Fractional charge oscillations in quantum dots with quantum spin Hall effect

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(Received 12 December 2016; published 16 May 2017)

We show that correlated two-particle backscattering can induce fractional charge oscillations in a quantum dot built at the edge of a two-dimensional topological insulator by means of magnetic barriers. The result nicely complements recent works where those fractional oscillations were predicted in the strong-coupling regime. Moreover, since by rotating the magnetization of the barriers a fractional charge can be trapped in the dot via the Jackiw-Rebbi mechanism, the system we analyze offers the opportunity to study the interplay between this noninteracting charge fractionalization and the fractionalization due to two-particle backscattering. We demonstrate that the number of fractional oscillations of the charge density depends on the magnetization angle. In fact, a rotating magnetization can add or subtract fractional charges from the dot continuously. Finally, we address the renormalization induced by two-particle backscattering on the spin density, which is characterized by a dominant oscillation with a length twice as large as the charge-density oscillations.

DOI: 10.1103/PhysRevB.95.205418

I. INTRODUCTION

As nanotechnology and material science advance, phenomena that are peculiar to high-energy physics can be brought down to the energy scale of condensed-matter physics. Apart from the celebrated Anderson-Higgs mechanism in superconductors [1], a paradigmatic example is graphene. Its low-energy properties are well described by Dirac-like Hamiltonians [2]: Klein tunneling has been predicted theoretically [3] and observed experimentally [4], and *Zitterbewegung* is believed to matter for the motion of its electrons [5]. More recently, it has been shown that the chiral anomaly [6–8] is crucial in the understanding of the electromagnetic response of Weyl semimetals [9–20] and the behavior of two-dimensional topological insulators in the presence of magnetic barriers [21]. Another connection between high-energy and condensed-matter physics is charge fractionalization due to the Jackiw-Rebbi mechanism [22], which has been shown to play a role in polyacetylene [23]. Fractional charges with charge $e/2$ also have been proposed recently to appear in carbon nanotubes under the influence of nonuniform strain and magnetic fields [24]. More general fractional charges, corresponding to complex solitons [25], are hosted by magnetically defined quantum dots defined at the edges of two-dimensional topological insulators [21,26–28], even in the presence of weak interactions [27,28].

A different type of charge fractionalization is known to take place in strongly interacting condensed-matter systems. Apart from the well-established fractional quantum Hall effect in two spatial dimensions [29], in one dimension, the interplay between strong spin-orbit coupling and electron-electron interactions is predicted to lead to charge fractionalization [30–34] and, in the presence of superconductors, to parafermions [35–38]. In the absence of superconductors, a powerful tool to investigate the emergence of fractionalization phenomena is represented by the study of the density oscillations: The fractional charge oscillations that emerge can compete with Wigner oscillations and, eventually, be dominant when Wigner oscillations have a less favorable scaling exponent or when they are absent [30,31]. Here, by fractional charge oscillations we mean oscillations of the electron charge with a number of peaks exceeding the number of electrons so that, qualitatively speaking, each peak only accounts for a fraction of an electron. From a technical point of view, fractional charge oscillations are due to sine-Gordon-type terms in Luttinger liquid Hamiltonians [39,40], and the mathematical treatments, which usually are performed, rely on strong-coupling limits. This issue represents a weakness in comparison to the Luttinger liquid theory of Wigner oscillations. In fact, the onset of Wigner oscillations in one-dimensional quantum dots can be captured very well by a Luttinger liquid theory enriched by a first-order perturbation theory in umklapp scattering. The results obtained in this context are in very good agreement with numerical results obtained on the one-dimensional Hubbard model by means of density-matrix renormalization-group (RG) analysis [41].

The aim of this paper is twofold: On one hand we establish the presence of fractional charge oscillations in a quantum spin Hall quantum dot in the presence of two-particle backscattering without relying on strong-coupling approximations but by means of a simple perturbative approach. For completeness, we also show that the spin density of the system only acquires small corrections of wave-vectors $2k_F$ and $6k_F$, k_F being the Fermi momentum, which are strongly suppressed by their scaling exponents. These corrections do not significantly alter the $2k_F$ spin oscillation characterizing the system in the absence of two-particle backscattering. On the other hand, the system we inspect is characterized by fractional charges induced via the Jackiw-Rebbi mechanism by the magnetic barriers defining the quantum dot. It, hence, represents an ideal playground for studying the interplay between the two different kinds of charge fractionalization. In this context we show how the relative magnetization of the barriers affects the charge and spin-density profiles. We further discuss how the dot can be filled or emptied with fractional charges of any portion by a rotation of the magnetic barrier.

