Spontaneous magnetization of quantum XY spin model in joint presence of quenched and annealed disorder

Anindita Bera,[1,2,3] Debraj Rakshit,[2,3,4] Aditi Sen(De),[2,3] and Ujjwal Sen[2,3]
[1]*Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata 700 009, India*
[2]*Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India*
[3]*Homi Bhabha National Institute, Anushaktinagar, Mumbai, Maharashtra 400 094, India*
[4]*Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, PL-02-668 Warsaw, Poland*
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We investigate equilibrium statistical properties of the isotropic quantum XY spin-1/2 model in an external magnetic field when the interaction and field parts are subjected to quenched or annealed disorder or both. The randomness present in the system are termed annealed or quenched depending on the relation between two different time scales—the time scale associated with the equilibration of the randomness and the time of observation. Within a mean-field framework, we study the effects of disorders on spontaneous magnetization, both by perturbative and numerical techniques. Our primary interest is to understand the differences between quenched and annealed cases, and also to investigate the interplay when both of them are present in a system. We find that the magnetization survives in the presence of a unidirectional random field, irrespective of its nature, i.e., whether it is quenched or annealed. However, the field breaks the circular symmetry of the magnetization, and the system magnetizes in specific directions, parallel or transverse to the applied magnetic field. Interestingly, while the transverse magnetization is affected by the annealed disordered field, the parallel one remains unfazed by the same. Moreover, the annealed disorder present in the interaction term does not affect the system's spontaneous magnetization and the corresponding critical temperature, irrespective of the presence or absence of quenched or annealed disorder in the field term. We carry out a comparative study of these and all other different combinations of the disorders in the interaction and field terms, and point out their generic features.

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I. INTRODUCTION

Disorder is an unavoidable feature of condensed matter and atomic many-body systems, in both classical as well as quantum domains [1]. There are long-standing quests to understand several nontrivial quantum phenomena caused by the presence of disorder. Some of the prominent examples in the quantum case are disorder induced localizations [2–4], high T_c superconductivity [5], and novel quantum phases [6–8].

For disordered parameters in a physical system, two typical situations may arise depending on the interrelation between the two fundamental time scales associated with the disordered parameter, viz. the time scale over which the disorder configuration equilibrates and the time scale associated with observation of the physical quantities of interest [1,6,9–12]. For the cases where the system's disorder configuration remains effectively frozen throughout the entire observation process, the disorder is considered to be "quenched". In such cases, one obtains a "functional" free energy as the logarithm of the partition function, for a given realization of the quenched random parameter. This functional free energy is utilized to obtain functional spin-averaged observables for the given realization of the quenched random parameter. The quenched averaged observables are obtained by averaging, over the distribution of the quenched disorder, of the functional observables. However, there can be a separate situation where the observation takes place during the equilibration process of the disorder parameters, so that the time scale associated with the configurational change is of the order or within a few orders of the observation time. In such cases, the randomness of the corresponding system parameters should be considered to be "annealed" and the partition function has to be averaged over several random realizations, so that the annealed averaged free energy is obtained by taking the logarithm of the averaged partition function.

There has been continuous efforts to understand effects of disorder in quantum systems [13–18]. A particular reason of recent interest in such systems is also due to the fact that current technology allows us to realize artificial randomness in a controlled way, in for example ultracold atoms trapped in optical lattices [13]. A considerable amount of effort has been dedicated to investigate disordered systems that are quenched [4,15–18], and in particular to understand the effects of such disorder on the universal dynamics in the vicinity of quantum phase transitions [15], a useful test which is given by the Harris criterion [19]. Unlike quenched disorder, the universality class of a phase transition is usually not affected by the presence of annealed disorder, as the partition function after averaging over random realizations can be replaced by one corresponding to an effective model which is free from the disorder. In other instances, quenched disorder spin systems have been studied to understand "glassy" properties in type II superconductors [20], to demonstrate breakdown of thermalization in the presence of disorder [4], to achieve quantum advantages due to the introduction of the disorder [16], and to explore disorder induced quantum phenomena such as "order from disorder" [16,21]. Significant works have also been carried out for studying the consequences of annealed disorder in the spin systems as well [22,23]. Moreover, efforts have been directed towards understanding system properties at equilibrium when the nature of the disorder changes from quenched to annealed [11].

In this respect, an important question, which to our knowledge is yet to be dealt with, is how the thermodynamical quantities respond in joint presence of quenched and annealed disorders. It is for example possible to inquire about the properties of a quantum spin system governed by the Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{field}}$, where the disorder introduced in one set of parameters, say the couplings in $\mathcal{H}_{\text{int}}$, remain quenched during the observation process, while the equilibrating time scales of another set, say the field parameters in $\mathcal{H}_{\text{field}}$, is of the same or near order as the observation time scale, so that the latter collection forms an annealed set of parameters. The main focus of this paper is to study the equilibrium properties, such as magnetization and critical temperatures, of such systems, and compare between them and with systems having only quenched or only annealed disorder or systems devoid of disorder.

Spontaneous magnetization in higher-dimensional quantum XY model in the presence of an unidirectional quenched random field has been considered with a lot of interest in recent times. It has been shown that spontaneous magnetization perishes when a small random magnetic field with appropriate symmetry is introduced in the XY spin systems [17,24,25]. However, it persists in the absence of the appropriate symmetry of the external random field [18,26]. Interestingly, it has been shown that a uniaxial random field may help the system to magnetize even in two dimensions [18,27]. Recently, mean-field approach [28,29] has been adapted to look into the aspects of spontaneous magnetizations and critical scalings in quenched disordered spin models [26].

