Exactly soluble local bosonic cocycle models, statistical transmutation, and simplest time-reversal symmetric topological orders in 3+1 dimensions

Xiao-Gang Wen
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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We propose a generic construction of exactly soluble *local bosonic models* that realize various topological orders with gappable boundaries. In particular, we construct an exactly soluble bosonic model that realizes a (3+1)-dimensional [(3+1)D] Z_2-gauge theory with emergent fermionic Kramers doublet. We show that the emergence of such a fermion will cause the nucleation of certain topological excitations in space-time without pin$^+$ structure. The exactly soluble model also leads to a statistical transmutation in (3+1)D. In addition, we construct exactly soluble bosonic models that realize 2 types of time-reversal symmetry-enriched Z_2 topological orders in 2+1 dimensions, and 20 types of simplest time-reversal symmetry-enriched topological (SET) orders which have only one nontrivial pointlike and stringlike topological excitation. Many physical properties of those topological states are calculated using the exactly soluble models. We find that some time-reversal SET orders have pointlike excitations that carry Kramers doublet, a fractionalized time-reversal symmetry. We also find that some Z_2 SET orders have stringlike excitations that carry anomalous (nononsite) Z_2 symmetry, which can be viewed as a fractionalization of Z_2 symmetry on strings. Our construction is based on cochains and cocycles in algebraic topology, which is very versatile. In principle, it can also realize emergent topological field theory beyond the twisted gauge theory.

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I. INTRODUCTION

A sign of a comprehensive understanding of a type of phases of matter is being able to classify all of them. We understand that the crystal orders are due to spontaneous symmetry breaking [1] of the translation and the rotation symmetry. This leads to the classification of all 230 crystal orders in three dimensions using group theory. Now, we realized that the phases of matter beyond symmetry-breaking theory are due to long-range entanglement [2–4] for topologically ordered phases [5–7], and due to symmetry-protected short-range entanglement [8,9] for symmetry-protected trivial (SPT) phases [8,10,11]. This leads to complete classification of many topological phases. Using projective representations [9], we can classify all (1+1)-dimensional [(1+1)D] gapped phases for bosonic and fermionic systems with any symmetry [12–15]. We can also classify all (2+1)-dimensional [(2+1)D] gapped liquid [16,17] phases for bosonic and fermionic systems with any finite unitary symmetry using unitary modular tensor categories [18,19], G-crossed unitary modular tensor categories [20], and/or unitary braided fusion categories over Rep(G) or sRep(G^f) [21,22]. Those phases are symmetry-breaking phases, topologically ordered phases, SPT phases (such as odd-integer-spin Haldane phase [23,24] and topological insulators [25–30]), symmetry-enriched topological (SET) orders, etc. So far, we still do not have a classification of (3+1)-dimensional [(3+1)D] gapped liquid phases, although we know that it is closely related to unitary four-category theory with one object [31,32].

With those powerful classification results, we would like to have a systematic construction of those topological phases. Ideally, we would like to have a universal construction that can realize any given topological phases. There are very systematic ways to construct exactly soluble models [32–41] based on tensor network [31]. Using unitary fusion categories as input, Turaev-Viro state-sum [34] and Levin-Wen string-net models allow us to realize all (2+1)D bosonic topological orders with gappable boundary. Using finite group G and group 4-cohomoly classes $\omega_4 \in \mathcal{H}^4(G;/\mathbb{R}/\mathbb{Z})$ as input, Dijkgraaf-Witten models allow us to realize all (3+1)D bosonic topological orders whose pointlike excitations are all bosons [42]. Using premodular categories as input, Walker-Wang models can also realize a large class of (3+1)D bosonic topological orders. But Walker-Wang models cannot realize all Dijkgraaf-Witten models. A further generalization of Walker-Wang models in Refs. [32,39] allow us to include all Dijkgraaf-Witten models as well. Such systematic constructions were also generalized to fermion systems [37,40,41,43,44].

The above constructions are very systematic, but also very complicated and hard to use. Despite their complexity, it is still not clear if they can realize all (3+1)D topological orders or not. [We already know that they cannot realize all (2+1)D topological orders.] In this paper, we are going to develop a simpler systematic construction. Our constructed models are not a subset of any one of the above-mentioned tensor network constructions. But, our construction also does include any one of the above-mentioned tensor network constructions, as a subset.

We will start with topological invariants for topological orders. Then, we will use cochain theory and cohomology theory [33,45,46] to construct exactly soluble local bosonic models whose ground states have topological orders described by the corresponding topological invariants. In other words, the low-energy effective field theory of those local bosonic models is the desired topological field theory. (Here, a *local bosonic model* is defined as a quantum model whose total Hilbert space has a tensor product decomposition $\mathcal{H}_{\text{tot}} = \bigotimes_i \mathcal{H}_i$ where $\mathcal{H}_i$ is a finite-dimensional local Hilbert space for site i, and the Hamiltonian is local respect to such a tensor product decomposition.) Many mathematical techniques developed for cohomology theory and algebraic topology will help us to do concrete calculations with our models.

One class of topological invariants is given by volume-independent partition function $Z^{\text{top}}(M^d)$ on manifolds with vanishing Euler number and Pontryagin number [31] $\chi(M^d) = P(M^d) = 0$. For invertible topological orders [31,47] and for SPT orders [11,48] (which have no nontrivial bulk topological excitations), such topological invariants are pure phases [31,47,49–52]

$$Z^{\text{top}}(M^d, a^{\text{sym}}) = e^{i2\pi \int_{M^d} W(\mathrm{w}_i, a^G) + k\omega_d}, \tag{1}$$

where w_i is the ith Stiefel-Whitney class, a^G the flat connection that describes symmetry G twist [53–56], and ω_d the gravitational Chern-Simons term. For example, a (2+1)D Z_n SPT state labeled by $k \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ is characterized by its SPT invariant [49–51,54,55] (see Sec. V B)

$$Z^{\text{top}}(M^{2+1}, a^{Z_n}) = e^{ik\frac{2\pi}{n}\int_{M^{2+1}} a^{Z_n} \cup \mathcal{B}_n a^{Z_n}}, \tag{2}$$

where a^{Z_2} becomes a 1-cochain and $\mathcal{B}_n$ is Bockstein homomorphism equation (43).

For other noninvertible topological orders (which have nontrivial bulk topological excitations), their topological invariants can be sums of phases

$$Z^{\text{top}}(M^d, a^{\text{sym}}) = \sum_{c \in H^*(M^d;\mathbb{M})} e^{i2\pi \int_{M^d} W(c, \mathrm{w}_i, a^{\text{sym}}) + k\omega_d},$$

where c are cohomology classes. Our constructed local bosonic model is designed to produce such form of topological invariants. The construction is very versatile and many exactly soluble local bosonic models can be constructed systematically to produce all the topological invariants of the above form (with $k = 0$). Some of those models have emergent gauge theories or emergent Dijkgraaf-Witten theories [33]. Other models have emergent "twisted" gauge theories beyond Dijkgraaf-Witten type.

In this paper, we will discuss many different types of gauge theories. To avoid confusion, here we will explain the terminology that will be used in this paper. We will use untwisted (UT) gauge theory to refer to the usual lattice gauge theories (without any twist) [57]. We will use all-boson (AB) gauge theory to refer to the lattice gauge theories (may be twisted) where all the pure gauge charges are bosons. We will use emergent-fermion (EF) gauge theory to refer to the lattice gauge theory (may be twisted) where some pure gauge charges are fermions. We will use the term G gauge theory to refer gauge theory with G gauge group. The Dijkgraaf-Witten theories [33] are AB gauge theories. This is because the Dijkgraaf-Witten theories can be viewed as the G-SPT states with the gauged symmetry G [53], all the gauge charges are bosonic in Dijkgraaf-Witten theories.

We will also discuss (3+1)D topological theories beyond Dijkgraaf-Witten theories. Many of those theories do not contain gauge fields, and it is hard to call them gauge theories. However, the pointlike topological excitations in those theories have the same fusion rule as (3+1)D gauge theories, i.e., fuse like the irreducible representations of a group G. So, we will still call those (3+1)D topological theories as gauge theory, which include EF gauge theories. Certainly, the EF gauge theories are not Dijkgraaf-Witten theories in (3+1)D.

We would like to mention that there are many related constructions of topological field theories using 1-form, 2-form gauge fields, etc. [46,58–64]. In contrast to those works, the cocycle models constructed in this paper are defined on lattice instead of continuous manifold. Also, cocycle models are not gauge theories. They are local bosonic models without any gauge redundancy. In other words, the emergent topological field theories studied in this paper are free of all anomalies. In comparison, some 1-form, 2-form gauge field theories defined on continuous manifold can be anomalous since they may not be mergeable from local lattice theories [31,50,65].

In this paper, we will use $\overset{n}{=}$ to mean equal up to a multiple of n, and use $\overset{d}{=}$ to mean equal up to df (i.e., up to a coboundary). We will use $[f]_n$ to mean mod(f,n) and $\langle l,m\rangle$ to mean the greatest common divisor of l and m ($\langle 0,m\rangle \equiv m$). We also introduce some modified δ functions

$$\delta_n(x) = \begin{cases} 1, & \text{if } x \overset{n}{=} 0 \\ 0, & \text{otherwise} \end{cases} \qquad \bar{\delta}(x) = \begin{cases} 1, & \text{if } x \overset{d}{=} 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\bar{\delta}_n(x) = \begin{cases} 1, & \text{if } x \overset{n,d}{=} 0 \\ 0, & \text{otherwise.} \end{cases}$$

II. A SUMMARY OF RESULTS

The cocycle models introduced in this paper not only can realize many types of topological orders, SPT orders, and SET orders, they are exactly soluble in the sense that that their partition function can be calculated exactly on any space-time manifold [34]. Those models are realizable by commuting projectors. Because the models are exactly soluble, we can use them to compute many physical properties of those topological phases, such as ground-state degeneracies, fractional quantum numbers on pointlike and stringlike topological excitations, braiding statistics, topological partition functions, dimension reduction, etc.

A. Symmetry fractionalization on stringlike defects in SPT states

One way to probe SPT order is to measure fractional quantum number carried by symmetry twist defect (see Fig. 1). For example, consider a (2+1)D Z_n SPT state which is labeled by k. In [55] it was shown that a symmetry-twist defect can carry a Z_n quantum number $2k$ (i.e., each defect will carry a

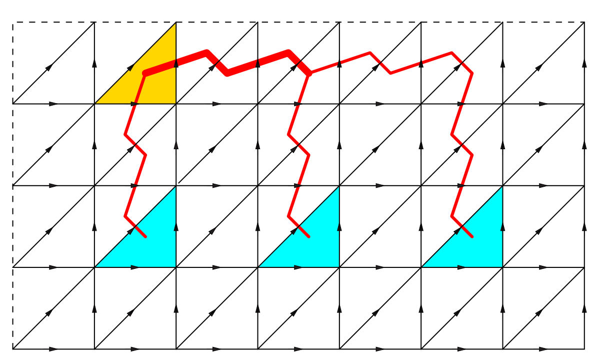

FIG. 1. Three *identical* Z_3 symmetry twist defects (blue triangles) on a torus. The red line is the symmetry twist line. A symmetry twist defect is an end of symmetry twist line.

fractional Z_n quantum number $\frac{2k}{n}$). We can use this property to measure the (2+1)D Z_n SPT order.

Similar results also appear in higher dimensions. Consider a (3+1)D $Z_n \times \tilde{Z}_n$ SPT state which is labeled by $k_1 = 0,\ldots,n-1$ and $k_2 = 0,\ldots,n-1$. A Z_n symmetry-twist defect will be a line defect in (3+1)D. We show that such a line defect must be gapless or symmetry breaking, which behaves just like the edge state of some (2+1)D SPT state. These phenomena can be viewed as symmetry fractionalization on defect lines.

To see the edge state of which (2+1)D SPT state that the defect line carries, we need to specify that the (3+1)D $Z_n \times \tilde{Z}_n$ SPT state is described by the following SPT invariant: $Z^{\text{top}} = e^{i\frac{2\pi}{n}\int_{M^{3+1}} k_1 a^{\tilde{Z}_n}\cup a^{Z_n}\cup \mathcal{B}_n a^{\tilde{Z}_n} + k_2 a^{\tilde{Z}_n}\cup a^{Z_n}\cup \mathcal{B}_n a^{Z_n}}$. Then, a Z_n symmetry-twist defect line will carry the edge state of a (2+1)D $Z_n \times \tilde{Z}_n$ SPT state characterized by the SPT invariant $Z^{\text{top}} = e^{i\frac{2\pi}{n}\int_{M^{2+1}} k_1 a^{\tilde{Z}_n}\cup \mathcal{B}_n a^{\tilde{Z}_n} - k_2 a^{Z_n}\cup \mathcal{B}_n a^{\tilde{Z}_n}}$ (see Sec. III D 3).

To be more precise, the Z_n symmetry-twist defect line in (3+1)D has a nononsite (anomalous) $Z_n \times \tilde{Z}_n$ symmetry [10,11,65] along the defect line. This (1+1)D anomalous symmetry makes the defect line to be either gapless or symmetry breaking [10]. This result generalizes the one in [55]. This (1+1)D anomalous symmetry can be viewed as the symmetry fractionalization on the strings.

The (1+1)D anomalous symmetry also appears on the edge of (2+1)D SPT state. The (1+1)D anomalous symmetry on the Z_n symmetry-twist defect line happens to be the same (1+1)D anomalous symmetry on the edge of a (2+1)D $Z_n \times \tilde{Z}_n$ SPT state characterized by the SPT invariant $Z^{\text{top}} = e^{i\frac{2\pi}{n}\int_{M^{2+1}} k_1 a^{\tilde{Z}_n}\cup \mathcal{B}_n a^{\tilde{Z}_n} - k_2 a^{Z_n}\cup \mathcal{B}_n a^{\tilde{Z}_n}}$.

Pointlike and stringlike symmetry-twist defects are extrinsic defects in the SPT states. The above results indicate that extrinsic defects in the SPT states can carry fractional quantum numbers or anomalous symmetry. We would like to remark here we need to distinguish extrinsic defects from excitations which are intrinsic. The pointlike or stringlike excitations, by definition, can all be trapped by potential traps of the same dimension. For example, a pointlike excitation at $\boldsymbol{x}_0$ can be trapped by a potential $V(\boldsymbol{x})$, which is nonzero only near $\boldsymbol{x}_0$. Those pointlike or stringlike excitations in SPT states do not have symmetry fractionalization. In contrast, extrinsic defects cannot be trapped by potentials of the same dimension. For example, a pointlike symmetry-twist defect in (2+1)D can only be trapped by a "potential" (a change of Hamiltonian) that is nonzero along a line, where the pointlike defect is trapped at an end of the line.

B. Statistical transmutation in (3+1)D

We have constructed a (3+1)D exactly soluble local bosonic model

$$Z(M^{3+1}) = \sum_{\substack{b\in C^2(M^{3+1};\mathbb{Z}_2)\\ db\overset{2}{=}0}} e^{i\pi\int_{M^{3+1}} b\cup b}, \tag{3}$$

where b is a 2-cochain field (see Sec. III A for a definition of cochain field) and $\cup$ the cup product of cochains. The model has an emergent fermion, and its low-energy effective theory is a EF Z_2-gauge theory. Such kind of EF Z_2-gauge theory has been constructed in terms of strings in (3+1)D [36,66]. Here, we give a construction in terms of membranes (see Sec. III E) [60,63].

As a corollary of the above construction, we find a statistical transmutation in (3+1)D lattice M^{3+1} (expressed in terms of partition function):

$$Z(M^{3+1}) = \sum_{\substack{b\in C^2(M^{3+1};\mathbb{Z}_2)\\ db\overset{2}{=}*j}} e^{i\pi\int_{M^{3+1}} b\cup b}, \tag{4}$$

where j is a cycle corresponding to the world line of a bosonic scalar particle, and $*j$ is the 3-cocycle corresponding to the Poincaré dual of j. The term $\pi\int_{M^{3+1}} b\cup b$ changes the statistics of the particle from bosonic to fermionic. This is similar to the statistical transmutation in (2+1)D by Chern-Simons term. Note that the condition $db \overset{2}{=} *j$ means $db = *j$ mod 2 which can be enforced using energy penalty $e^{-U\int_{M^{3+1}} |db-*j|^2}$.

Is the transmuted fermion a spin-up–spin-down doublet? To address this issue, we would like to mention that the term $\pi\int_{M^{3+1}} b\cup b$ is compatible with time-reversal symmetry. If the total model has a time-reversal symmetry, then the particle dressed by the b field, i.e., the fermion, will be a time-reversal singlet, which corresponds to a scalar fermion. However, this behavior can be adjusted by changing the topological term to $\pi\int_{M^{3+1}}(d\tilde{g}\cup d\tilde{g} + b)\cup b$, where $\tilde{g}_i$ is a $\mathbb{Z}_2$-0-cochain field which is a pseudoscalar. So, the new statistical transmutation is given by

$$Z(M^{3+1}) = \sum_{\substack{\tilde{g}\in C^0(M^{3+1};\mathbb{Z}_2)\\ b\in C^2(M^{3+1};\mathbb{Z}_2), db\overset{2}{=}*j}} e^{i\pi\int_{M^{3+1}}[d\tilde{g}\cup d\tilde{g}+b]\cup b}. \tag{5}$$

The second type of statistical transmutation can still change the statistics of the particle from bosonic to fermionic, but now the fermion, dressed by b and $\tilde{g}$ fields, will be a Kramers doublet which corresponds to a spin-$\frac{1}{2}$ fermion (see Sec. VI C 4).

C. (2+1)D time-reversal symmetric topological orders

We have constructed $2^3 = 8$ time-reversal symmetric local bosonic models in (2+1)D [see Eq. (203)]:

$$\begin{aligned} Z_{k_0k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(M^3) &= \sum_{\{\tilde{g}_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}\}} e^{i\pi\int_{M^3} b^{\mathbb{Z}_2}\cup(da^{\mathbb{Z}_2}-k_0\mathcal{B}_2 d\tilde{g}^{\mathbb{Z}_2})} \\ &\quad\times e^{i\pi\int_{M^3} k_1 a^{\mathbb{Z}_2}\cup a^{\mathbb{Z}_2}\cup a^{\mathbb{Z}_2}+k_2 d\tilde{g}^{\mathbb{Z}_2}\cup d\tilde{g}^{\mathbb{Z}_2}\cup a^{\mathbb{Z}_2}}, \end{aligned} \tag{6}$$

where $\tilde{g}_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}$ are $\mathbb{Z}_2$-valued 0-cochain, 1-cochain, and 2-cochain fields (see Sec. III A), and $k_{0,1,2} = 0,1$. Also, the time-reversal symmetry is described by group Z_2^T with $T^2 = 1$, whose action is given by $(\tilde{g}_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}) \to [\text{mod}(\tilde{g}_i^{\mathbb{Z}_2}+1,2),a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}]$ plus the complex conjugation. {The above model also has an additional Z_2' symmetry generated by $(\tilde{g}_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}) \to [\text{mod}(\tilde{g}_i^{\mathbb{Z}_2}+1,2),a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}]$ without the complex conjugation.} We see that $\tilde{g}_i^{\mathbb{Z}_2}$ is a pseudoscalar field. The above eight models realize five types of time-reversal SET orders.

The four constructed models (labeled by k_00k_2) reduce to the Z_2 topological order described by UT Z_2-gauge theory

TABLE I. The (2+1)D time-reversal (T) symmetric topological orders from four 1-cocycle models in Eq. (6). They have three or four types of pointlike topological excitations. d_i's and s_i's are the quantum dimensions and spins of those excitations. A quantum dimension $d=2$ means that the excitation has two internal degrees of freedom. $2_\pm$ means that the two internal degrees of freedom form a $T^2=1$ time-reversal doublet or a $T^2=-1$ Kramers doublet. Spin $s=\frac{1}{2}$ corresponds to a fermion and $s=\frac{1}{4}$ a semion. Spin $s=\frac{3}{4}$ is the time-reversal conjugate of a semion. The fourth and fifth columns are volume-independent partition functions $Z^{\text{top}}_{M^3}$ with $M^3=S^1\times\Sigma_g, S^1\times\Sigma_g^{\text{non}}$, where Σ_g is the genus g Riemannian surface and Σ_g^{non} is the genus g nonorientable surface.

$k_0k_1k_2$	$(d_1,d_e,\dots)$	$(s_1,s_e,\dots)$	$Z^{\text{top}}_{S^1\times\Sigma_g}$	$Z^{\text{top}}_{S^1\times\Sigma_g^{\text{non}}}$	Comments
000	(1,1,1,1)	$(0,0,0,\frac{1}{2})$	4^g	2^g	Z_2-gauge theory (three bosons and one fermion)
001	$(1,1,2_-,2_-)$	$(0,0,0,\frac{1}{2})$	4^g	$2^{g-1}[1+(-)^g]$	A boson and a fermion are Kramers doublets
10∗	$(1,2_-,1,2_-)$	$(0,0,0,\frac{1}{2})$	4^g	$2^{g-1}[1+(-)^g]$	The same SET order as above
010	$(1,1,2_+)$	$(0,0,[\frac{1}{4},\frac{3}{4}])$	4^g	2^{g-1}	Two semions form a $T^2=1$ time-reversal doublet
011	$(1,1,2_-)$	$(0,0,[\frac{1}{4},\frac{3}{4}])$	4^g	2^{g-1}	Two semions form a $T^2=-1$ Kramers doublet
11∗	$(1,2_-,2_-,2_+)$	$(0,0,[\frac{1}{4},\frac{3}{4}],[\frac{1}{4},\frac{3}{4}])$	4^g	$2^{g-2}[1+(-)^g]$	A boson is Kramers doublet

after we break the time-reversal symmetry (see top three rows in Table I). But, three of them have identical topological orders. Thus, the four models only give us two types of time-reversal symmetric Z_2-gauge theories [67]. They correspond to two types of time-reversal symmetric Z_2-gauge theories. Those four models are obtained by gauging the Z_2 subgroup in two of the four $Z_2\times Z_2^T$ SPT states and by gauging the Z_2 subgroup of Z_4^T SPT states (Z_4^T has $T^2=-1$). There is another type of time-reversal symmetric Z_2-gauge theory where the time-reversal transformation exchanges the Z_2 charge and Z_2 vortex [68]. Such a theory is missing from the table.

The other three of five constructed time-reversal SET orders correspond to three types of time-reversal symmetric double-semion theories [35,36] (see bottom three rows in Table I). Those theories are obtained by gauging the Z_2 subgroup in two of the four $Z_2\times Z_2^T$ SPT states. Two of four constructed models (labeled by k_01k_2) have identical topological orders. They give us three types of time-reversal symmetric double-semion theories.

It the interesting to note that one of the time-reversal symmetric double-semion topological orders (the last row in Table I) contains four types of pointlike excitations: (1) a trivial type which is a time-reversal singlet; (2) a bosonic Kramers doublet (denoted by quantum dimension, i.e., internal degrees of freedom, $d=2_-$); (3) a $T^2=1$ time-reversal doublet formed by two semions with spin $\frac{1}{4}$ and $\frac{3}{4}$ (denoted by quantum dimension $d=2_+$); (4) a $T^2=-1$ Kramers doublet formed by two semions with spin $\frac{1}{4}$ and $\frac{3}{4}$ (denoted by quantum dimension $d=2_-$).

D. (3+1)D time-reversal symmetric Z_2-gauge theories

We also have constructed $2^6=64$ local bosonic models in (3+1)D which can realize 20 types of simplest topological orders with time-reversal symmetry (see the black rows in Table II). Those topological orders are simplest since they have only one type of nontrivial pointlike topological excitation and one type of nontrivial stringlike topological excitation. The pointlike topological excitations in those (3+1)D SET orders can be Kramers doublet (which corresponds to the fractionalization [69,70] of time-reversal symmetry) and can be fermionic. If we break the time-reversal symmetry, 16 of the 20 SET orders reduce to the (3+1)D Z_2 topological order described by the UT Z_2-gauge theory, and the other 4 of the 20 SET orders reduce to the (3+1)D topological order described by the EF Z_2-gauge theory.

Those 64 bosonic models are given by [see Eq. (218)]:

$$Z_{k_1k_2k_3k_4k_5k_6}(M^4)=\sum_{\{\tilde g_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}\}} e^{i\pi\int_{M^4} a^{\mathbb{Z}_2}\cup[db^{\mathbb{Z}_2}+k_1a^{\mathbb{Z}_2}\cup a^{\mathbb{Z}_2}\cup a^{\mathbb{Z}_2}+(k_1+k_2)d\tilde g\cup d\tilde g\cup d\tilde g]} \times e^{i\pi\int_{M^4}[k_4b^{\mathbb{Z}_2}+(k_3+k_4)d\tilde g\cup d\tilde g]\cup b^{\mathbb{Z}_2}+k_5d\tilde g\cup d\tilde g\cup d\tilde g\cup d\tilde g+k_6\mathrm{w}_2\cup\mathrm{w}_2}, \tag{7}$$

where $k_I=0,1$, $b^{\mathbb{Z}_2}$ is a $\mathbb{Z}_2$ 2-cocycle field, $a^{\mathbb{Z}_2}$ a $\mathbb{Z}_2$ 1-cocycle field, and $\tilde g_i$ a pseudoscalar field which changes under the time-reversal transformation $\tilde g_i\to \text{mod}(\tilde g_i+1,2)$. The above local bosonic models have a time-reversal symmetry: the action amplitude is invariant under the combined transformation of $\tilde g_i\to\text{mod}(\tilde g_i+1,2)$ and complex conjugation. The models also have a Z_2' symmetry: the action amplitude is invariant under $\tilde g_i\to\text{mod}(\tilde g_i+1,2)$ (without the complex conjugation).

But, the above model is exactly soluble only when $k_1k_4=0$. Those 48 exactly soluble models produce the rows in Table II. The models described by the boldface rows in Table II produce topological orders that are identical to some of the other rows. Those identities come from the relations between the Stiefel-Whitney classes on four-dimensional space-time [see Eq. (D14)]:

$$\mathrm{w}_1\cup\mathrm{w}_2=0,\quad \mathrm{w}_1\cup\mathrm{w}_3=0,$$
$$\mathrm{w}_1\cup\mathrm{w}_1\cup\mathrm{w}_1\cup\mathrm{w}_1+\mathrm{w}_2\cup\mathrm{w}_2+\mathrm{w}_4=0. \tag{8}$$

We see that $\mathrm{w}_1\cup\mathrm{w}_1+\mathrm{w}_2=0$ implies $\mathrm{w}_1\cup\mathrm{w}_1\cup\mathrm{w}_1+\mathrm{w}_2\cup\mathrm{w}_1=\mathrm{w}_1\cup\mathrm{w}_1\cup\mathrm{w}_1=0$. Thus, $\delta(\mathrm{w}_1^3)\delta(\mathrm{w}_1^2+\mathrm{w}_2)=\delta(\mathrm{w}_1^2+\mathrm{w}_2)$, which implies that the first and the second rows in the

TABLE II. The (3+1)D time-reversal (T) symmetric Z_2-gauge theories emerged from lattice bosonic models $Z_{k_1k_2k_3k_4k_5k_6}$ in Eq. (7). Each row corresponds to a root family which contains a few ($N_{\rm dis}$) T-symmetric topological orders labeled by $k_5,k_6 = 0,1$. Topological orders in the same root family differ only by Z_2^T SPT states. d_i's and s_i's are the quantum dimensions and spins of pointlike excitations. The models described by the boldface rows produce topological orders that are identical to some of the other rows. A quantum dimension $d = 2$ means that the excitation has two internal degrees of freedom. 2_- means that the two internal degrees of freedom form a Kramers doublet with $T^2 = -1$. The fourth column is the volume-independent partition function on space-time M^4, where w_i is the ith Stiefel-Whitney class.

$k_1k_2k_3k_4k_5k_6$	(d_1,d_2)	(s_1,s_2)	$Z^{\rm top} = \frac{\lvert H^1(M^4;\mathbb{Z}_2)\rvert e^{i\pi\int_{M^4} k_5 \mathrm{w}_1^4}}{\lvert H^0(M^4;\mathbb{Z}_2)\rvert e^{i\pi\int_{M^4} k_6 \mathrm{w}_2^2}} \times$	$N_{\rm dis}$	As gauged SPT state
$0000k_5k_6$	(1,1)	(0,0)	1	4	Bosonic $Z_2 \times Z_2^T$ trivial state
$0100*k_6$	(1,1)	(0,0)	$\delta(\mathrm{w}_1^3)$	2	Bosonic $Z_2 \times Z_2^T$ SPT state
$1000k_5k_6$	(1,1)	(0,0)	$\delta(\mathrm{w}_3)$	4	Bosonic $Z_2 \times Z_2^T$ SPT state
$1100*k_6$	(1,1)	(0,0)	$\delta(\mathrm{w}_3+\mathrm{w}_1^3)$	2	Bosonic $Z_2 \times Z_2^T$ SPT state
$0001k_5*$	$(1,2_-)$	$(0,\frac{1}{2})$	$\delta(\mathrm{w}_2)$	2	Free fermion Z_4^T SPT state
$0101**$	$(1,2_-)$	$(0,\frac{1}{2})$	$\delta(\mathrm{w}_1^3)\delta(\mathrm{w}_2)$	1	unknown
$0010*k_6$	$(1,2_-)$	(0,0)	$\delta(\mathrm{w}_1^2)$	2	Bosonic Z_4^T SPT state
$\mathbf{0110*k_6}$	$\mathbf{(1,2_-)}$	**(0,0)**	$\boldsymbol{\delta(\mathrm{w}_1^3)\delta(\mathrm{w}_1^2)=\delta(\mathrm{w}_1^2)}$	**2**	
$1010*k_6$	$(1,2_-)$	(0,0)	$\delta(\mathrm{w}_3)\delta(\mathrm{w}_1^2)$	2	Bosonic Z_4^T SPT state
$\mathbf{1110*k_6}$	$\mathbf{(1,2_-)}$	**(0,0)**	$\boldsymbol{\delta(\mathrm{w}_3+\mathrm{w}_1^3)\delta(\mathrm{w}_1^2)=\delta(\mathrm{w}_3)\delta(\mathrm{w}_1^2)}$	**2**	
$0011**$	(1,1)	$(0,\frac{1}{2})$	$\delta(\mathrm{w}_1^2+\mathrm{w}_2)$	1	Fermionic $Z_2^f \times Z_2^T$ trivial state
$\mathbf{0111**}$	**(1,1)**	$\mathbf{(0,\frac{1}{2})}$	$\boldsymbol{\delta(\mathrm{w}_1^3)\delta(\mathrm{w}_1^2+\mathrm{w}_2)=\delta(\mathrm{w}_1^2+\mathrm{w}_2)}$	**1**	

fourth block in Table II have the same partition function and thus correspond to the same theory.

We note that the four types of (3+1)D Z_2^T SPT states [11,71] can be labeled by $k_5,k_6 = 0,1$ and are characterized by the SPT invariant $Z(M^4) = e^{i\pi\int_{M^4} k_5\mathrm{w}_1^4 + k_6\mathrm{w}_2^2}$. The Z_2^T SPT state $(k_5k_6) = (10)$ is the one described by group cohomolgy $\mathcal{H}^4(Z_2^T;(\mathbb{R}/\mathbb{Z})_T)$ [11], and has a time-reversal symmetric boundary described by an anomalous Z_2-gauge theory where the Z_2 charge e and the Z_2 vortex m are both Kramers doublet, while the e and m bound state ε is a time-reversal singlet fermion [71]. The Z_2^T SPT state $(k_5k_6) = (01)$ is beyond $\mathcal{H}^4[Z_2^T;(\mathbb{R}/\mathbb{Z})_T]$, and has a time-reversal symmetric boundary described by an anomalous Z_2 gauge theory where e, m, and ϵ are all fermions.

The model with the same $k_1k_2k_3k_4$ but different k_5k_6 only differs by stacking those four Z_2^T SPT states. We call two time-reversal SET orders that differ only by stacking of Z_2^T SPT states as to have the same root since those SETs have identical bulk pointlike and stringlike excitations. We find that the 20 SET orders belong to 9 root families. This is because stacking the four Z_2^T SPT states does not always produce four distinct time-reversal SET phases since the partition function may vanish on space-time with nontrivial $\mathrm{w}_1 \cup \mathrm{w}_1 \cup \mathrm{w}_1 \cup \mathrm{w}_1$, $\mathrm{w}_2 \cup \mathrm{w}_2$, or $\mathrm{w}_1 \cup \mathrm{w}_1 \cup \mathrm{w}_1 \cup \mathrm{w}_1 + \mathrm{w}_2 \cup \mathrm{w}_2$. The number $N_{\rm dis}$ of distinct time-reversal SET phases in each root family is given in Table II. The nine root families correspond to nine types of (3+1)D time-reversal symmetric Z_2-gauge theories.

From Table II and from the discussions in Sec. VI C 5, we also see the physical meaning of each topological term labeled by $k_1k_2k_3k_4$:

(1) $k_4 = 1$ makes the pointlike excitations to be fermions.

(2) $k_4 + k_3 = 1$ mod 2 makes the pointlike excitations to be Kramers doublet.

(3) $k_1 + k_2 = 1$ mod 2 makes the stringlike excitations to carry an anomalous Z_2' symmetry that appear on the boundary of a (2+1)D Z_2' SPT state. Such an anomalous (nononsite) Z_2' symmetry is given by $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z{}^{\frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}}$, where $\sigma_i^z = (-)^{\tilde g_i}$ and $\frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2} = CZ(\sigma_I^z,\sigma_{I+1}^z)$ is the controlled-Z gate acting on the two qubits σ_I and σ_{I+1}.

Certainly, when $k_1 + k_2 = 0$ mod 2, the string will not have anomalous symmetry, and are in general gapped and symmetric.

There are also many other ways to realize time-reversal symmetric Z_2-gauge theories. For example, one can use nonlinear σ-model field theory to realize many of the above time-reversal SETs with bosonic pointlike excitations [72]. More generally, one may start with $Z_2 \times Z_2^T$ bosonic SPT states. There are eight such states since $\mathcal{H}^4[Z_2 \times Z_2^T;(\mathbb{R}/\mathbb{Z})_T] = \mathbb{Z}_2^{\oplus 3}$ [11]. Gauging the Z_2 symmetry gives us eight time-reversal symmetric Z_2 topological orders. But, some of them only differ by a Z_2^T SPT state. We only obtain four root states (i.e., four time-reversal symmetric Z_2-gauge theories), that correspond to the first four rows in Table II. We can also start with Z_4^T bosonic SPT states. There are two such states since $\mathcal{H}^4[Z_4^T;(\mathbb{R}/\mathbb{Z})_T] = \mathbb{Z}_2$. After gauging the unitary Z_2 subgroup of Z_4^T, we obtain two time-reversal symmetric Z_2-gauge theories (see the two black rows in the third block in Table II). Those two root states have a property that stacking with the $(k_5k_6) = (10)$ Z_2^T SPT state gives us the same root states back. The other two root states (the boldface rows) are identical to the two black rows in the third block.

The second block in Table II contains two root states. The first one can be obtained by gauging Z_4^T fermionic SPT states, which is also known as the $T^2 = -1$ fermionic topological superconductor [73,74]. There are at least 16 Z_4^T fermionic SPT states labeled by $\nu = 0,1,\ldots,15$ [75–77]. Gauging the fermion-parity Z_2^f subgroup in Z_4^T fermionic SPT states will produce several time-reversal symmetric topological orders that contain Kramers-doublet fermions. The stringlike excitations (i.e., the Z_2^f vortex lines or the Z_2^f symmetry-twist defect line) in those topological orders must be gapless unless $\nu =$ even, if the time-reversal symmetry is not broken [78]. In comparison, the strings in the three time-reversal symmetric

TABLE III. Volume-independent partition function $Z^{\text{top}}(M^4)$ for the constructed local bosonic models, on closed four-dimensional space-time manifolds. The space-time M^4 considered here satisfies $\chi(M^4) = P_1(M^4) = 0$, which makes $Z^{\text{top}}(M^4)$ to be a topological invariant [31]. The topological invariants listed below are also the ground-state degeneracy on the corresponding spatial manifold M^3_{space}. Here, $L^3(p)$ is the three-dimensional lens space and $F^4 = (S^1 \times S^3)\#(S^1 \times S^3)\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. F^4 is not spin. The different models are labeled by k_I which all have a range $k_I = 0,1,\ldots,n-1$.

Models\\M^4	T^4	$T^2 \times S^2$	$S^1 \times L^3(p)$	F^4	Low-energy effective theory
$Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^4)$	n^3	n	$\langle n,p\rangle$	n	UT Z_n-gauge theory
$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}(M^4)$	$\langle 2k,n\rangle^3$	$\langle 2k,n\rangle$	$\langle 2k,n,p\rangle$	$\langle 2k,n\rangle$ if $\frac{2kn}{\langle 2k,n\rangle^2} =$ even 0 if $\frac{2k_2n}{\langle 2k_2,n\rangle^2} =$ odd	$Z_{\langle 2k,n\rangle}$-gauge theory with fermions iff $\frac{2kn}{\langle 2k,n\rangle^2} =$ odd
$Z^{\text{top}}_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4)$	n^6	n^2	$\langle n,p\rangle\langle n,p,k_1,k_2\rangle$ if p has no repeated prime factors	n^2	$Z_n \times Z_n$ Dijkgraaf-Witten theory
$Z^{\text{top}}_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M^4)$	$n^3\langle 2k_2,n\rangle^3$	$n\langle 2k_2,n\rangle$	$\langle n,p\rangle\langle 2k_2,n,p,\frac{k_1p}{\langle n,p\rangle}\rangle$	$n\langle 2k_2,n\rangle$ if $\frac{2k_2n}{\langle 2k_2,n\rangle^2} =$ even 0 if $\frac{2k_2n}{\langle 2k_2,n\rangle^2} =$ odd	$(Z_{\frac{n\langle 2k_2,n\rangle}{\langle k_1,2k_2,n\rangle}} \times Z_{\langle k_1,2k_2,n\rangle})$-gauge theory with fermions iff $\frac{2k_2n}{\langle 2k_2,n\rangle^2} =$ odd

topological orders described by the two rows in the second block do not carry any anomalous time-reversal symmetry. In other words, the excitations on the strings can be gapped even if we do not break the time-reversal symmetry.