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The outline of this paper is as follows: In Sec. II we start by presenting the model for the unconfined quantum spin Hall system. Particular attention is devoted to the two-particle backscattering term, which is demonstrated to be important even away from half-filling. The effects of confinement by magnetic barriers is analyzed in Sec. III. Then, in Sec. IV we address the effect of two-particle backscattering on the charge density. We show that fractional oscillations emerge and that their wavelengths depend on the fractional charge trapped in the dot. In Sec. V we address the spin-density oscillations, which also depend, in an interesting way, on the Jackiw-Rebbi charge. The conclusions are drawn in Sec. VI.

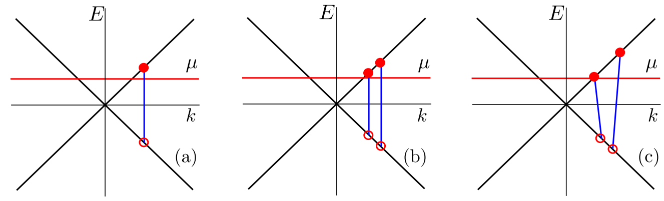

FIG. 1. Schematic of the dispersion relation of a helical one-dimensional system where the chemical potential μ is indicated. The virtual transitions associated with (a) magnetic fields, (b) two-particle backscattering at $q = 0$, and (c) two-particle backscattering for $q \neq 0$ are shown.

II. MODEL AND TWO-PARTICLE BACKSCATTERING

The main ingredient of our model is a one-dimensional edge of a two-dimensional topological insulator. We fix our reference frame so that spin-up/spin-down electrons move right/left. We adopt, for now, periodic boundary conditions on a length $\mathcal{L}$. The Hamiltonian H_0 reads [42,43]

$$H_0 = \int_0^{\mathcal{L}} dx\, \Psi^\dagger(x)(-i v_F \sigma_3 \partial_x)\Psi(x), \tag{1}$$

where $\Psi(x) = (\psi_+, \psi_-)^T$ is the Fermi spinor with $\pm$ as the spin projection, v_F is the Fermi velocity, and σ_3 is the third Pauli matrix in the usual representation. When contact density-density interactions are added, the Hamiltonian becomes a helical Luttinger liquid with Hamiltonian,

$$H = \frac{1}{2\pi}\int_0^{\mathcal{L}} dx\, uK(\partial_x\theta)^2 + \frac{u}{K}(\partial_x\phi)^2 + \frac{\pi u}{LK}(N_+^2 + N_-^2), \tag{2}$$

where u is the velocity of the bosonic excitations, K is the Luttinger parameter ($K < 1$ for repulsive interactions, and $K = 1$ in the absence of interactions), θ and ϕ are the Luttinger bosonic fields, and $N_\pm$ are operators counting spin-up/spin-down electrons, respectively. In terms of the bosonic fields, the Fermi fields read

$$\psi_\pm(x) = \frac{U_\pm}{\sqrt{2\pi\alpha}} e^{\pm i[(2\pi N_\pm x)/L]} e^{-i[\pm\phi(x)-\theta(x)]}, \tag{3}$$

where α is the Luttinger liquid cutoff and $U_\pm$ are Klein factors.

When axial spin symmetry is broken, a new interaction term, which becomes relevant in the RG sense for $K < 1/2$, can emerge [42], namely, two-particle backscattering H_{2p}. Explicitly, in the fermionic language, one has

$$H_{2p} = g_{2p}\int_0^L dx\, \psi_+^\dagger(\partial_x\psi_+^\dagger)(\partial_x\psi_-)\psi_- + \text{H.c.}, \tag{4}$$

where g_{2p} is the coupling constant. The process amounts to flipping two spins and hence to backscatter two electrons. It represents the only time-reversal invariant nonchiral interaction term that can be added to Eq. (2) [42]. In order to better understand the physical process involved, it is useful to expand the Fermi operators on the eigenstates of H_0 with wave-vector $k = 2n\pi/\mathcal{L}$, n being an integer. Explicitly, we use

$$\psi_\pm(x) = \sum_k \frac{e^{ikx}}{\sqrt{\mathcal{L}}} c_{k,\pm}, \tag{5}$$

with $c_{k,s}$ as fermionic annihilation operators. We then obtain

$$H_{2p} = \frac{g_{2p}}{\mathcal{L}} \sum_{k_1,k_2,q} k_2(k_2+q) c_{k_1,+}^\dagger c_{k_2,+}^\dagger c_{k_2+q,-} c_{k_1-q,-} + \text{H.c.} \tag{6}$$