In this work, the models we examine exhibit spontaneous magnetization in the absence of disorder, and the primary interest of our work is to understand the effect on the spontaneous magnetization and the corresponding critical temperatures due to the *joint* presence of two types of disorder, viz. quenched and annealed. We study the quantum spin-1/2 XY models where the system Hamiltonian has two parameters that are considered to be disordered. They are, respectively, the interaction and the field parameters. The interaction parameter is considered to be annealed disordered or quenched disordered or ordered. The transverse magnetic field is again chosen from these three options. We analyze the patterns of spontaneous magnetization when the interaction-field pair is in any of the nine possible combinations with respect to their disorder. The quantum spin-1/2 XY model in higher dimensions and especially with disorder cannot be solved analytically. Hence, numerical or approximate methods have to be employed for such investigations. In this work, the mean-field method is used for the study. Presence of randomness in the interaction strength preserves isotropic symmetry of the system, while a small random field, even with zero mean, breaks the same. In spite of the presence of the disordered fields, the system does magnetize in either the parallel or perpendicular direction to the applied random field. We derive analytical expressions for the critical temperatures and near-critical magnetizations for all the cases. Our analysis reveals, for example, that although an annealed disordered field affects the transverse magnetization, the magnetization parallel to the applied field remains unaltered by the same. Furthermore, an annealed disordered interaction does not have any effect on the system's spontaneous magnetization and the corresponding critical temperature, irrespective of the presence or absence of quenched or annealed disorder in the field. The magnetization survives quenched randomness, although it always gets shrunk, whether or not there is an accompanying annealed disorder in another parameter of the system. However, we find that there can be situations where the critical temperature is not affected by the presence of quenched disorder.

The rest of the paper is arranged as follows. In Sec. II we present a general recapitulation of the mechanism for obtaining the annealed and the quenched averaged values of physical observables. In Sec. III we introduce the system and its mean-field treatment. We also discuss about the various situations depending on the nature of the disorder parameters and the segment of the Hamiltonian in which the disorder is located. In Sec. IV we present a detailed analysis for a quantum spin-1/2 model in joint presence of quenched and annealed disorders. Section V tabulates the analytical expressions for the critical temperatures and scalings of the magnetizations near the critical points for the different types of disorder in the quantum spin-1/2 model. We conclude in Sec. VI.

II. ANNEALED AND QUENCHED DISORDERS

In this section we briefly discuss the mechanism for computing the annealed and quenched averaged values of the observables. As mentioned earlier, the distinction between quenched and annealed disorders is determined by relative comparison of two different time scales of the physical system under consideration, viz. the relaxation time associated with the equilibration of the disorders, say τ_1, and the time necessary for the required observation on the system, say τ_2. For the cases where τ_1 is of the same or near order of magnitude of τ_2, the statistical properties of the system at equilibrium is obtained via annealed averaging, which is calculated by averaging of the partition function $\mathcal{Z}$ over several random realizations. The free energy for a system with the annealed disorder is given as

$$\mathcal{F} = -(1/\beta)\ln\langle\mathcal{Z}\rangle, \tag{1}$$

where $\beta = 1/(\kappa_B\mathcal{T})$, with κ_B being the Boltzmann constant and $\mathcal{T}$ being the absolute temperature. Here, and in the rest of the paper, the notation $\langle\cdot\rangle$ shall imply an average of the argument over the relevant disorder degrees of freedom. However, if $\tau_1 \gg \tau_2$, i.e., the impurities remain trapped in random but fixed positions during the observation time, the statistical properties of the system at equilibrium is obtained via quenched averaging. In case of quenched averaging, the logarithm of the partition function for a given realization of the quenched disorder parameters, instead of the partition function itself, is considered for finding the functional spin-averaged observables for the given realization of the quenched disorder. The averaging over the quenched random variable is performed at this stage to obtain the quenched averaged observable. Let us note here that for both types of disorders, as well as for the ordered systems, the observation time is assumed to be much longer than the relaxation of the spin degrees of freedom.

Let us consider a general Hamiltonian $\mathcal{H} = \mathcal{H}(\{a_i\},\{q_j\})$, where $\{a_i\}$ and $\{q_j\}$ are two sets of system parameters that are respectively annealed and quenched disordered. We introduce the functional partition function $\mathcal{Z}(\{a_i\},\{q_j\},\{\lambda_k\}) = \text{Tr}\,[e^{-\beta\{\mathcal{H}(\{a_i\},\{q_j\})+\sum_k\lambda_k\mathcal{A}_k\}}]$, where the term $\sum_k\lambda_k\mathcal{A}_k$ is an auxiliary function. The auxiliary function is used later for obtaining the expectation values of the operator $\mathcal{A}_i$ by taking derivatives with respect to λ_i at $\lambda_k = 0\ \forall k$, where $\mathcal{Z}$ has been implicitly assumed to be a differentiable function of the λ_k's. The functional free energy $\mathcal{F}$, after performing configurational averaging over the annealed disorder for an arbitrary fixed realization of the $\{q_j\}$, reads

$$\mathcal{F}(\{q_j\},\{\lambda_k\}) = -\frac{1}{\beta}\ln\left\{\int\prod_i da_i\mathcal{P}_i(a_i)\mathcal{Z}(\{a_i\},\{q_j\},\{\lambda_k\})\right\}, \tag{2}$$

