[\hat{M}_\mathrm{S}, \hat{G}_{\omega\mathrm{S}}] = 0, \tag{C3}$$

which looks similar to Eq. (17). Here, $\hat{M}_\mathrm{S} = \tilde{\omega}\hat{X}_{30} + \Delta\hat{X}_{10}$, $\tilde{\omega} \simeq \omega + D_\mathrm{S}\varkappa_\mathrm{S}/d_\mathrm{S}$. The quantity $\gamma \equiv D_\mathrm{S}\varkappa_\mathrm{S}/d_\mathrm{S}$ is a damping in the superconductor S which is induced due to inverse proximity effect. This factor is small compared to Δ if the condition $D_\mathrm{S}\varkappa_\mathrm{S}/d_\mathrm{S} \simeq \langle 1 - \mathcal{R}\rangle\sqrt{\Delta/\tau_\mathrm{S}} \ll \Delta$ is fulfilled, where $\langle 1 - \mathcal{R}\rangle$ is an average transmission coefficient of electron passage through the S/N interface, which is supposed to be small (see Ref. [11]), and $\mathcal{R}$ is the reflection coefficient. We assumed that $d_\mathrm{S} \simeq \xi_\mathrm{S}$, where ξ_S is the correlation length in the superconductor S and τ_S is the momentum relaxation time there. However, even if the condition above is not fulfilled, but the thickness d_S is larger than ξ_S, the results obtained remain valid with a reduced value of Δ at the S/N interface $[\Delta(0) < \Delta(d_\mathrm{S})]$.

APPENDIX D: FORMULA FOR AN ac CURRENT

We write Eq. (14) for the ac current $\delta I(t)$ (for brevity we set $\check{\Gamma} = \check{1}$ so that there is no difference between matrix $\check{G}$ and matrix $\check{\bar{G}}$) in the form

$$\delta I(t) = (16\kappa R_\square e)^{-1} \int dt_1 \{[\delta\check{g}(t,t_1),\check{G}_0(t_1 - t)]^K + [\check{g}_0(t - t_1),\delta\check{G}(t_1,t)]^K\}_{30}. \tag{D1}$$

Consider the first term in Eq. (D1) (the second term can be recast in the same way):

$$\delta I_1(t)(16\kappa R_\square e) = \{[\delta\check{g}(t,t_1) \cdot \check{G}_0(t_1 - t) - \check{G}_0(t - t_1) \cdot \delta\check{g}(t_1,t)]^K\}_{30} \tag{D2}$$

$$= \int dt_1 \int d\epsilon \int \frac{d\epsilon_1}{2\pi} \int \frac{d\epsilon_2}{2\pi} \{[\delta\check{g}(\epsilon,\epsilon_1) \cdot \check{G}_0(\epsilon_2) \exp[-i\epsilon t + i\epsilon_1 t_1 - i\epsilon_2(t_1 - t)]$$

$$- \check{G}_0(\epsilon) \cdot \delta\check{g}(\epsilon_1,\epsilon_2) \exp[-i\epsilon(t - t_1) - i\epsilon_1 t_1 + i\epsilon_2 t]]^K\}_{30}$$

$$= \int d\epsilon \int \frac{d\epsilon_1}{2\pi} \{[\delta\check{g}(\epsilon,\epsilon_1) \cdot \check{G}_0(\epsilon_1) - \check{G}_0(\epsilon) \cdot \delta\check{g}(\epsilon,\epsilon_1)]^K\}_{30} \exp[-i(\epsilon - \epsilon_1)t].$$

Taking into account that $\delta\check{g}(\epsilon,\epsilon_1) = 2\pi\delta(\epsilon - \epsilon_1 - \Omega)\delta\check{g}$, we obtain

$$\delta I(t) = (16\kappa R_\square e)^{-1} \exp[-i\Omega t]I_\Omega, \tag{D3}$$

where

$$I_\Omega = (16\kappa R_\square e)^{-1} \int d\bar{\epsilon} \{[\delta\check{g} \cdot \check{G}_0(\epsilon_-) - \check{G}_0(\epsilon_+) \cdot \delta\check{g}]^K + [\check{g}(\epsilon_+) \cdot \delta\check{G} - \delta\check{G} \cdot \check{g}(\epsilon_-)]^K\}_{30}. \tag{D4}$$

The current I_Ω is related to the admittance according to the standard expression $I_\Omega = Y(\Omega)V_\Omega$.