It is important to note that H_{2p} commutes with the total momentum since it respects translational invariance, whereas it does not commute with the noninteracting Hamiltonian, just as the usual Coulomb interactions. There is, however, an important difference with respect to the usual interactions: When density-density interactions are considered, the terms associated with $q = 0$ and $q = 2k_F$ (with k_F as the Fermi momentum) mix noninteracting levels, which are very close in energy, independently of the chemical potential. On the other hand, when the chemical potential μ is not at the Dirac point, all the virtual transitions that two-particle backscattering can induce couple states which, with respect to the noninteracting Hamiltonian, are at least 4μ apart in energy [see Figs. 1(b) and 1(c)]. The importance of two-particle backscattering is hence expected to be reduced but not immediately negligible as the chemical potential is tuned away from the Dirac point. An intuitive way to convince ourselves that H_{2p} should be taken into account, even when the system is away from half-filling, is to consider the effects of a very similar, although much simpler, operator: a uniform ferromagnetic coupling in the x direction. Specifically, consider the contribution to the Hamiltonian,

$$H_B = B\int_0^{\mathcal{L}} dx\, \psi_+^\dagger(x)\psi_-(x) + \text{H.c.} = B\sum_k c_{k,+}^\dagger c_{k,-} + \text{H.c.} \tag{7}$$

Exactly as H_{2p}, H_B conserves the total momentum and, when the chemical potential is tuned away from the Dirac point, the energy threshold for virtual states is nonzero (it is 2μ instead of 4μ since only a single electron is now involved). Still, it is very simple to diagonalize the Hamiltonian $H_0 + H_B$ and to convince ourselves that all eigenstates and all eigenvectors are modified by H_B, although, of course, as the chemical potential is tuned away from the Dirac point, the effects of the magnetic coupling decrease. A scheme of the virtual transitions induced by the ferromagnetic coupling is shown in Fig. 1(a).

Moreover, it is worth pointing out that two-particle backscattering in helical systems is essentially a spin-spin

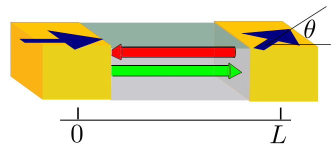

FIG. 2. Schematic of the quantum dot. Two magnetic barriers, whose in-plane magnetization differs by an angle θ, confine the helical liquid on the segment $[0,L]$.

interaction since one has

$$H_{2p} \propto g_{2p} \int_0^L dx\, s_x(x)^2 + 2\psi_+^\dagger(x)\psi_+(x)\psi_-^\dagger(x)\psi_-(x), \quad (8)$$

where $s_x(x) = \psi_+^\dagger(x)\psi_-(x) + \text{H.c.}$ so that, if the system is characterized by a nontrivial spin texture, two-particle backscattering is expected to induce interesting modifications thereof.

III. THE HELICAL QUANTUM DOT

The model we now inspect is an interacting quantum dot built at the edge of a two-dimensional topological insulator. Namely, a small part of length $L \ll \mathcal{L}$ of the system inspected in the previous section hosts a quantum dot. The confinement mass is provided by two magnetic barriers, whose magnetization is assumed to lie on the plane of the quantum spin Hall bar. The angular difference between the magnetization of the two barriers is assumed to be θ. For a scheme, see Fig. 2. As previously discussed in the literature [26], a fractional background charge of $\theta/2\pi$ is trapped in the dot via the Jackiw-Rebbi mechanism. It is worth mentioning that the fractional charge is not pinned at the boundaries of the dot, but it is homogeneously distributed as the whole quantum dot represents the mass kink of the Dirac equation describing the helical edge. Moreover, the two components of the spinor in the dot region are not independent. They satisfy the boundary conditions $\psi_-(x) = -i\psi_+(-x)$ [27] with

$$\psi_+(x) := \frac{U}{\sqrt{2\pi\alpha}} e^{i(\pi x/L)[N-(1/2)+(\theta/2\pi)]} e^{i\phi(x)}. \quad (9)$$