where $\mathcal{P}_i(a_i)$ represent the probability density functions of the annealed parameters. The thermodynamic functional average of the observable $\mathcal{A}_k$, averaged over the spin and annealed disorder degrees of freedom, for the fixed $\{q_j\}$, is $\partial\mathcal{F}/\partial\lambda_k|_{\{\lambda_i\}=0}$. The spin- as well as annealed and quenched disorder-averaged $\mathcal{A}_k$ is

$$\int\prod_j dq_j\mathcal{P}_j(q_j)\frac{\partial\mathcal{F}}{\partial\lambda_k}|_{\{\lambda_i\}=0}, \tag{3}$$

where $\mathcal{P}_j(q_j)$ denotes the probability density functions of the quenched disordered parameters.

III. SYSTEM HAMILTONIAN AND MEAN-FIELD TREATMENT

We investigate the isotropic quantum XY spin model in an external magnetic field with disorder in the interaction part or in the field part or in both, within a mean-field approximation. We are primarily interested in drawing a comparative analysis on the effect of disorder on spontaneous magnetization and their scalings near the critical point as a function of temperature, with different possible combinations of disorders. For example, a possible combination is quenched disorder in the coupling terms and annealed disorder in the field terms. In this section we introduce the system and its mean-field treatment.

The general form of the Hamiltonian of the ferromagnetic quantum XY model in the presence of disorders in both the interaction and the coupling parts is given by $\mathcal{H}_{\rm XY}(\tilde{\eta}_{ij},\eta_i) = \mathcal{H}_{\rm int}(\tilde{\eta}_{ij}) + \mathcal{H}_{\rm ext}(\eta_i)$, where

$$\mathcal{H}_{\rm int}(\tilde{\eta}_{ij}) = -\sum_{(i,j)\in S}(\mathcal{J}' + \tilde{\epsilon}\tilde{\eta}_{ij})\big[\sigma_x^i\sigma_x^j + \sigma_y^i\sigma_y^j\big],$$
$$\mathcal{H}_{\rm ext}(\eta_i) = -\epsilon\sum_{i=1}^{\mathcal{N}}\eta_i\sigma_y^i. \tag{4}$$

Here the coupling constant $\mathcal{J}' > 0$. The indices i and j denote the sites of an arbitrary d-dimensional lattice and $\sigma_i^\alpha, \alpha = x,y$, are the Pauli matrices at the ith site. $\mathcal{N}$ is the total number of lattice sites. The set S denotes a subset of the set of all (unordered) pairs of lattice sites. Both $\tilde{\epsilon}$ and ϵ are non-negative parameters, having the dimension of energy that quantify the strengths of the corresponding random parameters. The unidirectional random field is chosen to be directed along the y axis. $\tilde{\eta}_{ij}$ are independent and identically distributed (dimensionless) Gaussian random variables with zero mean and variance $1/f$. η_i are the same but with unit variance. The constant f is a dimensionless quantity that depends on the Hamiltonian and the lattice on which it is defined. We shall discuss it further in the next paragraph.

In such a system, the mean-field approximation can be obtained as follows. In the Hamiltonian, we replace the operator σ_x^j by a real number m_x, and σ_y^j by m_y. That is, we approximate the interaction term by $-\sum_{(i,j)\in S}(\mathcal{J}' + \tilde{\epsilon}\tilde{\eta}_{ij})(m_x\sigma_x^i + m_y\sigma_y^i)$, where m_x and m_y are spin- as well as disorder-averaged magnetizations of the system at absolute temperature $\mathcal{T}$. m_x and m_y are therefore mean-field variables, as yet unknown, to be obtained from the self-consistency equations of the mean-field theory. The interaction term can be further rewritten as $-\sum_{i=1}^{\mathcal{N}}(\mathcal{J} + \tilde{\epsilon}\tilde{\eta}_i)(m_x\sigma_x^i + m_y\sigma_y^i)$, where $\mathcal{J} = \mathcal{J}'f$ and $\tilde{\eta}_i$ are Gaussian random variables with zero mean and unit variance. Note therefore that f is the number of different j's for a given i in the set S. For nearest-neighbor interactions in one-dimension, $f = 1$, while for the same in the two-dimensional square lattice, $f = 2$. Hence, within the mean-field approximation, the Hamiltonian $\mathcal{H}_{\rm XY}$ can be written as

$$\mathcal{H}_{\rm XY}(\tilde{\eta},\eta) = -(\mathcal{J} + \tilde{\epsilon}\tilde{\eta})(m_x\sigma_x + m_y\sigma_y) - \epsilon\eta\sigma_y. \tag{5}$$

Note that in Eq. (5) the Hamiltonian corresponding to the ordered system can be obtained by simply setting $\tilde{\epsilon} = \epsilon = 0$. η and $\tilde{\eta}$ are independent Gaussian random variables with zero mean and unit variance. The quantities ϵ and $\tilde{\epsilon}$ are chosen to be small compared to $\mathcal{J}$. The functional partition function of the system in the canonical equilibrium state at absolute temperature $\mathcal{T}$ is given by

$$\mathcal{Z}(\tilde{\eta},\eta,\{\lambda_k\}) = {\rm Tr}[e^{-\beta\{\mathcal{H}_{\rm XY}(\tilde{\eta},\eta)+\sum_k\lambda_k\mathcal{A}_k\}}]. \tag{6}$$

Note that for the cases, where disorder is present in either the interaction part or the field part, the functional partition function reduces to $\mathcal{Z}(\tilde{\eta},\{\lambda_k\})$ $[\mathcal{Z}(\eta,\{\lambda_k\})]$ for $\epsilon = 0$ $[\tilde{\epsilon} = 0]$.