All the states in the second block have a property that stacking with the $(k_5k_6) = (01)$ bosonic Z_2^T SPT state (characterized by the SPT invariant $e^{i\pi\int_{M^4} k_5 \text{w}_2 \cup \text{w}_2}$) does not change their Z_2^T SET orders. Similarly, the Z_4^T fermionic topological superconductors also have the property that stacking with the $(k_5k_6) = (01)$ Z_2^T SPT state does not change the SPT order. For example, the $\nu = 0$ Z_4^T fermionic topological superconductor has a boundary with two types of quasiparticles $\{1,c\}$, where 1 is the trivial type and c is a Kramers-doublet fermion. The $(k_5k_6) = (01)$ Z_2^T SPT state has boundary with four types of quasiparticles $\{1,f_1,f_2,\varepsilon\}$, where f_1 and f_2 are Kramers-doublet fermions and ε is a time-reversal singlet fermion. Also, $f_{1,2}$ and ε have π-mutual statistics among them. The stacking of the two states has a boundary with quasiparticles $\{1,f_1,f_2,\varepsilon\} \times \{1,c\}$. We may condense the time-reversal singlet boson f_2c. Then, the new boundary state has quasiparticles $\{1,c\}$. The quasiparticle f_2 is also not confined, but it is equivalent to c since the two only differ by a condensed boson. Thus, the stacking of the $\nu = 0$ state and the $(k_5k_6) = (01)$ state can have the same boundary as the $\nu = 0$ state. The stacking of $(k_5k_6) = (01)$ state does not change the SPT order in Z_4^T fermionic topological superconductor.

The two states that correspond to the first row in the second block differ by stacking with the $(k_5k_6) = (10)$ Z_2^T SPT state characterized by the SPT invariant $e^{i\pi\int_{M^4} k_5 \text{w}_1 \cup \text{w}_1 \cup \text{w}_1 \cup \text{w}_1}$. For Z_4^T fermionic topological superconductors, stacking with the $(k_5k_6) = (10)$ state will shift ν by 8 [75,78,79]. This suggests that the two states are the $\nu = 0$ and the 8 Z_4^T fermionic topological superconductor with gauged Z_2^f symmetry [75,78,79].

On the other hand, for the time-reversal symmetric topological order described by the second row in the second block, stacking with any Z_2^T SPT states does not change its Z_2^T SET order. It is not clear if the Z_2^T SET order can be viewed as the Z_2^f-gauged $\nu = \pm 4$ Z_4^T fermionic topological superconductor or not.

E. Vanishing of the volume-independent partition function

We have calculated many volume-independent partition functions, and find they vanish some times. In general, a partition function may have a form

$$Z(M^d) = e^{-c_d L^d - c_{d-1} L^{d-1} - \cdots - c_0 L^0 - c_{-1} L^{-1} - \cdots}, \tag{9}$$

where L is the linear size of M^d. If the ground state does not contain pointlike, stringlike, etc., defects, then $c_1 = c_2 = \cdots = c_{d-1} = 0$. In this case,

$$Z^{\text{top}}(M^d) \equiv \lim_{L\to\infty} \frac{Z(M^d)}{e^{-c_d L^d}} = e^{-c_0 L^0} \tag{10}$$

is the volume-independent partition function. When the calculated volume-independent partition function vanishes, it does not mean the partition function to vanish nor the theory to be anomalous. It just means that $c_i > 0$, for some $0 < i < d$. This implies that the given space-time topology M^d induces pointlike, stringlike, etc., topological excitations.

We have calculated volume-independent partition functions for many constructed systems and for many space-time manifolds (see Tables I, II, and III). From those results, we conjecture the following: *A local bosonic model with emergent fermion always has vanishing volume-independent partition function $Z^{top}(M^d) = 0$ if the orientable M^d is not spin.*

In the presence of time-reversal symmetry, we have the following results: *(1) A local bosonic model with emergent Kramers doublet fermions always has vanishing volume-independent partition function $Z^{top}(M^d) = 0$ if M^d is not* pin$^+$ *(i.e.,* $\text{w}_2 \neq 0$*). (2) A local bosonic model with emergent time-reversal singlet fermions always has vanishing volume-independent partition function $Z^{top}(M^d) = 0$ if M^d is not* pin$^-$ *(i.e.,* $\text{w}_2 + \text{w}_1^2 \neq 0$*). (3) A local bosonic model with emergent Kramers doublet bosons always has vanishing volume-independent partition function $Z^{top}(M^d) = 0$ if $\text{w}_1^2 \neq 0$ on M^d.* (See Appendix E for a brief introduction of spin, pin$^+$, and pin$^-$ manifolds.) Those properties have been used to develop cobordism theory for fermionic SPT states [80].

In the rest of this paper, we will present detailed constructions and calculation.

III. A GENERIC CONSTRUCTION OF EXACTLY SOLUBLE BOSONIC LATTICE MODELS ON SPACE-TIME LATTICE

In this section, we are going to introduce a general way to construct exactly soluble local bosonic models. Those models are written in terms of path integral on space-time lattice. Those models are also designed to have topologically ordered ground states. In other words, those models have emergent topological field theory at low energies.

First, we will briefly review the related mathematics. Then, we will construct models that realize some well-known topological orders, such as those described by discrete gauge theories, and by Dijkgraaf-Witten theories. After that, we will construct models that realize more general topological orders whose low-energy effective theories are beyond Dijkgraaf-Witten theories. We will also compute the volume-independent partition functions for those constructed models on several choices of space-time manifolds. The results are summarized in Table III.

A. Space-time complex, cochains, and cocycles

Our local bosonic models will be defined on a space-time lattice. A space-time lattice is a triangulation of the d-dimensional space-time, which is denoted as M^d_{latt}. We will also call the triangulation M^d_{latt} as a space-time complex. A cell in the complex is called a simplex. We will use $i,j,\dots$ to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled by $(i,j),(j,k),\dots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i,j,k),(j,k,l),\dots$.

A cochain f_n is an assignment of values in $\mathbb{M}$ to each n-simplex, for example, a value $f_{n;i,j,\dots,k} \in \mathbb{M}$ for n-simplex $(i,j,\dots,k)$. So *a cochain f_n can be viewed as a bosonic field on the space-time lattice*. In this paper, we will use such cochain bosonic field to construct our models.

In this paper, we will assume $\mathbb{M}$ to be a ring which supports addition and multiplication operations, as well as scaling by an integer:

$$x + y = z, \quad x * y = z, \quad mx = y,$$
$$x,y,z \in \mathbb{M}, \quad m \in \mathbb{Z}. \tag{11}$$

We see that $\mathbb{M}$ can also be viewed a $\mathbb{Z}$ module (i.e., a vector space with integer coefficient) that also allows a multiplication operation. In this paper we will view $\mathbb{M}$ as a $\mathbb{Z}$ module. The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \overset{\text{as set}}{=} \mathbb{M}_1 \times \mathbb{M}_2. \tag{12}$$

We like to remark that a simplex $(i,j,\dots,k)$ can have two different orientations. We can use $(i,j,\dots,k)$ and $(j,i,\dots,k) = -(i,j,\cdots,k)$ to denote the same simplex with opposite orientations. The value $f_{n;i,j,\dots,k}$ assigned to the simplex with opposite orientations should differ by a sign: $f_{n;i,j,\dots,k} = -f_{n;j,i,\dots,k}$. So, to be more precise f_n is a linear map $f_n : n\text{-simplex} \to \mathbb{M}$. We can denote the linear map as $\langle f_n, n\text{-simplex}\rangle$, or

$$\langle f_n,(i,j,\dots,k)\rangle = f_{n;i,j,\dots,k} \in \mathbb{M}. \tag{13}$$

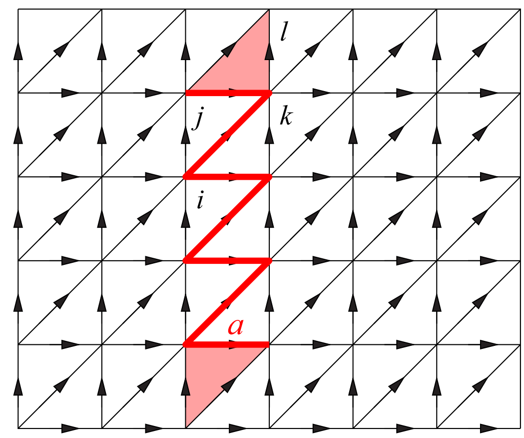

FIG. 2. A 1-cochain a has a value 1 on the red links: $a_{ik} = a_{jk} = 1$ and a value 0 on other links: $a_{ij} = a_{kl} = 0$. da is nonzero on the shaded triangles: $(da)_{jkl} = a_{jk} + a_{kl} - a_{jl}$. For such 1-cochain, we also have $a \cup a = 0$. So, when viewed as a $\mathbb{Z}_2$-valued cochain, $\mathcal{B}_2 a \neq a \cup a \bmod 2$.

More generally, a *cochain* f_n is a linear map of n-chains:

$$f_n : n\text{-chains} \to \mathbb{M} \tag{14}$$

or (see Fig. 2)

$$\langle f_n, n\text{-chain}\rangle \in \mathbb{M}, \tag{15}$$

where a *chain* is a composition of simplices. For example, a 2-chain can be a 2-simplex: (i,j,k), a sum of two 2-simplices: $(i,j,k) + (j,k,l)$, a more general composition of 2-simplices: $(i,j,k) - 2(j,k,l)$, etc. The map f_n is linear with respect to such a composition. For example, if a chain is m copies of a simplex, then its assigned value will be m times that of the simplex. $m = -1$ corresponds to an opposite orientation.

The total space-time lattice M^d_{latt} corresponds to a d-chain. We will use the same M^d_{latt} to denote it. Viewing f_d as a linear map of d-chains, we can define an "integral" over M^d_{latt}:

$$\int_{M^d_{\text{latt}}} f_d \equiv \langle f_d, M^d_{\text{latt}}\rangle. \tag{16}$$

In this paper, we usually take $\mathbb{M}$ to be integer $\mathbb{Z}$ or mod n integer $\mathbb{Z}_n = \{0,1,\dots,n-1\}$. So, not only the field $f_{n;i,j,\dots,k}$ is defined on a discrete space-time lattice, even the value of the field is discrete. We will use $C^n(M^d_{\text{latt}};\mathbb{M})$ to denote the set of all n-cochains on M^d_{latt}. $C^n(M^d_{\text{latt}};\mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-values fields (or paths) on M^d_{latt}. Note that $C^n(M^d_{\text{latt}};\mathbb{M})$ is an Abelian group under the $+$ operation.

We can define a derivative operator d acting on an n-cochain f_n, which give us an $n+1$-cochain (see Fig. 2)

$$\langle df_n,(i_0 i_1 i_2 \dots i_{n+1})\rangle = \sum_{m=0}^{n+1} (-)^m \langle f_n,(i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1})\rangle, \tag{17}$$

where $i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}$ is the sequence $i_0 i_1 i_2 \dots i_{n+1}$ with i_m removed, and $i_0, i_1, i_2 \dots i_{n+1}$ are the ordered vertices of the $(n+1)$-simplex $(i_0 i_1 i_2 \dots i_{n+1})$.

A cochain $f_n \in C^n(M^d_{\text{latt}};\mathbb{M})$ is called a *cocycle* if $df_n = 0$. The set of cocycles is denoted as $Z^n(M^d_{\text{latt}};\mathbb{M})$. A cochain

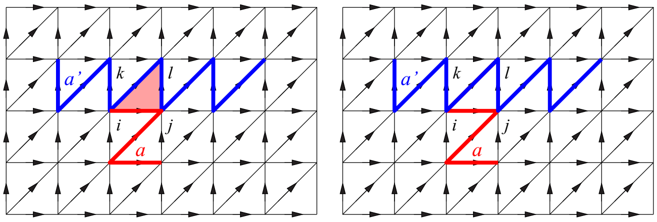

FIG. 3. A 1-cochain a has a value 1 on the red links. Another 1-cochain a' has a value 1 on the blue links. On the left, $a \cup a'$ is nonzero on the shade triangles: $(a \cup a')_{ijl} = a_{ij} a'_{jl} = 1$, while on the right, $a' \cup a$ is zero. Thus, $a \cup a' + a' \cup a$ is not a coboundary.

f_n is called a *coboundary* if there exists a cochain f_{n-1} such that $df_{n-1} = f_n$. The set of coboundaries is denoted as $B^n(M^d_{\text{latt}};\mathbb{M})$. Both $Z^n(M^d_{\text{latt}};\mathbb{M})$ and $B^n(M^d_{\text{latt}};\mathbb{M})$ are Abelian groups as well. Since $d^2 = 0$, a coboundary is always a cocycle: $B^n(M^d_{\text{latt}};\mathbb{M}) \subset Z^n(M^d_{\text{latt}};\mathbb{M})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles $[f_n]$ form the so-called cohomology group denoted as

$$H^n\big(M^d_{\text{latt}};\mathbb{M}\big) = Z^n\big(M^d_{\text{latt}};\mathbb{M}\big)/B^n\big(M^d_{\text{latt}};\mathbb{M}\big). \tag{18}$$

$H^n(M^d_{\text{latt}};\mathbb{M})$, as a group quotient of $Z^n(M^d_{\text{latt}};\mathbb{M})$ by $B^n(M^d_{\text{latt}};\mathbb{M})$, is also an Abelian group.

From two cochains f_m and h_n, we can construct a third cochain p_{m+n} via the cup product (see Fig. 3)

$$\begin{aligned} p_{m+n} &= f_m \cup h_n, \\ \langle p_{m+n},(i_0 \dots i_{m+n})\rangle &= \langle f_m,(i_0 i_1 \dots i_m)\rangle \\ &\quad \times \langle h_n,(i_m i_{m+1} \dots i_{m+n})\rangle. \end{aligned} \tag{19}$$

The cup product has the following property (see Fig. 3):

$$d(f_m \cup h_n) = (dh_n) \cup f_m + (-)^n h_n \cup (df_m). \tag{20}$$

We see that $f_m \cup h_n$ is a cocycle if both f_m and h_n are cocycles. If both f_m and h_n are cocycles, then $f_m \cup h_n$ is a coboundary if one of f_m and h_n is a coboundary. So, the cup product is also an operation on cohomology groups $\cup : H^m(M^d;\mathbb{M}) \times H^n(M^d;\mathbb{M}) \to H^{m+n}(M^d;\mathbb{M})$. When both f_m and h_n are cocycles, we also have

$$f_m \cup h_n = (-)^{mn} h_n \cup f_m + \text{coboundary}. \tag{21}$$

In the rest of this paper, we abbreviate the cup product $a \cup b$ as ab by dropping $\cup$. Also, we will use $\mathbb{Z}_n = \{0,1,\dots,n-1\}$ and $Z_n = \{1, e^{i\frac{2\pi}{n}}, e^{i2\frac{2\pi}{n}}, \dots, e^{i(n-1)\frac{2\pi}{n}}\}$ to denote the same Abelian group. In $\mathbb{Z}_n$, the group multiplication is mod-n "+" and in Z_n, the group multiplication is "$*$".

B. $\mathbb{Z}_n$-1-cocycle model and emergent Z_n-gauge theory

1. Model construction

Using the above mathematical formalism, let us construct a local bosonic model on a space-time lattice M^{d+1}_{latt}, where the local degrees of freedom live on the links. The possible values on each link are $a^{\mathbb{Z}_n}_{ij} = 0,1,\dots,n-1 \in \mathbb{Z}_n$.

The action amplitude $e^{-S_{\text{cell}}}$ for a $(d+1)$-simplex $(ij\dots l)$ is a complex function of $a^{\mathbb{Z}_n}_{ij}$: $e^{-L_{ij\dots l}(\{a^{\mathbb{Z}_n}_{ij}\})}$. The total action amplitude e^{-S} for a configuration (or a path) is given by

$$e^{-S(\{a^{\mathbb{Z}_n}_{ij}\})} = \prod_{(ij\dots l)} e^{-L_{ij\dots l}(\{a^{\mathbb{Z}_n}_{ij}\})}, \tag{22}$$

where $\prod_{(ij\dots l)}$ is the product over all the $(d+1)$-simplices $(ijkl)$. Our local bosonic model is defined by the following imaginary-time path integral (or partition function)

$$Z_{\mathbb{Z}_n\text{a}} = \sum_{\{a^{\mathbb{Z}_n}_{ij}\}} e^{-S(\{a^{\mathbb{Z}_n}_{ij}\})} = \sum_{\{a^{\mathbb{Z}_n}_{ij}\}} e^{-\sum_{(ij\dots l)} L_{ij\dots l}(\{a^{\mathbb{Z}_n}_{ij}\})}, \tag{23}$$

where $\sum_{\{a^{\mathbb{Z}_n}_{ij}\}}$ is a sum over all paths (i.e., the path integral).

We may view $a^{\mathbb{Z}_n}_{ij}$ as $\mathbb{Z}_n$-valued 1-cochain on the space-time complex M^3_{latt}:

$$a^{\mathbb{Z}_n}_{ij} = \langle a^{\mathbb{Z}_n},(ij)\rangle, \quad a^{\mathbb{Z}_n} \in C^1\big(M^3_{\text{latt}},\mathbb{Z}_n\big). \tag{24}$$

The Lagrangian $L_{ij\dots l}(\{a^{\mathbb{Z}_n}_{ij}\})$ will produce an emergent low-energy Z_2-gauge theory (i.e., have a Z_2 topological order) if we choose it to be

$$\begin{aligned} L_{ij\dots l}\big(\{a^{\mathbb{Z}_n}_{ij}\}\big) &= +\infty \quad &&\text{if } (da^{\mathbb{Z}_n}) \neq 0 \text{ on } (ij\dots l), \\ L_{ij\dots l}\big(\{a^{\mathbb{Z}_n}_{ij}\}\big) &= 0 &&\text{if } (da^{\mathbb{Z}_n}) = 0 \text{ on } (ij\dots l). \end{aligned} \tag{25}$$

So, the action amplitude $e^{-L_{ij\dots l}(\{a^{\mathbb{Z}_n}_{ij}\})}$ is nonzero only when $a^{\mathbb{Z}_n}$ is a cocycle, and the nonzero value is always 1. In other words, our local bosonic model is described by an action $S(a^{\mathbb{Z}_n}) = 0$ when $a^{\mathbb{Z}_n}$ is a cocycle, and $S(a^{\mathbb{Z}_n}) = +\infty$ when $a^{\mathbb{Z}_n}$ is not a cocycle. We see that the configurations described by noncocycles cost an infinity energy. We will call the local bosonic model described by the above $L_{ij\dots l}$ as a $\mathbb{Z}_n$-1-cocycle model.

2. Topological partition functions

The partition function $Z_{\mathbb{Z}_n\text{a}}(M^{d+1}_{\text{latt}})$ of the $\mathbb{Z}_n$-1-cocycle model can be calculated exactly, which is given by the number of 1-cocycles $|Z^1(M^{d+1}_{\text{latt}};\mathbb{Z}_n)|$, where $|S|$ denotes that number elements in set S. The number of 1-cochains is given by $|H^1(M^{d+1}_{\text{latt}};\mathbb{Z}_n)|$ times the number of 0-cochains whose derivatives are nonzero. The number of 0-cochains whose derivatives are nonzero is the number of 0-cochains, $|C^0(M^{d+1}_{\text{latt}};\mathbb{Z}_n)|$, divided by $|H^0(M^{d+1}_{\text{latt}};\mathbb{Z}_n)|$. Since $|C^0(M^{d+1}_{\text{latt}};\mathbb{Z}_n)| = 2^{N_v}$, where N_v is the number of vertices (the "volume" of space-time), we find that the partition function is

$$\begin{aligned} Z_{\mathbb{Z}_n\text{a}}\big(M^{d+1}_{\text{latt}}\big) &= \big|Z^1\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big| \\ &= \big|H^1\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big| \frac{\big|C^0\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big|}{\big|H^0\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big|} \\ &= 2^{N_v} \frac{\big|H^1\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big|}{\big|H^0\big(M^{d+1}_{\text{latt}};\mathbb{Z}_n\big)\big|}. \end{aligned} \tag{26}$$

According to [31], the topological information is given by the volume-independent part of partition function, which is obtained by taking the limit $N_v \to 0$:

$$Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^{d+1}) = \frac{|H^1(M^{d+1};\mathbb{Z}_n)|}{|H^0(M^{d+1};\mathbb{Z}_n)|}. \tag{27}$$

The volume-independent partition function can be a topological invariant [31] if the Euler number and the Pontryagin number vanish: $\chi(M^{d+1}) = P(M^{d+1}) = 0$. Such topological invariant characterizes the topological order realized by the model. Since the Z_2-gauge theory will produce the same volume-independent partition function $Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^{d+1})$ in large system size and low-energy limit, this allows us to determine that the $\mathbb{Z}_n$-1-cocycle model realizes the Z_2 *topological order* [81,82], the topological order described UT Z_n-gauge theory.

In (2+1)D and for $n = 2$, the Z_2 topological order has two bosonic topological quasiparticles, Z_2 charge e and Z_2 vortex m, and one fermionic topological quasiparticles f which is a bound state of e and m. In higher dimensions, the Z_n topological order has n types of bosonic pointlike excitations: the Z_n charge $q = 0,1,\dots,n-1$. It also has n types of $(d-2)$-dimensional branelike excitations: the Z_n flux $m = 0,\frac{2\pi}{n},\dots,\frac{2\pi(n-1)}{n}$.

We note that the volume-independent partition function on space-time $S^1\times S^1\times S^{d-1} = T^2\times S^{d-1}$ is given by

$$Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(T^2\times S^{d-1}) = n. \tag{28}$$

Since the volume-independent partition function on $S^1\times M^d$ is equal to the ground-state degeneracy (GSD) on space M^d,

$$\text{GSD}(M^d) = Z^{\text{top}}(S^1\times M^d), \tag{29}$$

we find that the GSD of our $\mathbb{Z}_n$-1-cocycle model on space $S^1\times S^{d-1}$ is given by $\text{GSD}(S^1\times S^{d-1}) = n$.

It turns out that

for any topological order, $GSD(S^1\times S^{d-1})$ is always equal to the number of types of pointlike topological excitations.

Such a result can be understood by the particle-hole tunneling process in Fig. 4. Such a particle tunneling process changes one ground state to another degenerate one, and relates the number types of pointlike topological excitations to $\text{GSD}(S^1\times S^{d-1})$. It is also true that

for any topological order, $GSD(S^1\times S^{d-1})$ is always equal to the number of types of $(d-2)$-dimensional branelike topological excitations [83].

(The notion of types of topological excitations, in particular high-dimensional topological excitations was discussed in [31]. It is very tricky to define the types of high-dimensional topological excitations.) This can be understood by a similar brane tunneling process around S^{d-1}.

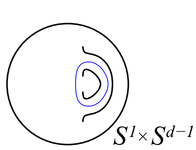

FIG. 4. A particle-hole tunneling process is a process where we create a particle-hole pair, move the particle around a noncontractible loop, and then annihilate the particle and the hole. The GSD on a d-dimensional space $S^1\times S^{d-1}$ is generated by the particle-hole tunneling process described by the blue loop. Thus, each degenerate ground state corresponds to a type of particle, and $\text{GSD}(S^1\times S^{d-1}) =$ number of types of pointlike excitations. Similarly, $\text{GSD}(S^k\times S^{d-k}) =$ number of types of $(k-1)$-dimensional excitations.

In general,

in d-dimensional space, the number of types of $(k-1)$-dimensional branelike excitations is equal to the number of types of $(d-k-1)$-dimensional branelike excitations, and they both equal to $GSD(S^k\times S^{d-k})$.

3. Boundary effective theory

Using the cocycle model, we can also easily study the properties of the boundary. Consider a space-time M^{d+1} whose boundary is $N^d = \partial M^d$. What is the low-energy effective theory of our $\mathbb{Z}_n$-1-cocycle model on the boundary N^d? To be more concrete, what is the partition function for the boundary effective theory? Here, we propose that the partition function for the boundary effective theory is simply given by

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(N^d) = Z_{\mathbb{Z}_n\text{a}}(M^{d+1}). \tag{30}$$

However, the above definition has a problem: the same N^d can be viewed as boundary of different space-time manifolds $N^d = \partial M^{d+1} = \partial\tilde{M}^{d+1}$. In general,

$$Z_{\mathbb{Z}_n\text{a}}(M^{d+1}) \neq Z_{\mathbb{Z}_n\text{a}}(\tilde{M}^{d+1}) \tag{31}$$

so the above definition of $Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(N^d)$ is not self-consistent.

In order for the definition to be self-consistent, we require that

$$Z_{\mathbb{Z}_n\text{a}}(M^{d+1}) = Z_{\mathbb{Z}_n\text{a}}(\tilde{M}^{d+1}) \tag{32}$$

for all M^{d+1} and $\tilde{M}^{d+1}$ with $\partial M^{d+1} = \partial\tilde{M}^{d+1}$. This implies that the bulk model on M^{d+1} has no topological order. So the boundary effective theory is well defined by itself iff the bulk theory on M^{d+1} has no topological order. This is exactly the gravitational-anomaly-free condition discussed in [31,50,65].

Since the bulk $\mathbb{Z}_n$-1-cocycle model has a nontrivial topological order, the boundary effective theory is anomalous. This implies that the boundary effective partition function $Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(N^d)$ not only depends on N^d, it also depends on how N^d is extended to one higher dimension, i.e., depends on M^{d+1}. The definition (30) correctly reflects such anomaly effect, and thus is a proper definition. However, to stress the dependence on the extension, we rewrite Eq. (30) as

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(N^d,M^{d+1}) = Z_{\mathbb{Z}_n\text{a}}(M^{d+1}). \tag{33}$$

Even though the boundary partition function depends on the bulk extension, it is still very useful in determining boundary low-energy properties, such as if the boundary gapped or not. Let us first choose $N^d = S^d$ and choose its extension to be $M^{d+1} = B^{d+1}$, where B^{d+1} is a $(d+1)$-dimensional ball. We find the boundary partition function to be

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(S^d,B^{d+1}) = \frac{1}{n} n^{N_v^{\text{bndr}}} n^{N_v^{\text{blk}}}, \tag{34}$$

where N_v^{bndr} is the number of vertices on the boundary S^d and N_v^{blk} is the number of vertices *inside* the ball B^{d+1}. The partition function only depends on the "volume" of the boundary and does not depend on the shape of the boundary. This implies that the boundary theory is gapped.

Next, let us choose $N^d = S^1_t\times S^{d-1}$, where we use S^1_t to represent the closed time direction. We choose its extension to be $M^{d+1} = S^1_t\times B^d$. We find the boundary partition function to be

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{d-1}, S^1_t \times B^d\big) = n^{N_v^{\text{bndr}}} n^{N_v^{\text{blk}}}. \tag{35}$$

We see that the volume-independent boundary partition function is

$$Z^{\text{bndr,top}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{d-1}, S^1_t \times B^d\big) = 1. \tag{36}$$

This implies that the gapped boundary has no ground-state degeneracy (for the boundary S^{d-1}). For example, there is no symmetry breaking.

To see if the boundary carries an anomalous topological order, let us choose $N^d = S^1_t \times S^{k+1} \times S^{d-2-k}$ and choose its extension to be $M^d = S^1_t \times S^{k+1} \times B^{d-1-k}$. Since the tunneling process of k-dimensional branelike topological excitations around S^{k+1} on the boundary corresponds to a noncontractible loop in the bulk $S^1_t \times S^{k+1} \times B^{d-1-k}$, the tunneling process will generate a map between different degenerate ground states. In contrast, the brane tunnel process around S^{d-2-k} on the boundary corresponds to a contractible "loop" in the bulk $S^1_t \times S^{k+1} \times B^{d-1-k}$ and does not generate nontrivial map between degenerate ground states. Therefore, similar to the bulk case, $Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(S^1_t \times S^{k+1} \times S^{d-2-k}, S^1_t \times S^{k+1} \times B^{d-1-k})$ can tell us the number of types of k-dimensional branelike topological excitations on the boundary.

For our $\mathbb{Z}_n$-1-cocycle model, we found the volume-independent partition function to be

$$\begin{aligned} &Z^{\text{bndr,top}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{k+1} \times S^{d-2-k}, S^1_t \times S^{k+1} \times B^{d-1-k}\big) \\ &= \frac{\big|H^1\big(S^1_t \times S^{k+1} \times B^{d-1-k}; \mathbb{Z}_n\big)\big|}{\big|H^0\big(S^1_t \times S^{k+1} \times B^{d-1-k}; \mathbb{Z}_n\big)\big|} = \begin{cases} 1, & k > 0 \\ n, & k = 0. \end{cases} \end{aligned} \tag{37}$$

Thus, the boundary theory contains n types of pointlike excitations, and no nontrivial branelike excitations of dimensions greater then 0.

The n types of pointlike topological excitations on the boundary contain a trivial type and $n-1$ nontrivial type. When $n > 1$, the existence of nontrivial topological excitations on the boundary implies that the boundary carries a nontrivial topological order (which is anomalous). This agrees with the previous known result [31,84].

A given bulk model can have many types of boundaries. For our $\mathbb{Z}_n$-1-cocycle model, the bulk contain n types of pointlike topological excitations and n types of $(d-2)$-dimensional branelike topological excitations. One type of the boundary is formed by the brane condensation. Such a boundary has n types of pointlike topological excitations only. Another type of boundary is formed by the particle condensation. Such a boundary has n types of $(d-2)$-dimensional branelike topological excitations only. We see that our boundary of $\mathbb{Z}_n$-1-cocycle model is the first type induced by the condensation of branes. We will call such boundary as "free boundary" since the 1-cocycle field has a free boundary condition on the boundary.

To realize the second type of the boundary, we need to use the fixed boundary condition by setting the 1-cocycle field to be $a^{\mathbb{Z}_n}_{ij} = 0$ on the boundary. Again, $Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}(S^1_t \times S^{k+1} \times S^{d-2-k}, S^1_t \times S^{k+1} \times B^{d-1-k})$ can tell us the number of types of k-dimensional branelike topological excitations on the boundary. To compute such partition function, we notice that when $k < d-2$, the 1-cocycle $a^{\mathbb{Z}_n}$ can be written as $a^{\mathbb{Z}_n} = dg^{\mathbb{Z}_n}$ is a $\mathbb{Z}_n$-valued 0-cochain which vanishes on the boundary. The correspondence between $a^{\mathbb{Z}_n}_{ij}$ and $g^{\mathbb{Z}_n}_i$ is one to one. This is because even when $k = 0$, $\oint_{S^{k+1}} a^{\mathbb{Z}_n} = 0$ since we have fixed $a^{\mathbb{Z}_n} = 0$ on the boundary. Thus,

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{k+1} \times S^{d-2-k}, S^1_t \times S^{k+1} \times B^{d-1-k}\big) = n^{N_v^{\text{blk}}}. \tag{38}$$

The volume-independent partition function is

$$\begin{aligned} &Z^{\text{bndr,top}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{k+1} \times S^{d-2-k}, S^1_t \times S^{k+1} \times B^{d-1-k}\big) \\ &\quad = 1, \quad \text{for } k < d-2. \end{aligned} \tag{39}$$

Thus, there are no nontrivial k-dimensional branelike excitations on the boundary for $k < d-2$. There are no nontrivial pointlike excitations on the boundary which is the $k = 0$ case included above. When $k = d-2$, $S^{d-2-k} = S^0$ is a set of two points. In this case, the boundary contains two disconnected pieces. We may set the 0-cochain field $g^{\mathbb{Z}_n} = 0$ on one piece. But, we need to set $g^{\mathbb{Z}_n} = \text{const}$ on the other piece. We find that

$$Z^{\text{bndr}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{d-1} \times S^0, S^1_t \times S^{d-1} \times B^1\big) = n n^{N_v^{\text{blk}}} \tag{40}$$

or the volume-independent one

$$Z^{\text{bndr,top}}_{\mathbb{Z}_n\text{a}}\big(S^1_t \times S^{d-1} \times S^0, S^1_t \times S^{d-1} \times B^1\big) = n. \tag{41}$$

There are n types of $(d-2)$-dimensional branelike excitations on the boundary. The $a^{\mathbb{Z}_n} = 0$ boundary gives us the second type of boundary formed by condensing the pointlike excitations.

C. Twisted (2+1)D $\mathbb{Z}_n$-1-cocycle model and emergent Dijkgraaf-Witten theory

1. Model construction

To construct another local bosonic model that realizes a different topological order, we may choose L_{ijkl} to be

$$\begin{aligned} L_{ijkl} &= +\infty \quad \text{if } (da^{\mathbb{Z}_n}) \neq 0, \\ L_{ijkl} &= -ik\frac{2\pi}{n}(a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n})(i,j,k,l) \quad \text{if } (da^{\mathbb{Z}_n}) = 0. \end{aligned} \tag{42}$$

Here, we have used Bockstein homomorphism $\mathcal{B}_n$: $H^m(M^d;\mathbb{Z}_n) \to H^{m+1}(M^d;\mathbb{Z}_n)$,

$$\begin{aligned} &\mathcal{B}_n x \overset{n}{=} \frac{1}{n}dx, \\ &x \in H^m(M^d;\mathbb{Z}_n),\ \mathcal{B}_n x \in H^{m+1}(M^d;\mathbb{Z}_n). \end{aligned} \tag{43}$$

To understand the Bockstein homomorphism, we note that x in the above is a cocycle with $\mathbb{Z}_n$. If we view it as a cochain with integer coefficient $\mathbb{Z}$, then dx is a cochain whose values are always multiples of n. Thus, $\frac{1}{n}dx$ is a valid cochain with integer coefficient. In fact, it is $(m+1)$-cocycle with integer coefficient. After a mod n reduction, $\frac{1}{n}dx$ mod n becomes a $(m+1)$-cocycle with $\mathbb{Z}_n$ coefficient. This is why $\mathcal{B}_n$ is a map from $H^m(M^d;\mathbb{Z}_n)$ to $H^{m+1}(M^d;\mathbb{Z}_n)$. Therefore, $\mathcal{B}_n a^{\mathbb{Z}_n}$ is a 2-cocycle and $a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}$ is a 3-cocycle. Here, we use such a 3-cocycle to construct the action L_{ijkl}.

The total action amplitude $e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})}$ is given by

$$e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} = e^{ik\frac{2\pi}{n}\int_{M^3_{\text{latt}}} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}} \quad (44)$$

for $da^{\mathbb{Z}_n} = 0$, and $e^{-S(\{a_{ij}^{\mathbb{Z}_n}\})} = 0$ for $da^{\mathbb{Z}_n} \neq 0$. The partition function is given by

$$Z_{k;a\mathcal{B}a\mathbb{Z}_n}(M^3_{\text{latt}}) = \sum_{\{a_{ij}^{\mathbb{Z}_n}\},da^{\mathbb{Z}_n}=0} e^{ik\frac{2\pi}{n}\int_{M^3_{\text{latt}}} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}}. \quad (45)$$

Such a partition function defines the twisted (2+1)D $\mathbb{Z}_n$-1-cocycle model.

The volume-independent part of partition function (45) is given by

$$Z^{\text{top}}_{k;a\mathcal{B}a\mathbb{Z}_n}(M^3) = \frac{\sum_{a^{\mathbb{Z}_n}\in H^1(M^3;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^3} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}}}{|H^0(M^3;\mathbb{Z}_n)|}. \quad (46)$$

Since the Euler number on odd-dimensional closed manifolds vanishes, the above volume-independent partition function is a topological invariant.

2. *Topological partition functions*

In this section, we are going to calculate some topological invariants. On $M^3 = S^3$, $S^1\times S^2$, or $T^3 = S^1\times S^1\times S^1$, $\mathcal{B}_n a^{\mathbb{Z}_n} = 0$ and the topological term $k\frac{2\pi}{n}\int_{M^3} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}$ vanishes. We find

$$\begin{aligned} Z^{\text{top}}_{k;a\mathcal{B}a\mathbb{Z}_n}(S^3) &= \frac{1}{n},\\ Z^{\text{top}}_{k;a\mathcal{B}a\mathbb{Z}_n}(S^1\times S^2) &= 1,\\ Z^{\text{top}}_{k;a\mathcal{B}a\mathbb{Z}_n}(T^3) &= n^2. \end{aligned} \quad (47)$$

From $Z^{\text{top}}_{1;a\mathcal{B}a\mathbb{Z}_n}(M^2\times S^1)$, we can determine the ground-state degeneracy (GSD) on M^2:

$$Z^{\text{top}}_{1;a\mathcal{B}a\mathbb{Z}_n}(M^2\times S^1) = \text{GSD}_{1;a\mathcal{B}a\mathbb{Z}_n}(M^2). \quad (48)$$

Using

$$\begin{aligned} Z^{\text{top}}_{1;a\mathcal{B}a\mathbb{Z}_n}(S^2\times S^1) &= 1,\\ Z^{\text{top}}_{1;a\mathcal{B}a\mathbb{Z}_n}(T^2\times S^1) &= n^2, \end{aligned} \quad (49)$$

we find that the GSD on a sphere S^2 is 1 and the GSD on a torus $T^2 = S^1\times S^1$ is n^2.

To obtain the topological invariant that detects the topological term, we put the system on the lens space $L^3(p)$ (see Appendix F 4). We find from

$$\begin{aligned} H_1(L^3(p),\mathbb{Z}) &= \mathbb{Z}_p,\\ H_2(L^3(p),\mathbb{Z}) &= 0,\\ H_3(L^3(p),\mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (50)$$

that [using Eq. (A8)]

$$\begin{aligned} H^1(L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_{\langle p,n\rangle} = \{a\},\\ H^2(L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_{\langle p,n\rangle} = \{b\},\\ H^3(L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_n = \{c\}, \end{aligned} \quad (51)$$

where we have also listed the generators $\{a,b,c\}$. Here, $\langle l,m\rangle$ is the greatest common divisor of l and m, and $\langle 0,m\rangle \equiv m$. In Appendix F 4, we have computed the cohomology ring $H^*(L^3(p),\mathbb{Z}_n)$ [see Eq. (F32)]:

$$a^2 = \frac{n^2 p(p-1)}{2\langle p,n\rangle^2}b,\quad ab = \frac{n}{\langle p,n\rangle}c,\quad b^2 = ac = 0. \quad (52)$$

We have also computed the Bockstein homomorphism

$$\mathcal{B}_n a = \frac{p}{\langle p,n\rangle}b. \quad (53)$$

We can parametrize $a^{\mathbb{Z}_n}$ as

$$a^{\mathbb{Z}_n} = \alpha a,\quad \alpha\in\mathbb{Z}_{\langle n,p\rangle} \quad (54)$$

and find that

$$Z^{\text{top}}_{k;a\mathcal{B}a\mathbb{Z}_n}[L^3(p)] = \frac{1}{n}\sum_{\alpha=0}^{\langle n,p\rangle-1} e^{i2\pi\alpha^2\frac{kp}{\langle n,p\rangle^2}}. \quad (55)$$

We find the above topological invariant is identical to the topological invariant of (2+1)D Z_n Dijkgraaf-Witten theory on lens space $L^3(p)$ for any p [see Eq. (168)]. In fact, one can show that the $\mathbb{Z}_n$-1-cocycle model realize the (2+1)D Z_n Dijkgraaf-Witten theory [33] [see discussions below Eq. (169)]. In other words, the above topological invariant is the topological invariant of a Z_n-gauge theory twisted by a quantized topological term [85] $k\frac{2\pi}{n}\int_{M^3} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}$. The quantized topological term corresponds to a group cocycle in $\mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$. It is the simplest Dijkgraaf-Witten theory. Such a Dijkgraaf-Witten theory can be obtained by gauging the Z_n symmetry of a Z_n SPT state [53]. When $k = 0$, our model realizes the Z_n topological order described by UT Z_n-gauge theory. For $(n,k) = (2,1)$, our model realizes the double-semion topological order [35,36,53].