Note that now a single Klein factor U and a single number operator N are sufficient. Since we will only inspect situations with a fixed number of electrons in the quantum dot, the operator N and its average on any state considered will be indicated with the same symbol. The interacting Hamiltonian H of the quantum dot reads

$$H = H_0 + H_{2p}, \quad (10)$$

where

$$H_0 = \frac{\pi v_F}{KL}\sum_{n>0} n a_n^\dagger a_n + \frac{\pi v_F}{2K^2 L}\left(N + \frac{\theta}{2\pi}\right)^2,$$

with bosonic annihilation operators a_n. Moreover,

$$H_{2p} = -\frac{g_{2p}}{2(\pi\alpha)^2}\int_0^L \mathcal{H}_{2p},$$

$$\mathcal{H}_{2p} = \cos\left[\frac{4\pi x}{L}\left(N - \frac{1}{2} + \frac{\theta}{2\pi}\right) - 4\varphi(x) - 4f(x)\right], \quad (11)$$

with $\varphi(x) = [\phi(-x) - \phi(x)]$, $f(x) = [\phi(x),\phi(-x)]/(4i)$, and

$$\phi(x) = \sum_{n>0} \frac{e^{-(\alpha n\pi/2L)}}{\sqrt{n}}\left[\frac{1}{\sqrt{K}}\cos\left(\frac{n\pi x}{L}\right) + i\sqrt{K}\sin\left(\frac{n\pi x}{L}\right)\right] a_n + \text{H.c.} \quad (12)$$

As a complete basis of states we will use the eigenstates $|\underline{n}\rangle = |N,\{n_j\}_{j>0}\rangle$ of H_0. These states are characterized by the number N of electrons in the dot and the occupation numbers of the bosonic modes. Explicitly $\langle N,\{n_j\}_{j>0}|N|N,\{n_j\}_{j>0}\rangle = N$ and $\langle N,\{n_j\}_{j>0}|a_k^\dagger a_k|N,\{n_j\}_{j>0}\rangle = n_k$. The state with N electrons in the dot and no bosonic excitations, explicitly given by $|N,\{n\}_{j>0}\rangle = |N,\{0\}_{j>0}\rangle$, is indicated with the symbol $|N\rangle$. The quantum dot described so far has interesting properties even in the absence of H_{2p}, notably, a fractional charge $\theta/2\pi$ is trapped into it. Moreover, the spin helix characterizing the usual unconfined helical Luttinger liquid is here pinned by the magnetic impurities. This pinning gives rise to nontrivial spin oscillations, whose typical wave-vector $k_s = (2\pi x/L)[N - 1/2 + \theta/(2\pi)]$ depends on the fractional charge trapped in the system. However, the zero-temperature average charge density is flat due to the absence of Friedel oscillations in the original helical Luttinger liquid. It is worth mentioning that, in contrast to spin-orbit-coupled one-dimensional quantum dots [44], there are no bumps of the electron density localized at the barriers. In the next sections, we address how two-particle backscattering affects the charge and spin densities of the quantum dot.

IV. PARTICLE DENSITY

Due to spin-momentum locking, the density-density correlation functions of the helical Luttinger liquid, in the absence of two-particle backscattering, do not show signatures of Friedel and Wigner oscillations [45,46]. The presence of impurities does not alter this behavior [27]. The density operator $\rho(x) = \psi_+^\dagger(x)\psi_+(x) + \psi_+^\dagger(-x)\psi_+(-x)$ reads

$$\rho(x) = \frac{N}{L} + \frac{\theta}{2L\pi} - \frac{i\sqrt{K}}{L}\sum_{n>0}\gamma_n(a_n^\dagger - a_n), \quad (13)$$

with

$$\gamma_n = \sqrt{n}e^{-(n\alpha\pi/2L)}\cos\frac{n\pi x}{L}. \quad (14)$$

The point splitting procedure is employed in the derivation of Eq. (13), and the twisted boundary conditions given in Eq. (9) must be taken into account, see, e.g., Appendix G of Ref. [47]. The expectation value $\bar{\rho}_0(x)$ of $\rho(x)$ on the N-electron ground-state $|N\rangle$ of H_0 is given by

$$\bar{\rho}_0(x) = \langle N|\rho(x)|N\rangle = \frac{N}{L} + \frac{\theta}{2L\pi}. \quad (15)$$

It is here worth noting three properties of the electron density. (i) Despite confinement and interactions, the average electron density in the absence of two-particle backscattering is fractional due to the Jackiw-Rebbi mechanism but flat, meaning that there are no bound states at the barriers. The fractional charge is homogeneously distributed inside the dot.

(ii) The total charge in the quantum dot is fractional. The whole quantum dot is, in fact, the mass domain generating fractional charges in the Jackiw-Rebbi model. The fractional charge contained in the dot is a sharp quantum number and does not affect the integerness of the total charge in the topological insulator as a whole [21]. (iii) No finite-size oscillations of the electron density are present in the absence of two-particle backscattering since the confinement does not couple to the charge density but rather to the spin density.