Now let us consider three different categories:

Category (i): Both the interaction as well as the field terms are annealed disordered, or any of them is so, while the other is ordered.

Category (ii): Both the interaction and the field terms are quenched disordered, or any one of them is so, while the other is ordered.

Category (iii): The interaction and field terms are, respectively, quenched and annealed disordered or vice versa.

As mentioned earlier, for the cases within the first category, the free energy is obtained by performing a disorder average over the partition function. In contrast, for the cases within the second category, the functional free energy is obtained for a given realization of the quenched disorder(s), which is followed by finding the relevant derivatives with respect to the λ_k and taking the limit as $\{\lambda_k\} \to 0$. These derivatives provide the functional spin averages for the observables $\mathcal{A}_k$ for the given realization of the disorder. Performing a disorder average of these functional spin averages of the observables provides us with the quenched averaged physical observables.

For the situations within the third category, first a configurational averaging of the partition function over the annealed parameters for fixed realization of quenched randomness is performed. This is followed by taking the logarithm to obtain the functional free energy for the fixed realization of the quenched randomness. Derivatives with respect to the relevant λ_k and subsequent limits $\{\lambda_k\} \to 0$ provide the functional spin- and annealed disorder-averaged observables. Quenched disorder averaging is performed only at this last stage to obtain the annealed as well as quenched disorder-averaged observables. As an example, let us consider a quantum spin magnetic system in the presence of quenched randomness in the parameter associated with the interaction part $\tilde{\eta}$ and annealed randomness in the parameter associated with the field part η. In this case, the functional free energy after performing the annealed disorder averaging for a fixed realization of quenched randomness is given by

$$\mathcal{F}(\tilde{\eta},\{\lambda_k\}) = -\frac{1}{\beta\sqrt{2\pi\Delta}}\ln\Bigg\{\int_{-\infty}^{\infty} d\eta e^{-\frac{\eta^2}{2\Delta}} \times \mathrm{Tr}[e^{-\beta\{\mathcal{H}_{\mathrm{XY}}(\tilde{\eta},\eta)+\sum_k\lambda_k\mathcal{A}_k\}}]\Bigg\}, \quad (7)$$

where Δ represents the standard deviation of the Gaussian distributed annealed disorder with zero mean. For convenience, we alternatively represent Eq. (7) as

$$\mathcal{F}(\tilde{\eta},\{\lambda_k\}) = -\frac{1}{\beta}\ln\langle\mathrm{Tr}[e^{-\beta\{\mathcal{H}_{\mathrm{XY}}(\tilde{\eta},\eta)+\sum_k\lambda_k\mathcal{A}_k\}}]\rangle_\eta. \quad (8)$$

The functional spin- and annealed disorder-averaged value for the observable $\mathcal{A}_k$ is

$$\frac{\partial}{\partial\lambda_k}\mathcal{F}(\tilde{\eta},\{\lambda_k\})|_{\{\lambda_k\}\to 0}. \quad (9)$$

The annealed as well as quenched disorder-averaged $\mathcal{A}_k$ is

$$\frac{1}{\sqrt{2\pi\tilde{\Delta}}}\int_{-\infty}^{\infty} d\tilde{\eta} e^{-\frac{\tilde{\eta}^2}{2\tilde{\Delta}}}\frac{\partial}{\partial\lambda_k}\mathcal{F}(\tilde{\eta},\{\lambda_k\})|_{\{\lambda_k\}\to 0}, \quad (10)$$

which we will represent, for short, as

$$\left\langle\frac{\partial}{\partial\lambda_k}\mathcal{F}(\tilde{\eta},\{\lambda_k\})|_{\{\lambda_k\}\to 0}\right\rangle_{\tilde{\eta}}, \quad (11)$$

where $\tilde{\Delta}$ represents the standard deviation of the Gaussian distributed quenched disorder with zero mean. In the following section we present a detailed analysis of the spontaneous magnetization and critical scalings of the quantum spin-1/2 spin model for this representative case by the formalism described here.

IV. QUANTUM XY SPIN-1/2 MODEL IN JOINT PRESENCE OF QUENCHED AND ANNEALED DISORDERS

We now investigate the behavior of spontaneous magnetizations of the isotropic quantum spin-1/2 XY model in the presence of both quenched and annealed disorders. This corresponds to the category (iii) of the preceding section. Our system Hamiltonian is given by Eq. (5). The field and interaction parts are subjected to annealed and quenched disorders, respectively. Starting from Eq. (7) and following straightforward algebraic steps, the components of magnetization along the x and y axes can be obtained by solving for common zeros of the following pair of functions:

$$f_x^{\tilde{\epsilon},\epsilon}(\vec{m}) = \langle[\langle\cosh(\beta k)\rangle_\eta]^{-1}\langle m_x(\mathcal{J}+\tilde{\epsilon}\tilde{\eta})\sinh(\beta k)/k\rangle_\eta\rangle_{\tilde{\eta}} - m_x \quad (12)$$