We like to remark that the twisted (2+1)D $\mathbb{Z}_n$-1-cocycle model and Dijkgraaf-Witten theory are different. Dijkgraaf-Witten theory is a gauge theory where two $a_{ij}^{\mathbb{Z}_n}$ configurations differing by a Z_n-gauge transformation are regarded as the same configuration. In other words, two $a_{ij}^{\mathbb{Z}_n}$ configurations differing by a coboundary are regarded as the same configuration. Thus, the Dijkgraaf-Witten Z_n-gauge theory may be called twisted $\mathbb{Z}_n$-1-cohomology model. In our twisted (2+1)D $\mathbb{Z}_n$-1-cocycle model, different $a_{ij}^{\mathbb{Z}_n}$ configurations are always different with no gauge redundancy. So, the cocycle model is not a gauge theory but a local bosonic system. However, the cocycle model has an emergent gauge theory at low energies which is described by Dijkgraaf-Witten theory.

D. Twisted (3+1)D ($\mathbb{Z}_n\oplus\mathbb{Z}_n$)-1-cocycle model and emergent Dijkgraaf-Witten theory

1. *Model construction*

In this section, we like to design a (3+1)D local bosonic model that realizes the Dijkgraaf-Witten twisted gauge theory at low energies. Since $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z}) = 0$, there is no Z_n Dijkgraaf-Witten gauge theory in (3+1)D. So, here we try to realize the $Z_n\times Z_n$ Dijkgraaf-Witten gauge theory. Such theory exists since $\mathcal{H}^4(Z_n\times Z_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n\oplus\mathbb{Z}_n$.

To realize $Z_n\times Z_n$ gauge theory, we construct a $(\mathbb{Z}_n\oplus\mathbb{Z}_n)$-1-cocycle theory on (3+1)D space-time lattice. The local degrees of freedom of the model correspond to two 1-cochains $a_1^{\mathbb{Z}_n}, a_2^{\mathbb{Z}_n} \in C^2(M^4_{\text{Latt}};\mathbb{Z}_n)$ (i.e., the local degrees of freedom are described by $\mathbb{Z}_n \oplus \mathbb{Z}_n$ on each 1-simplex). The partition function on an oriented space-time M^4_{latt} is given by [59,63]

$$Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4_{\text{latt}}) = \sum_{\{a^{\mathbb{Z}_n}_{I;ij}\}, da_I^{\mathbb{Z}_n}=0} e^{i\frac{2\pi}{n}\int_{M^4_{\text{latt}}} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n}\mathcal{B}_n a_1^{\mathbb{Z}_n}}, \quad (56)$$

where $k_1,k_2 = 0,1,\ldots,n-1$, We have assumed that the configuration with $da_I^{\mathbb{Z}_n} \neq 0$, $I = 1,2$, have infinite energy and do not contribute to the partition function. The term $\frac{1}{n}\int_{M^4_{\text{latt}}} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n}\mathcal{B}_n a_1^{\mathbb{Z}_n}$ corresponds to a cocycle $(k_1,k_2) \in \mathcal{H}^4(Z_n \times Z_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n \oplus \mathbb{Z}_n$.

There are other possible choices of the action amplitude, such as

$$e^{ik\frac{2\pi}{n}\int_{M^4_{\text{latt}}} \mathcal{B}_n a_1^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n}}. \quad (57)$$

But,

$$\int_{M^4_{\text{latt}}} \mathcal{B}_n a_1^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n} = \int_{M^4_{\text{latt}}} \frac{1}{n} da_1^{\mathbb{Z}_n} \frac{1}{n} da_2^{\mathbb{Z}_n} = 0, \quad (58)$$

if M^4_{latt} is orientable. So, such a term always vanishes. Yet another possible choice is $\int_{M^4_{\text{latt}}} a_1^{\mathbb{Z}_n}(a_2^{\mathbb{Z}_n})^3$. But when $n=2$, it is the same as $\int_{M^4_{\text{latt}}} a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n}$, and when $n =$ odd, it vanishes. So, here we do not discuss it further.

2. Topological partition functions

When $k_1,k_2 = 0$, the partition function is given by the square of the number of 1-cocycles, $|Z^1(M^4_{\text{latt}};\mathbb{Z}_n)|^2$. $|Z^1(M^4_{\text{latt}};\mathbb{Z}_n)|$ is $|H^1(M^4_{\text{latt}};\mathbb{Z}_n)|$ times the number of 0-cochains whose derivatives are nonzero. The number of 0-cochains whose derivatives are nonzero is the number of 0-cochains $[|C^0(M^4_{\text{latt}};\mathbb{Z}_n)| = n^{N_v}]$ divided by $|H^0(M^4_{\text{latt}};\mathbb{Z}_n)|$. Thus, the partition function is

$$\begin{aligned} Z_{0,0;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4_{\text{latt}}) &= |Z^1(M^4_{\text{Latt}};\mathbb{Z}_n)|^2 \\ &= |H^1(M^4_{\text{latt}};\mathbb{Z}_n)|^2 \frac{|C^0(M^4_{\text{Latt}};\mathbb{Z}_n)|^2}{|H^0(M^4_{\text{Latt}};\mathbb{Z}_n)|^2} \\ &= n^{N_v}\frac{|H^1(M^4_{\text{Latt}};\mathbb{Z}_n)|^2}{|H^0(M^4_{\text{Latt}};\mathbb{Z}_n)|^2}. \end{aligned} \quad (59)$$

The volume-independent topological partition function is given by

$$Z_{0,0;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4_{\text{latt}}) = \frac{|H^1(M^4_{\text{latt}};\mathbb{Z}_n)|^2}{|H^0(M^4_{\text{latt}};\mathbb{Z}_n)|^2}. \quad (60)$$

When $k_1,k_2 \neq 0$, the volume-independent topological partition function is given by

$$Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4_{\text{latt}}) = \sum_{a_I^{\mathbb{Z}_n}\in H^1(M^4;\mathbb{Z}_n)} \frac{e^{i\frac{2\pi}{n}\int_{M^4_{\text{latt}}} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n}\mathcal{B}_n a_1^{\mathbb{Z}_n}}}{|H^0(M^4;\mathbb{Z}_n)|^2}, \quad (61)$$

where $|H^1(M^4_{\text{latt}};\mathbb{Z}_n)|^2$ is replaced by the summation of phase factors.

Now, let us compute $Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4)$ on several M^4. On $M^4 = S^1\times S^1\times S^1\times S^1 = T^4$ or $M^4 = S^2\times S^1\times S^1 = S^2\times T^2$, $\mathcal{B}_n a_I^{\mathbb{Z}_n} = 0$. Thus, $Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4) = Z_{0,0;aa'\mathcal{B}a'\mathbb{Z}_n}(M^4)$ on those manifolds. Using

$$H^1(T^4;\mathbb{Z}_n) = 4\mathbb{Z}_n, \quad H^1(T^2\times S^2;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}, \quad (62)$$

we find that (see Table III)

$$\begin{aligned} Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(T^4) &= n^6, \\ Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(S^2\times T^2) &= n^2. \end{aligned} \quad (63)$$

On $M^4 = S^1\times L^3(p)$, from

$$H_1(L^3(p),\mathbb{Z}) = \mathbb{Z}_p,\ H_2(L^3(p),\mathbb{Z}) = 0, \quad H_3(L^3(p),\mathbb{Z}) = \mathbb{Z}, \quad (64)$$

we find that [using (A4)]

$$\begin{aligned} H_1(S^1\times L^3(p),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_p, \\ H_2(S^1\times L^3(p),\mathbb{Z}) &= \mathbb{Z}_p, \\ H_3(S^1\times L^3(p),\mathbb{Z}) &= \mathbb{Z}, \\ H_4(S^1\times L^3(p),\mathbb{Z}) &= \mathbb{Z}. \end{aligned} \quad (65)$$

This allows us to obtain [using (A8)]

$$\begin{aligned} H^1(S^1\times L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_n\oplus\mathbb{Z}_{\langle p,n\rangle} = \{a_1,a\}, \\ H^2(S^1\times L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_{\langle p,n\rangle}\oplus\mathbb{Z}_{\langle p,n\rangle} = \{a_1a,b\}, \\ H^3(S^1\times L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_n\oplus\mathbb{Z}_{\langle p,n\rangle} = \{c,a_1b\}, \\ H^4(S^1\times L^3(p),\mathbb{Z}_n) &= \mathbb{Z}_n = \{a_1c\}, \end{aligned} \quad (66)$$

where we have also listed the generators, where a_1 comes from S^1 and a,b,c from $L^3(p)$. Here, $\langle l,m\rangle$ is the greatest common divisor of l and m, and $\langle 0,m\rangle \equiv m$.

In Appendix F 4, we have computed the cohomology ring $H^*(S^1\times L^3(p),\mathbb{Z}_n)$ [see (F32)]:

$$a_1^2 = 0,\ a^2 = \frac{n^2p(p-1)}{2\langle p,n\rangle^2}b,\ ab = \frac{n}{\langle p,n\rangle}c,\ b^2 = ac = 0. \quad (67)$$

We have also computed the Bockstein homomorphism

$$\mathcal{B}_n a = \frac{p}{\langle p,n\rangle}b, \quad \mathcal{B}_n a_1 = 0. \quad (68)$$

We see that for $\langle n,p\rangle = 1$, $a = b = 0$, and thus $\mathcal{B}_n a_I^{\mathbb{Z}_n} = 0$. Therefore, $\int_{M^4_{\text{latt}}} k_1 a_1^{\mathbb{Z}_n} a_2^{\mathbb{Z}_n}\mathcal{B}_n a_2^{\mathbb{Z}_n} + k_2 a_2^{\mathbb{Z}_n} a_1^{\mathbb{Z}_n}\mathcal{B}_n a_1^{\mathbb{Z}_n} = 0$. So $Z_{k_1k_2;aa\beta a\mathbb{Z}_n}[S^1\times L^3(p)] = 1$.

For $\langle n,p\rangle \neq 1$, we can parametrize $a_I^{\mathbb{Z}_n}$ as

$$a_I^{\mathbb{Z}_n} = \alpha_I a_1 + \tilde\alpha_I a, \quad \alpha_I \in \mathbb{Z}_n,\ \tilde\alpha_I \in \mathbb{Z}_{\langle n,p\rangle}. \quad (69)$$

Using Eqs. (66), (67), and (68), we find that

$$Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}[S^1\times L^3(p)] = \frac{1}{n^2}\sum_{\alpha_{1,2}\in\mathbb{Z}_n,\tilde\alpha_{1,2}\in Z_{\langle n,p\rangle}} e^{i\frac{2\pi p}{\langle p,n\rangle^2}[k_1(\alpha_1\tilde\alpha_2^2 - \tilde\alpha_1\alpha_2\tilde\alpha_2) + k_2(\alpha_2\tilde\alpha_1^2 - \tilde\alpha_2\alpha_1\tilde\alpha_1)]}$$

$$= s^2 \sum_{\tilde\alpha_{1,2}=0}^{m-1} \delta_m\big(k_1\tilde\alpha_2^2 - k_2\tilde\alpha_1\tilde\alpha_2\big)\delta_m\big(k_2\tilde\alpha_1^2 - k_1\tilde\alpha_1\tilde\alpha_2\big),$$

$$s = \left\langle \frac{p}{\langle n,p\rangle},\langle n,p\rangle\right\rangle, \quad m = \langle n,p\rangle/s. \tag{70}$$

When p has no repeated prime factor, the above sum has a simple expression

$$Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}[S^1\times L^3(p)] = \langle n,p\rangle\langle n,p,k_1,k_2\rangle = s^2 m\langle m,k_1,k_2\rangle. \tag{71}$$

On $M^4 = F^4 \equiv (S^1\times S^3)\#(S^1\times S^3)\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, we note that the cup product of 1-cocycles is always zero (see Appendix F 5). Thus, $Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(F^4) = Z_{0,0;aa'\beta a'\mathbb{Z}_n}(F^4) = n^2$.

3. Dimension reduction

Last, let us consider $M^4 = M^3\times S^1$, where M^4 and M^3 are assumed to be closed manifolds. We write $a_I^{\mathbb{Z}_n}$ as

$$a_I^{\mathbb{Z}_n} = a_{I,M^3}^{\mathbb{Z}_n} + a_{I,S^1}^{\mathbb{Z}_n}, \tag{72}$$

where $a_{I,M^3}^{\mathbb{Z}_n}$ lives on M^3 and $a_{I,S^1}^{\mathbb{Z}_n}$ on S^1. We also fix $\oint_{S^1} a_{I,S^1}^{\mathbb{Z}_n} = \alpha_I\in\mathbb{Z}$. The partition function now has a form

$$Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^3\times S^1,\alpha_1,\alpha_2) = \frac{1}{|H^0(M^3;\mathbb{Z}_n)|^2}\sum_{a_{I,M^3}^{\mathbb{Z}_n}\in H^1(M^3;\mathbb{Z}_n)} e^{i\frac{2\pi}{n}\int_{M^3}(k_1\alpha_2-k_2\alpha_1)a_{1,M^3}^{\mathbb{Z}_n}\mathcal{B}_n a_{2,M^3}^{\mathbb{Z}_n}} \times e^{i\frac{2\pi}{n}\int_{M^3}k_1\alpha_1 a_{2,M^3}^{\mathbb{Z}_n}\mathcal{B}_n a_{2,M^3}^{\mathbb{Z}_n} + k_2\alpha_2 a_{1,M^3}^{\mathbb{Z}_n}\mathcal{B}_n a_{1,M^3}^{\mathbb{Z}_n}}. \tag{73}$$

In fact, α_I in the above happens to label the different sectors. We find the topological theory in each sector from the partition function $Z_{k_1k_2;aa'\mathcal{B}a'\mathbb{Z}_n}(M^3\times S^1,\alpha_1,\alpha_2)$. As we can see, they are (2+1)D Dijkgraaf-Witten theories.

Since the Dijkgraaf-Witten theories can be viewed as gauged SPT states [53], the dimension reduction of the Dijkgraaf-Witten theories implies a similar dimension reduction of SPT states: If we compact a (3+1)D $Z_n^{(1)}\times Z_n^{(2)}$ SPT state to (2+1)D via a circle S^1, and add a symmetry twist around S^1 described by $e^{i2\pi\alpha_1/n}$ for the $Z_n^{(1)}$ and $e^{i2\pi\alpha_2/n}$ for the $Z_n^{(2)}$, then the resulting (2+1)D SPT state is a stacking of a $Z_n^{(1)}$ SPT state labeled by $k_2\alpha_2\in\mathcal{H}^3(Z_n^{(1)},\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$, a $Z_n^{(2)}$ SPT state labeled by $k_1\alpha_1\in\mathcal{H}^3(Z_n^{(2)},\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$, and a $Z_n^{(1)}\times Z_n^{(2)}$ SPT state labeled by $k_1\alpha_2 - k_2\alpha_1\in\mathcal{H}^3(Z_n^{(1)}\times Z_n^{(2)},\mathbb{R}/\mathbb{Z})$ [51,59].

This implies that the symmetry-twist defect line (twisted by $e^{i2\pi\alpha_1/n}$ for the $Z_n^{(1)}$ and $e^{i2\pi\alpha_2/n}$ for the $Z_n^{(2)}$) in the (3+1)D $Z_n^{(1)}\times Z_n^{(2)}$ SPT state [labeled by $(k_1,k_2)\in\mathcal{H}^4(Z_n^{(1)}\times Z_n^{(2)},\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n\oplus\mathbb{Z}_n$], will carry gapless (1+1)D excitations along the symmetry-twist defect line described by the boundary of the $(k_2\alpha_2)$th $Z_n^{(1)}$ SPT state, the $(k_1\alpha_1)$th $Z_n^{(2)}$ SPT state, and the $(k_1\alpha_2 - k_2\alpha_1)$th $Z_n^{(1)}\times Z_n^{(2)}$ SPT state, provided that the $Z_n^{(1)}\times Z_n^{(2)}$ symmetry is not broken. This result generalized the one in [55].

E. Twisted (3+1)D $\mathbb{Z}_n$-2-cocycle model and emergence of fermions

1. Model construction

In this section, we will study $\mathbb{Z}_n$-2-cocycle theory on (3+1)D space-time lattice. The local degrees of freedom of the model correspond to 2-cochains $b^{\mathbb{Z}_n}\in C^2(M_{\text{Latt}}^4;\mathbb{Z}_n)$ (i.e., the local degrees of freedom are described by $\mathbb{Z}_n$ on each 2-simplex). The partition function is given by, for $k = 0,1,\dots,n-1$ [60],

$$Z_{k;b^2\mathbb{Z}_n}\big(M_{\text{latt}}^4\big) = \sum_{\{b_{ij}^{\mathbb{Z}_n}\},db^{\mathbb{Z}_n}=0} e^{ik\frac{2\pi}{n}\int_{M_{\text{latt}}^4}(b^{\mathbb{Z}_n})^2} \tag{74}$$

(i.e., the configuration with $db^{\mathbb{Z}_n}\neq 0$ has infinite energy). Note that the source (or "charge") of the 2-cocycle field b is a $\mathbb{Z}_n$ string. When $k=0$, it describes a $\mathbb{Z}_n$-2-cocycle theory. When $k\neq 0$, it describes a twisted $\mathbb{Z}_n$-2-cocycle theory.

When $k=0$, the partition function is given by the number of 2-cocycles $|Z^2(M_{\text{latt}}^4;\mathbb{Z}_n)|$, which is $|H^2(M_{\text{latt}}^4;\mathbb{Z}_n)|$ times the number of 1-cochains whose derivatives are nonzero. The number of 1-cochains whose derivatives are nonzero is the number of 1-cochains $[|C^1(M_{\text{latt}}^4;\mathbb{Z}_n)| = n^{N_e}]$ divided by $|H^1(M_{\text{latt}}^4;\mathbb{Z}_n)|$ and by the number of 0-cochains whose derivatives are nonzero. The number of 0-cochains whose derivatives are nonzero is the number of 0-cochains $[|C^0(M_{\text{Latt}}^4;\mathbb{Z}_n)| = n^{N_v}]$ divided by $|H^0(M_{\text{Latt}}^4;\mathbb{Z}_n)|$. Thus, the partition function is

$$\begin{aligned}Z_{0;b^2\mathbb{Z}_n}\big(M_{\text{latt}}^4\big) &= \big|Z^2\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big| \\ &= \big|H^2\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|\frac{\big|C^1\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|}{\big|H^1\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|} \\ &\quad\times\frac{\big|H^0\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|}{\big|C^0\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|} \\ &= n^{N_e-N_v}\frac{\big|H^2\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|\big|H^0\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|}{\big|H^1\big(M_{\text{latt}}^4;\mathbb{Z}_n\big)\big|}.\end{aligned} \tag{75}$$

The volume-independent topological partition function is given by

$$Z_{0;b^2\mathbb{Z}_n}^{\text{top}}(M^4) = \frac{|H^2(M^4;\mathbb{Z}_n)||H^0(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|}. \tag{76}$$

When $k\neq 0$, the volume-independent topological partition function is given by

$$Z_{k;b^2\mathbb{Z}_n}^{\text{top}}(M^4) = \frac{|H^0(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|}\sum_{b^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4}(b^{\mathbb{Z}_n})^2}, \tag{77}$$

where $\sum_{b^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4}(b^{\mathbb{Z}_n})^2}$ replaces $|H^2(M^4;\mathbb{Z}_n)|$.

2. Topological partition functions

Now, let us compute topological invariants (see Table III). On $M^4 = T^4$, the cohomology ring $H^*(T^4;\mathbb{Z}_n)$ is generated by a_I, $I = 1,2,3,4$, where $a_I\in H^1(T^4;\mathbb{Z}_n) = 4\mathbb{Z}_n$. Using the cohomology ring equation (F5) in Appendix F, we can parametrize $b^{\mathbb{Z}_n}$ as

$$b^{\mathbb{Z}_n} = \beta_{IJ} a_I a_J, \quad \beta_{IJ} = -\beta_{JI} \in \mathbb{Z}_n. \tag{78}$$

Thus,

$$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}(T^4) = \frac{1}{n^3} \sum_{\beta_{IJ}\in\mathbb{Z}_n} e^{ik\frac{2\pi}{n}(\beta_{12}\beta_{34}-\beta_{13}\beta_{24}+\beta_{14}\beta_{23})}. \tag{79}$$

Using $\sum_{\beta_1,\beta_2\in\mathbb{Z}_n} e^{ik\frac{2\pi}{n}2\beta_1\beta_2} = \langle 2k,n\rangle n$, we find that

$$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}(T^4) = \langle 2k,n\rangle^3. \tag{80}$$

On $M^4 = S^2 \times T^2$, the cohomology ring $H^*(T^2 \times S^2;\mathbb{Z}_n)$ is generated by a_I, $I = 1,2$ and b, where $a_I \in H^1(T^2 \times S^2;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$ and $b \in H^2(T^2 \times S^2;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$. Using the cohomology ring equation (F7) in Appendix F, we can parametrize $b^{\mathbb{Z}_n}$ as

$$b^{\mathbb{Z}_n} = \beta_1 a_1 a_2 + \beta_2 b, \quad \beta_1,\beta_2 \in \mathbb{Z}_n. \tag{81}$$

Thus,

$$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}(S^2 \times T^2) = \frac{1}{n} \sum_{\beta_1,\beta_2\in\mathbb{Z}_n} e^{ik\frac{2\pi}{n}2\beta_1\beta_2} = \langle 2k,n\rangle. \tag{82}$$

On $M^4 = S^1 \times L^3(p)$, we need to use the cohomology ring $H^*[S^1 \times L^3(p);\mathbb{Z}_n]$ as described in Eqs. (66), (67), and (68). For $\langle n,p\rangle = 1$, $Z^{\text{top}}_{k;b^2\mathbb{Z}_n}[S^1 \times L^3(p)] = 1$. For $\langle n,p\rangle \neq 1$, we can parametrize $b^{\mathbb{Z}_n}$ as

$$b^{\mathbb{Z}_n} = \beta_1 a a_1 + \beta_2 b, \quad \beta_1,\beta_2 \in \mathbb{Z}_{\langle n,p\rangle}. \tag{83}$$

Using $aa_1b = \frac{n}{\langle n,p\rangle} a_1 c$ and $(aa_1)^2 = b^2 = 0$, we find that

$$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}[S^1 \times L^3(p)] = \frac{1}{\langle n,p\rangle} \sum_{\beta_1,\beta_2=0}^{\langle n,p\rangle-1} e^{i2k\frac{2\pi}{\langle n,p\rangle}\beta_1\beta_2} = \langle 2k,n,p\rangle. \tag{84}$$

On $M^4 = F^4$, we need to use the cohomology ring $H^*(F^4;\mathbb{Z}_n)$ as described in Appendix F 5. We can parametrize $b^{\mathbb{Z}_n}$ as

$$b^{\mathbb{Z}_n} = \beta_1 b_1 + \beta_2 b_2, \quad \beta_1,\beta_2 \in \mathbb{Z}_n, \tag{85}$$

where b_1,b_2 are generators of $H^2(F^4;\mathbb{Z}_n)$. Using $b_1^2 = -b_2^2 = v$ and $b_1b_2 = 0$, we find that

$$Z^{\text{top}}_{k;b^2\mathbb{Z}_n}(F^4) = \frac{1}{n} \sum_{\beta_1,\beta_2=0}^{n-1} e^{ik\frac{2\pi}{n}(\beta_1^2-\beta_2^2)} = \begin{cases} \langle 2k,n\rangle & \text{if } \frac{2kn}{\langle 2k,n\rangle^2} = \text{even}; \\ 0, & \text{if } \frac{2kn}{\langle 2k,n\rangle^2} = \text{odd}. \end{cases} \tag{86}$$

The above results are summarized in Table III.

3. *Pointlike and stringlike topological excitations*

When $k \neq 0$, the twisted (3+1)D $\mathbb{Z}_n$-2-cocycle theory realizes a topological order that is not described by Z_n-gauge theory nor by the group-cocycle-twisted Dijkgraaf-Witten theory since the group-cohomology $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z}) = 0$. Here, we will show that

the (3+1)D twisted $\mathbb{Z}_n$-2-cocycle theory realizes a (3+1)D $Z_{\langle 2k,n\rangle}$-gauge theory. The $Z_{\langle 2k,n\rangle}$-gauge theory is a EF $Z_{\langle 2k,n\rangle}$-gauge theory if $2kn/\langle 2k,n\rangle^2 = odd$, and it is a UT $Z_{\langle 2k,n\rangle}$-gauge theory if $2kn/\langle 2k,n\rangle^2 = even$.

The reduction from Z_n to $Z_{\langle 2k,n\rangle}$ by the twist can be seen from the GSD of the model. The GSD on $S^1 \times S^2$ counts the number of types of pointlike topological excitations, and the number of types of stringlike topological excitations. From Eq. (82), we see that twisted $\mathbb{Z}_n$-2-cocycle model gives rise to a topological order with $\langle 2k,n\rangle$ types of pointlike topological excitations and $\langle 2k,n\rangle$ types of stringlike topological excitations.

It is interesting to see that the twisted model describes an invertible topological order when $\langle 2k,n\rangle = 1$. Since all (3+1)D invertible topological orders are trivial topological orders, thus

the twisted $\mathbb{Z}_n$-2-cocycle model describes a trivial product state when $\langle 2k,n\rangle = 1$.

Naively, the twisted $\mathbb{Z}_n$-2-cocycle model should have n types of pointlike topological excitations and n types of stringlike topological excitations. But actually, there are only $\langle 2k,n\rangle$ types of pointlike topological excitations and $\langle 2k,n\rangle$ types of stringlike topological excitations. Other excitations are confined.

To understand the unconfined topological excitations in level-$k\mathbb{Z}_n$-2-cocycle model, we note that we can view $b^{\mathbb{Z}_n}$ as the field strength 2-form of a U(1) gauge theory

$$2\pi b^{\mathbb{Z}_n} = f, \tag{87}$$

where the 2π factor comes from the different quantization convention $\int_{M^2_{\text{closed}}} b^{\mathbb{Z}_n} = \text{integer}$ and $\int_{M^2_{\text{closed}}} f = 2\pi \times$ integer. In this case, the pointlike topological excitations correspond to the monoples in the U(1) gauge theory. Such a U(1) gauge theory is described by the partition function

$$Z_{k,\text{U}(1)}(M^4) = \int D[a] e^{i\frac{\Theta}{8\pi^2}\int_{M^4} ff + \cdots}, \tag{88}$$

where $\Theta = \frac{4\pi k}{n}$, and $\cdots$ represents additional interactions. Without the additional interactions, the particle like excitation in the U(1) gauge theory are labeled by two integers (q,m) where $m = M$ is the magnetic charge. The U(1) charge of (q,m) is given by $Q_{q,m} = q + \frac{\Theta}{2\pi}m$. The statistics of particle (q,m) is determined by $e^{i\theta} = (-)^{mq}$, where $e^{i\theta} = 1$ corresponds to boson and $e^{i\theta} = -1$ corresponds to fermion. Let us express the statistics in terms of physical quantities (Q,M): $e^{i\theta} = (-)^{MQ-\frac{\Theta}{2\pi}M^2}$. We see that when $\Theta = 0$ or $\Theta = 2\pi$, both Q and M are integers, but the statistics of particles with charge (Q,M) are different for $\Theta = 0$ and $\Theta = 2\pi$. Thus, changing Θ by 2π will lead to a different U(1) gauge theory. Changing Θ by 4π will give us the same U(1) gauge theory. This is consistent with the mod n periodicity of k.

For $\Theta = \frac{4\pi k}{n}$, we note that the $(q,m) = (-2k,n)$ particle has a vanishing U(1) charge and is a boson. We can use the additional interactions to condense such a dyon [86]. Such a condensation will make the U(1) gauge theory to be our $\mathbb{Z}_n$-2-cocycle theory. This is because a change of $\int_{M^2_{\text{closed}}} b^{\mathbb{Z}_n}$ by n is a trivial change, which means a change of $\int_{M^2_{\text{closed}}} f$ by

$2\pi n$ should also be a trivial change in the U(1) gauge theory. This is achieved by condensing n units of magnetic charge that is carried by $(q,m) = (-2k,n)$ particle.

Since the condensing particles have a nonzero magnetic charge, in the condensed phase, all the particles with nonzero U(1) charge, $Q_{q,m} \neq 0$, are confined. Thus, the unconfined pointlike topological excitations are given by $(q,m) = l(\frac{-2k}{\langle 2k,n\rangle}, \frac{n}{\langle 2k,n\rangle})$, with $l = 0,1,\dots,\langle 2k,n\rangle - 1$. We see that the GSD on $S^1 \times S^2$ corresponds to the number of types of pointlike topological excitations. We also note that when $2kn/\langle 2k,n\rangle^2 =$ odd (such as $k = 1$, $n = 2$), some pointlike topological excitations are fermions. When $2kn/\langle 2k,n\rangle^2 =$ even, all pointlike topological excitations are bosons.

In the condensed phase, the electric flux lines are quantized as $\int_{M^2_{\text{closed}}} d\boldsymbol{S}\cdot\boldsymbol{E} = \frac{1}{n}\times m$, $m \in \mathbb{Z}$. They are the stringlike topological excitations. Moving a pointlike excitation labeled by l around a stringlike excitation labeled by m give rise a phase $e^{i\frac{lm}{\langle 2k,n\rangle}}$. So, the strings labeled by m and $m + \langle 2k,n\rangle$ are indistinguishable. This suggests that we have $\langle 2k,n\rangle$ type of stringlike excitations.

We find that the pointlike and stringlike topological excitations in the level-k $\mathbb{Z}_n$-2-cocycle model are very similar to those in $Z_{\langle 2k,n\rangle}$-gauge theory, except that the odd $Z_{\langle 2k,n\rangle}$ charges are fermions when $2kn/\langle 2k,n\rangle^2 =$ odd. The emergence of fermions is supported by the vanishing of volume-independent partition function on a nonspin manifold $F^4 = (S^1 \times S^3)\#(S^1 \times S^3)\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ [see Eq. (86)], which happens exactly at $2kn/\langle 2k,n\rangle^2 =$ odd.

It was first pointed out in the string-net theory [36] that a (3+1)D gauge theory can be twisted which makes some gauge charge described by "odd" representations to be fermionic. But, when we use cocycles in $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ to twist a G-gauge theory [33], the pointlike topological excitations are always boson [87]. Thus, the level-k $\mathbb{Z}_n$-2-cocycle model is a different realization of the twist discussed in the (3+1)D string-net theory.

4. Including excitations in the path integral

We know that the pointlike excitations are described by the world lines M^1_{WL} in space-time. A world line M^1_{WL} can be viewed as a $\mathbb{Z}_n$-valued 1-cycle, which is dual to a 3-coboundary $C^{\mathbb{Z}_n}_{\text{WL}}$. In the twisted $\mathbb{Z}_n$-2-cocycle model, such a pointlike excitation is described by the 2-cochain field $b^{\mathbb{Z}_n}$ that satisfies $db^{\mathbb{Z}_n} = \tilde{p}C^{\mathbb{Z}_n}_{\text{WL}}$, where $\tilde{p}$ is the charge of the pointlike excitation. The world sheet can be viewed as $\mathbb{Z}_n$-valued 2-cycles M^2_{WS} in the space-time lattice. Therefore, in the presence of pointlike topological excitations described by $C^{\mathbb{Z}_n}_{\text{WL}}$ and stringlike topological excitations described by M^2_{WS}, the partition function becomes

$$Z_{k;b^2\mathbb{Z}_n}\left(M^4_{\text{latt}};\tilde{p}C^{\mathbb{Z}_n}_{\text{WL}},sM^2_{\text{WS}}\right) = \sum_{\{b^{\mathbb{Z}_n}_{ijk}\},db^{\mathbb{Z}_n}=\tilde{p}C^{\mathbb{Z}_n}_{\text{WL}}} e^{ik\frac{2\pi}{n}\int_{M^4_{\text{latt}}}(b^{\mathbb{Z}_n})^2 + is\frac{2\pi}{n}\int_{M^2_{\text{WS}}} b^{\mathbb{Z}_n}}, \quad (89)$$

where s is the charge of the stringlike excitation.

We first solve $db^{\mathbb{Z}_n} = \tilde{p}C^{\mathbb{Z}_n}_{\text{WL}} \bmod n$ as

$$b^{\mathbb{Z}_n} \overset{n}{=} \tilde{p}b^{\mathbb{Z}_n}_{\text{WL}} + b^{\mathbb{Z}_n}_0 + da^{\mathbb{Z}_n}, \quad (90)$$

where $b^{\mathbb{Z}_n}_{\text{WL}}$ is a fixed 2-cochain field that satisfies $db^{\mathbb{Z}_n}_{\text{WL}} = C^{\mathbb{Z}_n}_{\text{WL}}$ and $b^{\mathbb{Z}_n}_0 \in H^2(M^4;\mathbb{Z}_n)$. We can rewrite the partition function as

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}\left(M^4;\tilde{p}C^{\mathbb{Z}_n}_{\text{WL}},sM^2_{\text{WS}}\right) &\propto e^{i\frac{2\pi}{n}\int_{M^4} k\tilde{p}^2(b^{\mathbb{Z}_n}_{\text{WL}})^2} e^{i\frac{2\pi}{n}\int_{M^2_{\text{WS}}} s\tilde{p}b^{\mathbb{Z}_n}_{\text{WL}}} \sum_{\{a^{\mathbb{Z}_n}_{ij}\},b^{\mathbb{Z}_n}_0\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4} 2\tilde{p}(b^{\mathbb{Z}_n}_0 + da^{\mathbb{Z}_n})b^{\mathbb{Z}_n}_{\text{WL}}} \\ &= e^{i\frac{2\pi k\tilde{p}^2}{n}\int_{M^4}(b^{\mathbb{Z}_n}_{\text{WL}})^2} e^{i\frac{2\pi s\tilde{p}}{n}\int_{M^2_{\text{WS}}} b^{\mathbb{Z}_n}_{\text{WL}}} \sum_{b^{\mathbb{Z}_n}_0\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4} b^{\mathbb{Z}_n}_0(2\tilde{p}b^{\mathbb{Z}_n}_{\text{WL}} + b^{\mathbb{Z}_n}_0)} \sum_{\{a^{\mathbb{Z}_n}_{ij}\}} e^{i2k\tilde{p}\frac{2\pi}{n}\int_{M^4} a^{\mathbb{Z}_n}C^{\mathbb{Z}_n}_{\text{WL}}}. \end{aligned} \quad (91)$$

Let D^3_{WS} be the extension of M^2_{WS}, i.e., $\partial D^3_{\text{WS}} = M^2_{\text{WS}}$. Then, we can rewrite $\int_{M^2_{\text{WS}}} b^{\mathbb{Z}_n}_{\text{WL}} = \int_{D^3_{\text{WS}}} db^{\mathbb{Z}_n}_{\text{WL}} = \int_{D^3_{\text{WS}}} C^{\mathbb{Z}_n}_{\text{WL}}$. In fact, $\int_{D^3_{\text{WS}}} C^{\mathbb{Z}_n}_{\text{WL}} = \text{Int}(D^3_{\text{WS}},M^1_{\text{WL}})$ is the intersection number between D^3_{WS} and M^1_{WL}, which in turn is the linking number between M^2_{WS} and M^1_{WL}: $\text{Lnk}(M^2_{\text{WS}},M^1_{\text{WL}})$.

Using the Poincaré duality we can also rewrite $\int_{M^4} a^{\mathbb{Z}_n}C^{\mathbb{Z}_n}_{\text{WL}}$ as $\int_{M^1_{\text{WL}}} a^{\mathbb{Z}_n}$. Then,

$$\sum_{\{a^{\mathbb{Z}_n}_{ij}\}} e^{i2k\tilde{p}\frac{2\pi}{n}\int_{M^4} a^{\mathbb{Z}_n}C^{\mathbb{Z}_n}_{\text{WL}}} = \sum_{\{a^{\mathbb{Z}_n}_{ij}\}} e^{i2k\tilde{p}\frac{2\pi}{n}\int_{M^1_{\text{WL}}} a^{\mathbb{Z}_n}} \neq 0 \quad (92)$$

only when $[2k\tilde{p}]_n = 0$, i.e., when $\tilde{p}$ is quantized as $\tilde{p} = p\frac{n}{\langle 2k,n\rangle}$, $p \in \mathbb{Z}_{\langle 2k,n\rangle}$. If $\tilde{p}$ is not quantized as the above, the corresponding pointlike excitation is confined.