When two-particle backscattering is added, the state $|N\rangle$ is not the ground state of the theory anymore. The first-order correction $\delta\bar{\rho}(x)=\bar{\rho}(x)-\bar{\rho}_0(x)$ of the electron-density $\bar{\rho}(x)$ averaged over the ground state of $H=H_0+H_{2p}$ reads

$$\delta\bar{\rho}(x)=2\,\mathrm{Re}\sum_{|\underline{n}\rangle\neq|N\rangle}\frac{\langle N|\rho(x)|\underline{n}\rangle\langle\underline{n}|H_{2p}|N\rangle}{E_{|N\rangle}-E_{|\underline{n}\rangle}}$$
$$=d_0\int_0^L dy\left\{\sin\left[\frac{4\pi y\left(N+\frac{\theta}{2\pi}-\frac{1}{2}\right)}{L}-4f(y)\right]\right.$$
$$\left.\times\left[f\left(\frac{x+y}{2}\right)-f\left(\frac{x-y}{2}\right)\right]D(y)^{4K}\right\}. \quad (16)$$

Here, $d_0=2g_{2p}K^2/[(\pi\alpha)^2v_F]$, $E_{|\underline{n}\rangle}=\langle\underline{n}|H_0|\underline{n}\rangle$, and

$$D(x)=\frac{\sinh\left(\frac{\pi\alpha}{2L}\right)}{\sqrt{\sinh^2\left(\frac{\pi\alpha}{2L}\right)+\sin^2\left(\frac{\pi x}{L}\right)}} \quad (17)$$

is a damping factor. Note that only states $|\underline{n}\rangle$ representing states with N electrons in the dot give nonzero contributions to the sum. The resulting density corrections for different interaction strengths and angles are shown in Fig. 3. The more important

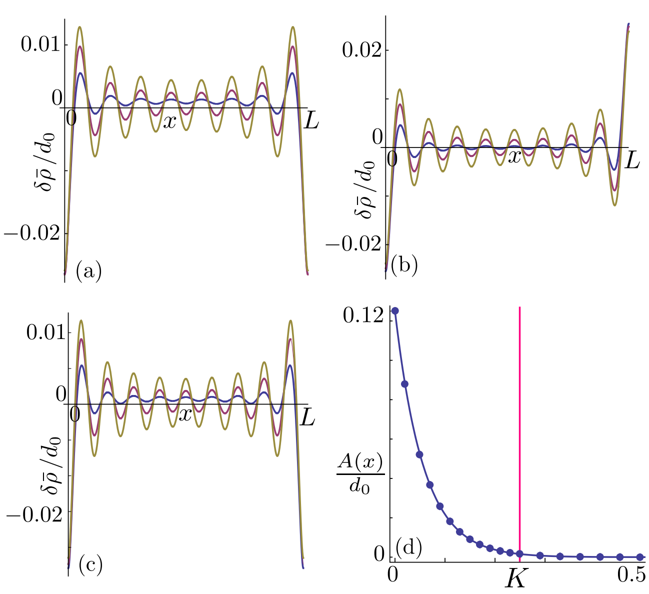

FIG. 3. Two-particle backscattering-induced oscillations $\delta\bar{\rho}(x)$ as a function x for $N=4$, $K=0.5$ (blue line), $K=0.3$ (violet line), and $K=0.2$ (brown line) and (a) $\theta=0$, (b) $\theta=\pi/2$, and (c) $\theta=\pi$. (d) The amplitude $A(x)$ of the oscillations as a function of the Luttinger parameter K (dots); the blue line plotted is $A(x)=D(L/2)^{(4K)}$, and the pink line is $K=1/4$.