[1] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).

[2] A. A. Abrikosov, L. P. Gor'kov, and I. M. Khalatnikov, Sov. Phys. JETP **8**, 182 (1958).

[3] A. A. Abrikosov and L. P. Gor'kov, Sov. Phys. JETP **8**, 1090 (1959).

[4] R. E. Glover and M. Tinkham, Phys. Rev. **104**, 844 (1956).

[5] D. M. Ginsberg and M. Tinkham, Phys. Rev. **118**, 990 (1960).

[6] M. Tinkham, *Introduction to Superconductivity*, 2nd ed., Dover Books on Physics (Dover, Mineola, NY, 2004).

[7] A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP **34**, 1144 (1972).

[8] A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP **20**, 762 (1965).

[9] W. L. McMillan, Phys. Rev. **175**, 537 (1968).

[10] A. A. Golubov and M. Y. Kupriyanov, Sov. Phys. JETP **69**, 805 (1989).

[11] A. Volkov, A. Zaitsev, and T. Klapwijk, Physica C: Superconduct. **210**, 21 (1993).

[12] K. K. Likharev, Rev. Mod. Phys. **51**, 101 (1979).

[13] A. Barone and G. Paternò, in *Physics and Applications of the Josephson Effect* (Wiley-VCH Verlag, New York, 1982), pp. 1–24.

[14] I. Kulik, I. Yanson, and P. Gluck, *Josephson Effect in Superconducting Tunneling Structures* (John Wiley & Sons, New York, 1972).

[15] G. N. Gol'tsman, O. Okunev, G. Chulkova, A. Lipatov, A. Semenov, K. Smirnov, B. Voronov, A. Dzardanov, C. Williams, and R. Sobolewski, Appl. Phys. Lett. **79**, 705 (2001).

[16] P. K. Day, H. G. LeDuc, B. A. Mazin, A. Vayonakis, and J. Zmuidzinas, Nature **425**, 817 (2003).

[17] R. M. J. Janssen, J. J. A. Baselmans, A. Endo, L. Ferrari, S. J. C. Yates, A. M. Baryshev, and T. M. Klapwijk, Appl. Phys. Lett. **103**, 203503 (2013).

[18] J. Clarke and F. K. Wilhelm, Nature **453**, 1031 (2008).

[19] S. N. Artemenko, A. F. Volkov, and A. V. Zaitsev, Sov. Phys. JETP **49**, 924 (1979).

[20] A. Moor and A. F. Volkov, Phys. Rev. B **95**, 134518 (2017).

[21] F. Kos, S. E. Nigg, and L. I. Glazman, Phys. Rev. B **87**, 174521 (2013).

[22] P. Virtanen, F. S. Bergeret, J. C. Cuevas, and T. T. Heikkilä, Phys. Rev. B **83**, 144514 (2011).

[23] K. S. Tikhonov and M. V. Feigel'man, Phys. Rev. B **91**, 054519 (2015).

[24] A. Moor, A. F. Volkov, and K. B. Efetov, Phys. Rev. Lett. **118**, 047001 (2017).

[25] R. V. Carlson and A. M. Goldman, Phys. Rev. Lett. **34**, 11 (1975).

[26] A. Schmid and G. Schön, Phys. Rev. Lett. **34**, 941 (1975).

[27] S. N. Artemenko and A. F. Volkov, Sov. Phys. JETP **42**, 1130 (1975).

[28] G. Schön, in *Nonequilibrium Superconductivity*, edited by D. N. Langenberg and A. I. Larkin, Modern Problems in Condensed Matter Sciences Vol. 12 (Elsevier, Amsterdam, 1986), pp. 589–640.

[29] S. N. Artemenko, A. F. Volkov, and A. V. Zaitsev, JETP Lett. **27**, 113 (1978).

[30] S. N. Artemenko and A. F. Volkov, Sov. Phys. Usp. **22**, 295 (1979).

[31] The triplet odd-frequency (TOF) component is often called the Berezinskii's type of superconductivity [96]. In our opinion, this term is not suitable for the description of the TOF

superconducting correlations that arise in S/F systems. Berezin-skii has suggested a new type of superconductivity caused by a special (hypothetical) type of coupling which is described by a frequency-dependent potential. On the other hand, the TOF correlations arise even in conventional BCS superconductors in the presence of a magnetic field acting on spins of electrons (Zeemansplitting). In this case one can hardly speak about a new type of superconductivity.