Thus, the above partition function for unconfined excitations can be rewritten as

$$\begin{aligned} Z_{k;b^2\mathbb{Z}_n}&\left(M^4;\frac{pn}{\langle 2k,n\rangle}C^{\mathbb{Z}_n}_{\text{WL}},sM^2_{\text{WS}}\right) \\ &\propto e^{isp\frac{2\pi}{\langle 2k,n\rangle}\text{Lnk}(M^2_{\text{WS}},M^1_{\text{WL}})} e^{i\pi p^2\frac{2nk}{\langle 2k,n\rangle^2}\int_{M^4}(b^{\mathbb{Z}_n}_{\text{WL}})^2} \\ &\times \sum_{b^{\mathbb{Z}_n}_0\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4} b^{\mathbb{Z}_n}_0(\frac{2pn}{\langle 2k,n\rangle}b^{\mathbb{Z}_n}_{\text{WL}} + b^{\mathbb{Z}_n}_0)}. \end{aligned} \quad (93)$$

The above expression tells us the braiding statistics of pointlike excitations and stringlike excitations. Let us assume $H^2(M^4;\mathbb{Z}_n) = 0$. In this case $\int_{M^4} b^{\mathbb{Z}_n}_{\text{WL}}b^{\mathbb{Z}_n}_{\text{WL}}$ is an integer, and $p^2\frac{2nk}{\langle 2k,n\rangle^2}$ is also an integer. Thus, $e^{i\pi p^2\frac{2nk}{\langle 2k,n\rangle^2}\int_{M^4} b^{\mathbb{Z}_n}_{\text{WL}}b^{\mathbb{Z}_n}_{\text{WL}}}$ is always 1 when $p^2\frac{2nk}{\langle 2k,n\rangle^2} =$ even and $e^{i\pi p^2\frac{2nk}{\langle 2k,n\rangle^2}\int_{M^4} b^{\mathbb{Z}_n}_{\text{WL}}b^{\mathbb{Z}_n}_{\text{WL}}}$ can be -1 when $p^2\frac{2nk}{\langle 2k,n\rangle^2} =$ odd. This factor determines the statistics of the pointlike excitations since $b_{\rm WL}^{\mathbb{Z}_n}$ is determined by the particle world line $M_{\rm WL}^1$. Comparing with the results obtained in the last section, we find that *when the factor* $e^{i\pi p^2 \frac{2nk}{\langle 2k,n\rangle^2}\int_{M^4} b_{WL}^{\mathbb{Z}_n} b_{WL}^{\mathbb{Z}_n}}$ *can be* -1 *(depending one the braiding of the world line* M_{WL}^1*), then the corresponding particle is a fermion.* This means that when $p^2\frac{2nk}{\langle 2k,n\rangle^2} =$ odd, the charge p particle is a fermion.

The factor $e^{isp\frac{2\pi}{\langle 2k,n\rangle}{\rm Lnk}(M_{\rm WS}^2,M_{\rm WL}^1)}$ determines the mutual statistics (i.e., the Aharonov-Bohm phase) between pointlike and stringlike excitations. We see that it is the usual mutual statistics of $Z_{\langle 2k,n\rangle}$-gauge theory. We also see that there is no nontrivial braiding statistics between stringlike excitations. This confirms our result in the last section that

the $\mathbb{Z}_n$*-2-cocyle model produces a low-energy effective* $Z_{\langle 2k,n\rangle}$*-gauge theory. It is a UT* $Z_{\langle 2k,n\rangle}$*-gauge theory if* $\frac{2nk}{\langle 2k,n\rangle^2} =$ *even, and a EF* $Z_{\langle 2k,n\rangle}$*-gauge theory if* $\frac{2nk}{\langle 2k,n\rangle^2} =$ *odd.*

The term $\sum_{b_0^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4} b_0^{\mathbb{Z}_n}(\frac{2pn}{\langle 2k,n\rangle}b_{\rm WL}^{\mathbb{Z}_n}+b_0^{\mathbb{Z}_n})}$ tells us when the partition function will vanish in the presence of emergent fermions, i.e., when $\frac{2nk}{\langle 2k,n\rangle^2} =$ odd. Let us assume there is no world line and $\frac{2nk}{\langle 2k,n\rangle^2} =$ odd. In this case, the above factor becomes $\sum_{b_0^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4}(b_0^{\mathbb{Z}_n})^2}$. We note that $\frac{2nk}{\langle 2k,n\rangle^2} =$ odd implies that k and $\frac{n}{2}$ are both odd integers. Since n is even and $\frac{n}{2}$ is odd, we have $\mathbb{Z}_n = \mathbb{Z}_{n/2}\oplus\mathbb{Z}_2$. Therefore, $b_0^{\mathbb{Z}_n}$ can be expressed as

$$b_0^{\mathbb{Z}_n} = 2b_0^{\mathbb{Z}_{n/2}} + \frac{n}{2}b_0^{\mathbb{Z}_2}. \tag{94}$$

We obtain

$$\begin{aligned}&\sum_{b_0^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)} e^{ik\frac{2\pi}{n}\int_{M^4}(b_0^{\mathbb{Z}_n})^2}\\ &= \sum_{b_0^{\mathbb{Z}_{n/2}}\in H^2(M^4;\mathbb{Z}_{n/2})} e^{ik\frac{2\pi}{n}\int_{M^4}4(b_0^{\mathbb{Z}_{n/2}})^2}\\ &\quad\times \sum_{b_0^{\mathbb{Z}_2}\in H^2(M^4;\mathbb{Z}_2)} e^{ik\frac{2\pi}{n}\int_{M^4}(\frac{n}{2})^2(b_0^{\mathbb{Z}_2})^2}\\ &= \sum_{b_0^{\mathbb{Z}_{n/2}}\in H^2(M^4;\mathbb{Z}_{n/2})} e^{i2k\frac{2\pi}{n/2}\int_{M^4}(b_0^{\mathbb{Z}_{n/2}})^2} \sum_{b_0^{\mathbb{Z}_2}\in H^2(M^4;\mathbb{Z}_2)} e^{i\pi\int_{M^4}(b_0^{\mathbb{Z}_2})^2}.\end{aligned} \tag{95}$$

The factor $\sum_{b_0^{\mathbb{Z}_2}\in H^2(M^4;\mathbb{Z}_2)} e^{i\pi\int_{M^4}(b_0^{\mathbb{Z}_2})^2}$ can be rewritten as

$$\sum_{b_0^{\mathbb{Z}_2}\in H^2(M^4;\mathbb{Z}_2)} e^{i\pi\int_{M^4}(b_0^{\mathbb{Z}_2})^2} = \sum_{b_0^{\mathbb{Z}_2}\in H^2(M^4;\mathbb{Z}_2)} e^{i\pi\int_{M^4}{\rm w}_2 b_0^{\mathbb{Z}_2}} \tag{96}$$

since M^4 is orientable. Now, we see that

$Z_{k;b^2\mathbb{Z}_n}(M^4) = 0$ *when* ${\rm w}_2 \neq 0$ *(i.e., when the orientable* M^4 *is not spin), if there is an emergence of fermions.*

F. (3+1)D twisted $\mathbb{Z}_n$a$\mathbb{Z}_n$b model

1. Model construction

In this section, we are going to construct a local bosonic model on space-time lattice $M_{\rm latt}^4$. Our model is a mixture of $\mathbb{Z}_n$-1-cocycle model and $\mathbb{Z}_n$-2-cocycle model. The local degrees of freedom of our model are $\mathbb{Z}_n$ indices $a_{ij}^{\mathbb{Z}_n}$ on the links and $b_{ijk}^{\mathbb{Z}_n}$ on the triangles. We view $a_{ij}^{\mathbb{Z}_n}$ as a 1-cochain in $C^1(M_{\rm latt}^4;\mathbb{Z}_n)$ and $b_{ijk}^{\mathbb{Z}_n}$ as a 2-cochain in $C^2(M_{\rm latt}^4;\mathbb{Z}_n)$.

Using the Bockstein homomorphism for $\mathbb{Z}_n$, $\mathcal{B}_n : H^m(M^d;\mathbb{Z}_n)\to H^{m+1}(M^d;\mathbb{Z}_n)$, the partition function of our model is defined as

$$Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M_{\rm latt}^4) = \sum_{\substack{\{a_{ij}^{\mathbb{Z}_n}\}\\ da^{\mathbb{Z}_n}=0}}\sum_{\substack{\{b_{ijk}^{\mathbb{Z}_n}\}\\ db^{\mathbb{Z}_n}=0}} e^{i\frac{2\pi}{n}\int_{M_{\rm latt}^4} k_1 b^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}+k_2 b^{\mathbb{Z}_n}b^{\mathbb{Z}_n}}. \tag{97}$$

The volume-independent topological partition function is given by

$$Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}^{\rm top}(M^4) = \sum_{\substack{a^{\mathbb{Z}_n}\in H^1(M^4;\mathbb{Z}_n)\\ b^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)}} \frac{e^{i\frac{2\pi}{n}\int_{M^4} k_1 b^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}+k_2 b^{\mathbb{Z}_n}b^{\mathbb{Z}_n}}}{|H^1(M^4;\mathbb{Z}_n)|}. \tag{98}$$

2. Topological partition functions

On $M^4 = T^4$ or $M^4 = S^2\times T^2$, $\mathcal{B}_n a^{\mathbb{Z}_n} = 0$. Thus, the partition function is a product of the partition function of the $\mathbb{Z}_n$a model in Sec. III B and the partition function of the $\mathbb{Z}_n$b model in Sec. III E. We find that (see Table III)

$$\begin{aligned}Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}^{\rm top}(T^4) &= n^3\langle 2k_2,n\rangle^3,\\ Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}^{\rm top}(S^2\times T^2) &= n\langle 2k_2,n\rangle.\end{aligned} \tag{99}$$

On $M^4 = S^1\times L^3(p)$, for $\langle n,p\rangle = 1$, we find that $\int_{M^4} b^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n} = \int_{M^4} b^{\mathbb{Z}_n}b^{\mathbb{Z}_n} = 0$, since $H^2[S^1\times L^3(p);\mathbb{Z}_n] = 0$. So, $Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}^{\rm top}[S^1\times L^3(p)] = 1$. For $\langle n,p\rangle \neq 1$, we can parametrize $a^{\mathbb{Z}_n}$, $b^{\mathbb{Z}_n}$ as

$$\begin{aligned}a^{\mathbb{Z}_n} &= \alpha_1 a_1 + \alpha_2 a,\quad \alpha_1\in\mathbb{Z}_n,\ \alpha_2\in\mathbb{Z}_{\langle n,p\rangle},\\ b^{\mathbb{Z}_n} &= \beta_1 a_1 a + \beta_2 b,\quad \beta_1,\beta_2\in\mathbb{Z}_{\langle n,p\rangle}.\end{aligned} \tag{100}$$

Using $\mathcal{B}_n a = \frac{p}{\langle n,p\rangle}b$, $\mathcal{B}_n a_1 = 0$, $a_1ab = \frac{n}{\langle n,p\rangle}ca_1$, and $b^2 = (a_1a)^2 = 0$ [see Eqs. (66), (67), and (68)], we find that

$$\begin{aligned}&Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}^{\rm top}[S^1\times L^3(p)]\\ &= \sum_{\alpha_1\in\mathbb{Z}_n;\alpha_2,\beta_1,\beta_2\in Z_{\langle n,p\rangle}} \frac{e^{i\frac{2\pi}{\langle n,p\rangle}(k_1\frac{p}{\langle n,p\rangle}\alpha_2\beta_1+2k_2\beta_1\beta_2)}}{n\langle n,p\rangle}\\ &= \sum_{\alpha_2,\beta_2\in Z_{\langle n,p\rangle}} \frac{\delta_{\langle n,p\rangle}\left(k_1\frac{p}{\langle n,p\rangle}\alpha_2+2k_2\beta_2\right)}{\langle n,p\rangle}\\ &= \langle n,p\rangle\left\langle 2k_2,k_1\frac{p}{\langle n,p\rangle},\langle n,p\rangle\right\rangle.\end{aligned} \tag{101}$$

On $M^4 = F^4$, we note that the Bockstein homomorphism $\mathcal{B}_n$ maps all 1-cocycles to 0. Thus, the partition function is a product of the partition function of the $\mathbb{Z}_n$a model in Sec. III B and the partition function of the $\mathbb{Z}_n$b model in Sec. III E:

$$Z^{\text{top}}_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(F^4) = \begin{cases} n\langle 2k_2,n\rangle & \text{if } \frac{2k_2 n}{\langle 2k_2,n\rangle^2} = \text{even}, \\ 0 & \text{if } \frac{2k_2 n}{\langle 2k_2,n\rangle^2} = \text{odd}. \end{cases} \quad (102)$$

3. Pointlike and stringlike topological excitations

Here, we are going to study more physical properties of the $\mathbb{Z}_n$a$\mathbb{Z}_n$b model. The GSD of our model on space M^3 is given by $\text{GSD}_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M^3) = Z^{\text{top}}_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_2}(S^1\times M^3)$. If we choose $M^3 = S^1\times S^2$, we find that

$$\text{GSD}_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(S^1\times S^2) = n\langle 2k_2,n\rangle. \quad (103)$$

The GSD on $S^1\times S^2$ implies that there are $n\langle 2k_2,n\rangle$ types of pointlike and stringlike excitations regardless the value of k_1. This result is unexpected since one may guess the number of types of pointlike and stringlike excitations are n^2. The reduction is due to confinement as will be explained below.

Again, we will view $b^{\mathbb{Z}_n}$ as the field strength 2-form of a U(1) gauge theory

$$2\pi b^{\mathbb{Z}_n} = f. \quad (104)$$

We will also view $\mathcal{B}_n a^{\mathbb{Z}_n}$ as the field strength 2-form of another U(1) gauge theory

$$2\pi \mathcal{B}_n a^{\mathbb{Z}_n} = f'. \quad (105)$$

So, the twisted (3+1)D $b\mathcal{B}a$-$bb\mathbb{Z}_n$ model can be viewed as U(1) × U′(1) gauge theory with some proper condensations. The U(1) × U′(1) gauge theory has a form

$$Z_{1,\text{U}^2(1)}(M^4) = \int D[a]D[a']e^{i\frac{\Theta_1}{4\pi^2}\int_{M^4} ff' i\frac{\Theta_2}{8\pi^2}\int_{M^4} ff+\cdots}, \quad (106)$$

with $\Theta_1 = k_1\frac{2\pi}{n}$ and $\Theta_2 = k_2\frac{4\pi}{n}$.

Let us consider a more general $\text{U}^\kappa(1)$ model

$$Z = \int \prod_I D[a_I]e^{i\frac{2\pi}{8\pi^2}\int_{M^4} f_I\Lambda_{IJ}f_J+\cdots}, \quad (107)$$

where Λ_{IJ} is a symmetric rational matrix. On the boundary, the action amplitude becomes

$$e^{i\frac{1}{4\pi}\int_{\partial M^4}\Lambda_{IJ}a_I da_J+\cdots}. \quad (108)$$

We see that 2π flux of a_J carries a_I charge $Q_I = \Lambda_{IJ}$.

Before the condensation, the pointlike excitations are labeled by (q,m,q',m'). The magnetic charges for the two U(1) gauge fields are $M = m$ and $M' = m'$. Using the above result with

$$\Lambda = \begin{pmatrix} 0 & \frac{k_1}{n} \\ \frac{k_1}{n} & \frac{2k_2}{n} \end{pmatrix}, \quad (109)$$

we see that the electric charges for the two U(1) gauge fields are $Q = q + \frac{k_1}{n}m'$ and $Q' = q' + \frac{k_1}{n}m + \frac{2k_2}{n}m'$. The statistics of the (q,m,q',m') excitation is $e^{i\theta} = (-)^{qm+q'm'}$.

Next, we condense $(q,m,q',m') = (-k_1,0,-2k_2,n)$ excitations that have $Q = Q' = 0$. Since $(M,M') = (0,n)$ for such excitations, it breaks the second U′(1) to Z_n (in the dual picture). We also condense $(q,m,q',m') = (n,0,0,0)$ particles with $(Q,Q',M,M') = (n,0,0,0)$. It breaks the first U(1) to Z_n. The unconfined particles must have $M = Q' = 0$, i.e., $q' = m = 0$. Thus, the unconfined particles are generated by $(q,m,q',m') = (1,0,0,0)$ with $(Q,M,Q',M') = (1,0,0,0)$ and $(q,m,q',m') = (0,0,-\frac{2k_2}{\langle 2k_2,n\rangle},\frac{n}{\langle 2k_2,n\rangle})$ with $(Q,M,Q',M') = (\frac{k_1}{\langle 2k_2,n\rangle},0,0,\frac{n}{\langle 2k_2,n\rangle})$. We see that the pointlike excitations are labeled by (p,p') [a bound state of p type $(q,m,q',m') = (1,0,0,0)$ and p' type $(q,m,q',m') = (0,0,-\frac{2k_2}{\langle 2k_2,n\rangle},\frac{n}{\langle 2k_2,n\rangle})$ excitations]. Two particles that differ by a condensing particle are regarded as equivalent. Thus, (p,p') labels have the following equivalent relation:

$$(p+n,p') \sim (p,p') \sim (p-k_1,p'+\langle 2k_2,n\rangle). \quad (110)$$

So, there are $n\langle 2k_2,n\rangle$ distinct types of pointlike excitations. The type $(q,m,q',m') = (1,0,0,0)$ excitation is a boson. The type $(q,m,q',m') = (0,0,-\frac{2k_2}{\langle 2k_2,n\rangle},\frac{n}{\langle 2k_2,n\rangle})$ excitation has a statistics $(-)^{\frac{2k_2 n}{\langle 2k_2,n\rangle^2}}$.

We note that the pointlike excitations are labeled by the integer points (p,p') in a two-dimensional unit cell with basis vectors $(n,0)$ and $(-k_1,\langle 2k_2,n\rangle)$. We put the two basis vectors together to form a matrix $\begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2,n\rangle\end{pmatrix}$. The fusion of the pointlike excitations is described by an Abelian group $G\begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2,n\rangle\end{pmatrix}$ characterized by the matrix. In general, the fusion rule of the pointlike excitations is not given by $Z_n\times Z_{\langle 2k_2,n\rangle}$.

The stringlike excitations are generated by the $2\pi/n$ magnetic flux line of the first U(1) and the $1/n$-unit electric flux line of the second U′(1). So, the generic stringlike excitations are labeled by (s,s'). Two strings that can join are regarded as equivalent [31]. Note we can attach a (q,m,q',m') excitation to change string (s,s') to an equivalent one, which generates the following equivalence relation:

$$(s,s') \sim (s+nm,s'+nq'+k_1m+2k_2m'). \quad (111)$$

The above can be rewritten as

$$(s+n,s'+k_1) \sim (s,s') \sim (s,s'+\langle 2k_2,n\rangle). \quad (112)$$

We see that there are $n\langle 2k_2,n\rangle$ distinct types of stringlike excitations.

The fusion of the stringlike excitations is described by an Abelian group $G\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2,n\rangle\end{pmatrix}$. It turns out that the fusion of the pointlike excitations and the fusion of the stringlike excitations are described by the same Abelian group

$$G\begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2,n\rangle\end{pmatrix} = G\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2,n\rangle\end{pmatrix}. \quad (113)$$

In general, two integer matrices M_1 and M_2 describe the same Abelian group if $M_2 = WM_1U$ where U,W are invertible integer matrices. In this case, we say $M_1\sim M_2$. Let $\begin{pmatrix} m_1 & 0 \\ 0 & m_2\end{pmatrix}$ be the Smith normal form of $\begin{pmatrix} n & 0 \\ k_1 & \langle 2k_2,n\rangle\end{pmatrix}$, i.e.,

$$W\begin{pmatrix} n & 0 \\ k_1 & \langle 2k_2,n\rangle\end{pmatrix}U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2\end{pmatrix}. \quad (114)$$

This implies that

$$U_T\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2,n\rangle \end{pmatrix} W^T = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{115}$$

We see that

$$\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2,n\rangle \end{pmatrix} \sim \begin{pmatrix} n & 0 \\ k_1 & \langle 2k_2,n\rangle \end{pmatrix} \sim \begin{pmatrix} n & 0 \\ -k_1 & \langle 2k_2,n\rangle \end{pmatrix}. \tag{116}$$

Via direct numerical calculation, we find that

$$G\begin{pmatrix} n & k_1 \\ 0 & \langle 2k_2,n\rangle \end{pmatrix} = Z_{\frac{n\langle n,2k_2\rangle}{\langle n,k_1,2k_2\rangle}} \times Z_{\langle n,k_1,2k_2\rangle}. \tag{117}$$

The mutual braiding phase between a type-(p,p') pointike excitation and a type-(s,s') stringlike excitation is given by

$$\theta = 2\pi\Big(\frac{ps}{n} - \frac{p's'}{\langle 2k_2,n\rangle} + \frac{p'sk_1}{n\langle 2k_2,n\rangle}\Big). \tag{118}$$

Since both pointlike excitations and stringlike excitations have a fusion described by $Z_{\frac{n\langle n,2k_2\rangle}{\langle n,k_1,2k_2\rangle}} \times Z_{\langle n,k_1,2k_2\rangle}$, we will call the corresponding theory a $Z_{\frac{n\langle n,2k_2\rangle}{\langle n,k_1,2k_2\rangle}} \times Z_{\langle n,k_1,2k_2\rangle}$ fusion theory. When $\frac{2k_2n}{\langle 2k_2,n\rangle^2}=$ odd, some pointlike excitations are fermions.

4. Including excitations in the path integral

In the $\mathbb{Z}_n$a$\mathbb{Z}_n$b model, there are two kinds of pointlike excitations described by the world lines M^1_{WL} and N^1_{WL}, which are $\mathbb{Z}$-valued 1-cycles. Let 3-coboundary $C^{\mathbb{Z}}_{\text{WL}}$ be the Poincaré dual of M^1_{WL}. Then, the pointlike excitation that corresponds to M^1_{WL} is described the 2-cochain field $b^{\mathbb{Z}_n}$ that satisfies

$$db^{\mathbb{Z}_n} \overset{n}{=} \tilde p_1 C^{\mathbb{Z}}_{\text{WL}}, \tag{119}$$

where $\tilde p_1$ is the charge of the pointlike excitation.

The $\mathbb{Z}_n$a$\mathbb{Z}_n$b model also contains two kinds of stringlike excitations described by the world sheets M^2_{WS} and N^2_{WS} in space-time. The world sheet N^2_{WS} can be viewed as a $\mathbb{Z}$-valued 2-cycle, which is dual to a $\mathbb{Z}$-valued 2-coboundary $B^{\mathbb{Z}}_{\text{WS}}$. Such a stringlike excitation is described the 1-cochain field $a^{\mathbb{Z}_n}$ that satisfies

$$da^{\mathbb{Z}_n} \overset{n}{=} s_2 B^{\mathbb{Z}}_{\text{WS}}, \tag{120}$$

where s_2 is the charge of the stringlike excitation. Therefore, in the presence of pointlike topological excitations described by $C^{\mathbb{Z}}_{\text{WL}}$, N^1_{WL} and stringlike topological excitations described by M^2_{WS}, $B^{\mathbb{Z}}_{\text{WS}}$, the partition function becomes

$$\begin{aligned} &Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}\big(M^4,\tilde p_1C^{\mathbb{Z}_n}_{\text{WL}},p_2N^1_{\text{WL}},s_1M^2_{\text{WS}},s_2B^{\mathbb{Z}}_{\text{WS}}\big)\\ &= \sum_{\substack{\{a^{\mathbb{Z}_n}_{ij}\}\\ da^{\mathbb{Z}_n}\overset{n}{=}s_2B^{\mathbb{Z}}}} \sum_{\substack{\{b^{\mathbb{Z}_n}_{ijk}\}\\ db^{\mathbb{Z}_n}\overset{n}{=}\tilde p_1C^{\mathbb{Z}}_{\text{WL}}}} e^{i\frac{2\pi}{n}\int_{M^4}k_1b^{\mathbb{Z}_n}\mathcal{B}_na^{\mathbb{Z}_n}+k_2(b^{\mathbb{Z}_n})^2}\\ &\quad\times e^{i\frac{2\pi}{n}(p_2\int_{N^1_{\text{WL}}}a^{\mathbb{Z}_n}+s_1\int_{M^2_{\text{WS}}}b^{\mathbb{Z}_n})}, \end{aligned} \tag{121}$$

where p_2 is the charge of the pointlike excitation and s_1 the charge of the stringlike excitation. However, the above partition function is not well defined. It is well defined only when $b^{\mathbb{Z}_n}$ and $a^{\mathbb{Z}_n}$ are cocycles. When $b^{\mathbb{Z}_n}$ and $a^{\mathbb{Z}_n}$ are not cocycles, the partition function is not invariant under the shift $b^{\mathbb{Z}_n}\to b^{\mathbb{Z}_n}+n\tilde b^{\mathbb{Z}_n}$ and/or $a^{\mathbb{Z}_n}\to a^{\mathbb{Z}_n}+n\tilde a^{\mathbb{Z}_n}$.

So, to remove such ambiguity, we write $b^{\mathbb{Z}_n}$ and $a^{\mathbb{Z}_n}$ as

$$\begin{aligned} b^{\mathbb{Z}_n} &= \tilde p_1 b^{\mathbb{Z}}_{\text{WL}} + b^{\mathbb{Z}_n}_0 + d\tilde a^{\mathbb{Z}_n},\\ a^{\mathbb{Z}_n} &= s_2 a^{\mathbb{Z}}_{\text{WS}} + a^{\mathbb{Z}_n}_0 + dg^{\mathbb{Z}_n}. \end{aligned} \tag{122}$$

Here, $b^{\mathbb{Z}}_{\text{WL}}$ is a fixed $\mathbb{Z}$-valued 2-cochain field that satisfies

$$db^{\mathbb{Z}}_{\text{WL}} = C^{\mathbb{Z}}_{\text{WL}}, \tag{123}$$

and $b^{\mathbb{Z}_n}_0 \in H^2(M^4;\mathbb{Z}_n)$. Also, $a^{\mathbb{Z}}_{\text{WS}}$ is a fixed $\mathbb{Z}$-valued 1-cochain field that satisfies

$$da^{\mathbb{Z}}_{\text{WS}} = B^{\mathbb{Z}}_{\text{WS}}, \tag{124}$$

and $a^{\mathbb{Z}_n}_0 \in H^1(M^4;\mathbb{Z})$. $a^{\mathbb{Z}_n}_0$, $g^{\mathbb{Z}_n}$, $b^{\mathbb{Z}_n}_0$, $\tilde a^{\mathbb{Z}_n}$ are $\mathbb{Z}_n$ valued. The partition function on orientable M^4 is defined by summing over those $\mathbb{Z}_n$-valued fields:

$$\begin{aligned} &Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}\big(M^4,\tilde p_1C^{\mathbb{Z}_n}_{\text{WL}},p_2N^1_{\text{WL}},s_1M^2_{\text{WS}},s_2B^{\mathbb{Z}}_{\text{WS}}\big)\\ &= \sum_{\{\tilde a^{\mathbb{Z}_n}_{ij},g^{\mathbb{Z}_n}_i\}} \sum_{\substack{a^{\mathbb{Z}_n}_0\in H^1(M^4;\mathbb{Z}_n)\\ b^{\mathbb{Z}_n}_0\in H^2(M^4;\mathbb{Z}_n)}} e^{i\frac{2\pi}{n}\int_{M^4}k_2(\tilde p_1b^{\mathbb{Z}}_{\text{WL}}+b^{\mathbb{Z}_n}_0+d\tilde a^{\mathbb{Z}_n})^2} e^{i\frac{2\pi}{n}\int_{M^4}k_1(\tilde p_1b^{\mathbb{Z}}_{\text{WL}}+b^{\mathbb{Z}_n}_0)\mathcal{B}_n(s_2a^{\mathbb{Z}}_{\text{WS}}+a^{\mathbb{Z}_n}_0)} e^{i\frac{2\pi}{n}[p_2\int_{N^1_{\text{WL}}}(s_2a^{\mathbb{Z}}_{\text{WS}}+a^{\mathbb{Z}_n}_0)+s_1\int_{M^2_{\text{WS}}}(\tilde p_1b^{\mathbb{Z}}_{\text{WL}}+b^{\mathbb{Z}_n}_0)]}. \end{aligned} \tag{125}$$

We note that

$$\begin{aligned} e^{i\frac{2\pi}{n}\int_{M^4}k_2(\tilde p_1b^{\mathbb{Z}}_{\text{WL}}+b^{\mathbb{Z}_n}_0+d\tilde a^{\mathbb{Z}_n})^2} &= e^{i\frac{2\pi}{n}\int_{M^4}k_2\tilde p_1^2(b^{\mathbb{Z}}_{\text{WL}})^2} e^{i\frac{2\pi}{n}\int_{M^4}k_2(b^{\mathbb{Z}}_0)^2} e^{i\frac{2\pi}{n}\int_{M^4}2k_2\tilde p_1b^{\mathbb{Z}}_{\text{WL}}d\tilde a^{\mathbb{Z}_n}}\\ &= e^{i\frac{2\pi}{n}\int_{M^4}k_2\tilde p_1^2(b^{\mathbb{Z}}_{\text{WL}})^2} e^{i\frac{2\pi}{n}\int_{M^4}k_2(b^{\mathbb{Z}}_0)^2} e^{-i\frac{2\pi}{n}\int_{M^4}2k_2\tilde p_1C^{\mathbb{Z}}_{\text{WL}}\tilde a^{\mathbb{Z}_n}}\\ &= e^{i\frac{2\pi}{n}\int_{M^4}k_2\tilde p_1^2(b^{\mathbb{Z}}_{\text{WL}})^2} e^{i\frac{2\pi}{n}\int_{M^4}k_2(b^{\mathbb{Z}}_0)^2} e^{-i\frac{2\pi}{n}\int_{M^1_{\text{WL}}}2k_2\tilde p_1\tilde a^{\mathbb{Z}_n}}. \end{aligned} \tag{126}$$

Also,

$$\begin{aligned} e^{i\frac{2\pi}{n}\int_{M^4}k_1(\tilde p_1b^{\mathbb{Z}}_{\text{WL}}+b^{\mathbb{Z}_n}_0)\mathcal{B}_n(s_2a^{\mathbb{Z}}_{\text{WS}}+a^{\mathbb{Z}_n}_0)} &= e^{i\frac{2\pi}{n^2}\int_{M^4}k_1\tilde p_1s_2b^{\mathbb{Z}}_{\text{WL}}B^{\mathbb{Z}}_{\text{WS}}} e^{i\frac{2\pi}{n^2}\int_{M^4}k_1s_2b^{\mathbb{Z}_n}_0B^{\mathbb{Z}}_{\text{WS}}} e^{i\frac{2\pi}{n^2}\int_{M^4}k_1\tilde p_1C^{\mathbb{Z}}_{\text{WL}}a^{\mathbb{Z}_n}_0} e^{i\frac{2\pi}{n}\int_{M^4}k_1b^{\mathbb{Z}_n}_0\mathcal{B}_na^{\mathbb{Z}_n}_0}\\ &= e^{i\frac{2\pi}{n^2}\int_{M^4}k_1\tilde p_1s_2b^{\mathbb{Z}}_{\text{WL}}B^{\mathbb{Z}}_{\text{WS}}} e^{i\frac{2\pi}{n^2}\int_{N^2_{\text{WS}}}k_1s_2b^{\mathbb{Z}_n}_0} e^{i\frac{2\pi}{n^2}\int_{M^1_{\text{WL}}}k_1\tilde p_1a^{\mathbb{Z}_n}_0} e^{i\frac{2\pi}{n}\int_{M^4}k_1b^{\mathbb{Z}_n}_0\mathcal{B}_na^{\mathbb{Z}_n}_0}. \end{aligned} \tag{127}$$

We can rewrite the partition function as

$$\begin{aligned} Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}\left(M^4,\tilde{p}_1C_{\mathrm{WL}}^{\mathbb{Z}_n},p_2N_{\mathrm{WL}}^1,s_1M_{\mathrm{WS}}^2,s_2B_{\mathrm{WS}}^{\mathbb{Z}}\right) \propto{}& e^{i\frac{2\pi}{n}\int_{M^4}k_2\tilde{p}_1^2(b_{\mathrm{WL}}^{\mathbb{Z}})^2}e^{i\frac{2\pi}{n^2}\int_{M^4}k_1\tilde{p}_1s_2b_{\mathrm{WL}}^{\mathbb{Z}}B_{\mathrm{WS}}^{\mathbb{Z}}}e^{i\frac{2\pi}{n}[p_2\int_{N_{\mathrm{WL}}^1}s_2a_{\mathrm{WS}}^{\mathbb{Z}}+s_1\int_{M_{\mathrm{WS}}^2}\tilde{p}_1b_{\mathrm{WL}}^{\mathbb{Z}}]}\\ &\times\sum_{\substack{a_0^{\mathbb{Z}_n}\in H^1(M^4;\mathbb{Z}_n)\\ b_0^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)}}e^{i\frac{2\pi}{n}\int_{M^4}k_2(b_0^{\mathbb{Z}})^2}e^{i\frac{2\pi}{n}\int_{M^4}k_1b_0^{\mathbb{Z}_n}\mathcal{B}_na_0^{\mathbb{Z}_n}}e^{i\frac{2\pi}{n^2}\int_{N_{\mathrm{WS}}^2}k_1s_2b_0^{\mathbb{Z}_n}}e^{i\frac{2\pi}{n^2}\int_{M_{\mathrm{WL}}^1}k_1\tilde{p}_1a_0^{\mathbb{Z}_n}}\\ &\times e^{i\frac{2\pi}{n}[p_2\int_{N_{\mathrm{WL}}^1}a_0^{\mathbb{Z}_n}+s_1\int_{M_{\mathrm{WS}}^2}b_0^{\mathbb{Z}_n}]}\sum_{\{\tilde{a}_{ij}^{\mathbb{Z}_n},g_i^{\mathbb{Z}_n}\}}e^{-i\frac{2\pi}{n}\int_{M_{\mathrm{WL}}^1}2k_2\tilde{p}_1\tilde{a}^{\mathbb{Z}_n}}. \end{aligned} \tag{128}$$

Using the Poincaré duality, we can rewrite $\int_{M^4}b_{\mathrm{WL}}^{\mathbb{Z}_n}B_{\mathrm{WS}}^{\mathbb{Z}_n}$ as $\int_{N_{\mathrm{WS}}^2}b_{\mathrm{WL}}^{\mathbb{Z}_n}$. Let D_{WS}^3 be the extension of N_{WS}^2, i.e., $\partial D_{\mathrm{WS}}^3=N_{\mathrm{WS}}^2$. Then, we can rewrite $\int_{N_{\mathrm{WS}}^2}b_{\mathrm{WL}}^{\mathbb{Z}_n}=\int_{D_{\mathrm{WS}}^3}db_{\mathrm{WL}}^{\mathbb{Z}_n}=\int_{D_{\mathrm{WS}}^3}C_{\mathrm{WL}}^{\mathbb{Z}_n}$. In fact, $\int_{D_{\mathrm{WS}}^3}C_{\mathrm{WL}}^{\mathbb{Z}_n}=\mathrm{Int}(D_{\mathrm{WS}}^3,M_{\mathrm{WL}}^1)$ is the intersection number between D_{WS}^3 and M_{WL}^1 which is the linking number between N_{WS}^2 and M_{WL}^1: $\mathrm{Lnk}(N_{\mathrm{WS}}^2,M_{\mathrm{WL}}^1)$.

Also, $\sum_{\{\tilde{a}_{ij}^{\mathbb{Z}_n}\}}e^{i\frac{2\pi}{n}\int_{M_{\mathrm{WL}}^1}2k_2\tilde{p}_1\tilde{a}^{\mathbb{Z}_n}}\neq 0$ only when $[2k_2\tilde{p}_1]_n=0$, or when $\tilde{p}_1$ is quantized as $\tilde{p}_1=p_1\frac{n}{\langle 2k_2,n\rangle}$, $p_1\in\mathbb{Z}_{\langle 2k,n\rangle}$. If $\tilde{p}_1$ is not quantized as the above, the corresponding pointlike excitation is confined.

Thus, the above partition function for unconfined like excitations can be rewritten as

$$\begin{aligned} &Z_{k_1k_2;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}\left(M^4,\frac{np_1C_{\mathrm{WL}}^{\mathbb{Z}_n}}{\langle 2k_2,n\rangle},p_2N_{\mathrm{WL}}^1,s_1M_{\mathrm{WS}}^2,s_2B_{\mathrm{WS}}^{\mathbb{Z}_n}\right)\\ &\quad= e^{i\pi\frac{2nk_2p_1^2}{\langle 2k_2,n\rangle^2}\int_{M^4}b_{\mathrm{WL}}^{\mathbb{Z}_n}b_{\mathrm{WL}}^{\mathbb{Z}_n}}e^{i\frac{2\pi s_1p_1}{\langle 2k_2,n\rangle}\mathrm{Lnk}(M_{\mathrm{WS}}^2,M_{\mathrm{WL}}^1)}e^{i\frac{2\pi s_2p_1k_1}{n\langle 2k_2,n\rangle}\mathrm{Lnk}(N_{\mathrm{WS}}^2,M_{\mathrm{WL}}^1)}e^{i\frac{2\pi s_2p_2}{n}\mathrm{Lnk}(N_{\mathrm{WS}}^2,N_{\mathrm{WL}}^1)}\sum_{\substack{a_0^{\mathbb{Z}_n}\in H^1(M^4;\mathbb{Z}_n)\\ b_0^{\mathbb{Z}_n}\in H^2(M^4;\mathbb{Z}_n)}}e^{i\frac{2\pi}{n}\int_{M^4}k_2(b_0^{\mathbb{Z}})^2}\\ &\qquad\times e^{i\frac{2\pi}{n}\int_{M^4}k_1b_0^{\mathbb{Z}_n}\mathcal{B}_na_0^{\mathbb{Z}_n}}e^{i\frac{2\pi k_1s_2}{n^2}\int_{N_{\mathrm{WS}}^2}b_0^{\mathbb{Z}_n}}e^{i\frac{2\pi p_1k_1}{n\langle 2k_2,n\rangle}\int_{M_{\mathrm{WL}}^1}a_0^{\mathbb{Z}_n}}e^{i\frac{2\pi}{n}[p_2\int_{N_{\mathrm{WL}}^1}a_0^{\mathbb{Z}_n}+s_1\int_{M_{\mathrm{WS}}^2}b_0^{\mathbb{Z}_n}]}. \end{aligned} \tag{129}$$

The factors

$$\begin{aligned} &e^{i\pi\frac{2nk_2p_1^2}{\langle 2k_2,n\rangle^2}\int_{M^4}b_{\mathrm{WL}}^{\mathbb{Z}_n}b_{\mathrm{WL}}^{\mathbb{Z}_n}}e^{i\frac{2\pi s_1p_1}{\langle 2k_2,n\rangle}\mathrm{Lnk}(M_{\mathrm{WS}}^2,M_{\mathrm{WL}}^1)},\\ &e^{i\frac{2\pi s_2p_1k_1}{n\langle 2k_2,n\rangle}\mathrm{Lnk}(N_{\mathrm{WS}}^2,M_{\mathrm{WL}}^1)}e^{i\frac{2\pi s_2p_2}{n}\mathrm{Lnk}(N_{\mathrm{WS}}^2,N_{\mathrm{WL}}^1)} \end{aligned} \tag{130}$$

in the above expression determine the braiding statistics of pointlike and stringlike excitations. We see that *there is no nontrivial braiding for stringlike excitations*. But, *there are nontrivial mutual statistics (i.e., the Aharonov-Bohm phase) between pointlike and stringlike excitations*. Also, *when* $\frac{2nk_2}{\langle 2k_2,n\rangle^2}=$ *odd, the theory contains fermions*.