and

$$f_y^{\tilde{\epsilon},\epsilon}(\vec{m}) = \langle[\langle\cosh(\beta k)\rangle_\eta]^{-1}\langle[m_y(\mathcal{J}+\tilde{\epsilon}\tilde{\eta})+\epsilon\eta] \times \sinh(\beta k)/k\rangle_\eta\rangle_{\tilde{\eta}} - m_y, \quad (13)$$

where $k=\sqrt{[m_x(\mathcal{J}+\tilde{\epsilon}\tilde{\eta})]^2+[m_y(\mathcal{J}+\tilde{\epsilon}\tilde{\eta})+\epsilon\eta]^2}$ and $\vec{m}=(m_x,m_y)$. We perform a perturbative analysis for solving the coupled set of equations formed by equating the functions $f_x^{\tilde{\epsilon},\epsilon}$ and $f_y^{\tilde{\epsilon},\epsilon}$ to zero. The perturbative approach helps us to derive the exact analytical expressions for the near-critical temperature and scaling of magnetization. Moreover, we carry out numerical analysis, which helps us to look into effects of disorder in the system properties as functions of temperature, in near-critical as well as far-from-critical regimes.

A. Critical point and scaling of magnetization near criticality

When strengths of the random parameters are small, it turns out that perturbative analyses yield a great deal of insight about the system's behavior. Such analyses, in particular, provide quantitative values of critical temperatures and near-critical scalings of magnetization. Bivariate Taylor series expansions of Eqs. (12) and (13) around $\epsilon/\mathcal{J}$ and $\tilde{\epsilon}/\mathcal{J}$ at $\epsilon=0$ and $\tilde{\epsilon}=0$ give the leading order behaviors of $f_x^{\tilde{\epsilon},\epsilon}(\vec{m})$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m})$ as

$$f_x^{\tilde{\epsilon},\epsilon}(\vec{m}) = a_x + \frac{1}{2}\mathcal{J}^2 b_x\left(\frac{\tilde{\epsilon}}{\mathcal{J}}\right)^2 + \frac{1}{2}\mathcal{J}^2 c_x\left(\frac{\epsilon}{\mathcal{J}}\right)^2 + \cdots \quad (14)$$

and

$$f_y^{\tilde{\epsilon},\epsilon}(\vec{m}) = a_y + \frac{1}{2}\mathcal{J}^2 b_y\left(\frac{\tilde{\epsilon}}{\mathcal{J}}\right)^2 + \frac{1}{2}\mathcal{J}^2 c_y\left(\frac{\epsilon}{\mathcal{J}}\right)^2 + \cdots, \quad (15)$$

where

$$a_x = \frac{m_x}{m}\tanh[\beta\mathcal{J}m] - m_x, \quad (16)$$

$$a_y = \frac{m_y}{m}\tanh[\beta\mathcal{J}m] - m_y, \quad (17)$$

$$b_x = \frac{-2\beta^2 m_x m\tanh[\beta\mathcal{J}m]}{\cosh[\beta\mathcal{J}m]^2}, \quad (18)$$

$$b_y = \frac{-2\beta^2 m_y m\tanh[\beta\mathcal{J}m]}{\cosh[\beta\mathcal{J}m]^2}, \quad (19)$$

$$c_x = \frac{m_x m_y^2}{\mathcal{J}m^4}\left[\frac{3\tanh[\beta\mathcal{J}m]}{\mathcal{J}m} + \beta(\tanh[\beta\mathcal{J}m]^2 - 3)\right] + \frac{m_x}{\mathcal{J}m^2}\left[\frac{\beta}{\cosh[\beta\mathcal{J}m]^2} - \frac{\tanh[\beta\mathcal{J}m]}{\mathcal{J}m}\right], \quad (20)$$

$$c_y = \frac{m_x^2 m_y}{\mathcal{J}m^4}\left[-\frac{3\tanh[\beta\mathcal{J}m]}{\mathcal{J}m} + \beta(3 - \tanh[\beta\mathcal{J}m]^2)\right], \quad (21)$$

with $m=|\vec{m}|=\sqrt{m_x^2+m_y^2}$.

The ordered system with vanishing $\tilde{\epsilon}$ and ϵ has a continuous (circular) symmetry, which implies that magnetization behaves uniformly in all possible directions. The continuous symmetry of the system is broken in the presence of the unidirectional annealed disorder. The possible directions of magnetizations can be deduced forthwith via a contour analysis [26]. This is done by identifying the zero-contour lines corresponding to the functions $f_x^{\tilde{\epsilon},\epsilon}(\vec{m})$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m})$ [see Eqs. (14) and (15)], and the intersection points of the lines are solutions of the magnetization. See Fig. 1. Contour analysis suggest two pos-

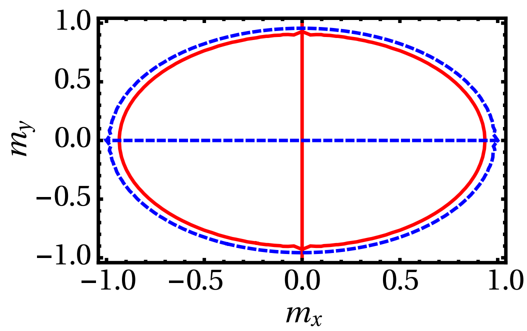

FIG. 1. Contour plot showing directions of magnetization in the presence of the disorder. Zero contour lines corresponding to $f_x^{\tilde{\epsilon},\epsilon}(\vec{m})$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m})$ in Eqs. (14) (solid red) and (15) (dotted blue) for $\epsilon/\mathcal{J} = 0.2$, $\tilde{\epsilon}/\mathcal{J} = 0.05$, and $\mathcal{J}\beta = 2$, as functions of m_x and m_y. All quantities are dimensionless.