feature is the emergence of oscillations with wave-vector $k_C\sim 4k_F=4N\pi/L$, in accordance with the strong-coupling limit. Since the system is a one-channel Luttinger liquid, $4k_F$ is not the wave vector of the usual Wigner oscillations, which would be characterized by the wave-vector $2k_F$ [48,49]. Hence, the number of peaks associated with a $4k_F$ oscillation is twice as large as the number of electrons in the dot. The presence of a dominant $4k_F$ oscillation therefore represents a very peculiar feature of the helical quantum dot in the presence of two-particle backscattering. These oscillations can be identified as a signature of the emergence of fractional charges of charge $e/2$ in the dot. These $e/2$ fractional charges are only due to strong interactions leading to two-particle backscattering and are distinct from the Jackiw-Rebbi charge induced by the magnetic barriers. There is, however, an interplay between the interaction-induced fractional charge oscillations and the Jackiw-Rebbi charges. In fact, half a period (one maximum of the density) is gained when the magnetic barrier is rotated. Although it is beyond the scope of this paper, one can speculate that, in the strong-coupling regime, when the barrier is rotated by $\pm\pi$, a sharp wave packet with fractional charge $e/2$ is introduced/expelled from the quantum dot. This behavior is due to the interplay of the chiral anomaly, responsible for the θ dependence of the particle number in the dot, and strong interactions. As expected, the oscillations become more pronounced as interactions are increased. Surprisingly, even in the finite-size setup under investigation, the Luther-Emery point [50] $K=1/4$ plays a crucial role: Intuitively, this point is special because it marks the beginning of the repulsive regime for the effective charge $1/2$ fermions [51]. We have numerically obtained the difference between the relative maximum and the relative minimum of the electron density, which are closer to $x=L/2$ (the center of the dot). This difference, normalized to d_0, is referred to as $A(x)$. The numerical points are indicated by the dots in Fig. 3(d). The line is proportional to $D(L/2)^{4K}$. Since in the usual spinless Luttinger liquid the same quantity scales as $D(L/2)^K$ [52], the fractional charge oscillations of the density can be interpreted as Wigner oscillations of the new fermions, which are noninteracting at $K=1/4$. A very drastic simplification of the formula in Eq. (16) can be obtained in the limits $\alpha\to 0$ and $K\to 0$. Although these limits are outside of the validity of perturbation theory, they allow us to clearly identify the $4k_F$ nature of the oscillation: Whereas the damping factor tends to 1 due to the scaling exponent, the term $f[(x+y)/2]-f[(x-y)/2]$ becomes piecewise linear, and the integral can easily be performed analytically. The correction to the electron density $\delta\bar{\rho}_\infty$ in this limit reads

$$\delta\bar{\rho}_\infty=-d_0\left(\frac{\cos\left[\frac{4\pi x}{L}\left(N+\frac{\theta}{2\pi}\right)\right]}{\frac{8}{L}\left(N+\frac{\theta}{2\pi}\right)}-\frac{\pi\,\sin(2\theta)}{8(2N\pi+\theta)^2}\right). \quad (18)$$

This formula is plotted for clarity in Fig. 4 as a function of x and θ. The influence of the chiral anomaly is clear: As θ is increased, the number of particles and the number of peaks in the density are increased as well. Moreover, the result in Eq. (18) is in qualitative accordance with both the strong-coupling regime and the physical intuition since the amplitude of the correction is reduced as the chemical potential is tuned away from the Dirac point.

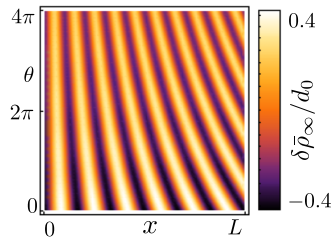

FIG. 4. Density plot of $\delta\bar{\rho}_\infty$ for $N = 3$ in units of d_0 as a function of x and of angle θ.

Let us briefly summarize the physics inspected up to this point. A fractional charge of arbitrary value can be trapped between two magnetic barriers implanted on a noninteracting quantum spin Hall system. This fractional charge is not pinned at the edges of the dot, but it is uniformly distributed in the dot [21,26]. When spin preserving electron-electron interactions are considered, this picture is not significantly modified [27]. When axial spin symmetry is broken in the bulk of the topological insulator, two-particle backscattering can emerge. For strong enough interactions, this new term makes the relevant quasiparticles describing the quantum dot fractional with charge $e/2$ [30,33]. In the quantum dot under inspection, fractionalization of the quasiparticles is captured by the electron density. For parallel magnetization of the barriers, which means no Jackiw-Rebbi fractional charge, the density shows a number of peaks which is twice the number of electrons trapped in the dot, or, in other words, is equal to the number of the fractional quasiparticles generated by two-particle backscattering. Note that in the usual one-dimensional systems the ground-state charge oscillations range from half (one-fourth in the case of carbon nanotubes) of the number of electrons, called Friedel oscillations, to a number of oscillations that equals the number of electrons, called Wigner oscillations. We are, hence, facing a different scenario: a Wigner oscillation of fractional quasiparticles. Moreover the Jackiw-Rebbi fractional charges of any value enrich the physics. By rotating the magnetization of one of the barriers, individual fractional quasiparticles can be manipulated as can be seen from the shift of the peaks of the electron density. As a by-product, one can note that, when the Jackiw-Rebbi fractional charge is $e/2$ in the presence of two-particle backscattering, the dot is in a state which is well described as a Wigner oscillation of an odd number of fractional charges with charge $e/2$.