IV. COMPARISON BETWEEN THE (3+1)D $\mathbb{Z}_n$-2-COCYCLE MODEL AND (3+1)D Z_n-1-COCYCLE MODEL

There is a well-known duality between the (3+1)D Z_n-1-cocycle theory (with emergent Z_n-gauge theory) and the above (3+1)D $\mathbb{Z}_n$-2-cocycle theory with $k=0$. In the following, we will compare the two theories in detail. We find that the two theories are equivalent, if they are viewed as pure topological theory without any symmetry. So, both (3+1)D Z_n-1-cocycle theory and (3+1)D $\mathbb{Z}_n$-2-cocycle theory realize the same topological order described by UT Z_n-gauge theory. However, if we view the two theories as topological theory with time-reversal symmetry or parity symmetry, then the two theories are not equivalent. In other words, the two models realize the same topological orders, but different symmetry-enriched topological orders (with time-reversal symmetry or parity symmetry).

A. Duality

To see the above-mentioned duality, let us describe the lattice Hamiltonian of the two theories. We consider a 3D cubic lattice whose sites are labeled by i. To obtain a $\mathbb{Z}_n$-1-cocycle theory, we put a $\mathbb{Z}_n$ degrees of freedom $a_{ij}^{\mathbb{Z}_n}=0,1,\dots,n-1=-a_{ji}^{\mathbb{Z}_n}$ on each nearest-neighbor link (ij). Let $U_{ij}=e^{i\frac{2\pi}{n}a_{ij}^{\mathbb{Z}_n}}$ and V_{ij} is an operator that raises $a_{ij}^{\mathbb{Z}_n}$ by one: $V_{ij}|a_{ij}^{\mathbb{Z}_n}=m\rangle=|a_{ij}^{\mathbb{Z}_n}=m+1\rangle$. Noting that the $\mathbb{Z}_n$-1-cocycle theory is a theory of closed $\mathbb{Z}_n$ loops at low energy, we find that the lattice Hamiltonian for the $\mathbb{Z}_n$-1-cocycle theory will be

$$\begin{aligned} H_{\mathbb{Z}_n\mathrm{a}}&=-\sum_i(Q_i+Q_i^\dagger)-\sum_{(ijkl)}(B_{ijkl}+B_{ijkl}^\dagger),\\ Q_i&=\prod_{j\text{ next to }i}U_{ij},\\ B_{ijkl}&=V_{ij}V_{jk}V_{kl}V_{li}, \end{aligned} \tag{131}$$

where $\sum_i$ sum over all sites and $\sum_{(ijkl)}$ sum over all squares $(ijkl)$. The $-(Q_i+Q_i^\dagger)$ terms enforce the closed-loop condition and the $-(B_{ijkl}+B_{ijkl}^\dagger)$ terms are the loop hopping and/or loop creation/annihilation terms.

To obtain a $\mathbb{Z}_n$-2-cocycle theory, we put a $\mathbb{Z}_n$ degrees of freedom $b_{ijkl}^{\mathbb{Z}_n}=0,1,\dots,n-1=-b_{lkji}^{\mathbb{Z}_n}$ on each square $(ijkl)$. But, this is equivalent to put a $\mathbb{Z}_n$ degrees of freedom $a_{IJ}^{\mathbb{Z}_n}=0,1,\dots,n-1=-a_{JI}^{\mathbb{Z}_n}$ on each link (IJ) of the dual lattice. The dual lattice of a cubic lattice is also a cubic lattice.

The $\mathbb{Z}_n$-2-cocycle theory is a theory of closed $\mathbb{Z}_n$ membranes at low energy. Thus, the lattice Hamiltonian for the $\mathbb{Z}_n$-2-cocycle theory with $k=0$ is

$$H_{0;b^2\mathbb{Z}_n} = -\sum_I (Q_I + Q_I^\dagger) - \sum_{(IJKL)} (B_{IJKL} + B_{IJKL}^\dagger),$$
$$Q_I = \prod_{J \text{ next to } I} V_{IJ},$$
$$B_{IJKL} = U_{IJ}U_{JK}U_{KL}U_{LI}. \quad (132)$$

The $-(B_{IJKL} + B_{IJKL}^\dagger)$ terms enforce the closed-membrane condition and the $-(Q_I + Q_I^\dagger)$ are the membrane hopping and/or membrane creation/annihilation terms. The two Hamiltonians $H_{\mathbb{Z}_n\text{a}}$ and $H_{0;b^2\mathbb{Z}_n}$ are equivalent under a local unitary transformation that exchanges U and V. This implies that the two theories are really equivalent.

B. Topological invariants for orientable space-time

To compare the two theories at Lagrangian level, we note that the volume-independent topological partition function for (3+1)D $\mathbb{Z}_n$-1-cocycle theory is given by

$$Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^4) = \frac{|H^1(M^4;\mathbb{Z}_n)|}{|H^0(M^4;\mathbb{Z}_n)|}, \quad (133)$$

while the volume-independent topological partition function for (3+1)D $\mathbb{Z}_n$-2-cocycle theory (with $k_1 = k_2 = 0$) is given by

$$Z^{\text{top}}_{00;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M^4) = \frac{|H^0(M^4;\mathbb{Z}_n)||H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|}. \quad (134)$$

So, their ratio is given by

$$\frac{Z^{\text{top}}_{00;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M^4)}{Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^4)} = \frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2}. \quad (135)$$

In Appendix C, we will show that for orientable close space-time M^4,

$$\frac{Z^{\text{top}}_{00;b\mathcal{B}a\text{-}bb\mathbb{Z}_n}(M^4)}{Z^{\text{top}}_{\mathbb{Z}_n\text{a}}(M^4)} = n^{\chi(M^4)}, \quad (136)$$

where $\chi(M^4)$ is the Euler number. The volume-independent topological partition functions of the two models are different, which may lead one to conclude that the $\mathbb{Z}_n$-1-cocycle model and the $\mathbb{Z}_n$-2-cocycle model realize different topological orders. However, in [31], it was conjectured that two (3+1)D topological partition functions $Z_1^{\text{top}}(M^4)$ and $Z_2^{\text{top}}(M^4)$ describe the same L-type topological orders iff their ratio has a form

$$\frac{Z_1^{\text{top}}(M^4)}{Z_2^{\text{top}}(M^4)} = \rho^{\chi(M^4)}\lambda^{P_1(M^4)}, \quad (137)$$

where $P_1(M^4)$ is the Pontryagin number of M^4. Therefore, the above result implies that the $\mathbb{Z}_n$-1-cocycle model and the $\mathbb{Z}_n$-2-cocycle model realize the same topological order.

C. Ground-state degeneracy for nonorientable spaces

Now, we turn to study the ground-state degeneracy of the two models. To calculate the GSD on closed space manifold M^3, we compute the volume-independent partition function on $M^3 \times S^1$ space-time:

$$\text{GSD}(M^3) = Z^{\text{top}}(M^3 \times S^1). \quad (138)$$

We see that the ground-state degeneracy of the two models is the same on orientable spaces M^3 since their partition functions are the same on orientable space-times $M^3 \times S^1$.

However, for nonorientable space M^3, the GSDs of the two models can be different. For example, let us assume the space to be $M^3 = S^1 \times \text{KB}$, where KB is the Klein bottle. We note that

$$H_2(\text{KB};\mathbb{Z}) = 0, \quad H_1(\text{KB};\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2, \quad H_0(\text{KB};\mathbb{Z}) = \mathbb{Z} \quad (139)$$

and

$$H_2(S^1\times\text{KB};\mathbb{Z}) = H_2(\text{KB};\mathbb{Z}) \oplus H_1(\text{KB};\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2;$$
$$H_1(S^1\times\text{KB};\mathbb{Z}) = H_1(\text{KB};\mathbb{Z}) \oplus \mathbb{Z} = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}. \quad (140)$$

Then, using the universal coefficient theorem (A8), we find that

$$H^2(S^1\times\text{KB};\mathbb{Z}_n) = \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n,2\rangle}^{\oplus 2};$$
$$H^1(S^1\times\text{KB};\mathbb{Z}) = \mathbb{Z}_n^{\oplus 2}\mathbb{Z}_{\langle n,2\rangle}. \quad (141)$$

Thus,

$$\text{GSD}_{0;b^2\mathbb{Z}_n}(S^1\times\text{KB}) = n\langle n,2\rangle^2,$$
$$\text{GSD}_{\mathbb{Z}_n\text{a}}(S^1\times\text{KB}) = n^2\langle n,2\rangle. \quad (142)$$

When $n > 2$, the GSDs of the two theories are different. Since the difference only appears in nonorientable manifolds,

the $\mathbb{Z}_n$-2-cocycle model and the $\mathbb{Z}_n$-1-cocyle model realize two different time-reversal symmetry-enriched topological orders.

This is consistent with the fact that the two theories realize the same topological order if we ignore the time-reversal symmetry.

Both topological orders have pointlike excitations labeled by $i \in \mathbb{Z}_n$ and stringlike excitations labeled by $s \in \mathbb{Z}_n$. But, they transform differently under time reversal. For the $\mathbb{Z}_n$-1-cocyle theory $(i,s) \to (i,-s)$ under time reversal. For the $\mathbb{Z}_n$-2-cocycle theory $(i,s) \to (-i,s)$ under time reversal. Both the $\mathbb{Z}_n$-1-cocycle theory and the $\mathbb{Z}_n$-2-cocycle theory are described by the same Hamiltonian (131). But, the time-reversal symmetry is realized differently. In the $\mathbb{Z}_n$-1-cocyle theory, we assume $|a_{ij}^{\mathbb{Z}_n}\rangle$, the eigenstates of U_{ij}, are invariant under time reversal. Thus, $(U_{ij},V_{ij}) \to (U_{ij}^\dagger,V_{ij})$ under time reversal. In the $\mathbb{Z}_n$-2-cocycle theory, we assume that the eigenstates of V_{ij} are invariant under time reversal. Thus, $(U_{ij},V_{ij}) \to (U_{ij},V_{ij}^\dagger)$ under time reversal.

V. NON-ABELIAN COCYCLE MODELS

So far, we have constructed many local bosonic models: the cocycle models. But, in those constructions, the local degrees of freedom are always described by an Abelian group,

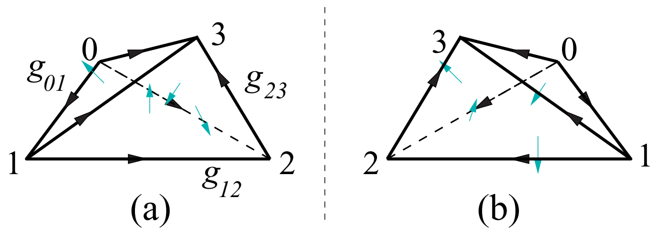

FIG. 5. Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

such as $\mathbb{Z}_n$. In this section, we will use group cocycles in group cohomology theory (see Appendix G) to generalize the cocycle models so that the local degrees of freedom are described by a non-Abelian group G. To use group cocycles to construct the cocycle models, we need to map the group cocycles in group cohomology theory to topological cocycles in topological cohomology theory. To obtain such a map, we need to first introduce the branching structure in space-time lattice.

A. Branching structure of space-time lattice

In order to define a generic lattice theory on the space-time complex M^d_{latt} using group cocycles, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure [11,48,88]. A branching structure is a choice of orientation of each link in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 5).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, etc. So, the simplex in Fig. 5(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subsimplices) a canonical orientation. Figure 5 illustrates two 3-simplices with opposite canonical orientations compared with the three-dimensional space in which they are embedded. The blue arrows indicate that canonical orientations of the 2-simplices. The black arrows indicate that canonical orientations of the 1-simplices.

B. Group-vertex models that realize G-SPT orders

References [10,11,48] have constructed exactly soluble local bosonic models using homogeneous group cocycles (see Appendix G) of group G to realize G-SPT orders. Those models are actually cocycle models on space-time lattice. In this section, we will review those results using the cocycle notation introduced above.

The local degrees of freedom of our model are now group elements living on the vertices of the orientable space-time lattice M^d_{latt}: $g_i \in G$. Let $\nu_n(g_0,\dots,g_n)$ be a homogeneous group n-cocycle: $\nu_n(g_0,\dots,g_n) \in \mathcal{H}^n(G,\mathbb{R}/\mathbb{Z})$. From ν_n, we can construct a topological n-cocycle $\tilde\nu_n$ on M^d_{latt}:

$$\tilde\nu_n(i_0,i_1,\dots,i_n) = \nu_n(g_{i_0},g_{i_1},\dots,g_{i_n}), \tag{143}$$

where $(i_0,i_1,\dots,i_n)$ is an n-simplex with the canonical orientation and the vertex ordering $i_0 < i_1 \cdots < i_n$. Below, we will drop the $\sim$ and denote $\tilde\nu_n(i_0,i_1,\dots,i_n)$ as $\nu_n(g_{i_0},g_{i_1},\dots,g_{i_n})$.

Using such mapping, we can construct a group-vertex model on orientable space-time M^d_{latt}:

$$Z_{\nu_d}\big(M^d_{\text{latt}}\big) = \sum_{\{g_i\}} e^{i2\pi\int_{M^d_{\text{latt}}}\nu_d(\{g_i\})}. \tag{144}$$

Since $\nu_d(\{fg_i\}) = \nu_d(\{g_i\})$, $f \in G$, the group-vertex model has a global onsite G symmetry. Since $e^{i2\pi\int_{M^d}\nu_d(\{g_i\})} = 1$ on any closed orientable manifold M^d. We find that the constructed model is gapped. We also see that

$$Z_{\nu_d}\big(M^d_{\text{latt}}\big) = |G|^{N_v}. \tag{145}$$

So, the volume-independent partition function $Z^{\text{top}}_{\nu_d}(M^d) = 1$, for all closed orientable manifolds M^d, which implies that the model does not have any topological order regardless the choice of the group cocycle ν_d. $Z^{\text{top}}_{\nu_d}(M^d) = 1$ also implies that the group-vortex model does not break the G symmetry [as one can see from the ground-state degeneracy on closed orientable space manifold M^{d-1}_{space}: $\text{GSD}^{\text{top}}_{\nu_d}(M^{d-1}_{\text{space}}) = Z^{\text{top}}_{\nu_d}(S^1 \times M^{d-1}_{\text{space}}) = 1$].

But, $Z^{\text{top}}_{\nu_d}(M^d) = 1$ also means that volume-independent partition function fails to detect SPT orders. In fact, we do not even know whether the lattice models with different ν_d's belong to different SPT phases, if we just look at $Z^{\text{top}}_{\nu_d}(M^d)$.

To detect SPT order via the partition function [49,50,54,55], we need to add the symmetry twist [53] in space-time. A symmetry twist is described by $a_{ij} \in G$ on each link (i.e., 1-simplex), that satisfy

$$a_{ij} = a^{-1}_{ji}, \quad a_{ij}a_{jk}a_{ki} = 1. \tag{146}$$

Such a a_{ij} configuration defines a so-called "flat G connection" on space-time M^d. In the presence of symmetry twist, the partition function becomes

$$Z_{\nu_d}\big(M^d_{\text{latt}},a_{ij}\big) = \sum_{\{g_i\}} e^{i2\pi\int_{M^d_{\text{latt}}}\nu^{\text{g}}_d(\{g_i\},\{a_{ij}\})}, \tag{147}$$

where

$$\begin{aligned}\nu^{\text{g}}_d(\{g_i\},\{a_{ij}\}) &\equiv \nu^{\text{g}}_d\big(g_{i_0},g_{i_1},\dots,g_{i_d};a_{i_0i_1},a_{i_1i_2},\dots\big)\\ &\equiv \nu_d\big(g_{i_0},a_{i_0i_1}g_{i_1},a_{i_0i_1}a_{i_1i_2}g_{i_2},\dots\big).\end{aligned} \tag{148}$$

Clearly, the partition function $Z_{\nu_d}(M^d_{\text{latt}},a_{ij})$ is invariant under the gauge transformation

$$\begin{aligned} g_i \to f_ig_i, &\quad a_{ij} \to f_ia_{ij}f_j^{-1};\\ \nu^{\text{g}}_d\big(\{f_ig_i\},\{f_ia_{ij}f_j^{-1}\}\big) &= \nu^{\text{g}}_d(\{g_i\},\{a_{ij}\});\\ Z_{\nu_d}\big(M^d_{\text{latt}},a_{ij}\big) &= Z_{\nu_d}\big(M^d_{\text{latt}},f_ia_{ij}f_j^{-1}\big).\end{aligned} \tag{149}$$

So, the partition function $Z_{\nu_d}(M^d_{\text{latt}},a_{ij})$ only depends on the gauge-equivalent class of the flat connection a_{ij}.

The volume-independent partition functions $Z^{\text{top}}_{\nu_d}(M^d_{\text{latt}},a_{ij})$ are the so-called SPT invariants that suppose to fully characterize the SPT order [49–51,54,55]. Using a gauge transformation to change $g_i \to 1$, we find the SPT invariant to be given by

$$Z^{\text{top}}_{\nu_d}(M^d_{\text{latt}}, a_{ij}) = e^{i2\pi \int_{M^d_{\text{latt}}} \nu^{\text{g}}_d(\{g_i=1\},\{a_{ij}\})} = e^{i2\pi \int_{M^d_{\text{latt}}} \omega_d(\{a_{ij}\})}, \tag{150}$$

where ω_d is the inhomogeneous group cocycle that corresponds to the homogeneous group cocycle ν_d [see Eq. (G9)]. The above expression allows us to compute the SPT invariant.

In the following, we will list some of the SPT invariants for some simple SPT states:

(1) The Z_n SPT states in (2+1)D are classified by $\mathcal{H}^3(Z_n;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$. For a Z_n SPT state labeled by $k \in \mathbb{Z}_n$, its SPT invariant is

$$Z^{\text{top}}_k(M^3, a^{\mathbb{Z}_n}) = e^{ik\frac{2\pi}{n}\int_{M^3} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}}. \tag{151}$$

(2) The $Z_n \times \tilde{Z}_n$ SPT states in (3+1)D are classified by $\mathcal{H}^4(Z_n \times \tilde{Z}_n;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n^{\oplus 2}$. For a $Z_n \times \tilde{Z}_n$ SPT state labeled by $i(k_1,k_2) \in \mathbb{Z}_n^{\oplus 2}$, its SPT invariant is

$$Z^{\text{top}}_{k_1,k_2}(M^4, a^{\mathbb{Z}_n}, \tilde{a}^{\mathbb{Z}_n}) = e^{i\frac{2\pi}{n}\int_{M^4} k_1 a^{\mathbb{Z}_n}\tilde{a}^{\mathbb{Z}_n}\mathcal{B}_n\tilde{a}^{\mathbb{Z}_n} + k_2\tilde{a}^{\mathbb{Z}_n}a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}}. \tag{152}$$

C. Group-vertex models that realize Z_2^T SPT orders

To construct a local bosonic model that realizes the time-reversal Z_2^T SPT order, we consider a $\mathbb{Z}_2$-group-vertex model: $g_i \in \mathbb{Z}_2 = \{0,1\}$. The $\mathbb{Z}_2$-group-vertex model on orientable space-time M^d_{latt} is given by

$$Z_{\nu_d}(M^d_{\text{latt}}) = \sum_{\{g_i\}} e^{i2\pi\int_{M^d_{\text{latt}}} \nu_d(\{g_i\})}, \tag{153}$$

where the homogeneous $\mathbb{Z}_2$-group cocycle $\nu_d(\{g_i\}) \in \mathcal{H}^d(\mathbb{Z}_2,(\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2})$ satisfies

$$\nu_d(\{g_i+1\}) = -\nu_d(\{g_i\}) \bmod 1. \tag{154}$$

The extra "−" sign implies that the $\mathbb{Z}_2$ group has a nontrivial action on $\mathbb{R}/\mathbb{Z}$ which is indicated by the subscript $\mathbb{Z}_2$ in $(\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2}$. For example, in (1+1)D,

$$\nu_2(g_0,g_1,g_2) = \tfrac{1}{2}[g_1-g_0]_2[g_2-g_1]_2. \tag{155}$$

Since the $\mathbb{Z}_2$ action corresponds to the time-reversal (or orientation reversal) transformation, to obtain partition function with the symmetry twist, we need to put the system on nonorientable space-time and to introduce a $\mathbb{Z}_2$-valued 1-cocycle a_{ij} to describe orientation reversal:

$$Z_{\nu_d}(M^d_{\text{latt}}) = \sum_{\{g_i\}} e^{i2\pi\int_{M^d_{\text{latt}}} \nu^{\text{g}}_d(\{g_i\},\{a_{ij}\})}, \tag{156}$$

where

$$\nu^{\text{g}}_d(\{g_i\},\{a_{ij}\}) \equiv \nu^{\text{g}}_d(g_{i_0},g_{i_1},\dots,g_{i_d};a_{i_0i_1},a_{i_1i_2},\dots) \equiv \nu_d(g_{i_0},a_{i_0i_1}+g_{i_1},a_{i_0i_1}+a_{i_1i_2}+g_{i_2},\dots). \tag{157}$$

Here, a_{ij} is the $\mathbb{Z}_2$ flat connection that describes the orientation of the manifold (see Fig. 6). In other words, if the orientation does not change around a loop C, then $\sum_{(ij)\in C} a_{ij} = \oint_C a = 0$; if the orientation changes around a loop C, then $\sum_{(ij)\in C} a_{ij} = \oint_C a = 1$ (see Fig. 6). The above definition implies that a_{ij} is a $\mathbb{Z}_2$-valued 1-cocycle $a \in C^1(M^d;\mathbb{Z}_2)$. In fact, $a = \text{w}_1$.

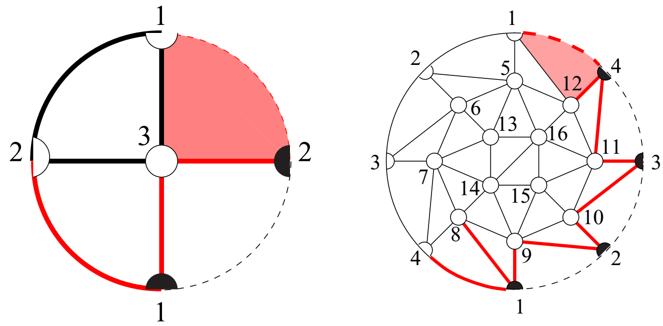

FIG. 6. Two triangulations of $\mathbb{R}P^2$ where the opposite points on the boundary are identified. One triangulation has 3 vertices and the other has 16 vertices. The open dots represent $\tilde{g}_i = 0$ and the filled dots represent $\tilde{g}_i = 1$ at the vertices. $\tilde{g}_i$ is multivalued since it takes different values on the same vertex, such as vertex 1 and vertex 2. The black links represent $a_{ij} = (d\tilde{g})_{ij} = \tilde{g}_i - \tilde{g}_j = (\text{w}_1)_{ij} = 0$ and the red links represent $a_{ij} = (d\tilde{g})_{ij} = \tilde{g}_i - \tilde{g}_j = (\text{w}_1)_{ij} = 1 \bmod 2$. The unshaded triangles represent $(d\tilde{g}d\tilde{g})_{ijk} = (\text{w}_1^2)_{ijk} = \mathcal{B}_2 d\tilde{g} = \mathcal{B}_2\text{w}_1 = 0$ and the shaded triangle represents $(d\tilde{g}d\tilde{g})_{ijk} = (\text{w}_1^2)_{ijk} = \mathcal{B}_2 d\tilde{g} = \mathcal{B}_2\text{w}_1 = 1$. {By definition, $(d\tilde{g}d\tilde{g})_{ijk} = [(\tilde{g}_i-\tilde{g}_j)(\tilde{g}_j-\tilde{g}_k)]_2$ and $(\mathcal{B}_2\text{w}_1)_{ijk} = [\frac{(\text{w}_1)_{ij}+(\text{w}_1)_{jk}-(\text{w}_1)_{ik}}{2}]_2$ where i,j,k are ordered as $i<j<k$.} We see that $\int_{\mathbb{R}P^2}\text{w}_1^2 = \int_{\mathbb{R}P^2} d\tilde{g}d\tilde{g} = 1$.

We can use a multivalued $\mathbb{Z}_2$-gauge transformation to make $a_{ij} = 0$, which changes the single-valued g_i to multivalued $\tilde{g}_i$. If the orientation changes around a loop C, $\tilde{g}$ will have to take different values on the same vertex somewhere on C (see Fig. 6). We see that to realize Z_2^T SPT order, the local bosonic degrees of freedom must couple to space-time orientation. In other words, $(-)^{\tilde{g}_i}$ is a pseudoscalar, which changes sign under time-reversal and parity transformations. In this paper, we will also refer $\tilde{g}$ as a pseudoscalar field. Thus, if we view $\tilde{g}_i$ as a $\mathbb{Z}_2$-valued 0-cochain, we have (see Fig. 6)

$$a = \text{w}_1 = d\tilde{g}. \tag{158}$$

In terms of such multivalued $\tilde{g}_i$, the partition function can be written as

$$Z_{\nu_d}(M^d_{\text{latt}}) = \sum_{\{\tilde{g}_i\}} e^{i2\pi\int_{M^d_{\text{latt}}} \nu_d(\{\tilde{g}_i\})}. \tag{159}$$

The Z_2^T SPT invariant is given by the corresponding inhomogeneous cocycle ω_d:

$$Z^{\text{top}}_{\nu_d}(M^d_{\text{latt}}) = e^{i2\pi\int_{M^d_{\text{latt}}} \nu^{\text{g}}_d(\{g_i=1\},\{a_{ij}\})} = e^{i2\pi\int_{M^d_{\text{latt}}} \omega_d(\{a_{ij}\})}. \tag{160}$$

We can express $\omega_d(\{a_{ij}\})$ in terms of a_{ij} (see Fig. 6):

$$\omega_d(\{a_{ij}\}) = \begin{cases} \frac{1}{2}a^d & \text{if } d = \text{even}, \\ 0 & \text{if } d = \text{odd}. \end{cases} \tag{161}$$

Thus, the Z_2^T SPT invariant is given by

$$Z^{\text{top}}_{\nu_d}(M^d) = e^{i\pi\int_{M^d}\text{w}_1^d}. \tag{162}$$

From $w_1^d = Sq^1(w_1^{d-1}) = (d-1)w_1^d$, we see that $w_1^d = 0$ mod 2 automatically, when $d =$ odd. So, the above expression for the Z_2^T SPT invariant is valid for both $d =$ even and odd.

Last, we like to mention that, using multivalued $\tilde{g}_i$, we can also express the nontrivial homogeneous cocycle $\nu_d(\{\tilde{g}_i\})$ as (see Fig. 6)

$$\nu_d(\{\tilde{g}_i\}) = \begin{cases} \frac{1}{2}(d\tilde{g})^d & \text{if } d = \text{even}, \\ 0 & \text{if } d = \text{odd}, \end{cases} \tag{163}$$

since $a = d\tilde{g}$. This allows us to rewrite Eq. (159) as (see Fig. 6)

$$Z_{\nu_d}\big(M_{\text{latt}}^d\big) = \sum_{\{\tilde{g}_i\}} e^{i\pi \int_{M_{\text{latt}}^d} (d\tilde{g})^d} \tag{164}$$

for even d.

D. Group-link model and emergent Dijkgraaf-Witten gauge theory

Now, let us construct local bosonic models: group-link models, whose topological orders are described by Dijkgraaf-Witten gauge theory. The local degrees of freedom of the group-link model are group elements living on the links of the space-time lattice M_{latt}^d: $a_{ij} \in G$ that satisfies $a_{ij} = G_{ji}^{-1}$. Then, using the inhomogeneous group cocycle $\omega_d(\{a_{ij}\})$, we can construct a group-link model [33,83,85,89]

$$Z_{G,\omega_d}\big(M_{\text{latt}}^d\big) = \sum_{\substack{\{a_{ij}\} \\ a_{ij}a_{jk}a_{ki}=1}} e^{i2\pi \int_{M_{\text{latt}}^d} \omega_d(\{a_{ij}\}) - U\sum_{(ijk)} |a_{ij}a_{jk}a_{ki}-1|}, \tag{165}$$

where $\sum_{(ijk)}$ sums over all 3-simplices, and $U \to +\infty$.

Note that the above model is a local bosonic model, not the Dijkgraaf-Witten gauge theory. The Dijkgraaf-Witten gauge theory is defined by

$$Z_{G,\omega_d,\text{DW}}\big(M_{\text{latt}}^d\big) = \sum_{\substack{[\{a_{ij}\}] \\ a_{ij}a_{jk}a_{ki}=1}} e^{i2\pi \int_{M_{\text{latt}}^d} \omega_d(\{a_{ij}\}) - U\sum_{(ijk)} |a_{ij}a_{jk}a_{ki}-1|}, \tag{166}$$

where the summation $\sum_{[\{a_{ij}\}]}$ is over the gauge-equivalent class $[\{a_{ij}\}]$ of the configurations $\{a_{ij}\}$. In contrast, the summation $\sum_{\{a_{ij}\}}$ in the group-link model is over all the configurations $\{a_{ij}\}$ (without the gauge reduction). However, the volume-independent partition function of the two models is the same:

$$Z_{G,\omega_d,\text{DW}}^{\text{top}}\big(M_{\text{latt}}^d\big) = Z_{G,\omega_d}^{\text{top}}\big(M_{\text{latt}}^d\big). \tag{167}$$

So, the two models have the same emergent topological order.

As an example, let us compute the topological invariant for (2+1)D lens space $L^3(p)$, using the explicit CW complex decomposition in Fig. 7:

$$Z_{G,\omega_3}^{\text{top}}[L^3(p)] = \frac{1}{|G|^2} \sum_{\substack{g,h\in G \\ g^p=1}} e^{i2\pi \sum_{m=0}^{p-1} \omega_3(g,g^m h,h^{-1}gh)}. \tag{168}$$

For $G = \mathbb{Z}_n$, $\omega_3 \in \mathcal{H}^3(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ is labeled by $k \in \mathbb{Z}_n$:

$$\omega_3(g_1,g_2,g_3) = \frac{k}{n^2} g_1(g_2 + g_3 - [g_2+g_3]_n). \tag{169}$$

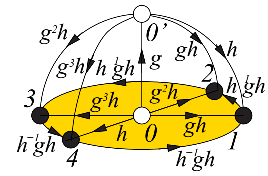

FIG. 7. The lens space $L^3(p)$ is obtained by identifying the bottom and the top disks after a $2\pi/p$ rotation, e.g., link (01) and link (0′2) are identified, link (02) and link (0′3) are identified, link (12) and link (23) are identified, etc.

We find that

$$Z_{\mathbb{Z}_n,k}^{\text{top}}[L^3(p)] = Z_{k;a\mathcal{B}a\mathbb{Z}_n}^{\text{top}}[L^3(p)]. \tag{170}$$

In fact, the topological term in the Z_n Dijkgraaf-Witten theory and the topological term in the $\mathbb{Z}_n$-1-cocycle model are directly related,

$$2\pi \int_{M^3} \omega_3(\{a^{\mathbb{Z}_n}\}) = k\frac{2\pi}{n} \int_{M^3} a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}, \tag{171}$$

as one can see from Eq. (169) and the explicit expression of $a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n}$:

$$\begin{aligned} \langle a^{\mathbb{Z}_n}\mathcal{B}_n a^{\mathbb{Z}_n},(ijkl)\rangle &= a_{ij}^{\mathbb{Z}_n}\langle \mathcal{B}_n a^{\mathbb{Z}_n},(jkl)\rangle; \\ \langle \mathcal{B}_n a^{\mathbb{Z}_n},(jkl)\rangle &= \frac{1}{n}\big(a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n} - a_{jl}^{\mathbb{Z}_n}\big) \\ &= \frac{1}{n}\big(a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n} - \big[a_{jk}^{\mathbb{Z}_n} + a_{kl}^{\mathbb{Z}_n}\big]_n\big). \end{aligned} \tag{172}$$

Therefore, the $\mathbb{Z}_n$-1-cocycle model realizes the Z_n Dijkgraaf-Witten theory.

E. Symmetric topological orders described by gauge theories

We can also construct local bosonic models (called mixed group-vertex group-link models) that will produce topological orders described by a G_{gauge}-gauge theory that also have a symmetry G_{symm}. In the mixed model, the local degrees of freedom of are group elements $g_i \in G_{\text{symm}}$ living on the links group elements $a_{ij} \in G_{\text{gauge}}$ living on the links of the space-time lattice M_{latt}^d. Then, using the homogeneous group cocycle $\nu_n(\{g_i\}) \in \mathcal{H}^n(G_{\text{symm}},\mathbb{R}/\mathbb{Z})$, and the inhomogeneous group cocycle $\omega_{d-n}(\{a_{ij}\}) \in \mathcal{H}^{d-n}(G_{\text{gauge}},\mathbb{R}/\mathbb{Z})$, we can construct the mixed model

$$Z_{\nu_n\omega_{d-n}}\big(M_{\text{latt}}^d\big) = \sum_{\{g_i,a_{ij}\},a_{ij}a_{jk}a_{ki}=1} e^{i2\pi \int_{M_{\text{latt}}^d} \nu_n(\{g_i\})\omega_{d-n}(\{a_{ij}\})}. \tag{173}$$

We can also construct a more general mixed model using inhomogeneous group cocycle $\omega_d \in \mathcal{H}^d(G_{\text{symm}} \times G_{\text{gauge}},\mathbb{R}/\mathbb{Z})$:

$$Z_{\nu_n\omega_{d-n}}\big(M_{\text{latt}}^d\big) = \sum_{\{g_i,a_{ij}\},a_{ij}a_{jk}a_{ki}=1} e^{i2\pi \int_{M_{\text{latt}}^d} \omega_d[\{(g_i^{-1}g_j,a_{ij})\}]}, \tag{174}$$

where (g_i,a_{ij}) is the group element of $G_{\text{symm}} \times G_{\text{gauge}}$.

We can construct an even more general mixed model using inhomogeneous group cocycle $\omega_d \in \mathcal{H}^d(G_{\text{PSG}},\mathbb{R}/\mathbb{Z})$ [85]:

$$Z_{\omega_d;G_{\text{PSG}}}\big(M^d_{\text{latt}}\big) = \sum_{\{g_i,a_{ij}\},a^{\text{PSG}}_{ij}a^{\text{PSG}}_{jk}a^{\text{PSG}}_{ki}=1} e^{i2\pi\int_{M^d_{\text{latt}}}\omega_d[\{(g_i^{-1}g_j,a_{ij})\}]}, \tag{175}$$

where G_{PSG} is a group that contains G_{gauge} as a normal subgroup such that $G_{\text{PSG}}/G_{\text{gauge}} = G_{\text{symm}}$, and $a^{\text{PSG}}_{ij} = (g_i^{-1}g_j,a_{ij})$ is the group element of G_{PSG} [69]. In other words, G_{PSG} is an extension of G_{symm} by G_{gauge}, which is also described by the following short exact sequence:

$$1 \to G_{\text{gauge}} \to G_{\text{PSG}} \to G_{\text{symm}} \to 1. \tag{176}$$

In this case, as discussed in [69], a gauge charge does not transform as a representation of G_{gauge}, but rather transforms as a representation of G_{PSG}. Under the symmetry transformation, the gauge charge transforms according to G_{PSG} (which is called the projective symmetry group). In fact, G_{PSG} describes the so-called "symmetry fractionalization".

If there is a symmetry twist described by $a^{\text{symm}}_{ij} \in G_{\text{symm}}$ on the links, then the partition function will be

$$Z_{\omega_d;G_{\text{PSG}}}\big(M^d_{\text{latt}},a^{\text{symm}}_{ij}\big) = \sum_{\{g_i,a_{ij}\},a_{ij}a_{jk}a_{ki}=1} e^{i2\pi\int_{M^d_{\text{latt}}}\omega_d[\{(g_i^{-1}a^{\text{symm}}_{ij}g_j,a_{ij})\}]}. \tag{177}$$

The above construction also applies to the situation where G_{symm} contains time-reversal symmetry. In that case, a^{symm}_{ij} will contain contributions from the change of the orientations of the manifold, and $\omega_d \in \mathcal{H}^d(G_{\text{PSG}},(\mathbb{R}/\mathbb{Z})_T)$ where time reversal $T \in G_{\text{PSG}}$ will have a sign-changing action on $\mathbb{R}/\mathbb{Z}$.

If we include $\mathbb{Z}_n$-2-cochain field $b^{\mathbb{Z}_n}$, we can construct new general local boson models with emergent symmetric topological order, such as [45]

$$Z_{b^{\mathbb{Z}_n}\omega_{d-2};G_{\text{PSG}}}\big(M^d_{\text{latt}},a^{\text{symm}}_{ij}\big) = \sum_{\substack{\{g_i,a_{ij},b^{\mathbb{Z}_n}_{ijk}\},db^{\mathbb{Z}_n}=0\\ a_{ij}a_{jk}a_{ki}=1}} e^{i2\pi\int_{M^d_{\text{latt}}} b^{\mathbb{Z}_n}\omega_{d-2}[\{(g_i^{-1}a^{\text{symm}}_{ij}g_j,a_{ij})\}]}, \tag{178}$$

where we have assumed that $n\omega_{d-2} = 0$. This model has an emergent $(Z_n \times G_{\text{gauge}})$-gauge theory with G_{symm} symmetry. When, $G_{\text{gauge}} = 1$, the Z_n charge may carry a projective representation of G_{symm}. When $G_{\text{symm}} = 1$, the Z_n charge may carry a projective representation of G_{gauge}. In general, the Z_n charge may carry projective representation of G_{PSG} (i.e., with mixed fractionalized symmetry G_{symm} charge and gauge G_{gauge} charge).