sible solutions: The system magnetizes either in the transverse direction of the external annealed field, i.e., $m_x \neq 0, m_y = 0$ (case I) or in the parallel direction of the random field, i.e., $m_x = 0, m_y \neq 0$ (case II). By setting $\vec{m} = (m\cos\phi, m\sin\phi)$, the transverse and the parallel magnetization correspond to $\phi = 0$ and $\pi/2$, respectively. For ease of reference, we will henceforth use $m_\perp$ for m_x ($m_\parallel$ for m_y) to refer to the transverse (parallel) magnetization.

In order to derive the expressions for the critical temperature and the scalings of the magnetizations near criticality, we perform another round of Taylor expansions in Eqs. (14) and (15) around $m = 0$. The leading order behavior of the functions $f_x^{\tilde{\epsilon},\epsilon}(\vec{m})$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m})$ for small m are given by

$$\begin{aligned} f_x^{\tilde{\epsilon},\epsilon}(\vec{m}) &= \left[-1 + \mathcal{J}\left(\beta - \frac{\epsilon^2\beta^3}{3}\right)\right] m\cos\phi \\ &\quad + \frac{1}{3!}\left[\frac{2}{5}\mathcal{J}\beta^3[\mathcal{J}^2(-5 + 4\epsilon^2\beta^2) - 15\tilde{\epsilon}^2]\right] \\ &\quad \times m^3\cos^3\phi + O(m^5) \end{aligned} \tag{22}$$

and

$$\begin{aligned} f_y^{\tilde{\epsilon},\epsilon}(\vec{m}) &= (-1 + \mathcal{J}\beta)m\sin\phi \\ &\quad + \frac{1}{3!}[-2J\beta^3(J^2 + 3\tilde{\epsilon}^2)]m^3\sin^3\phi + O(m^5). \end{aligned} \tag{23}$$

Now as discussed earlier in context of the contour analysis, the allowed values of ϕ are 0 (case I) and $\pi/2$ (case II). For transverse magnetization, $\phi = 0$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m})$ vanishes identically. Equation (22) leads us to

$$m_\perp = \pm\sqrt{5}\sqrt{\frac{3(\mathcal{J}\beta - 1) - \mathcal{J}\epsilon^2\beta^3}{\mathcal{J}\beta^3[\mathcal{J}^2(5 - 4\epsilon^2\beta^2) + 15\tilde{\epsilon}^2]}}. \tag{24}$$

The system magnetizes in the perpendicular direction only below a certain critical temperature. This can be obtained by setting $m_\perp = 0$, where the critical temperature is given by

$$\beta_{c,\perp} = \frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}. \tag{25}$$

Similarly, the parallel magnetization can be obtained by setting $\phi = \pi/2$ in Eqs. (22) and (23). For this case, the expression in Eq. (22) vanishes identically, and by equating the expression in Eq. (23) to zero, we obtain

$$m_\parallel = \pm\sqrt{3}\sqrt{\frac{\mathcal{J}\beta - 1}{\mathcal{J}\beta^3(\mathcal{J}^2 + 3\tilde{\epsilon}^2)}}. \tag{26}$$

Setting $m_\parallel = 0$, we find the critical temperature to be given by

$$\beta_{c,\parallel} = \frac{1}{\mathcal{J}}. \tag{27}$$

Note that from the set of Eqs. (24)–(27), one can recover the results for the ordered system by setting $\tilde{\epsilon} = 0$ and $\epsilon = 0$. We find that for case being studied in this section, i.e., for a system with quenched randomness in the interaction term and annealed randomness in the field term, both parallel and transverse magnetizations survive the onslaught of the defects in the system as modeled by the disordered parameters in the Hamiltonian. Interestingly, the critical temperatures are not affected by the presence of the quenched disorder (in the interaction terms). Moreover, the annealed randomness in the field term does not influence the parallel critical temperature, although it lowers the transverse critical temperature. Our analysis also reveals that both transverse and parallel magnetizations are lowered in magnitude compared to the ordered system due to the presence of quenched disorder. However, it is only the transverse magnetization on which the annealed randomness has an impact, and the effect is to reduce the magnetization, while the parallel magnetization remains unfazed in the presence of the annealed disorder in the field term.

B. Away from critical point

Away from the critical point, the perturbative approach fails. We numerically find out the roots of the coupled set of equations, obtained by setting the expressions in Eqs. (12) and (13) equal to zero, i.e., $f_x^{\tilde{\epsilon},\epsilon}(\vec{m}) = 0$ and $f_y^{\tilde{\epsilon},\epsilon}(\vec{m}) = 0$. We perform the configurational averaging for 8000 random realizations for each type of disorder, viz. η and $\tilde{\eta}$. As predicted by the perturbative approach, the numerical simulations also indicate two possible directions of magnetization—the system can either magnetize along the transverse direction of the external annealed disordered field or it can magnetize in the direction parallel to it.