[32] A. A. Golubov, M. Y. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).

[33] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Rev. Mod. Phys. **77**, 1321 (2005).

[34] A. I. Buzdin, Rev. Mod. Phys. **77**, 935 (2005).

[35] M. Eschrig, Phys. Today **64**(1), 43 (2011).

[36] M. G. Blamire and J. W. A. Robinson, J. Phys.: Condens. Matter **26**, 453201 (2014).

[37] M. Eschrig, Rep. Prog. Phys. **78**, 104501 (2015).

[38] J. Linder and J. W. A. Robinson, Nat. Phys. **11**, 307 (2015).

[39] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Phys. Rev. Lett. **86**, 4096 (2001).

[40] Similar ideas were suggested a little later in Ref. [97], where this effect is discussed qualitatively.

[41] R. S. Keizer, S. T. B. Goennenwein, T. M. Klapwijk, G. Miao, G. Xiao, and A. Gupta, Nature **439**, 825 (2006).

[42] I. Sosnin, H. Cho, V. T. Petrashov, and A. F. Volkov, Phys. Rev. Lett. **96**, 157002 (2006).

[43] M. S. Anwar, F. Czeschka, M. Hesselberth, M. Porcu, and J. Aarts, Phys. Rev. B **82**, 100501 (2010).

[44] M. S. Anwar and J. Aarts, Supercond. Sci. Technol. **24**, 024016 (2011).

[45] M. S. Anwar, M. Veldhorst, A. Brinkman, and J. Aarts, Appl. Phys. Lett. **100**, 052602 (2012).

[46] T. S. Khaire, M. A. Khasawneh, W. P. Pratt, Jr., and N. O. Birge, Phys. Rev. Lett. **104**, 137002 (2010).

[47] C. Klose, T. S. Khaire, Y. Wang, W. P. Pratt, Jr., N. O. Birge, B. J. McMorran, T. P. Ginley, J. A. Borchers, B. J. Kirby, B. B. Maranville, and J. Unguris, Phys. Rev. Lett. **108**, 127002 (2012).

[48] D. Sprungmann, K. Westerholt, H. Zabel, M. Weides, and H. Kohlstedt, Phys. Rev. B **82**, 060505(R) (2010).

[49] J. W. A. Robinson, J. D. S. Witt, and M. G. Blamire, Science **329**, 59 (2010).

[50] J. W. A. Robinson, G. B. Halász, A. I. Buzdin, and M. G. Blamire, Phys. Rev. Lett. **104**, 207001 (2010).

[51] M. S. Kalenkov, A. D. Zaikin, and V. T. Petrashov, Phys. Rev. Lett. **107**, 087003 (2011).

[52] A. Moor, A. F. Volkov, and K. B. Efetov, Phys. Rev. B **92**, 214510 (2015).

[53] C. Richard, A. Buzdin, M. Houzet, and J. S. Meyer, Phys. Rev. B **92**, 094509 (2015).

[54] A. F. Volkov, F. S. Bergeret, and K. B. Efetov, Phys. Rev. Lett. **90**, 117006 (2003).

[55] M. Eschrig, J. Kopu, J. C. Cuevas, and G. Schön, Phys. Rev. Lett. **90**, 137003 (2003).

[56] A. F. Volkov, Y. V. Fominov, and K. B. Efetov, Phys. Rev. B **72**, 184504 (2005).

[57] Z. Pajović, M. Božović, Z. Radović, J. Cayssol, and A. Buzdin, Phys. Rev. B **74**, 184509 (2006).

[58] V. Braude and Y. V. Nazarov, Phys. Rev. Lett. **98**, 077003 (2007).

[59] Y. Asano, Y. Tanaka, and A. A. Golubov, Phys. Rev. Lett. **98**, 107002 (2007).

[60] M. Houzet and A. I. Buzdin, Phys. Rev. B **76**, 060504(R) (2007).

[61] A. V. Galaktionov, M. S. Kalenkov, and A. D. Zaikin, Phys. Rev. B **77**, 094520 (2008).

[62] M. Eschrig and T. Löfwander, Nat. Phys. **4**, 138 (2008).