VI. TIME-REVERSAL SYMMETRIC TOPOLOGICAL ORDERS

In this section, we are going to construct exactly soluble local bosonic models that have time-reversal symmetry and emergent time-reversal symmetric topological orders. The time-reversal symmetry T is described by the symmetry group Z_2^T, which means $T^2 = 1$. We will first construct (2+1)D models and then (3+1)D models. All the (3+1)D models realize time-reversal symmetric Z_2-gauge theories at low energies.

A. (2+1)D time-reversal symmetric $\mathbb{Z}_2$-1-cocycle models

1. Model construction

We start with the $\mathbb{Z}_2$-1-cocycle models which produce time-reversal symmetry-enriched Z_2 topological orders and double-semion topological orders in (2+1)D. The partition function has a form

$$Z_{\mathbb{Z}_2aT}\big(M^3_{\text{latt}}\big) = \sum_{\{a^{\mathbb{Z}_2}_{ij}\},da^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3_{\text{latt}}} W(a^{\mathbb{Z}_2},\mathrm{w}_m)}. \tag{179}$$

The possible topological terms $W(a^{\mathbb{Z}_2},\mathrm{w}_m)$ are mixture of 1-cocycle $a^{\mathbb{Z}_2}$ and Stiefel-Whitney classes w_m. Here, $W(a^{\mathbb{Z}_2},\mathrm{w}_m)$ has its value in $\mathbb{Z}_2$. Thus, $e^{i\pi\int_{M^3_{\text{latt}}} W(a^{\mathbb{Z}_2},\mathrm{w}_m)} = \pm 1$ and there is time-reversal symmetry in our model. Also, since $W(a^{\mathbb{Z}_2},\mathrm{w}_m) \in C^3(M^3_{\text{latt}};\mathbb{Z}_2)$, $e^{i\pi\int_{M^3_{\text{latt}}} W(a^{\mathbb{Z}_2},\mathrm{w}_m)}$ is well defined even for nonorientable manifold M^3_{non} where $H_3(M^3_{\text{non}};\mathbb{Z}) = 0$ but $H_3(M^3_{\text{non}};\mathbb{Z}_2) = \mathbb{Z}_2$. We also note that for nonorientable manifold, M^3_{non} itself is a chain with boundary (i.e., M^3_{non} is not a cycle). Therefore, $\int_{M^3_{\text{non}}} db \neq 0$, for a 2-cochain b.

The possible topological terms are given by the combinations of the following six 3-cocycles:

$$\mathrm{w}_1^3, \quad \mathrm{w}_1\mathrm{w}_2, \quad \mathrm{w}_3,$$
$$(a^{\mathbb{Z}_2})^3, \quad \mathrm{w}_1(a^{\mathbb{Z}_2})^2, \quad \mathrm{w}_1^2 a^{\mathbb{Z}_2}. \tag{180}$$

From Appendix D 3, we find many relations between Stiefel-Whitney and the $\mathbb{Z}_2$-1-cocycle:

$$\mathrm{w}_1^2 = \mathrm{w}_2,\ \ \mathrm{w}_1\mathrm{w}_2 = \mathrm{w}_3 = 0,$$
$$\mathrm{w}_1(a^{\mathbb{Z}_2})^2 = \mathrm{Sq}^1[(a^{\mathbb{Z}_2})^2] = 2(a^{\mathbb{Z}_2})^3 = 0. \tag{181}$$

So, the most general time-reversal symmetric $\mathbb{Z}_2$-1-cocycle model that couples to Stiefel-Whitney classes is given by

$$Z_{k_1k_2;\mathrm{t}\mathbb{Z}_2aT}\big(M^3_{\text{latt}}\big) = \sum_{\{a^{\mathbb{Z}_2}_{ij}\},da^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3_{\text{latt}}} k_1 a^{\mathbb{Z}_2}\mathcal{B}_2 a^{\mathbb{Z}_2} + k_2\mathrm{w}_1^2 a^{\mathbb{Z}_2}}, \tag{182}$$

where $k_1,k_2 \in \mathbb{Z}_2$, and we have used $(a^{\mathbb{Z}_2})^3 = a^{\mathbb{Z}_2}\mathcal{B}_2 a^{\mathbb{Z}_2}$.

We like to remark that the Stiefel-Whitney class w_1 in the above path integral can be induced by a local degrees of freedom, a pseudoscalar $\tilde{g}_i$ introduced in Sec. V C. Using $\mathrm{w}_1 = d\tilde{g}_i - dg_i$, where g_i is $\mathbb{Z}_2$ single-valued 0-cochain, we can rewrite the above path integral as (the g_i dependence disappears)

$$Z_{k_1k_2;\mathrm{t}\mathbb{Z}_2aT}\big(M^3_{\text{latt}}\big) = \sum_{\{\tilde{g}_i,a^{\mathbb{Z}_2}_{ij}\},da^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3_{\text{latt}}} k_1 a^{\mathbb{Z}_2}\mathcal{B}_2 a^{\mathbb{Z}_2} + k_2\mathcal{B}_2 d\tilde{g} a^{\mathbb{Z}_2}} \tag{183}$$

which is a pure local bosonic model.

The above four local bosonic models with different values of k_1,k_2 give rise to four different time-reversal symmetry-enriched topological orders. If we break the time-reversal symmetry, the above local bosonic model will only give rise to two different topological orders labeled by k_1: the Z_2 topological order (i.e., the Z_2-gauge theory) for $k_1 = 0$ and the double-semion topological order for $k_1 = 1$.

2. Topological partition functions

Next, we will compute the volume-independent partition function, which is given by

$$Z^{\text{top}}_{k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(M^3)=\frac{1}{2}\sum_{a^{\mathbb{Z}_2}\in H^1(M^3;\mathbb{Z}_2)} e^{i\pi\int_{M^3}k_1a^{\mathbb{Z}_2}\mathcal{B}_2a^{\mathbb{Z}_2}+k_2\mathrm{w}_1^2a^{\mathbb{Z}_2}}. \tag{184}$$

On $M^3=S^1\times\Sigma_g$, $\int_{M^3}k_1a^{\mathbb{Z}_2}\mathcal{B}_2a^{\mathbb{Z}_2}+k_2\mathrm{w}_1^2a^{\mathbb{Z}_2}=0$. Thus,

$$Z^{\text{top}}_{k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(S^1\times\Sigma_g)=2^{2g}. \tag{185}$$

On $M^3=S^1\times\Sigma_g^{\text{non}}$, we note that the cohomology ring $H^*(S^1\times\Sigma_g^{\text{non}};\mathbb{Z}_2)$ has a basis

$$H^*(S^1\times\Sigma_g^{\text{non}};\mathbb{Z}_2)=\{a_0,a_i|_{i=1,\dots,g},a_0a_i,b,a_0b\} \tag{186}$$

with $a_0,a_i\in H^1(S^1\times\Sigma_g^{\text{non}};\mathbb{Z}_2)$ and $b\in H^2(S^1\times\Sigma_g^{\text{non}};\mathbb{Z}_2)$, which have the following cup product:

$$a_i^2=b,\quad a_0^2=a_ib=0. \tag{187}$$

The Stiefel-Whitney classes are given by

$$\mathrm{w}_1=\sum_{i=1}^g a_i,\quad \mathrm{w}_2=\mathrm{w}_1^2=[g]_2b, \tag{188}$$

and the Bockstein homomorphism is given by

$$\mathcal{B}_2a_i=(a_i)^2=b,\quad \mathcal{B}_2a_0=0. \tag{189}$$

Expanding

$$a^{\mathbb{Z}_2}=\sum_{\mu=0}^g\alpha_\mu a_\mu, \tag{190}$$

we find that

$$\begin{aligned}Z^{\text{top}}_{k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(S^1\times\Sigma_g^{\text{non}})&=\frac{1}{2}\sum_{a_\mu=0,1}e^{i\pi(k_1\alpha_0\sum_{i=1}^g\alpha_i+k_2g\alpha_0)}\\&=\sum_{a_i=0,1}\delta_2\left(k_1\sum_{i=1}^g\alpha_i+k_2g\right)\\&=(1-k_1)[k_2g+1]_2 2^g+k_12^{g-1}.\end{aligned} \tag{191}$$

The results are summarized in Table I.

We like to remark that $Z_2\times Z_2^T$ SPT states are classified by $\mathcal{H}^3[Z_2\times Z_2^T;(\mathbb{R}/\mathbb{Z})_T]=\mathbb{Z}_2^{\oplus 2}$. For a $Z_2\times Z_2^T$ SPT state labeled by $(k_1,k_2)\in\mathbb{Z}_2^{\oplus 2}$, its SPT invariant is given by $Z^{\text{top}}(M^3,a^{\mathbb{Z}_2})=e^{i\pi\int_{M^3}k_1a^{\mathbb{Z}_2}\mathcal{B}_2a^{\mathbb{Z}_2}+k_2\mathrm{w}_1^2a^{\mathbb{Z}_2}}$, where $a^{\mathbb{Z}_2}$ describes the Z_2 symmetry twist on M^3. Such SPT invariant happens to be the phase factor in Eq. (184), and the summation in Eq. (184) happens to be the summation of all possible Z_2 symmetry twists. This implies that the topological orders produced by the (2+1)D $\mathbb{Z}_2$-1-cocycle model can be regarded as the Z_2-gauged $Z_2\times Z_2^T$ SPT states.

3. Properties of excitations

When $k_1=0$, the (2+1)D $\mathbb{Z}_2$-1-cocycle model has an emergent Z_2 topological order described by a low-energy Z_2-gauge theory. It has four types of pointlike excitations: 1, e, m, $\varepsilon=em$, where ε is a fermion and others are bosons. When $k_1=1$, the cocycle model has an emergent double-semion topological order. It has four types of pointlike excitations: 1, e, m, ε, where e is a semion with spin $\frac{1}{4}$, and ε a semion with spin $-\frac{1}{4}$. 1 and e are bosons, and they carry Z_2 charge 0 and 1, respectively.

To obtain more properties of the excitations in those T-symmetric topological orders, let us consider dimension reduction. In general, when we reduce a stable phase $\mathcal{C}^d$ in d dimension to lower dimension d' via a compactification $M^d\to M^{d'}\times N^{d-d'}$, the resulting lower-dimensional phase on $M^{d'}$ may correspond to several stable phases $\mathcal{C}_i^{d'}$ with accidental degenerate energy [90]. We denote such dimension reduction as

$$\mathcal{C}^d=\bigoplus_i\mathcal{C}_i^{d'}, \tag{192}$$

and refer $\mathcal{C}_i^{d'}$'s as different sectors. The different sectors arise from different field configurations on $N^{d-d'}$. We like to ask the following: What are effective theories for those d'-dimensional systems in each sector?

To apply the above general picture to our case, let us assume the space-time to be $M^3=M^2\times S^1$ and S^1 is a small circle. We can view the (2+1)D $\mathbb{Z}_2$-1-cocycle models as a (1+1)D local bosonic system. Then, what is the effective theory for such (1+1)D systems?

To answer the above question, we can write $a^{\mathbb{Z}_2}$ as $a^{\mathbb{Z}_2}=a^{\mathbb{Z}_2}_{M^2}+a^{\mathbb{Z}_2}_{S^1}$, where $a^{\mathbb{Z}_2}_{M^2}$ are low-energy degrees of freedom only live on M^2 (i.e., constant in the S^1 direction), and $a^{\mathbb{Z}_2}_{S^1}$ are high-energy degrees of freedom only live on S^1 (i.e., constant in the M^2 directions). The different field configurations on S^1 are labeled by $\alpha=\int_S^1 a^{\mathbb{Z}_2}_{S^1}\in\mathbb{Z}_2$. So, the different sectors are also labeled by $\alpha=0,1$. The partition function on $M^2\times S^1$ becomes

$$\begin{aligned}&Z_{k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(M^2\times S^1)\\&=\sum_{\{a_{ij}^{\mathbb{Z}_2}\},da^{\mathbb{Z}_2}=0}e^{i\pi\int_{M^2\times S^1}k_1a^{\mathbb{Z}_2}\mathcal{B}_2a^{\mathbb{Z}_2}+k_2\mathcal{B}_2d\tilde{g}a^{\mathbb{Z}_2}}\\&=\sum_{\{a_{ij}^{\mathbb{Z}_2}\},da^{\mathbb{Z}_2}_{M^2}=0}e^{i\pi\alpha\int_{M^2}k_1\mathcal{B}_2a^{\mathbb{Z}_2}_{M^2}+k_2\mathcal{B}_2d\tilde{g}}.\end{aligned} \tag{193}$$

We see that in the sector $\alpha=0$, the resulting (1+1)D Z_2^T SPT order is trivial. In contrast, in the sector $\alpha=1$, the resulting (1+1)D Z_2^T SPT order is nontrivial. Usually, in (1+1)D, the gauge field $a^{\mathbb{Z}_2}_{M^2}$ fluctuates strongly. Here, we want to treat the (1+1)D system as reduced from the (2+1)D system as shown in Fig. 8. In this case, we can assume the gauge field $a^{\mathbb{Z}_2}_{M^2}$ to fluctuate weakly, and treat $a^{\mathbb{Z}_2}_{M^2}$ as a background probe field. Therefore, we can view the (1+1)D system as a system with $Z_2\times Z_2^T$ symmetry. Then, from the (1+1)D effective theory (193) which can be viewed as an SPT invariant [55], we see that in the sector $\alpha=1$ is described by a $Z_2\times Z_2^T$ SPT state labeled by (k_1,k_2), which agrees with the group cohomology result $\mathcal{H}^2(Z_2\times Z_2^T,\mathbb{R}/\mathbb{Z}_T)=\mathbb{Z}_2^{\oplus 2}$.

If $(k_1,k_2)=(0,1)$, the (1+1)D SPT state is a pure Z_2^T SPT state as indicated by the term $e^{i\pi\int_{M^2}k_2\mathcal{B}_2d\tilde{g}}$. Such SPT state has Kramers doublet at the chain end. In fact, the chain end has

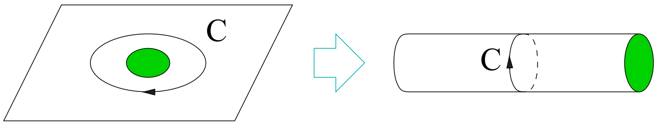

FIG. 8. In a dimension reduction from 2D space to 1D space (a cylinder), a hole in the 2D space becomes an end of the 1D space. The Z_2 vortex with $\int_C a^{Z_2} = 1$ in 2D space becomes the $\int_C a^{Z_2} = 1$ sector in the 1D space.

to sector with Z_2 charge 0 and with Z_2 charge 1. Both sectors are Kramers doublets. We may view the (1+1)D system with a chain end as a (2+1)D system with a hole as described in Fig. 8. Thus, the $\alpha = 1$ sector, corresponding to a π flux in (2+1)D. We see that a π flux carries a Kramers doublet regardless if it carries addition Z_2 charge or not. Similarly, the $\alpha = 0$ sector gives rise to trivial (1+1)D SPT state, and thus a π flux carries a time-reversal singlet regardless if it carries additional Z_2 charge or not. To summarize,

> *the (2+1)D $\mathbb{Z}_2$-1-cocycle model labeled by $(k_1,k_2) = (0,1)$ has four types of pointlike excitations 1, e, m, $\varepsilon = em$. The excitations m, ε carry π flux, while the excitations e, ε carry a Z_2 gauge 1. The excitations m, ε are Kramers doublets and the excitation ε is a fermion (see Table I).*

The time-reversal singlet has a quantum dimension $d = 1$ and the Kramers doublet has a quantum dimension $d = 2$. (Quantum dimension is the dimension of the Hilbert space for the internal degrees of freedom carried by a particle.) Thus, the four types of particles have the following quantum dimensions $(d_1,d_e,d_m,d_f) = (1,1,2_-,2_-)$, where the subscript $-$ indicates the Kramers doublet. A particle can also carry spin s, which is defined mod 1. A boson has spin 0 mod 1 and a fermion has spin $\frac{1}{2}$ mod 1. Thus, the four types of particles have the following spins $(s_1,s_e,s_m,s_f) = (0,0,0,\frac{1}{2})$ (see Table I).

If $(k_1,k_2) = (1,0)$, the cocycle model has four excitations: 1, e, m, ε. 1 and e transform as time-reversal singlet. m and ε transform into each other and form a time-reversal doublet. Since m and ε are always degenerate with time-reversal symmetry, we view them as a single type of excitation with quantum dimension 2. Thus,

> *the (2+1)D $\mathbb{Z}_2$-1-cocycle model labeled by $(k_1,k_2) = (1,0)$ has three types of pointlike excitations with quantum dimensions $(d_i) = (1,1,2)$ and spins $(s_i) = (0,0,[\frac{1}{4},\frac{3}{4}])$.*

Under the dimension reduction, the (1+1)D state in $\alpha = 1$ sector is a $Z_2 \times Z_2^T$ SPT state described by the SPT invariant $e^{i\pi\int_{M^2}\mathcal{B}_2 a^{\mathbb{Z}_2}_{M^2}}$. The chain end for such a $Z_2 \times Z_2^T$ SPT is a doublet with fraction Z_2 charge $\pm\frac{1}{2}$. Under the time reversal, the $+\frac{1}{2}$ and $-\frac{1}{2}$ Z_2 charge states get exchanged and $T^2 = 1$. Thus, the π flux in (2+1)D ground state will carry a doublet of $\pm\frac{1}{2} Z_2$ charges. There are two types of 0-flux excitations with 0 and 1 Z_2 charges. Those two types of excitations are time-reversal singlets. Thus, we denote that quantum dimensions for those excitations as $(d_i) = (1,1,2_+)$, where subscript $+$ indicates $T^2 = 1$ (see Table I).

If $(k_1,k_2) = (1,1)$, under the dimension reduction, the (1+1)D state in $\alpha = 1$ sector is a $Z_2 \times Z_2^T$ SPT state described by the SPT invariant $e^{i\pi\int_{M^2}\mathcal{B}_2 a^{\mathbb{Z}_2}_{M^2} + \mathcal{B}_2 d\tilde{g}}$. The chain end for such $Z_2 \times Z_2^T$ SPT states may contain four degenerate states formed by a doublet with fraction Z_2 charge $\pm\frac{1}{2}$ and a Kramers doublet. The time-reversal transformation is described by

$$T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} K = \sigma^1 \otimes i\sigma^2 K, \quad T^2 = -1, \tag{194}$$

where K is the antiunitary transformation, and $\sigma^{1,2,3}$ are the Pauli matrices. The Z_2 symmetry is generated by

$$Q = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = i\sigma^3 \otimes \sigma^0, \quad Q^2 = -1. \tag{195}$$

However, the four states can be split by a time-reversal and Z_2-symmetric perturbation

$$\delta H = \Delta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Delta\sigma^3 \otimes \sigma^3. \tag{196}$$

Thus, the chain end in general has a doublet with fractional Z_2 charge $\pm\frac{1}{2}$ which is also a $T^2 = -1$ Kramers doublet at the same time. As a result, the π flux in (2+1)D ground state carries a Kramers doublet with fractional Z_2 charge $\pm\frac{1}{2}$. We stress that there is no time-reversal symmetric perturbation that can give rise to $T^2 = 1$ doublet. To summarize,

> *the (2+1)D $\mathbb{Z}_2$-1-cocycle model labeled by $(k_1,k_2) = (1,1)$ has three types of pointlike excitations with quantum dimensions $(d_1,d_e,d_s) = (1,1,2_-)$ and spins $(s_1,s_e,s_s) = (0,0,[\frac{1}{4},\frac{3}{4}])$, where subscript "$-$" indicates $T^2 = -1$ (see Table I).*

B. (2+1)D time-reversal symmetric Z_4^T group cohomology models

1. Model construction

Using the group cocycles, we can construct more local bosonic models that can produce time-reversal symmetric (2+1)D (twisted) Z_2-gauge theories at low energy [see (177)]. In this section, we will discuss those models.

We put $\mathbb{Z}_2$ degrees of freedom on both vertices and links: $\tilde{g}_i \in \mathbb{Z}_2$ and $a_{ij}^{\mathbb{Z}_2} \in \mathbb{Z}_2$. Note that $\tilde{g}_i$ is a pseudoscalar as discussed in Sec. V C (see Fig. 6). Using

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1, \tag{197}$$

we can construct a $\mathbb{Z}_4$-1-cocycle field

$$a_{ij}^{\mathbb{Z}_4} = 2a_{ij}^{\mathbb{Z}_2} + (d\tilde{g})_{ij}. \tag{198}$$

Notice that $\mathcal{H}^3(\mathbb{Z}_4,(\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_4}) = 0$. Thus, there is no group cocycle term in the action amplitude. We obtain the following time-reversal symmetric model:

$$Z_{Z_4^T}(M^3) = \sum_{\substack{\{a_{ij}^{\mathbb{Z}_2},\tilde{g}_i\}, da^{\mathbb{Z}_4}=0 \\ a^{\mathbb{Z}_4}=2a^{\mathbb{Z}_2}+d\tilde{g}}} 1. \tag{199}$$

The condition $da^{\mathbb{Z}_4} = 0$ becomes (when we view the cochains as $\mathbb{Z}$ valued)

$$da^{\mathbb{Z}_4} = 2da^{\mathbb{Z}_2} + d(d\tilde{g}) \overset{4}{=} 0$$
$$\to da^{\mathbb{Z}_2} + \mathcal{B}_2 d\tilde{g} \overset{2}{=} 0. \qquad (200)$$

We can rewrite the above partition function as

$$Z_{Z_4^T}(M^3) = \sum_{\{a_{ij}^{\mathbb{Z}_2},\tilde{g}_i\}, da^{\mathbb{Z}_2} \overset{2}{=} \mathcal{B}_2 d\tilde{g}} 1. \qquad (201)$$

We see that such a model is different from the model (183) with $k_{1,2} = 0$. The condition $da^{\mathbb{Z}_2} = \mathcal{B}_2 d\tilde{g}$ encodes the nontrivial group extension (197).

Due to the relation $\mathcal{B}_2 d\tilde{g} = \mathcal{B}_2 \mathrm{w}_1 \overset{2}{=} \mathrm{w}_1^2$, $Z_{Z_4^T}(M^3) \neq 0$ only when $\mathrm{w}_1^2 = 0$ as $\mathbb{Z}_2$-valued cohomology class. Thus, we introduce

$$\bar{\delta}_m(c) = \begin{cases} 0 & \text{if } c \neq db \text{ mod } m, \\ 1 & \text{if } c = db \text{ mod } m. \end{cases} \qquad (202)$$

So $Z_{Z_4^T}(M^3)$ contains a factor $\bar{\delta}_2(\mathcal{B}_2\mathrm{w}_1)$. Furthermore, on space-time M^3 with $\mathcal{B}_2\mathrm{w}_1 = 0$, we have $da^{\mathbb{Z}_2} \overset{2}{=} 0$. In this case, we can combine the Z_2-1-cocycle model and the Z_4^T group cohomology model together:

$$\begin{aligned} Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) &= \sum_{\{\tilde{g}_i,a_{ij}^{\mathbb{Z}_2}\}, da^{\mathbb{Z}_2} \overset{2}{=} k_0\mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2}\mathcal{B}_2 d\tilde{g}} \\ &= \sum_{\{\tilde{g}_i,a_{ij}^{\mathbb{Z}_2},\tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3 + \tilde{a}^{\mathbb{Z}_2} da^{\mathbb{Z}_2} + k_0 \tilde{a}^{\mathbb{Z}_2}\mathcal{B}_2 d\tilde{g} + k_2 a^{\mathbb{Z}_2}\mathcal{B}_2 d\tilde{g}}. \end{aligned} \qquad (203)$$

When $k_0 = 0$, the above model reduces to the Z_2-1-cocycle model (183). When $k_0 = 1$ and $k_1 = k_2 = 0$, the above becomes the Z_4^T group cohomology model. The volume-independent partition function is given by

$$Z^{\mathrm{top}}_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) = \frac{\bar{\delta}_2(k_0\mathcal{B}_2\mathrm{w}_1)}{|H^0(M^3;\mathbb{Z}_2)|} \sum_{a^{\mathbb{Z}_2} \in H^1(M^3;\mathbb{Z}_2)} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3 + k_2 \mathrm{w}_1^2 a^{\mathbb{Z}_2}}. \qquad (204)$$

In the above, we have assumed that when $k_0\mathcal{B}_2\mathrm{w}_1$ is a coboundary, we will choose such a coboundary to be zero. We note that $Z^{\mathrm{top}}_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3)$ is simply given by $Z^{\mathrm{top}}_{k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3)$ (see Sec. VI A 2) with an extra $\bar{\delta}_2(k_0\mathcal{B}_2\mathrm{w}_1)$ term.

When $k_0 = 0$, the above model becomes the one studied in Sec. VI A, and the topological order that it produces can be viewed as a gauged $Z_2 \times Z_2^T$ SPT state.

2. Properties of excitations

When $k_0 = 1$, the nontrivial group extension makes the time-reversal transformation T to have a property that T^2 is a Z_2-gauge transformation. So, $T^2 = -1$ for a nontrivial Z_2 charge. In other words, the e particle with Z_2 charge 1 carries a Kramers doublet. e is also a boson, since if we break the time-reversal symmetry, the above model gives rise to the Z_2 or double-semion topological orders, where, in both cases, the Z_2 charge is a boson. We also note that when $k_0 = 1$, $k_2 = 0,1$ gives rise to the same model.

When $(k_0,k_1,k_2) = (1,0,*)$, the dimension reduction $M^3 \to M^2 \times S^1$ does not produce nontrivial $Z_2 \times Z_2^T$ SPT state in (1+1)D, thus, the Z_2 vortex m in (2+1)D is a time-reversal singlet and is a boson. The bound state of a Z_2 charge and a Z_2 vortex is a fermion that carries a Kramers doublet. The results are summarized in Table I.

When $(k_0,k_1,k_2) = (1,1,0)$, the dimension reduction $M^3 \to M^2 \times S^1$ produces a nontrivial $Z_2 \times Z_2^T$ SPT state in (1+1)D, thus, the Z_2 vortex m. In fact, the Z_2 vortex m is a $T^2 = 1$ time-reversal doublet that carries Z_2-gauge charge $\pm\frac{1}{2}$ [the same as discussed in Sec. VI A 3 for the $(k_0,k_1,k_2) = (0,1,0)$ case]. The Z_2-gauge-charge $\pm\frac{1}{2}$ doublet is formed by a semion with spin $s = \frac{1}{4}$ and a conjugate semion with spin $s = \frac{3}{4}$. The bound state of a Z_2 charge and a Z_2 vortex is ε which also forms a time-reversal doublet. But, ε is a $T^2 = -1$ Kramers doublet that carries Z_2-gauge charge $\pm\frac{1}{2}$. To summarize,

> *the (2+1)D $\mathbb{Z}_4^T$ group-cocycle model labeled by $(k_0,k_1,k_2) = (1,1,0)$ has four types of pointlike excitations with quantum dimensions $(d_1,d_e,d_m,d_\varepsilon) = (1,2_-,2_+,2_-)$ and spins $(s_1,s_e,s_m,s_\varepsilon) = (0,0,[\frac{1}{4},\frac{3}{4}],,[\frac{1}{4},\frac{3}{4}])$ (see Table I).*

For $(k_0,k_1,k_2) = (1,1,1)$, the results are the same as those for $(k_0,k_1,k_2) = (1,1,0)$, except that the properties of m and ε are exchanged. This is why $(k_0,k_1,k_2) = (1,1,0)$ and $(k_0,k_1,k_2) = (1,1,1)$ correspond to the same time-reversal SET order.

3. Including excitations in the path integral

Now, let us include the excitations in the partition function (203). Let M_e^1 be the $\mathbb{Z}_2$-valued 1-cycle that corresponds to the world line of the Z_2 charge e: $M_e^1 \in Z_1(M^4;\mathbb{Z}_2)$. Let M_m^1 be the $\mathbb{Z}_2$-valued 1-cycle that corresponds to the world line of the Z_2 vortex m: $M_m^1 \in Z_1(M^4;\mathbb{Z}_2)$. The Poincaré dual of M_e^1 is a $\mathbb{Z}_2$-valued 2-cocycle $B_e^{\mathbb{Z}_2}$ and the Poincaré dual of M_m^1 is a $\mathbb{Z}_2$-valued 2-cocycle $B_m^{\mathbb{Z}_2}$: $B_e^{\mathbb{Z}_2} \in Z^2(M^4;\mathbb{Z}_2)$ and $B_m^{\mathbb{Z}_2} \in Z^2(M^4;\mathbb{Z}_2)$. The partition function with excitations is given by

$$\begin{aligned} Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) &= \sum_{\{\tilde{g}_i,a_{ij}^{\mathbb{Z}_2}\}, da^{\mathbb{Z}_2} \overset{2}{=} k_0\mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2}\mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M_e^1} a^{\mathbb{Z}_2}} \\ &= \sum_{\{\tilde{g}_i,a_{ij}^{\mathbb{Z}_2},\tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2}(da^{\mathbb{Z}_2} + k_0\mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2})} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3 + k_2 a^{\mathbb{Z}_2}\mathcal{B}_2 d\tilde{g}} e^{i\pi \int_{M_e^1} a^{\mathbb{Z}_2}} \\ &= \sum_{\{\tilde{g}_i,a_{ij}^{\mathbb{Z}_2},\tilde{a}_{ij}^{\mathbb{Z}_2}\}} e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2} da^{\mathbb{Z}_2}} e^{i\pi \int_{M^3} k_1(a^{\mathbb{Z}_2})^3} e^{i\pi \int_{M^3} \tilde{a}^{\mathbb{Z}_2}(k_0\mathcal{B}_2 d\tilde{g} + B_m^{\mathbb{Z}_2}) + a^{\mathbb{Z}_2}(k_2\mathcal{B}_2 d\tilde{g} + B_e^{\mathbb{Z}_2})}. \end{aligned} \qquad (205)$$

Let us change the variables to

$$a^{\mathbb{Z}_2} \stackrel{2}{=} a_m^{\mathbb{Z}_2} + a_0^{\mathbb{Z}_2}, \quad \tilde{a}^{\mathbb{Z}_2} \stackrel{2}{=} a_e^{\mathbb{Z}_2} + \tilde{a}_0^{\mathbb{Z}_2}, \tag{206}$$

where $a_0^{\mathbb{Z}_2}, \tilde{a}_0^{\mathbb{Z}_2} \in C^1(M^2;\mathbb{Z}_2)$, and $a_m^{\mathbb{Z}_2}$, $a_e^{\mathbb{Z}_2}$ are fixed $\mathbb{Z}_2$-valued 1-cochains satisfying

$$da_m^{\mathbb{Z}_2} \stackrel{2}{=} B_m^{\mathbb{Z}_2} + k_0\mathcal{B}_2\mathrm{w}_1,\ \ da_e^{\mathbb{Z}_2} \stackrel{2}{=} B_e^{\mathbb{Z}_2} + k_2\mathcal{B}_2\mathrm{w}_1. \tag{207}$$

(Here, we have assumed that $B_m^{\mathbb{Z}_2} + k_0\mathcal{B}_2\mathrm{w}_1$ and $B_e^{\mathbb{Z}_2} + k_2\mathcal{B}_2\mathrm{w}_1$ are coboundaries.) Now, we can rewrite the partition function as

$$\begin{aligned} Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) &= \sum_{\tilde{g},a_0^{\mathbb{Z}_2},\tilde{a}_0^{\mathbb{Z}_2}} e^{i\pi\int_{M^3} a_e^{\mathbb{Z}_2}da_m^{\mathbb{Z}_2}+\tilde{a}_0^{\mathbb{Z}_2}da_0^{\mathbb{Z}_2}} e^{i\pi\int_{M^3} k_1(a_m^{\mathbb{Z}_2}+a_0^{\mathbb{Z}_2})^3} e^{i\pi\int_{M^3} a_e^{\mathbb{Z}_2}(k_0\mathcal{B}_2d\tilde{g}+B_m^{\mathbb{Z}_2})+a_m^{\mathbb{Z}_2}(k_2\mathcal{B}_2d\tilde{g}+B_e^{\mathbb{Z}_2})} \\ &= \sum_{\tilde{g},da_0^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3} a_e^{\mathbb{Z}_2}da_m^{\mathbb{Z}_2}} e^{i\pi\int_{M^3} k_1(a_m^{\mathbb{Z}_2}+a_0^{\mathbb{Z}_2})^3} e^{i\pi\int_{M^3} a_e^{\mathbb{Z}_2}(k_0\mathcal{B}_2d\tilde{g}+B_m^{\mathbb{Z}_2})+a_m^{\mathbb{Z}_2}(k_2\mathcal{B}_2d\tilde{g}+B_e^{\mathbb{Z}_2})}. \end{aligned} \tag{208}$$

Since $a_0^{\mathbb{Z}_2}$ becomes a cocycle, we can further simplify the factor $e^{i\pi\int_{M^3} k_1(a_m^{\mathbb{Z}_2}+a_0^{\mathbb{Z}_2})^3}$ using Eq. (21):

$$\begin{aligned} &e^{i\pi\int_{M^3} k_1(a_m^{\mathbb{Z}_2}+a_0^{\mathbb{Z}_2})^3} \\ &\quad = e^{i\pi\int_{M^3} k_1[(a_m^{\mathbb{Z}_2})^3+(a_0^{\mathbb{Z}_2})^3+(a_m^{\mathbb{Z}_2})^2a_0^{\mathbb{Z}_2}+a_m^{\mathbb{Z}_2}(a_0^{\mathbb{Z}_2})^2]}. \end{aligned} \tag{209}$$

The partition function now becomes

$$\begin{aligned} &Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) \\ &\quad = e^{i\pi\int_{M^3} a_m^{\mathbb{Z}_2}B_e^{\mathbb{Z}_2}} \sum_{\tilde{g},da_0^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3} k_0a_e^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_2a_m^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}} \\ &\qquad \times e^{i\pi\int_{M^3} k_1[(a_m^{\mathbb{Z}_2})^3+(a_0^{\mathbb{Z}_2})^3+(a_m^{\mathbb{Z}_2})^2a_0^{\mathbb{Z}_2}+a_m^{\mathbb{Z}_2}(a_0^{\mathbb{Z}_2})^2]}. \end{aligned} \tag{210}$$

The above partition function can be expressed in terms of linking numbers. Consider $\int_{M^3} B_e^{\mathbb{Z}}a_m^{\mathbb{Z}_2} = \int_{M_e^1} a_m^{\mathbb{Z}_2}$. If M_e^1 is a boundary $M_e^1 = \partial D_e^2$, then we can relate the above to the intersection number and the linking number:

$$\begin{aligned} \int_{M_e^1} a_m^{\mathbb{Z}_2} &= \int_{D_e^2} da_m^{\mathbb{Z}_2} = \int_{D_e^2} B_m^{\mathbb{Z}_2} + k_0\mathrm{w}_1^2 \\ &= \mathrm{Int}(D_e^2, M_m^1 + k_0M_\mathrm{w}^1) = \mathrm{Lnk}(M_e^1, M_m^1 + k_0M_\mathrm{w}^1), \end{aligned} \tag{211}$$

where M_w^1 is the $\mathbb{Z}_2$-valued 1-cycle which is the Poincaré dual of $\mathcal{B}_2\mathrm{w}_1$. Here, $\mathrm{Int}(D_e^2, M_m^1)$ is the intersection number between D_e^2 and M_m^1, and $\mathrm{Lnk}(M_e^1, M_m^1)$ the linking number between M_e^1 and M_m^1. The linking number satisfies

$$\mathrm{Lnk}(M_e^1, M_m^1) = \mathrm{Lnk}(M_m^1, M_e^1). \tag{212}$$

Using the linking number, we can rewrite the partition function as

$$\begin{aligned} &Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) \\ &\quad \propto e^{i\pi\int_{M^3} k_1(a_m^{\mathbb{Z}_2})^3} e^{i\pi\mathrm{Lnk}(k_2M_\mathrm{w}^1+M_e^1, M_m^1)} \\ &\qquad \times \sum_{\tilde{g},da_0^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3} k_0a_e^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_2a_m^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}} \\ &\qquad \times e^{i\pi\int_{M^3} k_1[(a_0^{\mathbb{Z}_2})^3+(a_m^{\mathbb{Z}_2})^2a_0^{\mathbb{Z}_2}+a_m^{\mathbb{Z}_2}(a_0^{\mathbb{Z}_2})^2]}. \end{aligned} \tag{213}$$

We like to stress that the above path integral has a time-reversal symmetry: it is invariant under a combined transformation $\tilde{g}_i \to [\tilde{g}_i + 1]_2$, $a_{0,ij}^{\mathbb{Z}_2} \to a_{0,ij}^{\mathbb{Z}_2}$, and complex conjugation.

The physical properties of excitations can be obtained from the above effective theory. Let us first assume $k_1 = 0$, and rewrite the partition function as

$$\begin{aligned} &Z_{k_0k_1k_2;\mathrm{t}\mathbb{Z}_2\mathrm{a}T}(M^3) \\ &\quad \propto \bar{\delta}_2(M_m^1 + k_0M_\mathrm{w}^1)\bar{\delta}_2(M_e^1 + k_2M_\mathrm{w}^1)e^{i\pi\mathrm{Lnk}(k_2M_\mathrm{w}^1+M_e^1, M_m^1)} \\ &\qquad \times \sum_{\tilde{g},da_0^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3} k_0a_e^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_2a_m^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}}, \end{aligned} \tag{214}$$

where we have restored the two δ functions. For simplicity, we will also assume $\mathrm{w}_1^2 = 0$, and choose $a_e^{\mathbb{Z}_2}$ to be the Poincaré dual of D_e^2 and $a_m^{\mathbb{Z}_2}$ to be the Poincaré dual of D_m^2. Here, D_e^2 and D_m^2 are the disks bonded by the world lines M_e^1 and M_m^1. The dynamical part of the partition function can be written as

$$\begin{aligned} &\sum_{\tilde{g}} e^{i\pi\int_{M^3} k_0a_e^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_2a_m^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}} \\ &\quad = \sum_{\tilde{g}} e^{i\pi\int_{D_e^2} k_0\mathcal{B}_2d\tilde{g}} e^{i\pi\int_{D_m^2} k_2\mathcal{B}_2d\tilde{g}} \\ &\quad \propto e^{i\pi\int_{D_e^2} k_0\mathcal{B}_2\mathrm{w}_1} e^{i\pi\int_{D_m^2} k_2\mathcal{B}_2\mathrm{w}_1}. \end{aligned}$$

From the above, we see that, when $k_0 = 1$, there is Z_2^T SPT state described by the SPT invariant $e^{i\pi\int_{D_e^2}\mathcal{B}_2\mathrm{w}_1}$ on D_e^2. In this case, the boundary of D_e^2, i.e., the e particle described by the world line $M_e^1 = \partial D_e^2$, will carry a Kramers doublet. This agrees with the result in Sec. VI B 2. Similarly, when $k_2 = 1$, there is Z_2^T SPT state described by the SPT invariant $e^{i\pi\int_{D_m^2}\mathcal{B}_2\mathrm{w}_1}$ on D_m^2, and the m particle will carry a Kramers doublet.