In Fig. 2 we show the results obtained from numerical analysis for the transverse magnetization, i.e., $m_\perp$ is nonzero and $m_\parallel = 0$ for $\epsilon/\mathcal{J} = 0.1$ and $\tilde{\epsilon}/\mathcal{J} = 0.15$. At high temperature (above the critical temperature), the system does not magnetize. For $\beta > \beta_{c,\perp}$, the system magnetizes in the direction which is transverse to the applied random field. For the ordered system, the critical temperature corresponds to $J\beta = 1$. The spontaneous magnetization persists in the presence of disorder, albeit with a reduced critical temperature. The behavior of parallel magnetization $m_{||}$ obtained from the numerical simulations also confirms the trends from the perturbative derivations.

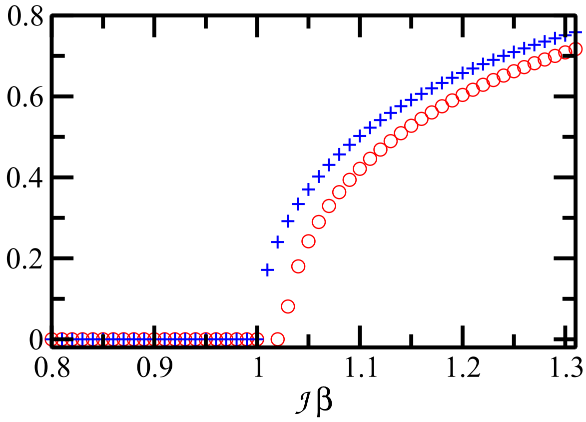

FIG. 2. Spontaneous transverse magnetization in the joint presence of quenched and annealed disorders. The plot shows numerical results for transverse magnetization in the presence of annealed disorder in the field term and quenched disorder in the interaction term, and is compared to the same in the pure system. Blue pluses correspond to the magnetization of the pure system and red circles correspond to the transverse magnetization in the presence of annealed disorder in the field term and quenched disorder in the interaction term, obtained by solving for roots of Eqs. (12) and (13) with $\epsilon/\mathcal{J} = 0.1$ and $\tilde{\epsilon}/\mathcal{J} = 0.15$. All quantities are dimensionless. The vertical axis represents the transverse magnetization for the disordered case, and the magnetization in the pure case.

V. OTHER COMBINATIONS OF QUENCHED AND ANNEALED DISORDERS

To perform comparative studies between different kinds of disordered systems, we shall now adopt similar techniques as in the preceding section. Let us now consider all possible combinations of the three categories as mentioned in Sec. III, obtained by considering different types of disorders in the interaction and the field terms of the Hamiltonian in Eq. (5). One of these cases has already been discussed in the last section. We summarize our results in Table I that considers all such possible combinations and also includes cases where disorder is absent. We introduce the following notations for convenience: $\langle\eta\rangle_a$ ($\langle\tilde{\eta}\rangle_a$) implies that the disorder in the field (interaction) term is annealed. Moreover, the same symbol also denotes the mean of the corresponding annealed distribution. On the other hand, $\langle\eta\rangle_q$ ($\langle\tilde{\eta}\rangle_q$) implies that the disorder in the field (interaction) term is quenched, and the same symbol also denotes the mean of the corresponding quenched distribution. The variances of the distributions of all the disordered random variables are taken to be unity. We also use following shorthand notation: $a_1 = \mathcal{J}\beta - 1, a_2 = 3(1 - \mathcal{J}\beta) + \mathcal{J}\epsilon^2\beta^3$, $a_3 = 1 - \mathcal{J}\beta + \mathcal{J}\beta^3\epsilon^2$, $b_1 = -5 + 4\epsilon^2\beta^2$, $b_2 = 4\epsilon^2\beta^2 - 1$, $b_3 = \mathcal{J}^2 + 3\tilde{\epsilon}^2$.

Here we briefly describe the results summarized in Table I. Case 1 corresponds to the case when the system is free from any kind of disorder. The isotropic quantum XY model in absence of any external field manifests a spontaneous magnetization which has a continuous circular symmetry. The spontaneous magnetization occurs below a critical temperature constrained by the condition $\beta = 1/\mathcal{J}$.

Let us now discuss the cases that belong to category (i), as described in Sec. III. As expected, the system retains its circular symmetry in the presence of an annealed disorder in the interaction part (case 2). Surprisingly, the presence of an annealed disorder in the interaction part neither disturbs the magnitude of the magnetization nor does it shift the critical temperature. However, when the clean system is subjected to an annealed randomness, only in the field term (case 3), the situation changes. The circular symmetry of the system is broken and the system now possesses magnetization in the direction either parallel or transverse to the applied field. Although the magnitude of the parallel magnetization $m_\parallel$ and corresponding critical temperature $(\kappa_B\beta_{c,\parallel})^{-1}$ remain unaltered due to this annealed field, the magnitude of $m_\perp$ as well as the corresponding critical temperature are lowered compared to the ordered system. We find that the results in case 4, where annealed disorder is present both in the interaction and the field terms, are identical with those of

TABLE I. A comparison of the magnetizations and the critical temperatures for the different combinations of disorders. $m_\perp$ and $m_\parallel$ denotes, respectively, the magnetizations transverse and parallel to the applied random field. $\beta_{c,\perp}$ and $\beta_{c,\parallel}$ are proportional to the inverse of the critical temperatures in the transverse and parallel directions, respectively.