The nonconservation of the charge in the dot as the magnetization of the barriers is rotated has a deep origin. The quantum dot investigated in this paper, but in the absence of electron-electron interactions and two-particle backscattering, has been shown [21] to be equivalent to a quantum ring with a linear dispersion relation pierced by a magnetic flux. The role of the magnetic flux is taken by angle θ. In the same way the chiral anomaly induces persistent currents in the quantum ring, angle θ is responsible for the nonconservation of the charge in the dot.

The experimental detection of the charge oscillations can be carried out by means of scanning gate microscopy [53–55], provided that the effect is strong enough. Although an estimation of the magnitude of the oscillations would require the consideration of the full two-dimensional topological insulator, which is beyond the scope of this paper, encouraging indications can be obtained from related systems. (i) Strongly interacting quantum spin Hall systems with Luttinger parameter $K \sim 0.25$ have been reported [56]. (ii) Two-particle backscattering has been shown to be able to open helical gaps in spin-orbit-coupled quantum wires [57]. (iii) It has been proposed in order to interpret experimental data to take into account parafermions when topological insulators are brought in proximity to superconductors [58]. Scenarios dominated by two-particle backscattering are hence experimentally relevant.

V. SPIN DENSITY

In this section, we examine the effect of two-particle backscattering on $s_x(x) = \Psi^\dagger(x)\sigma_x\Psi(x)$ in the quantum dot. The effects on $s_y(x) = \Psi^\dagger(x)\sigma_y\Psi(x)$ are similar but shifted by half an oscillation so that the rotating spin pattern discussed in Ref. [28] also characterizes the two-particle backscattering-induced corrections. The third component $s_z(x) = \Psi^\dagger(x)\sigma_z\Psi(x)$ is not affected at all by two-particle backscattering. The correction $\delta\bar{s}_x(x)$ to the average of $s_x(x)$ on the ground state of H_0 is given to first order in H_{2p} by

$$\delta\bar{s}_x(x) = 2\,\mathrm{Re}\sum_{|\underline{n}\rangle\neq|N\rangle}\frac{\langle N|s_x(x)|\underline{n}\rangle\langle\underline{n}|H_{2p}|N\rangle}{E_{|N\rangle}-E_{|\underline{n}\rangle}}. \tag{19}$$

Unfortunately, in the present case we were not able to find an explicit form for the corrections since the result of the calculation contains entangled series. Using the well-known relations between the matrix elements of the Fermi operator on the Luttinger liquid eigenstates and the Laguerre polynomials [59,60] we could, however, derive

$$\delta\bar{s}_x(x) = \frac{2g_{2p}}{(2\pi\alpha)^2}\mathrm{Re}\int_0^L dy\sum_{\{\mathbf{n}\}\neq\{\mathbf{0}\}}\frac{T(x,y,N,\theta,\{\mathbf{n}\})}{E_{\{\mathbf{n}\}}}. \tag{20}$$

Several quantities are here defined: $\{\mathbf{n}\}\neq\{\mathbf{0}\}$ is any succession of non-negative integers. From the physical point of view, the sum emerges from the necessity to consider every possible configuration of the bosonic field, that is, every possible occupation number n_p of the pth bosonic mode. The energy factor is given by $E_{\{\mathbf{n}\}} = \sum_{p=1}^{\infty}\frac{n_p p v_F}{KL}$. Furthermore, we have introduced

$$T(x,y,N,\theta,\{\mathbf{n}\}) = \sum_{\xi_1=\pm,\xi_2=\pm}\xi_1\exp\left(\frac{2\xi_1 i\pi x[N-1/2+\theta/(2\pi)]}{L}-2f(2x)\right)$$
$$\times A(x,\xi_1,\{\mathbf{n}\})\exp\left(\frac{4\xi_2 i\pi y[N-1/2+\theta/(2\pi)]}{L}-4f(2y)\right)A(y,2\xi,\{\mathbf{n}\})/(4i),$$

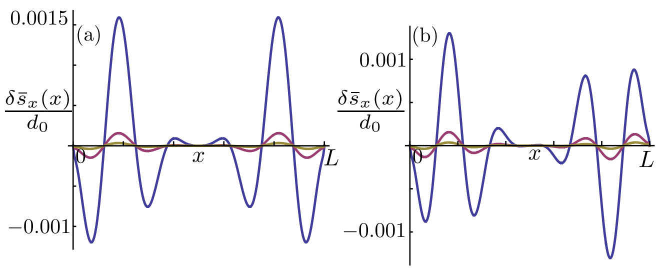

FIG. 5. Two-particle backscattering-induced oscillations $\delta\bar{s}_x(x)$ as a function x for $N = 4$, $K = 0.5$ (brown line), $K = 0.3$ (violet line), and $K = 0.2$ (blue line) and (a) $\theta = 0$ and (b) $\theta = \pi/2$.