The term $e^{i\pi\mathrm{Lnk}(M_e^1, M_m^1)}$ tells us that the e and m have a mutual π statistics between them. The absence of self-linking terms $e^{i\theta\mathrm{Lnk}(M_e^1, M_e^1)}$ and $e^{i\theta\mathrm{Lnk}(M_m^1, M_m^1)}$ implies that the e and m are bosons. We also see that the emergence of Kramers-doublet bosons cause the partition function to vanish on the space-time with $\mathrm{w}_1^2 \neq 0$. From the form of $\bar{\delta}_2(M_m^1 + k_0M_\mathrm{w}^1)\bar{\delta}_2(M_e^1 + k_2M_\mathrm{w}^1)$, we see that space-time with $\mathrm{w}_1^2 \neq 0$ will generate a m particle (or more precisely, a noncontractible world line of the m) if the bosonic e particle is a Kramers doublet. Similarly, space-time with $w_1^2 \neq 0$ will generate a e particle if the bosonic m particle is a Kramers doublet. In other words,

if there is an emergent bosonic Kramers doublet, then a space-time with $w_1^2 \neq 0$ *will create a world line of a particle that has a mutual* π *statistics with the bosonic Kramers doublet. The world line is equal to the Poincaré dual of* w_1^2.

Those results are summarized by the top three rows in Table I.

Next, we consider the case of $k_1 = 1$. The partition function now reads as

$$\begin{aligned} &Z_{k_0k_1k_2;\text{t}\mathbb{Z}_2\text{a}T}(M^3)\\ &\propto \bar{\delta}_2\big(M_m^1 + k_0M_{\text{w}}^1\big)e^{i\pi\int_{M^3}k_1(a_m^{\mathbb{Z}_2})^3}e^{i\pi\text{Lnk}(k_2M_{\text{w}}^1+M_e^1,M_m^1)}\\ &\times \sum_{\tilde{g},da_0^{\mathbb{Z}_2}=0} e^{i\pi\int_{M^3}k_0a_e^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_2a_m^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}}\\ &\times e^{i\pi\int_{M^3}(a_0^{\mathbb{Z}_2})^3+(a_m^{\mathbb{Z}_2})^2a_0^{\mathbb{Z}_2}+a_m^{\mathbb{Z}_2}(a_0^{\mathbb{Z}_2})^2}. \end{aligned} \tag{215}$$

Note that we only have one δ function in this case. The above result for the e particle is not changed: the e is still a boson, which carries Kramers doublet if $k_0 = 1$ and time-reversal singlet if $k_0 = 0$.

But, the result for the m particle is changed. The effective theory on D_m^2 now becomes

$$\sum_{\tilde{g},a_0^{\mathbb{Z}_2}} e^{i\pi\int_{D_m^2}k_2\mathcal{B}_2d\tilde{g}+(a_0^{\mathbb{Z}_2})^2}. \tag{216}$$

If we treat the emergent Z_2-gauge symmetry as a Z_2 symmetry, then the above can be viewed as a $Z_2 \times Z_2^T$ SPT state on D_m^2. The SPT state is characterized by SPT invariant $e^{i\pi\int_{D_e^2}k_2\mathcal{B}_2\text{w}_1+(a^{Z_2})^2}$ where a^{Z_2} is the symmetry twist of Z_2. As discussed in Sec. VI A 3, when $k_2 = 0$, the m particle will carry $\pm\frac{1}{2}$ Z_2-gauge charge, which forms a $T^2 = 1$ time-reversal doublet (labeled by 2_+). When $k_2 = 1$, the m particle will carry $\pm\frac{1}{2}$ Z_2-gauge charge, which forms a $T^2 = -1$ Kramers doublet (labeled by 2_-). The above applies for both $k_0 = 0,1$ cases.

For the bond state of e and m, the ε particle, the $Z_2 \times Z_2^T$ SPT state on the corresponding D_ε^2 is described by

$$\sum_{\tilde{g},a_0^{\mathbb{Z}_2}} e^{i\pi\int_{D_m^2}(k_0+k_2)\mathcal{B}_2d\tilde{g}+k_1(a_0^{\mathbb{Z}_2})^2}. \tag{217}$$

We see that the ε is always a $\pm\frac{1}{2}$ Z_2-gauge-charge doublet. It is a $T^2 = -1$ Kramers doublet (2_-) if $(k_0 + k_2) = 1$ and a $T^2 = 1$ time-reversal doublet (2_+) if $(k_0 + k_2) = 0$.

The statistics of the m particle is no longer bosonic due the self-braiding term $e^{i\pi\int_{M^3}k_1(a_m^{\mathbb{Z}_2})^3}$ (which can be viewed as the triple self-intersection of D_m^2). We note that $e^{i\pi\int_{M^3}k_1(a_m^{\mathbb{Z}_2})^3} = \pm 1$ respects the time-reversal symmetry. But, one expects m to be a semion described by the self-linking term $e^{i\frac{\pi}{2}\text{Lnk}(M_m^1,M_m^1)}$. In fact, the above self-linking term breaks the time-reversal symmetry, and does not describe the statistics of m which in our case is a particle with respect to the time-reversal symmetry. In other words, due to the time-reversal symmetry, m is not a semion.

In fact, m is a $T^2 = -1$ Kramers doublet or a $T^2 = 1$ time-reversal doublet formed by a semion (with spin $s = \frac{1}{4}$) and a conjugate semion (with spin $s = \frac{3}{4}$). The statistics of such a time-reversal symmetric doublet is not described by the self-linking term $e^{i\frac{\pi}{2}\text{Lnk}(M_m^1,M_m^1)}$ or the self-linking term $e^{-i\frac{\pi}{2}\text{Lnk}(M_m^1,M_m^1)}$. Our calculation suggests that the statistics of the time-reversal symmetric doublet is described by $e^{i\pi\int_{M^3}k_1(a_m^{\mathbb{Z}_2})^3}$: the triple self-intersection of D_m^2. Those results are summarized by the bottom three rows in Table I.

C. (3+1)D time-reversal symmetric model

1. Model construction

In this section, we are going to study a class of (3+1)D time-reversal symmetric local bosonic models, that can produce the simplest time-reversal symmetric topological orders. The (3+1)D time-reversal symmetric local bosonic models contain $\mathbb{Z}_2$-multivalued 0-cochain field $\tilde{g}_i$, $\mathbb{Z}_2$-valued 1-cochain field $a_{ij}^{\mathbb{Z}_2}$, and $\mathbb{Z}_2$-valued 2-cochain field $b_{ijk}^{\mathbb{Z}_2}$. Its path integral is given by

$$\begin{aligned} &Z_{k_1k_2k_3k_4k_5k_6}(M^4)\\ &= \sum_{\{\tilde{g}_i^{\mathbb{Z}_2},a_{ij}^{\mathbb{Z}_2},b_{ijk}^{\mathbb{Z}_2}\}} e^{i\pi\int_{M^4}b^{\mathbb{Z}_2}da^{\mathbb{Z}_2}}e^{i\pi\int_{M^4}(k_3+k_4)b^{\mathbb{Z}_2}\mathcal{B}_2d\tilde{g}+k_4(b^{\mathbb{Z}_2})^2}\\ &\times e^{i\pi\int_{M^4}k_1(a^{\mathbb{Z}_2})^4+(k_2+k_1)a^{\mathbb{Z}_2}(d\tilde{g})^3}e^{i\pi\int_{M^4}k_5(d\tilde{g})^4+k_6(\text{w}_2)^2}. \end{aligned} \tag{218}$$

The 0-cocycle field $\tilde{g}_i$ is a pseudoscalar as introduced in Sec. V C. It satisfies $d\tilde{g}_i = \text{w}_1 + dg$, where g_i is a $\mathbb{Z}_2$ single-valued 0-cochain field. Thus, $\mathcal{B}_2d\tilde{g}_i = \mathcal{B}_2\text{w}_1$. The above path integral defines the system for both closed and open space-time manifold M^4. But, in the following, we will assume M^4 to be closed. The index $k_I = 0,1$. So, there are $2^6 = 64$ different models.

We note that the above path integral has the time-reversal symmetry Z_2^T, i.e., invariant under the combined transformation of $\tilde{g}_i \to [\tilde{g}_i + 1]_2$ and complex conjugation. (Under the transformation $\tilde{g}_i \to \tilde{g}_i' = [\tilde{g}_i + 1]_2, d\tilde{g}_i = -d\tilde{g}_i'$.) This is a designed property. However, the path integral also has an extra Z_2' symmetry: $\tilde{g}_i \to [\tilde{g}_i + 1]_2$ (without the complex conjugation).

Let us also include the excitations in the path integral. We know that the pointlike excitations are described by the world lines in space-time. A world line M_{WL}^1 can be viewed as a $\mathbb{Z}_2$-valued 1-cycle, which is Poincaré dual to a $\mathbb{Z}_2$-valued 3-cochain $C_{\text{WL}}^{\mathbb{Z}_2}$. The stringlike excitations are described by the world sheet in space-time, which can be viewed as $\mathbb{Z}_2$-valued 2-cycles M_{WS}^2 in the space-time lattice, whose Poincaré dual is a $\mathbb{Z}_2$-valued 2-cocycle $B_{\text{WS}}^{\mathbb{Z}_2}$.

Just like the Z_2-gauge theory, we can include those excitations in path integral (218), by adding the Z_2-charge coupling term $e^{i\pi\int_{M_{\text{WL}}^1}a^{\mathbb{Z}_2}}$ and the Z_2-flux coupling term $e^{i\pi\int_{M_{\text{WS}}^2}b^{\mathbb{Z}_2}}$. Due to the Poincaré duality,

$$\begin{aligned} e^{i\pi\int_{M_{\text{WL}}^1}a^{\mathbb{Z}_2}} &= e^{i\pi\int_{M^4}C_{\text{WL}}^{\mathbb{Z}_2}a^{\mathbb{Z}_2}},\\ e^{i\pi\int_{M_{\text{WS}}^2}b^{\mathbb{Z}_2}} &= e^{i\pi\int_{M^4}B_{\text{WS}}^{\mathbb{Z}_2}b^{\mathbb{Z}_2}}. \end{aligned} \tag{219}$$

Thus, in the presence of pointlike topological excitations described by $C^{\mathbb{Z}_2}_{\text{WL}}$ and stringlike topological excitations described by B^2_{WS}, the partition function (218) becomes

$$Z_{k_1k_2k_3k_4k_5k_6}(M^4) = \sum_{\tilde g, a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} b^{\mathbb{Z}_2}(da^{\mathbb{Z}_2}+k_4 b^{\mathbb{Z}_2}+(k_3+k_4)\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}})} \times e^{i\pi \int_{M^4} k_1(a^{\mathbb{Z}_2})^4+[(k_2+k_1)(d\tilde g)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}} e^{i\pi \int_{M^4} k_5(d\tilde g)^4+k_6 \text{w}_2^2}. \tag{220}$$

2. Partition function

To understand the physical properties of those 64 models, we like to compute the corresponding partition functions on closed space-time M^4. However, unlike other models constructed in this paper, the above models are not exactly soluble. They are exactly soluble only in the cases $k_1 = 0$ or $k_4 = 0$. So, we will calculate the partition functions for those two cases.

When $k_4 = 0$, the action is linear in $b^{\mathbb{Z}_2}$, and we can integrate out $b^{\mathbb{Z}_2}$ first, which leads to a constraint

$$da^{\mathbb{Z}_2} \overset{2}{=} k_3\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}} = k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}, \tag{221}$$

where $\tilde g_0$ is a fixed $\mathbb{Z}_2$-multivalued 0-cochain such that

$$\tilde g - \tilde g_0 \overset{2}{=} g \tag{222}$$

is a $\mathbb{Z}_2$-single-valued 0-cochain g. We see that the partition function is zero when $k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}$ is not a coboundary. Thus, the partition function contains a factor $\bar\delta_2(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}})$. We may solve the $da^{\mathbb{Z}_2} \overset{2}{=} k_3\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}}$ constraint via the following ansatz:

$$a^{\mathbb{Z}_2} \overset{2}{=} a^{\mathbb{Z}_2}_{\text{WS}} + a^{\mathbb{Z}_2}_0, \tag{223}$$

where $a^{\mathbb{Z}_2}_{\text{WS}}$ is a $\mathbb{Z}_2$-valued 1-cochain that satisfies

$$da^{\mathbb{Z}_2}_{\text{WS}} \overset{2}{=} k_3\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}}, \tag{224}$$

and $a^{\mathbb{Z}_2}_0$ is a $\mathbb{Z}_2$-valued 1-cocycle field $a^{\mathbb{Z}_2}_0 \in Z^1(M^4;\mathbb{Z}_2)$. The partition function now becomes

$$Z_{k_1k_2k_30k_5k_6}(M^4) = \bar\delta_2\big(k_3\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}}\big) \sum_{\tilde g, da_0^{\mathbb{Z}_2} \overset{2}{=} 0} e^{i\pi \int_{M^4} k_1(a^{\mathbb{Z}_2}_{\text{WS}})^4+[(k_2+k_1)(d\tilde g)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_{\text{WS}}} \times e^{i\pi \int_{M^4} k_1(a^{\mathbb{Z}_2}_0)^4+[(k_2+k_1)(d\tilde g)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_0} e^{i\pi \int_{M^4} k_5(d\tilde g)^4+k_6\text{w}_2^2}. \tag{225}$$

Since $a^{\mathbb{Z}_2}_0$ is a cocycle, we can replace $d\tilde g$ by $d\tilde g_0$ in the last line above, and use many relations between $a^{\mathbb{Z}_2}_0$ and Stiefel-Whitney classes, such as (see Appendix D 4 where w_1 is replaced by $d\tilde g_0$)

$$(d\tilde g_0)^2\big(a^{\mathbb{Z}_2}_0\big)^2 \overset{2,d}{=} (d\tilde g_0)^3 a^{\mathbb{Z}_2}_0, \quad \text{w}_2\big(a^{\mathbb{Z}_2}_0\big)^2 \overset{2,d}{=} \text{w}_3 a^{\mathbb{Z}_2}_0, \quad \big(a^{\mathbb{Z}_2}_0\big)^4 \overset{2,d}{=} d\tilde g_0\big(a^{\mathbb{Z}_2}_0\big)^3, \quad \big[a^{\mathbb{Z}_2}_0\big)^2 + (d\tilde g_0)^2 + \text{w}_2\big]\big(a^{\mathbb{Z}_2}_0\big)^2 \overset{2,d}{=} 0, \tag{226}$$

to simplify the last line. Note that those relations are valid only when $a^{\mathbb{Z}_2}_0$ is a cocycle and when M^4 is closed. Therefore, we can rewrite the above partition function on closed M^4 as

$$\begin{aligned} Z_{k_1k_2k_30k_5k_6}(M^4) &= \bar\delta_2\big(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big) \sum_{\tilde g, da_0^{\mathbb{Z}_2} \overset{2}{=} 0} e^{i\pi \int_{M^4} k_1(a^{\mathbb{Z}_2}_{\text{WS}})^4+[(k_2+k_1)(d\tilde g)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_{\text{WS}}} e^{i\pi \int_{M^4}[k_1\text{w}_3+k_2(d\tilde g_0)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_0} e^{i\pi \int_{M^4} k_5(d\tilde g)^4+k_6\text{w}_2^2} \\ &= \bar\delta_2\big(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big)\bar\delta_2\big[k_1\text{w}_3 + k_2(d\tilde g_0)^3 + C^{\mathbb{Z}_2}_{\text{WL}}\big] e^{i\pi \int_{M^4} k_5(d\tilde g_0)^4+k_6\text{w}_2^2} \sum_{\tilde g} e^{i\pi \int_{M^4} k_1(a^{\mathbb{Z}_2}_{\text{WS}})^4+[(k_2+k_1)(d\tilde g)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_{\text{WS}}}. \end{aligned} \tag{227}$$

We note that $x^2 \overset{2}{=} \mathcal{B}_2 x$ for any Z_2-valued 1-cocycle x, and $d\tilde g = d\tilde g_0 + dg$. Thus,

$$\begin{aligned} e^{i\pi \int_{M^4}(k_2+k_1)(d\tilde g)^3 a^{\mathbb{Z}_2}_{\text{WS}}} &= e^{i\pi \int_{M^4}(k_2+k_1)d\tilde g \mathcal{B}_2 d\tilde g_0 a^{\mathbb{Z}_2}_{\text{WS}}} \\ &= e^{i\pi \int_{M^4}(k_2+k_1)d\tilde g_0 \mathcal{B}_2 d\tilde g_0 a^{\mathbb{Z}_2}_{\text{WS}}} e^{i\pi \int_{M^4}(k_2+k_1)dg \mathcal{B}_2 d\tilde g_0 a^{\mathbb{Z}_2}_{\text{WS}}} \\ &= e^{i\pi \int_{M^4}(k_2+k_1)(d\tilde g_0)^3 a^{\mathbb{Z}_2}_{\text{WS}}} e^{i\pi \int_{M^4}(k_2+k_1)g \mathcal{B}_2 d\tilde g_0(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}})}. \end{aligned} \tag{228}$$

Therefore, the volume-independent partition function is given by

$$Z^{\text{top}}_{k_1k_2k_30k_5k_6}(M^4) = \frac{|H^0(M^4;\mathbb{Z}_2)||H^2(M^4;\mathbb{Z}_2)|}{|H^1(M^4;\mathbb{Z}_2)|} e^{i\pi \int_{M^4} k_5(d\tilde g_0)^4+k_6\text{w}_2^2+k_1(a^{\mathbb{Z}_2}_{\text{WS}})^4+[(k_2+k_1)(d\tilde g_0)^3+C^{\mathbb{Z}_2}_{\text{WL}}]a^{\mathbb{Z}_2}_{\text{WS}}} \times \bar\delta_2\big(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big)\bar\delta_2\big[k_1\text{w}_3 + k_2(d\tilde g_0)^3 + C^{\mathbb{Z}_2}_{\text{WL}}\big]\delta_2\big[(k_2+k_1)\mathcal{B}_2 d\tilde g_0\big(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big)\big].$$

We note that $\tilde{g}_0$ is multivalued only on ∂M^4 (which is a nonzero even cycle when M^4 is not orientable). So, $\mathcal{B}_2 d\tilde{g}_0$ is nonzero only near the "boundary" ∂M^4 (see Fig. 6). Therefore, $\delta_2[(k_2 + k_1)\mathcal{B}_2 d\tilde{g}_0(k_3\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS})]$ is a boundary term.

When $k_1 = 0$, the action is linear in $a^{\mathbb{Z}_2}$. In this case, we can integrate out $a^{\mathbb{Z}_2}$ first, which leads to a constraint

$$db^{\mathbb{Z}_2} = k_2(d\tilde{g})^3 + C^{\mathbb{Z}_2}_{\rm WL}. \tag{229}$$

So, the partition function is zero when $k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}$ is not a coboundary. Thus, the partition function contains a factor $\bar{\delta}_2[k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}]$. We may solve the $db^{\mathbb{Z}_2} \overset{2}{=} k_2(d\tilde{g})^3 + C^{\mathbb{Z}_2}_{\rm WL}$ constraint via the following ansatz:

$$b^{\mathbb{Z}_2} \overset{2}{=} b^{\mathbb{Z}_2}_{\rm WL} + b^{\mathbb{Z}_2}_0, \tag{230}$$

where $b^{\mathbb{Z}_2}_{\rm WL}$ is a $\mathbb{Z}_2$-valued 2-cochain that satisfies

$$db^{\mathbb{Z}_2}_{\rm WL} \overset{2}{=} k_2(d\tilde{g})^3 + C^{\mathbb{Z}_2}_{\rm WL}, \tag{231}$$

and $b^{\mathbb{Z}_2}_0$ is a $\mathbb{Z}_2$-valued 2-cocycle field $b^{\mathbb{Z}_2}_0 \in Z^2(M^4;\mathbb{Z}_2)$. The partition function now becomes

$$Z_{0k_2k_3k_4k_5k_6}(M^4) = \bar{\delta}_2\big[k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}\big] \sum_{\tilde{g}, db^{\mathbb{Z}_2}_0 \overset{2}{=} 0} e^{i\pi\int_{M^4} b^{\mathbb{Z}_2}_{\rm WL}[k_4 b^{\mathbb{Z}_2}_{\rm WL} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B^{\mathbb{Z}_2}_{\rm WS}]} \times e^{i\pi\int_{M^4} b^{\mathbb{Z}_2}_0[k_4 b^{\mathbb{Z}_2}_0 + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B^{\mathbb{Z}_2}_{\rm WS}]} e^{i\pi\int_{M^4} k_5 (d\tilde{g}_0)^4 + k_6 \mathrm{w}_2^2}. \tag{232}$$

Since $b^{\mathbb{Z}_2}_0$ is a cocycle, we can replace $d\tilde{g}$ by $d\tilde{g}_0$ in the last line above, and use many relations between $b^{\mathbb{Z}_2}_0$ and Stiefel-Whitney classes, such as (see Appendix D 5)

$$d\tilde{g}_0\mathcal{B}_2 b^{\mathbb{Z}_2} = 0, \quad (b^{\mathbb{Z}_2})^2 + [(d\tilde{g}_0)^2 + \mathrm{w}_2]b^{\mathbb{Z}_2} = 0, \tag{233}$$

to simplify the last line. Therefore, we can rewrite the above partition function on closed M^4 as

$$\begin{aligned} Z_{0k_2k_3k_4k_5k_6}(M^4) &= \bar{\delta}_2\big[k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}\big] \sum_{\tilde{g}, db^{\mathbb{Z}_2}_0 \overset{2}{=} 0} e^{i\pi\int_{M^4} b^{\mathbb{Z}_2}_{\rm WL}[k_4 b^{\mathbb{Z}_2}_{\rm WL} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B^{\mathbb{Z}_2}_{\rm WS}]} e^{i\pi\int_{M^4} b^{\mathbb{Z}_2}_0[k_4\mathrm{w}_2 + k_3\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}]} e^{i\pi\int_{M^4} k_5(d\tilde{g}_0)^4 + k_6\mathrm{w}_2^2} \\ &= \bar{\delta}_2\big[k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}\big]\bar{\delta}_2\big(k_4\mathrm{w}_2 + k_3\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}\big) e^{i\pi\int_{M^4} k_5(d\tilde{g}_0)^4 + k_6\mathrm{w}_2^2} \sum_{\tilde{g}} e^{i\pi\int_{M^4} b^{\mathbb{Z}_2}_{\rm WL}[k_4 b^{\mathbb{Z}_2}_{\rm WL} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B^{\mathbb{Z}_2}_{\rm WS}]}. \end{aligned} \tag{234}$$

The above partition function can be simplified further. Let $\bar{b}^{\mathbb{Z}_2}_{\rm WL}$ be a fixed 2-cocycle that satisfies

$$d\bar{b}^{\mathbb{Z}_2}_{\rm WL} \overset{2}{=} k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}, \tag{235}$$

and let

$$b^{\mathbb{Z}_2}_{\rm WL} \overset{2}{=} \bar{b}^{\mathbb{Z}_2}_{\rm WL} + b^{\mathbb{Z}_2}_1. \tag{236}$$

In this case, $b^{\mathbb{Z}_2}_1$ satisfies

$$db^{\mathbb{Z}_2}_1 \overset{2}{=} k_2[(d\tilde{g})^3 - (d\tilde{g}_0)^3] \overset{2}{=} k_2[d\tilde{g}\mathcal{B}_2 d\tilde{g} - d\tilde{g}_0\mathcal{B}_2 d\tilde{g}_0] = k_2 d(g\mathcal{B}_2 d\tilde{g}_0), \tag{237}$$

where we have used $d\tilde{g} = d\tilde{g}_0 + dg$ and $x^2 \overset{2}{=} \mathcal{B}_2 x$ for any Z_2-valued 1-cocycle x. So, $b^{\mathbb{Z}_2}_{\rm WL}$ is given by

$$b^{\mathbb{Z}_2}_{\rm WL} \overset{2}{=} k_2 g\mathcal{B}_2 d\tilde{g}_0 + \bar{b}^{\mathbb{Z}_2}_{\rm WL}. \tag{238}$$

Now, we can rewrite the partition function (232) in the following form (using the relations obtained in Appendix D 5):

$$\begin{aligned} Z_{0k_2k_3k_4k_5k_6}(M^4) &= \bar{\delta}_2\big[k_2(d\tilde{g}_0)^3 + C^{\mathbb{Z}_2}_{\rm WL}\big]\bar{\delta}_2\big(k_4\mathrm{w}_2 + k_3\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}\big) e^{i\pi\int_{M^4} k_5(d\tilde{g}_0)^4 + k_6\mathrm{w}_2^2} e^{i\pi\int_{M^4}\bar{b}^{\mathbb{Z}_2}_{\rm WL}[k_4\bar{b}^{\mathbb{Z}_2}_{\rm WL} + (k_3+k_4)\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}]} \\ &\quad \times \sum_{g} e^{i\pi\int_{M^4} k_2 g\mathcal{B}_2 d\tilde{g}_0[k_4 g\mathcal{B}_2 d\tilde{g}_0 + (k_3+k_4)\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}]}. \end{aligned} \tag{239}$$

Using the fact [see Eq. (21)] $g\mathcal{B}_2 d\tilde{g}_0 g \overset{2,d}{=} g^2\mathcal{B}_2 d\tilde{g}_0 = g\mathcal{B}_2 d\tilde{g}_0$, we can simplify

$$\sum_{g} e^{i\pi\int_{M^4} k_2 g\mathcal{B}_2 d\tilde{g}_0[k_4 g\mathcal{B}_2 d\tilde{g}_0 + (k_3+k_4)\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}]} = \sum_{g} e^{i\pi\int_{M^4} k_2 g\mathcal{B}_2 d\tilde{g}_0[k_4\mathcal{B}_2 d\tilde{g}_0 + (k_3+k_4)\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}]} = \delta_2\big[k_2\mathcal{B}_2 d\tilde{g}_0\big(k_3\mathcal{B}_2 d\tilde{g}_0 + B^{\mathbb{Z}_2}_{\rm WS}\big)\big]. \tag{240}$$

Thus, the volume-independent partition function is given by

$$Z^{\text{top}}_{0k_2k_3k_4k_5k_6}(M^4) = \frac{|H^0(M^4;\mathbb{Z}_2)||H^2(M^4;\mathbb{Z}_2)|}{|H^1(M^4;\mathbb{Z}_2)|} e^{i\pi\int_{M^4} k_5(d\tilde g_0)^4 + k_6 \mathrm{w}_2^2 + \bar b^{\mathbb{Z}_2}_{\text{WL}}[k_4\bar b^{\mathbb{Z}_2}_{\text{WL}} + (k_3+k_4)\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}]} \times \bar\delta_2\big[k_2(d\tilde g_0)^3 + C^{\mathbb{Z}_2}_{\text{WL}}\big]\bar\delta_2\big(k_4\mathrm{w}_2 + k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big)\delta_2\big[k_2\mathcal{B}_2 d\tilde g_0\big(k_3\mathcal{B}_2 d\tilde g_0 + B^{\mathbb{Z}_2}_{\text{WS}}\big)\big].$$

3. Physical properties of ground states

Using the above partition functions, we can obtain many physical properties of ground states by setting $B^{\mathbb{Z}_2}_{\text{WS}} = C^{\mathbb{Z}_2}_{\text{WL}} = 0$. For simplicity, we will also assume that $\mathrm{w}_i = 0$ on M^4, so that we can choose $a^{\mathbb{Z}_2}_{\text{WS}} = \bar b^{\mathbb{Z}_2}_{\text{WL}} = 0$.

First, we see that the partition functions for different k_I's do not depend the shape or the metrics of space-time manifold M^4. So, the ground states of the 48 models with $k_1k_4 = 0$ are all gapped. The partition functions also do not depend on the triangulation of the space-time. So, the ground states are all gapped liquids [16,17]. If we choose space-time to be $M^4 = S^1 \times S^3$ where $\mathrm{w}_1 = \mathrm{w}_2 = \mathrm{w}_3 = 0$, we find the volume-independent partition functions to be $Z^{\text{top}}_{k_1k_2k_3k_4k_5k_6}(S^1\times S^3) = 1$. This means that the ground-state degeneracies on S^3 for the 48 models (with $k_1k_4 = 0$) are all equal to 1, and there is no spontaneous symmetry breaking of Z_2^T or Z_2'.

The volume-independent partition functions are not equal to 1 for other closed space-times with vanishing Euler number and Pontryagin number. For example, on $M^4 = T^2 \times S^2$ where $\mathrm{w}_1 = \mathrm{w}_2 = \mathrm{w}_3 = 0$, $Z^{\text{top}}_{k_1k_2k_3k_4k_5k_6}(T^2\times S^2) = 2$. Thus, those 48 models all realize nontrivial (3+1)D topological orders in their ground states. The twofold ground-state degeneracy on space $S^1 \times S^2$ tells us that the topological orders are simple since they all have only one nontrivial pointlike topological excitation and one nontrivial stringlike topological excitation. In fact, the emergent topological orders are Z_2 topological orders described by UT or EF Z_2-gauge theories with $a^{\mathbb{Z}_2}$ as the Z_2-gauge field. Because the ground states also have symmetries, we may view those topological orders as Z_2^T SET orders or as $Z_2' \times Z_2^T$ SET orders.

We remark that the action amplitude $e^{i\pi\int_{M^4} k_5(d\tilde g)^4 + k_6\mathrm{w}_2^2} = e^{i\pi\int_{M^4} k_5\mathrm{w}_1^4 + k_6\mathrm{w}_2^2}$ is the SPT invariant for the Z_2^T SPT states. So, different k_5,k_6 correspond to stacking with different Z_2^T SPT states.

4. Properties of pointlike excitations

First, if we break the time-reversal symmetry (i.e., only put the system on orientable space-time with $\mathrm{w}_1 = 0$), then our models with $k_1 = 0$ reduce to the $\mathbb{Z}_n$-2-cocycle model (74) with $n = 2$ and $k = k_4$. So, when $k_1 = 0$, the pointlike topological excitation in our model is a fermion if $k_4 = 1$, and a boson if $k_4 = 0$ (see Table II where a fermion is indicated by spin $s_2 = \frac{1}{2}$ and a boson by spin $s_2 = 0$).

When $k_4 = 0$ (and without time-reversal symmetry), our model reduces to the UT Z_2-gauge theory [note that $(a_0^{\mathbb{Z}_2})^4 = (a_0^{\mathbb{Z}_2})^3\mathrm{w}_1$, and $(a_0^{\mathbb{Z}_2})^4 = 0$ when $\mathrm{w}_1 = 0$]. So, the pointlike topological excitation in our model is a boson if $k_4 = 0$, even when $k_1 \neq 0$.

In the presence of time-reversal symmetry Z_2^T with $T^2 = 1$, the pointlike topological excitation may carry fractionalized time-reversal symmetry with $T^2 = -1$, i.e., it may carry Kramers doublet. In fact, in this section, we will consider both time-reversal symmetry and the extra Z_2' symmetry $\tilde g_i \to [\tilde g_i + 1]_2$ of our models. So, the total symmetry group is $Z_2' \times Z_2^T$. In this case, the multivaluedness of $\tilde g_i$ is not only due to the orientation twist around a loop, it is also due to the Z_2' symmetry twist around a loop. Thus,

$$d\tilde g = \mathrm{w}_1 + a'^{\mathbb{Z}_2}, \tag{241}$$

where $a'^{\mathbb{Z}_2}$ is the 1-cocycle that describes the Z_2' symmetry twist in space-time [53–56].

To see the time-reversal and Z_2' symmetry properties of the pointlike topological excitation, we first consider the $k_4 = 0$ case and start with the path integral (225). We like to stress that in our calculation to obtain Eq. (225), we did not use the relation $d\tilde g = \mathrm{w}_1$ which is not valid in the presence of Z_2' symmetry twist, which is necessary to consider Z_2' symmetry. The only term that involves the world line of the pointlike topological excitation is $e^{i\pi\int_{M^4} C^{\mathbb{Z}_2}_{\text{WL}}(a^{\mathbb{Z}_2}_{\text{WS}} + a_0^{\mathbb{Z}_2})}$, which can be expanded as

$$e^{i\pi\int_{M^4} C^{\mathbb{Z}_2}_{\text{WL}}(a^{\mathbb{Z}_2}_{\text{WS}} + a_0^{\mathbb{Z}_2})} = e^{i\pi\int_{M^1_{\text{WL}}} a^{\mathbb{Z}_2}_{\text{WS}} + a_0^{\mathbb{Z}_2}} = e^{i\pi\int_{D^2_{\text{WL}}} da^{\mathbb{Z}_2}_{\text{WS}}} = e^{i\pi\int_{D^2_{\text{WL}}} k_3\mathcal{B}_2 d\tilde g + B^{\mathbb{Z}_2}_{\text{WS}}}, \tag{242}$$

where D^2_{WL} is the two-dimensional submanifold whose boundary is the world line $\partial D^2_{\text{WL}} = M^1_{\text{WL}}$. The term $e^{i\pi\int_{M^1_{\text{WL}}} a_0^{\mathbb{Z}_2}}$ indicates that the pointlike excitation carries a unit of Z_2-gauge charge.

The term $e^{i\pi\int_{D^2_{\text{WL}}} B^{\mathbb{Z}_2}_{\text{WS}}} = e^{i\pi \text{Lnk}(M^1_{\text{WL}}, M^2_{\text{WS}})}$ is determined by the linking number $\text{Lnk}(M^1_{\text{WL}}, M^2_{\text{WS}})$ between the world line M^1_{WL} of the pointlike excitation and the world sheet M^2_{WS} of the stringlike excitation. It describes the π phase change as a pointlike excitation goes around the stringlike excitation.

The term $e^{i\pi\int_{D^2_{\text{WL}}} k_3\mathcal{B}_2 d\tilde g}$ gives rise to a $Z_2' \times Z_2^T$ SPT invariant

$$e^{i\pi\int_{D^2_{\text{WL}}} k_3\mathcal{B}_2 d\tilde g} \to e^{i\pi\int_{D^2_{\text{WL}}} k_3(\mathcal{B}_2\mathrm{w}_1 + \mathcal{B}_2 a'^{\mathbb{Z}_2})}, \tag{243}$$

which describes a (1+1)D $Z_2' \times Z_2^T$ SPT state on D^2_{WL} when $k_3 = 1$. Due to the term $\mathcal{B}_2\mathrm{w}_1$ in the SPT invariant, the boundary of the (1+1)D Z_2^T SPT state carries a Kramers doublet. Thus,

the world line, i.e., the pointlike excitation, carries a Kramers doublet if $k_3 = 1$ and carries a time-reversal singlet if $k_3 = 0$

(see Table II where a Kramers doublet is indicated by quantum dimension $d_2 = 2_-$ and a time-reversal singlet by quantum dimension $d_2 = 1$). Due to the term $\mathcal{B}_2 a'^{\mathbb{Z}_2}$, the Kramers doublet on the pointlike excitation is formed by

Z_2'-charge $\pm\frac{1}{2}$ states. So,

the Kramers doublet also carries fractional Z_2' charge $\pm\frac{1}{2}$.

We next consider the $k_1 = 0$ case and start with the path integral (232). Again, in our calculation to obtain Eq. (232), we did not use the relation $d\tilde{g} = \mathrm{w}_1$. The only term that involves the world line of the pointlike topological excitation is

$$e^{i\pi \int_{M^4} b_{\mathrm{WL}}^{\mathbb{Z}_2}[k_4 b_{\mathrm{WL}}^{\mathbb{Z}_2} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B_{\mathrm{WS}}^{\mathbb{Z}_2}]}. \tag{244}$$

Let us consider a particular world line which is a boundary: $M_{\mathrm{WL}}^1 = \partial D_{\mathrm{WL}}^2$, and write $b_{\mathrm{WL}}^{\mathbb{Z}_2}$ as

$$b_{\mathrm{WL}}^{\mathbb{Z}_2} \stackrel{2}{=} \bar{b}_{\mathrm{WL}}^{\mathbb{Z}_2} + b_{\mathrm{WL}}'^{\mathbb{Z}_2}, \tag{245}$$

where

$$db_{\mathrm{WL}}'^{\mathbb{Z}_2} = C_{\mathrm{WL}}^{\mathbb{Z}_2} \tag{246}$$

comes from the world line and $\bar{b}_{\mathrm{WL}}^{\mathbb{Z}_2}$ from the background Stiefel-Whitney class and other world lines. We obtain

$$\begin{aligned} &e^{i\pi \int_{M^4} b_{\mathrm{WL}}^{\mathbb{Z}_2}[k_4 b_{\mathrm{WL}}^{\mathbb{Z}_2} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B_{\mathrm{WS}}^{\mathbb{Z}_2}]} \\ &= e^{i\pi \int_{M^4} \bar{b}_{\mathrm{WL}}^{\mathbb{Z}_2}[k_4 \bar{b}_{\mathrm{WL}}^{\mathbb{Z}_2} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B_{\mathrm{WS}}^{\mathbb{Z}_2}]} \\ &\quad \times e^{i\pi \int_{M^4} b_{\mathrm{WL}}'^{\mathbb{Z}_2}[k_4 b_{\mathrm{WL}}'^{\mathbb{Z}_2} + (k_3+k_4)\mathcal{B}_2 d\tilde{g} + B_{\mathrm{WS}}^{\mathbb{Z}_2}]} \end{aligned} \tag{247}$$

which can be viewed as the effective action amplitude on the word line.

Compared with our previous result, we see that

the term $e^{i\pi \int_{M^4} k_4 (b_{\mathrm{WL}}'^{\mathbb{Z}_2})^2}$ should describe the Fermi statistics of the pointlike excitation when $k_4 = 1$.