Case	Interaction term	Field term	$m_\perp$	$m_\parallel$	$\beta_{c,\perp}$	$\beta_{c,\parallel}$
1	$\tilde{\eta} = 0$	$\eta = 0$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}}$	$\frac{1}{\mathcal{J}}$
2	$\langle\tilde{\eta}\rangle_a = 0$	$\eta = 0$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}}$	$\frac{1}{\mathcal{J}}$
3	$\tilde{\eta} = 0$	$\langle\eta\rangle_a = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{b_1\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}}$
4	$\langle\tilde{\eta}\rangle_a = 0$	$\langle\eta\rangle_a = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{b_1\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}}$
5	$\langle\tilde{\eta}\rangle_q = 0$	$\eta = 0$	$\pm\sqrt{3}\sqrt{\frac{a_1}{b_3\mathcal{J}\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{b_3\mathcal{J}\beta^3}}$	$\frac{1}{\mathcal{J}}$	$\frac{1}{\mathcal{J}}$
6	$\tilde{\eta} = 0$	$\langle\eta\rangle_q = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{b_1\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_3}{b_2\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{\mathcal{J}^3}$
7	$\langle\tilde{\eta}\rangle_q = 0$	$\langle\eta\rangle_q = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{\mathcal{J}\beta^3[b_1\mathcal{J}^2-15\tilde{\epsilon}^2]}}$	$\pm\sqrt{3}\sqrt{\frac{a_3}{\mathcal{J}\beta^3(b_2\mathcal{J}^2-3\tilde{\epsilon}^2)}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{\mathcal{J}^3}$
8	$\langle\tilde{\eta}\rangle_a = 0$	$\langle\eta\rangle_q = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{b_1\mathcal{J}^3\beta^3}}$	$\pm\sqrt{3}\sqrt{\frac{a_3}{b_2\mathcal{J}^3\beta^3}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{\mathcal{J}^3}$
9	$\langle\tilde{\eta}\rangle_q = 0$	$\langle\eta\rangle_a = 0$	$\pm\sqrt{5}\sqrt{\frac{a_2}{\mathcal{J}\beta^3[b_1\mathcal{J}^2-15\tilde{\epsilon}^2]}}$	$\pm\sqrt{3}\sqrt{\frac{a_1}{b_3\mathcal{J}\beta^3}}$	$\frac{1}{\mathcal{J}} + \frac{\epsilon^2}{3\mathcal{J}^3}$	$\frac{1}{\mathcal{J}}$

case 3. This is intuitively understandable from our analyses in cases 1 and 2, where the presence of an annealed disorder in the interaction term has no effect on the magnetizations and critical temperatures of the system in the mean-field limit.

Let us now look into the cases that belongs to category (ii) of Sec. III. Case 5 represents the situation when there is quenched randomness in the interaction term. The system preserves continuous symmetry of the spontaneous magnetizations, the critical temperature remains unaltered. However, the magnetization gets affected and shrinks in magnitude. For the case where interaction is ordered but the field is quenched disordered (case 6), the continuous symmetry is broken, the system exhibits transverse and parallel magnetizations, albeit with a lowered value compared to the magnetization in the clean system and requires lower temperatures to magnetize. Interestingly the effect of disorder is more pronounced in the parallel direction than in the transverse direction. Finally, we find that the behavior of the system with quenched randomness in both interaction and field parts (case 7) is qualitatively similar to the previous case.

Finally, we consider the cases in category (iii) of Sec. III. For the cases in this category, annealed and quenched disorders are both introduced in the system—one in the interaction term and another in the field term. One of such scenarios (case 9) was considered at length in Sec. IV. The other one is case 8. Consistent with what we have seen in previous cases, the annealed disordered interaction does not have any effect on the magnetizations and the critical temperatures. Any disorder effect in this case is only due to the quenched disorder present in the field term. Therefore, the perturbative formulas in this case are identical with case 6, where there was no disorder present in the interaction term.

VI. CONCLUSIONS

In summary, this work examines quantum spin-$1/2$ XY models with continuous and broken continuous isotropic symmetries within the mean-field framework, and investigates the effect on spontaneous magnetization due to the presence of disorders in external field or/and in the couplings. The disorders we consider can be annealed or quenched in nature.

A combined approach of perturbative analysis and numerical simulation has been adopted for characterizing the spontaneous magnetizations in the systems. We derive exact analytical expressions, within a perturbative approach, for the critical temperatures and near-critical scalings of magnetization corresponding to the various combinations of the disorders, and carry out a comparative study. The results obtained within the perturbative theory are found to match with those obtained from the numerical simulations. A key focus has been on systems that exhibit a joint presence of annealed and quenched disorders, and we discuss the corresponding effect on spontaneous magnetizations and their critical temperatures. We find that spontaneous magnetization persists in the presence of randomness in these models. The ordered system as well as the disordered systems with randomness only in couplings, exhibit magnetization for all possible orientations due to the continuous circular symmetry. The circular symmetry breaks down in the presence of an infinitesimal unidirectional disordered field. In the presence of the random field (with zero mean), which can be annealed or quenched, the system still exhibits magnetization for two selective orientations—parallel or transverse to the external field. The parallel magnetization remains untouched while the transverse one survives, but is decreased, with the introduction of an annealed disordered field. Moreover, the annealed disorder present in the interaction term does not affect the system's spontaneous magnetization and the corresponding critical temperature, irrespective of the presence or absence of quenched or annealed disorder in the field term.

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