[63] R. Grein, M. Eschrig, G. Metalidis, and G. Schön, Phys. Rev. Lett. **102**, 227005 (2009).

[64] F. Konschelle and A. Buzdin, Phys. Rev. Lett. **102**, 017001 (2009).

[65] L. Trifunovic and Z. Radović, Phys. Rev. B **82**, 020505(R) (2010).

[66] L. Trifunovic, Z. Popović, and Z. Radović, Phys. Rev. B **84**, 064511 (2011).

[67] A. M. Bobkov and I. V. Bobkova, Phys. Rev. B **84**, 054533 (2011).

[68] N. G. Pugach and A. I. Buzdin, Appl. Phys. Lett. **101**, 242602 (2012).

[69] M. Alidoust and K. Halterman, Appl. Phys. Lett. **105**, 202601 (2014).

[70] M. Alidoust, K. Halterman, and J. Linder, Phys. Rev. B **88**, 075435 (2013).

[71] I. Kulagina and J. Linder, Phys. Rev. B **90**, 054504 (2014).

[72] J. Linder and K. Halterman, Phys. Rev. B **90**, 104502 (2014).

[73] S. Hikino and S. Yunoki, Phys. Rev. B **92**, 024512 (2015).

[74] K. Halterman, O. T. Valls, and C.-T. Wu, Phys. Rev. B **92**, 174516 (2015).

[75] Y. Asano, A. A. Golubov, Y. V. Fominov, and Y. Tanaka, Phys. Rev. Lett. **107**, 087001 (2011).

[76] Y. Asano, M. Ozaki, T. Habe, A. A. Golubov, and Y. Tanaka, Phys. Rev. B **86**, 024510 (2012).

[77] Note misprints in Ref. [20]. The denominators in Eqs. (60) and (61) should be the same as that in Eq. (B6). The sign of I_Ω^{an} must be changed.

[78] N. B. Kopnin, *Theory of Nonequilibrium Superconductivity*, The International Series of Monographs on Physics (Clarendon Press, Oxford, UK, 2001).

[79] A. I. Larkin and Y. N. Ovchinnikov, *Nonequilibrium Superconductivity* (Elsevier, Amsterdam, 1986).

[80] J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

[81] W. Belzig, F. K. Wilhelm, C. Bruder, G. Schön, and A. D. Zaikin, Superlatt. Microstruct. **25**, 1251 (1999).

[82] F. S. Bergeret, A. Verso, and A. F. Volkov, Phys. Rev. B **86**, 214516 (2012).

[83] P. Machon, M. Eschrig, and W. Belzig, Phys. Rev. Lett. **110**, 047002 (2013).

[84] M. Eschrig, A. Cottet, W. Belzig, and J. Linder, New J. Phys. **17**, 083037 (2015).

[85] A. Moor, A. F. Volkov, and K. B. Efetov, Phys. Rev. B **93**, 104525 (2016).

[86] L. P. Gor'kov and G. M. Eliashberg, Sov. Phys. JETP **27**, 328 (1968).

[87] Y. Tanaka and A. A. Golubov, Phys. Rev. Lett. **98**, 037003 (2007).

[88] P. M. R. Brydon, Phys. Rev. B **80**, 224520 (2009).

[89] J.-F. Liu and K. S. Chan, Phys. Rev. B **82**, 184533 (2010).

[90] I. Margaris, V. Paltoglou, and N. Flytzanis, J. Phys.: Condens. Matter **22**, 445701 (2010).

[91] A. Moor, A. F. Volkov, and K. B. Efetov, Phys. Rev. B **92**, 180506(R) (2015).

[92] S. Mironov and A. Buzdin, Phys. Rev. B **92**, 184506 (2015).

[93] M. A. Silaev, I. V. Tokatly, and F. S. Bergeret, Phys. Rev. B **95**, 184508 (2017).

[94] I. V. Bobkova and Y. S. Barash, J. Exp. Theor. Phys. Lett. **80**, 494 (2004).

[95] S. Mironov and A. Buzdin, Phys. Rev. Lett. **118**, 077001 (2017).

[96] V. L. Berezinskii, JETP Lett. **20**, 287 (1974).

[97] A. Kadigrobov, R. I. Shekhter, and M. Jonson, Europhys. Lett. **54**, 394 (2001).