where

$$A(x,\chi,\{\mathbf{n}\}) = \prod_p \frac{\left[-2\chi e^{-(-\alpha p\pi/2L)}\sqrt{\frac{K}{p}}\sin\left(\frac{p\pi x}{L}\right)\right]^{n_p}}{\sqrt{n_p!}} D(x)^{\chi^2 K}. \quad (21)$$

The sums and the integrals can be performed numerically, and the results are presented in Fig. 5. The dominant oscillation is of the $2k_F$ type, although a small $6k_F$ component also is present. Moreover, the unfavorable scaling factors of the damping factor $D(x)$ flatten the oscillations near the center of the dot. Additionally, the correction to the spin oscillations is strongly influenced by the Jackiw-Rebbi charge, in accordance with the behavior of the average spin density. As the background fractional charge is added to the dot, the number of oscillations increases. When a full rotation of the magnetization of the barrier is performed, an additional peak is emerging in the spin-density profile. As expected, by increasing interactions, the peak-to-valley ratio of spin oscillations increases. However, whereas the charge oscillations induced by two-particle backscattering add up to a flat original density profile, the spin oscillations are superimposed onto a commensurate oscillation pattern.

The difference between the characteristic wavelength of the charge oscillations $4k_F$ and the spin oscillations $2k_F$ and $6k_F$ is essentially due to the original spin-momentum locking characterizing the quantum spin Hall state. In a perturbative scheme, in fact, given that two-particle backscattering is characterized by the wave-vector $4k_F$, the dominant oscillation of the charge density is $4k_F$ due to the absence of Friedel oscillations in the density operator, see Eq. (13). Similarly, no corrections with wavelength $4k_F$ are present in the $s_x(x)$ and $s_y(x)$ spin densities since no long-wave contribution is present in their operator forms. Such a contribution would be present in a usual Luttinger liquid. A further confirmation of the different wavelengths of the corrections can be obtained by means of the strong-coupling expansion of the Fermi operators as discussed in Ref. [30].

VI. CONCLUSIONS

In this paper, we have presented the effects of two-particle backscattering on the charge and spin densities of a quantum spin Hall quantum dot. First, we have characterized the different interaction terms. We have shown that, for sufficiently strong interactions, two-particle backscattering must be taken into account even when the system is tuned away from half-filling. Then, we have demonstrated, by means of a simple perturbation theory, that the charge density is strongly influenced by two-particle backscattering since it induces oscillations with a wave vector that depends on the Jackiw-Rebbi fractional charge trapped in the dot. The peak-to-valley ratio of the oscillations increases as the forward density-density interaction is increased, that is, when the Luttinger parameter decreases. These results are in striking contrast to the case in which two-particle backscattering can be neglected: In that case the electron density, despite confinement and electron-electron interactions, is flat. An analogous result also holds for the in-plane spin density, which, when two-particle backscattering is added, acquires a $6k_F$ component which is absent in the dot without two-particle backscattering. More generally, we have shown that the fractional charge oscillations recently obtained by means of perturbation schemes in the regime of strong two-particle backscattering are robust. They can also be obtained within the usual perturbative approaches. Our paper implies that a quantum spin Hall quantum dot displays very rich physics: Interaction-induced fractionalization and Jackiw-Rebbi fractional charges coexist and have a nontrivial interplay. Apart from the interest in the signatures of strong interactions in topological systems, the results inspected in this paper are relevant for the understanding of parafermions in topological insulators and spin-orbit-coupled quantum wires. Strong interactions of the form of the two-particle backscattering investigated here, in fact, play a fundamental role in the formation of parafermions. Moreover, although the parafermions which are relevant for quantum computation are boundary states, their manipulation necessarily involves their interaction with the bulk of the host material. In our paper, we have presented a study of the spectral properties of the nonsuperconducting part of a prototypical system for the emergence of parafermionic excitations.

ACKNOWLEDGMENTS

Financial support by the DFG (Grants No. SPP1666 and No. SFB1170 "ToCoTronics"), the Helmholtz Foundation (VITI), and the ENB Graduate school on "Topological Insulators" is gratefully acknowledged.

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