Using the fact that Poincaré dual of $b_{\mathrm{WL}}'^{\mathbb{Z}_2}$ is D_{WL}^2, we can express $e^{i\pi \int_{M^4} k_4 (b_{\mathrm{WL}}'^{\mathbb{Z}_2})^2}$ in terms of self-intersection number of D_{WL}^2:

$$e^{i\pi \int_{M^4} k_4 (b_{\mathrm{WL}}'^{\mathbb{Z}_2})^2} = e^{i\pi \mathrm{Int}(D_{\mathrm{WL}}^2, D_{\mathrm{WL}}^2)}. \tag{248}$$

We see that

the Fermi statistics in (3+1)D is described by the self-intersection number of the disk whose boundary is the world line of the fermion.

The term $e^{i\pi \int_{M^4} b_{\mathrm{WL}}'^{\mathbb{Z}_2} B_{\mathrm{WS}}^{\mathbb{Z}_2}}$ describes the π phase change as a pointlike excitation goes around the stringlike excitation.

Now, let us concentrate on

$$e^{i\pi \int_{M^4} (k_3+k_4) b_{\mathrm{WL}}'^{\mathbb{Z}_2} \mathcal{B}_2 d\tilde{g}} = e^{i\pi \int_{D_{\mathrm{WL}}^2} (k_3+k_4)\mathcal{B}_2 d\tilde{g}}, \tag{249}$$

where we have used the fact that Poincaré dual of $b_{\mathrm{WL}}'^{\mathbb{Z}_2}$ is D_{WL}^2. As discussed before, due to such a term will make

the pointlike excitation carries a Kramers doublet formed by fractional Z_2' charge $\pm\frac{1}{2}$, if $k_3 + k_4 \stackrel{2}{=} 1$ and carry a time-reversal singlet with integer Z_2' charge, if $k_3 + k_4 \stackrel{2}{=} 0$.

5. Properties of stringlike excitations

To obtain physical properties of string excitations, let us consider a dimension reduction $M^4 = M^3 \times S^1$ (for details, see Sec. VI A 3).

Let us first consider the case for $k_4 = 0$ and start from Eq. (225). We can choose $a_{\mathrm{WS}}^{\mathbb{Z}_2}$ to make $\int_{S^1} a_{\mathrm{WS}}^{\mathbb{Z}_2} = 0$. The two sectors after the reduction are labeled by $\alpha = \int_{S^1} a^{\mathbb{Z}_2} = \int_{S^1} a_0^{\mathbb{Z}_2}$. The effective theory on M^3 after the dimension reduction is given by

$$\begin{aligned} Z_{k_1k_2k_3k_4k_5k_6}(M^4) &= \bar{\delta}_2\big(k_3\mathcal{B}_2\mathrm{w}_1 + B_{\mathrm{WS}}^{\mathbb{Z}_2}\big) \sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^3} B_{\mathrm{WL}}^{\mathbb{Z}_2} a_{\mathrm{WS}}^{\mathbb{Z}_2}} \\ &\quad \times e^{i\pi \int_{M^3} B_{\mathrm{WL}}^{\mathbb{Z}_2} a_0^{\mathbb{Z}_2} + \alpha(k_2+k_1)(d\tilde{g})^3}, \end{aligned}$$

where $a_0^{\mathbb{Z}_2}$ now lives on M^3 and $B_{\mathrm{WL}}^{\mathbb{Z}_2}$ is the Poincaré dual of the world line in M^3.

For simplicity, let us choose the world line to make $B_{\mathrm{WL}}^{\mathbb{Z}_2} = 0$. The effective theory on M^3 now becomes (only the dynamical part)

$$Z_{k_1k_2k_3k_4k_5k_6}(M^4) = \sum_{\tilde{g}, da_0^{\mathbb{Z}_2} \stackrel{2}{=} 0} e^{i\pi \int_{M^3} \alpha(k_2+k_1)(d\tilde{g})^3}. \tag{250}$$

If we view the above effective theory as a (2+1)D theory with time-reversal Z_2^T symmetry that acts on $\tilde{g}_i$, then the above effective theory describe trivial Z_2^T SPT states since the SPT invariant

$$e^{i\pi \int_{M^3} \alpha(k_2+k_1)(d\tilde{g})^3} = e^{i\pi \int_{M^3} \alpha(k_2+k_1)\mathrm{w}_1^3} = 1 \tag{251}$$

becomes trivial in (2+1)D (see Appendix D 3). The (1+1)D boundary of the (2+1)D theory in the $\alpha = 1$ sector corresponds to the Z_2 vortex line. So, the above result implies that the Z_2 vortex line of our model just behaves like the Z_2 vortex line of UT Z_2-gauge theory regardless the values of k_I.

Our model actually has a $Z_2' \times Z_2^T$ symmetry. So, the (2+1)D effective theory can be viewed as a model with Z_2' symmetry. In this case, the model describes a nontrivial Z_2' SPT state, when $\alpha(k_2 + k_1) \neq 0$. To see this, we note that the Z_2' acts like $\tilde{g}_i \to [\tilde{g}_i + 1]_2$. So, to obtain the Z_2' SPT invariant, we need to gauge the Z_2' symmetry (see Sec. V B) by replacing $d\tilde{g}$ by $a'^{\mathbb{Z}_2}$:

$$e^{i\pi \int_{M^3} \alpha(k_2+k_1)(d\tilde{g})^3} = e^{i\pi \int_{M^3} \alpha(k_2+k_1)(a'^{\mathbb{Z}_2})^3}. \tag{252}$$

The above SPT invariant allows us to show our (2+1)D effective theory leads to a nontrivial Z_2' SPT state, which was first studied in [10]. Since the (1+1)D boundary of the (2+1)D theory in the $\alpha = 1$ sector corresponds to the Z_2 vortex line, so the above result implies that

the Z_2 vortex line of our model carries nontrivial edge excitations of Z_2' SPT state described by SPT invariant $e^{i\pi \int_{M^3} (k_2+k_1)(a'^{\mathbb{Z}_2})^3}$.

The above results about the Z_2 vortex line can be obtained by directly calculating the effective theory on the Z_2 vortex line. We start from the theory with excitations (225). Let the world sheet of the string (i.e., the Z_2 vortex line) M_{WS}^2 be the boundary of D_{WS}^3. For simplicity, let us assume that $\mathrm{w}_2 = \mathrm{w}_1 = 0$ and $a'^{\mathbb{Z}_2} = 0$ (i.e., no Z_2' symmetry twist) on M^4. In this case, $a_{\mathrm{WS}}^{\mathbb{Z}}$ can be chosen to be the Poincaré dual of D_{WS}^3.

The effective theory on the string comes from the factor $e^{i\pi \int_{M^4} (k_2+k_1) a_{\mathrm{WS}}^{\mathbb{Z}_2} (d\tilde{g})^3}$ in Eq. (225), which leads to the following effective theory:

$$Z = \sum_{\tilde{g}_i} e^{i\pi \int_{D_{\mathrm{WS}}^3} (k_2+k_1)(d\tilde{g})^3}. \tag{253}$$

If we identify $(-)^{\tilde{g}_i}$ as σ_i^z of spin-$\frac{1}{2}$, then the above action amplitude describes a (2+1)D spin-$\frac{1}{2}$ model with $Z_2' \times Z_2^T$ symmetry acting on $\tilde{g}_i$'s:

$$Z_2' : \prod_i \sigma_i^x, \quad Z_2^T : K \prod_i \sigma_i^x, \tag{254}$$

where K is the complex conjugation. The effective theory actually describes a nontrivial $Z_2' \times Z_2^T$ SPT state on D^3_{WS}. So, the effective theory on the world sheet M^2_{WS} is the effective boundary theory of the $Z_2' \times Z_2^T$ SPT state. In other worlds, the string will carry nontrivial boundary excitations of the (2+1)D $Z_2' \times Z_2^T$ SPT state. The nontrivialness of the excitations on the string is protected by the anomalous symmetry on the boundary [65]. This can be viewed as the symmetry fractionalization (or quantum number fractionalization) on strings. We have seen that on pointlike excitation, the $T^2 = 1$ Z_2^T time-reversal symmetry can be fractionalized into $T^2 = -1$ Kramers doublet. In contrast, on strings, the symmetry fractionalization is realized as the anomalous (i.e., nononsite) symmetry that constrains the effective theory for degrees of freedom on the strings.

So, the key to calculate the symmetry fractionalization is to calculate the nononsite (i.e., anomalous) symmetry on the strings. Let us do the calculation for the case $k_2 + k_1 = 1$, which leads to the following effective theory:

$$Z = \sum_{\tilde{g}_i} e^{i\pi \int_{D^3_{\text{WS}}} (d\tilde{g})^3}. \tag{255}$$

which describes a $Z_2' \times Z_2^T$ SPT state. The group cocycle that describes the $Z_2' \times Z_2^T$ SPT phase is in fact the topological term $\int_{D^3_{\text{WS}}} (d\tilde{g})^3$:

$$\nu_3(\tilde{g}_0,\tilde{g}_1,\tilde{g}_2,\tilde{g}_3) = -(\tilde{g}_0 - \tilde{g}_1)(\tilde{g}_1 - \tilde{g}_2)(\tilde{g}_2 - \tilde{g}_3). \tag{256}$$

The $Z_2 \times Z_2^T$ symmetry on the string is twisted by the group cocycle and becomes nononsite:

$$\begin{aligned} Z_2' &: \prod_I \sigma_I^x e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_{I+1})}, \\ Z_2^T &: K \prod_I \sigma_I^x e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_{I+1})}, \end{aligned} \tag{257}$$

where

$$\begin{aligned} e^{i\pi \nu_3(1,0,\tilde{g}_I,\tilde{g}_J)} &= (-)^{\tilde{g}_I(\tilde{g}_I - \tilde{g}_J)} = (-)^{\tilde{g}_I}(-)^{\tilde{g}_I\tilde{g}_J} \\ &= \sigma_I^z \frac{1 + \sigma_I^z + \sigma_J^z - \sigma_I^z\sigma_J^z}{2}. \end{aligned} \tag{258}$$

The effective Hamiltonian on the string respects the anomalous $Z_2' \times Z_2^T$ symmetry, which may take a form

$$H_{\text{str}} = \sum_I J_I^z \sigma_I^z \sigma_{I+1}^z + \sum_I K_I^x \left(\sigma_{I-1}^z \sigma_I^x \sigma_{I+1}^z - \sigma_I^x\right). \tag{259}$$

The ground state of such Hamiltonian is gapless or spontaneously breaks the Z_2' symmetry. So, when $k_1 + k_2 = 1$ and $k_4 = 0$, the strings carry nontrivial excitations described by the above Hamiltonian with an anomalous $Z_2' \times Z_2^T$ symmetry: $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}$ and $U_T = K \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}$.

Next, let us consider the case for $k_1 = 0$ and start from Eq. (234). The only term that involves the world sheet is $e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} B_{\text{WS}}^{\mathbb{Z}_2}}$, which can be rewritten as

$$\begin{aligned} e^{i\pi \int_{M^4} b_{\text{WL}}^{\mathbb{Z}_2} B_{\text{WS}}^{\mathbb{Z}_2}} &= e^{i\pi \int_{M^2_{\text{WS}}} b_{\text{WL}}^{\mathbb{Z}_2}} = e^{i\pi \int_{D^3_{\text{WS}}} db_{\text{WL}}^{\mathbb{Z}_2}} \\ &= e^{i\pi \int_{D^3_{\text{WS}}} k_2 (d\tilde{g})^3 + C_{\text{WL}}^{\mathbb{Z}_2}}. \end{aligned} \tag{260}$$

Repeating the above calculation, we see that

when $k_2 = 1$ and $k_1 = 0$, the strings carry nontrivial excitations with an anomalous $Z_2' \times Z_2^T$ symmetry: $U' = \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}$ and $U_T = K \prod_I \sigma_I^x \prod_I \sigma_I^z \frac{1+\sigma_I^z+\sigma_{I+1}^z-\sigma_I^z\sigma_{I+1}^z}{2}$.

We like to remark that, potentially, the strings may carry an anomalous $Z_2 \times Z_2' \times Z_2^T$ symmetry, where Z_2 is associated with $a^{\mathbb{Z}_2}$. From the above calculation, we see that the anomalous symmetry only comes from the Z_2' symmetry. There is no anomalous symmetry from Z_2.

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APPENDIX A: KÜNNETH FORMULA

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex X and chain complex X'. The Künneth formula is expressed in terms of the tensor product operation $\otimes_R$ and the torsion product operation Tor_1^R that act on R modules $\mathbb{M},\mathbb{M}',\mathbb{M}''$. Here, R is a ring and a R module is like a vector space over R (i.e., we can "multiply" a "vector" in $\mathbb{M}$ by an element of R, and two "vectors" in $\mathbb{M}$ can add.) The tensor product operation $\otimes_R$ has the following properties:

$$\begin{aligned} \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{M}' &\simeq \mathbb{M}' \otimes_{\mathbb{Z}} \mathbb{M}, \\ (\mathbb{M}' \oplus \mathbb{M}'') \otimes_R \mathbb{M} &= (\mathbb{M}' \otimes_R \mathbb{M}) \oplus (\mathbb{M}'' \otimes_R \mathbb{M}), \\ \mathbb{M} \otimes_R (\mathbb{M}' \oplus \mathbb{M}'') &= (\mathbb{M} \otimes_R \mathbb{M}') \oplus (\mathbb{M} \otimes_R \mathbb{M}''); \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M}, \\ \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n &= \mathbb{Z}_{\langle m,n\rangle}. \end{aligned} \tag{A1}$$

The torsion product operation Tor_1^R has the following properties:

$$\begin{aligned} \text{Tor}_R^1(\mathbb{M},\mathbb{M}') &\simeq \text{Tor}_R^1(\mathbb{M}',\mathbb{M}), \\ \text{Tor}_R^1(\mathbb{M}' \oplus \mathbb{M}'',\mathbb{M}) &= \text{Tor}_R^1(\mathbb{M}',\mathbb{M}) \oplus \text{Tor}_R^1(\mathbb{M}'',\mathbb{M}), \\ \text{Tor}_R^1(\mathbb{M},\mathbb{M}' \oplus \mathbb{M}'') &= \text{Tor}_R^1(\mathbb{M},\mathbb{M}') \oplus \text{Tor}_R^1(\mathbb{M},\mathbb{M}''), \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z},\mathbb{M}) &= \text{Tor}_{\mathbb{Z}}^1(\mathbb{M},\mathbb{Z}) = 0, \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}_n,\mathbb{M}) &= \{m \in \mathbb{M} | nm = 0\}, \\ \text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}_m,\mathbb{Z}_n) &= \mathbb{Z}_{\langle m,n\rangle}, \end{aligned} \tag{A2}$$

where $\langle m,n\rangle$ is the greatest common divisor of m and n. These expressions allow us to compute the tensor product $\otimes_R$ and the torsion product Tor_R^1. We will use abbreviated Tor to denote $\text{Tor}_{\mathbb{Z}}^1$.

The Künneth formula itself is given by (see [91], p. 247)

$$\begin{aligned} &H^d(X\times X',\mathbb{M}\otimes_R \mathbb{M}')\\ &\quad\simeq \left[\oplus_{k=0}^d H^k(X,\mathbb{M})\otimes_R H^{d-k}(X',\mathbb{M}')\right]\\ &\qquad\oplus\left[\oplus_{k=0}^{d+1}\text{Tor}_R^1(H^k(X,\mathbb{M}),H^{d-k+1}(X',\mathbb{M}'))\right]. \end{aligned} \tag{A3}$$

Here, R is a principal ideal domain and $\mathbb{M},\mathbb{M}'$ are R modules such that $\text{Tor}_R^1(\mathbb{M},\mathbb{M}')=0$. We also require either (1) $H_d(X;\mathbb{Z})$ and $H_d(X';\mathbb{Z})$ are finitely generated, or (2) $\mathbb{M}'$ and $H_d(X';\mathbb{Z})$ are finitely generated.

For more details on principal ideal domain and R module, see the corresponding Wiki articles. Note that $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generate R modules if $R=\mathbb{Z}$. The Künneth formula also works for topological cohomology where X and X' are treated as topological spaces.

For homology, there is a similar Künneth formula

$$\begin{aligned} &H_d(X\times X';\mathbb{Z})\\ &\quad\simeq\left[\oplus_{k=0}^d H_k(X;\mathbb{Z})\otimes H_{d-k}(X';\mathbb{Z})\right]\\ &\qquad\oplus\left[\oplus_{k=0}^{d-1}\text{Tor}(H_k(X;\mathbb{Z}),H_{d-k-1}(X';\mathbb{Z}))\right]. \end{aligned} \tag{A4}$$

As the first application of Künneth formula, we like to use it to calculate $H^*(X',\mathbb{M})$ from $H^*(X';\mathbb{Z})$, by choosing $R=\mathbb{M}'=\mathbb{Z}$. In this case, the condition $\text{Tor}_R^1(\mathbb{M},\mathbb{M}')=\text{Tor}_{\mathbb{Z}}^1(\mathbb{M},\mathbb{Z})=0$ is always satisfied. $\mathbb{M}$ can be $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}_n$, etc. So, we have

$$\begin{aligned} &H^d(X\times X',\mathbb{M})\\ &\quad\simeq\left[\oplus_{k=0}^d H^k(X,\mathbb{M})\otimes_{\mathbb{Z}} H^{d-k}(X';\mathbb{Z})\right]\\ &\qquad\oplus\left[\oplus_{k=0}^{d+1}\text{Tor}(H^k(X,\mathbb{M}),H^{d-k+1}(X';\mathbb{Z}))\right]. \end{aligned} \tag{A5}$$

The above is also valid for topological cohomology.

We can further choose X to be the space of one point in Eq. (A5), and use

$$H^d(X,\mathbb{M})) = \begin{cases}\mathbb{M} & \text{if } d=0,\\ 0 & \text{if } d>0,\end{cases} \tag{A6}$$

to reduce Eq. (A5) to

$$H^d(X,\mathbb{M})\simeq H^d(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{M}\oplus\text{Tor}(H^{d+1}(X;\mathbb{Z}),\mathbb{M}), \tag{A7}$$

where X' is renamed as X. The above is a form of the universal coefficient theorem which can be used to calculate $H^*(X,\mathbb{M})$ from $H^*(X;\mathbb{Z})$ and the module $\mathbb{M}$. The universal coefficient theorem works for topological cohomology where X is a topological space. In fact, we also have a similar universal coefficient theorem for homology

$$H_d(X,\mathbb{M})\simeq H_d(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{M}\oplus\text{Tor}(H_{d-1}(X;\mathbb{Z}),\mathbb{M}). \tag{A8}$$

Using the universal coefficient theorem, we can rewrite Eq. (A5) as

$$H^d(X\times X',\mathbb{M})\simeq\oplus_{k=0}^d H^k[X,H^{d-k}(X',\mathbb{M})]. \tag{A9}$$

The above is also valid for topological cohomology. We note that

$$H^0(X,\mathbb{M})=\mathbb{M}. \tag{A10}$$

There is also a universal coefficient theorem between homology and cohomolgy

$$H^d(X,\mathbb{M})\simeq\text{Hom}(H_d(X;\mathbb{Z}),\mathbb{M})\oplus\text{Ext}(H_{d-1}(X;\mathbb{Z}),\mathbb{M}). \tag{A11}$$

Here, Ext operation on modules is given by

$$\begin{aligned} \text{Ext}_R^1(\mathbb{M}'\oplus\mathbb{M}'',\mathbb{M}) &= \text{Ext}_R^1(\mathbb{M}',\mathbb{M})\oplus\text{Ext}_R^1(\mathbb{M}'',\mathbb{M}),\\ \text{Ext}_R^1(\mathbb{M},\mathbb{M}'\oplus\mathbb{M}'') &= \text{Ext}_R^1(\mathbb{M},\mathbb{M}')\oplus\text{Ext}_R^1(\mathbb{M},\mathbb{M}''),\\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z},\mathbb{M}) &= 0,\\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_n,\mathbb{M}) &= \mathbb{M}/n\mathbb{M},\\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_n,\mathbb{Z}) &= \mathbb{Z}_n,\\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_m,\mathbb{Z}_n) &= \mathbb{Z}_{\langle m,n\rangle}. \end{aligned} \tag{A12}$$

The Hom operation on modules is given by

$$\begin{aligned} \text{Hom}_R(\mathbb{M}'\oplus\mathbb{M}'',\mathbb{M}) &= \text{Hom}_R(\mathbb{M}',\mathbb{M})\oplus\text{Hom}_R(\mathbb{M}'',\mathbb{M}),\\ \text{Hom}_R(\mathbb{M},\mathbb{M}'\oplus\mathbb{M}'') &= \text{Hom}_R(\mathbb{M},\mathbb{M}')\oplus\text{Hom}_R(\mathbb{M},\mathbb{M}''),\\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{M}) &= \mathbb{M},\\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{M}) &= \{m\in\mathbb{M}\,|\,nm=0\},\\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{Z}) &= 0,\\ \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n) &= \mathbb{Z}_{\langle m,n\rangle}. \end{aligned} \tag{A13}$$

We will use abbreviated Ext and Hom to denote $\text{Ext}_{\mathbb{Z}}^1$ and $\text{Hom}_{\mathbb{Z}}$.

APPENDIX B: POINCARÉ DUALITY

Poincaré duality relates $H^k(M^d,R)$ and $H_{d-k}(M^d,R)$. We note that for a closed connected d-dimensional space M^d, $H_0(M^d;\mathbb{Z})=\mathbb{Z}$, $H_d(M^d;\mathbb{Z})=\mathbb{Z}$ if M^d is orientable, and $H_d(M^d;\mathbb{Z})=0$ if M^d is nonorientable. Similarly, $H^0(M^d;\mathbb{Z})=\mathbb{Z}$, $H^d(M^d;\mathbb{Z})=\mathbb{Z}$ if M^d is orientable, and $H^d(M^d;\mathbb{Z})=\mathbb{Z}_2$ if M^d is nonorientable.

Poincaré duality: If M is a closed R-orientable n-dimensional manifold with fundamental class $[M]\in H^n(M,R)$ (here R is a ring), then the map $D: H^k(M;R)\to H_{n-k}(M;R)$ defined by $D(\alpha)=[M]\cap\alpha$ is an isomorphism for all k.

The cup product pairing between $H^k(M^d,R)$ and $H^{d-k}(M^d,R)$ is nonsingular for closed R-orientable manifolds when R is a field, or when $R=\mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out. This implies that the free part of $H^k(M^d;\mathbb{Z})$ and $H^{d-k}(M^d;\mathbb{Z})$ has the same dimension.

APPENDIX C: THE FACTOR $\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2}$

To calculate the factor $\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2}$ we first use Eq. (A11) to show

$$\begin{aligned} &H^1(M^4;\mathbb{Z}_n)\\ &\quad=\text{Hom}(H_1(M^4;\mathbb{Z});\mathbb{Z}_n)\oplus\text{Ext}(H_0(M^4;\mathbb{Z}),Z_n) \end{aligned} \tag{C1}$$

$$= \text{Hom}(\text{f}H_1(M^4;\mathbb{Z});\mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_1(M^4;\mathbb{Z});\mathbb{Z}_n)$$
$$= \mathbb{Z}_n^{\oplus b_1} \oplus \text{Hom}(\text{t}H_1(M^4;\mathbb{Z});\mathbb{Z}_n), \quad \text{(C2)}$$

and

$$\begin{aligned} &H^2(M^4;\mathbb{Z}_n)\\ &= \text{Hom}(H_2(M^4;\mathbb{Z});\mathbb{Z}_n) \oplus \text{Ext}(H_1(M^4;\mathbb{Z}),Z_n)\\ &= \text{Hom}(\text{f}H_2(M^4;\mathbb{Z});\mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_2(M^4;\mathbb{Z});\mathbb{Z}_n)\\ &\quad \oplus \text{Ext}(\text{t}H_1(M^4;\mathbb{Z}),Z_n),\\ &= \mathbb{Z}_n^{\oplus b_2} \oplus \text{Hom}(\text{t}H_2(M^4;\mathbb{Z});\mathbb{Z}_n) \oplus \text{Hom}(\text{t}H_1(M^4;\mathbb{Z}),Z_n), \end{aligned} \quad \text{(C3)}$$

where “f” and “t” indicate the free and torsion parts of a discrete Abelian group and b_n is the dimension of $\text{f}H_n(M^4;\mathbb{Z})$ (i.e., the nth Betti number). Using $H^0(M^4;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus b_0}$, we find that

$$\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2} = n^{2b_0+b_2-2b_1}\frac{|\text{Hom}(\text{t}H_2(M^4;\mathbb{Z});\mathbb{Z}_n)|}{|\text{Hom}(\text{t}H_1(M^4;\mathbb{Z});\mathbb{Z}_n)|}. \quad \text{(C4)}$$

We note that, according to Eq. (A11)

$$H^2(M^d;\mathbb{Z}) = \text{Hom}(H_2(M^d;\mathbb{Z}),\mathbb{Z}) \oplus \text{Ext}(H_1(M^d;\mathbb{Z}),\mathbb{Z}). \quad \text{(C5)}$$

Since $\text{Hom}(\mathbb{Z}_n,\mathbb{Z}) = 0$, $\text{Ext}(\mathbb{Z}_n,\mathbb{Z}) = \mathbb{Z}_n$, and $\text{Ext}(\mathbb{Z},\mathbb{Z}) = 0$, we see that $\text{t}H^2(M^d;\mathbb{Z}) = \text{t}H_1(M^d;\mathbb{Z})$. We get

$$\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2} = n^{2b_0+b_2-2b_1}\frac{|\text{Hom}(\text{t}H_2(M^4;\mathbb{Z});\mathbb{Z}_n)|}{|\text{Hom}(\text{t}H^2(M^4;\mathbb{Z});\mathbb{Z}_n)|}. \quad \text{(C6)}$$

For four-dimensional closed orientable manifolds $b_1 = b_3$, $b_0 = b_4$, and $\chi(M^4) = \sum_{n=0}^4 (-)^n b_n$ is the Euler number of M^4. Using Poincaré duality $H^2(M^4;\mathbb{Z}) = H_2(M^4;\mathbb{Z})$, we can show that

$$\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2} = n^{\chi(M^4)}. \quad \text{(C7)}$$

When $n = 2$ we have a Poincaré duality $H^k(M^d;\mathbb{Z}_2) = H^{d-k}(M^d;\mathbb{Z}_2)$ for any closed manifold M^d regardless if M^d is orientable or not (since M^d is always Z_2 orientable). Thus,

$$\frac{|H^0(M^4;\mathbb{Z}_2)|^2|H^2(M^4;\mathbb{Z}_2)|}{|H^1(M^4;\mathbb{Z}_2)|^2} = \frac{|H^0(M^4;\mathbb{Z}_2)||H^2(M^4;\mathbb{Z}_2)||H^4(M^4;\mathbb{Z}_2)|}{|H^1(M^4;\mathbb{Z}_2)||H^3(M^4;\mathbb{Z}_2)|}. \quad \text{(C8)}$$

According to Eq. (A11),

$$\begin{aligned} H^k(M^d;\mathbb{Z}_2) &= \text{Hom}(H_k(M^d;\mathbb{Z}),\mathbb{Z}_2)\oplus\text{Ext}(H_{k-1}(M^d;\mathbb{Z}),\mathbb{Z}_2)\\ &= \mathbb{Z}_2^{\oplus b_k} \oplus \text{Hom}(\text{t}H_k(M^d;\mathbb{Z}),\mathbb{Z}_2)\\ &\quad \oplus \text{Hom}(\text{t}H_{k-1}(M^d;\mathbb{Z}),\mathbb{Z}_2). \end{aligned} \quad \text{(C9)}$$

This allows us to show

$$\frac{|H^0(M^4;\mathbb{Z}_2)|^2|H^2(M^4;\mathbb{Z}_2)|}{|H^1(M^4;\mathbb{Z}_2)|^2} = 2^{\chi(M^4)}, \quad \text{(C10)}$$

where we have used the fact that $\text{t}H_4(M^4;\mathbb{Z}) = 0$ for both orientable and nonorientable closed manifolds. On the other hand, the factor $\frac{|H^0(M^4;\mathbb{Z}_n)|^2|H^2(M^4;\mathbb{Z}_n)|}{|H^1(M^4;\mathbb{Z}_n)|^2}$ is in general not of the form $\rho^{\chi(M^4)}$ for nonorientable manifolds when $n > 2$.

APPENDIX D: RELATIONS BETWEEN COCYCLES AND STIEFEL-WHITNEY CLASSES ON A CLOSED MANIFOLD

The cocycles and the Stiefel-Whitney classes on a closed manifold satisfy many relations. In this appendix, we will show how to generate those relations.

1. Introduction to Stiefel-Whitney classes

The Stiefel-Whitney classes $\text{w}_i \in H^i(M^d;\mathbb{Z}_2)$ are defined for an $O(n)$ vector bundle on a d-dimensional space with $n \to \infty$. If the $O(\infty)$ vector bundle on d-dimensional space M^d happens to be the tangent bundle of M^d direct summed with a trivial ∞-dimensional vector bundle, then the corresponding Stiefel-Whitney classes are referred as the Stiefel-Whitney classes of the manifold M^d.

The Stiefel-Whitney classes of manifold behave well under the connected sum of manifolds. Let $\text{w}(M)$ be the total Stiefel-Whitney class of a manifold M. If we know $\text{w}(M)$ and $\text{w}(N)$, then we can obtain $\text{w}(M\#N)$:

$$\text{w}(M\#N) = \text{w}(M) + \text{w}(N) - 1. \quad \text{(D1)}$$

Under the product of manifolds, we have

$$\text{w}(M \times N) = \text{w}(M)\text{w}(N). \quad \text{(D2)}$$

The Stiefel-Whitney numbers are nonoriented cobordism invariant. All the Stiefel-Whitney numbers of a smooth compact manifold vanish iff the manifold is the boundary of some smooth compact manifold. Here, the manifold can be nonorientable.

The Stiefel-Whitney numbers and Pontryagin numbers are oriented cobordism invariant. All the Stiefel-Whitney numbers and Pontryagin numbers of a smooth compact orientable manifold vanish iff the manifold is the boundary of some smooth compact orientable manifold.

2. Relations between Stiefel-Whitney classes of the tangent bundle

For generic $O(\infty)$ vector bundle, the Stiefel-Whitney classes are all independent. However, the Stiefel-Whitney classes for a manifold (i.e., for the tangent bundle) are not independent and satisfy many relations.

To obtain those relations, we note that, for any $O(\infty)$ vector bundle, the total Stiefel-Whitney class $\text{w} = 1 + \text{w}_1 + \text{w}_2 + \cdots$ is related to the total Wu class $u = 1 + u_1 + u_2 + \cdots$ through the total Steenrod square [92]:

$$\text{w} = \text{Sq}(u), \quad \text{Sq} = 1 + \text{Sq}^1 + \text{Sq}^2 + \cdots. \quad \text{(D3)}$$

Therefore, $\mathrm{w}_n = \sum_{i=0}^{n} \mathrm{Sq}^i(u_{n-i})$. The Steenrod squares have the following properties:

$$\mathrm{Sq}^i(x_j) = 0, \quad i > j, \quad \mathrm{Sq}^j(x_j) = x_j x_j, \quad \mathrm{Sq}^0 = 1, \tag{D4}$$

for any $x_j \in H^j(M^d;\mathbb{Z}_2)$. Thus

$$u_n = \mathrm{w}_n + \sum_{i=1,2i\leqslant n} \mathrm{Sq}^i(u_{n-i}). \tag{D5}$$

This allows us to compute u_n iteratively, using Wu formula

$$\mathrm{Sq}^i(\mathrm{w}_j) = 0,\ i > j, \quad \mathrm{Sq}^i(\mathrm{w}_i) = \mathrm{w}_i\mathrm{w}_i,$$
$$\mathrm{Sq}^i(\mathrm{w}_j) = \mathrm{w}_i\mathrm{w}_j + \sum_{k=1}^{i} \frac{(j-i-1+k)!}{(j-i-1)!k!}\mathrm{w}_{i-k}\mathrm{w}_{j+k},\ i < j,$$
$$\mathrm{Sq}^1(\mathrm{w}_j) = \mathrm{w}_1\mathrm{w}_j + (j-1)\mathrm{w}_{j+1}, \tag{D6}$$

and the Steenrod relation

$$\mathrm{Sq}^n(xy) = \sum_{i=0}^{n} \mathrm{Sq}^i(x)\mathrm{Sq}^{n-i}(y). \tag{D7}$$

We find

$$u_0 = 1, \quad u_1 = \mathrm{w}_1, \quad u_2 = \mathrm{w}_1^2 + \mathrm{w}_2,$$
$$u_3 = \mathrm{w}_1\mathrm{w}_2, \quad u_4 = \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4,$$
$$u_5 = \mathrm{w}_1^3\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_4,$$
$$u_6 = \mathrm{w}_1^2\mathrm{w}_2^2 + \mathrm{w}_1^3\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_4 + \mathrm{w}_2\mathrm{w}_4,$$
$$u_7 = \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_3^2 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_4,$$
$$u_8 = \mathrm{w}_1^8 + \mathrm{w}_2^4 + \mathrm{w}_1^2\mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_4 + \mathrm{w}_1\mathrm{w}_3\mathrm{w}_4 + \mathrm{w}_4^2 + \mathrm{w}_1^3\mathrm{w}_5 + \mathrm{w}_3\mathrm{w}_5 + \mathrm{w}_1^2\mathrm{w}_6 + \mathrm{w}_2\mathrm{w}_6 + \mathrm{w}_1\mathrm{w}_7 + \mathrm{w}_8. \tag{D8}$$

If the $O(\infty)$ vector bundle on d-dimensional space M^d happens to be the tangent bundle of M^d, then the corresponding Wu class and the Steenrod square satisfy

$$\mathrm{Sq}^{d-j}(x_j) = u_{d-j}x_j \text{ for any } x_j \in H^j(M^d;\mathbb{Z}_2). \tag{D9}$$

We can generate many relations for cocycles and Stiefel-Whitney classes on a manifold using the above result:

(1) If we choose x_j to be a combination of Stiefel-Whitney classes, the above will generate many relations between Stiefel-Whitney classes.

(2) If we choose x_j to be a combination of Stiefel-Whitney classes and cocycles, the above will generate many relations between Stiefel-Whitney classes and cocycles.

(3) Since $\mathrm{Sq}^i(x_j) = 0$ if $i > j$, therefore, $u_i x_{d-i} = 0$ for any $x_{d-i} \in H^{d-i}(M^d;\mathbb{Z}_2)$ if $i > d-i$. Since $\mathbb{Z}_2$ is a field and according to the Poincaré duality, this implies that $u_i = 0$ for $2i > d$.

(4) $\mathrm{Sq}^n \dots \mathrm{Sq}^m(u_i) = 0$ if $2i > d$. This also gives us relations among Stiefel-Whitney classes.

3. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 1-cocycle in three dimensions

On a three-dimensional manifold, we can find many relations between Stiefel-Whitney classes: (1) $u_2 = \mathrm{w}_1^2 + \mathrm{w}_2 = 0$. (2) $u_3 = \mathrm{w}_1\mathrm{w}_2 = 0$. (3) $\mathrm{Sq}^1(u_2) = 0$. Using $\mathrm{Sq}^1(\mathrm{w}_i) = \mathrm{w}_1\mathrm{w}_i + (i+1)\mathrm{w}_{i+1}$, we find that $\mathrm{Sq}^1(\mathrm{w}_1^2 + \mathrm{w}_2) = \mathrm{Sq}^1(\mathrm{w}_1)\mathrm{w}_1 + \mathrm{w}_1\mathrm{Sq}^1(\mathrm{w}_1) + \mathrm{Sq}^1(\mathrm{w}_2) = \mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3 = 0$. This gives us three relations

$$\mathrm{w}_1^2 = \mathrm{w}_2,\ \mathrm{w}_1\mathrm{w}_2 = \mathrm{w}_3 = 0. \tag{D10}$$

Let $a^{\mathbb{Z}_2}$ be a $\mathbb{Z}_2$-valued 1-cocycle. We can also find a relation between the Stiefel-Whitney classes and $a^{\mathbb{Z}_2}$:

$$\mathrm{w}_1(a^{\mathbb{Z}_2})^2 = \mathrm{Sq}^1\big[(a^{\mathbb{Z}_2})^2\big] = 2(a^{\mathbb{Z}_2})^3 = 0. \tag{D11}$$

There are six possible 3-cocycles that can be constructed from the Stiefel-Whitney classes and the 1-cocycle $a^{\mathbb{Z}_2}$:

$$(\mathrm{w}_1)^3, \quad \mathrm{w}_1\mathrm{w}_2, \quad \mathrm{w}_3,$$
$$(a^{\mathbb{Z}_2})^3, \quad \mathrm{w}_1(a^{\mathbb{Z}_2})^2, \quad \mathrm{w}_1^2 a^{\mathbb{Z}_2}. \tag{D12}$$

From the above relations, we see that only two of them are nonzero:

$$(a^{\mathbb{Z}_2})^3, \quad \mathrm{w}_1^2 a^{\mathbb{Z}_2}. \tag{D13}$$

4. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 1-cocycle in four dimensions

The relations between the Stiefel-Whitney classes for four-dimensional manifold can be listed: (1) $u_3 = \mathrm{w}_1\mathrm{w}_2 = 0$. (2) $u_4 = \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4 = 0$. (3) $\mathrm{Sq}^1(u_3) = 0$, which implies $\mathrm{Sq}^1(\mathrm{w}_1\mathrm{w}_2) = \mathrm{Sq}^1(\mathrm{w}_1)\mathrm{w}_2 + \mathrm{w}_1\mathrm{Sq}^1(\mathrm{w}_2) = \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_3 = \mathrm{w}_1\mathrm{w}_3 = 0$, which can be summarized as

$$\mathrm{w}_1\mathrm{w}_2 = 0,\ \mathrm{w}_1\mathrm{w}_3 = 0,\ \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_4 = 0. \tag{D14}$$

We also have many relations between the Stiefel-Whitney classes and $a^{\mathbb{Z}_2}$: (1) $\mathrm{Sq}^1[(a^{\mathbb{Z}_2})^3] = (a^{\mathbb{Z}_2})^4 = \mathrm{w}_1(a^{\mathbb{Z}_2})^3$. (2) $\mathrm{Sq}^1(\mathrm{w}_1^2 a^{\mathbb{Z}_2}) = \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 = \mathrm{w}_1^3 a^{\mathbb{Z}_2}$. (3) $\mathrm{Sq}^1(\mathrm{w}_2 a^{\mathbb{Z}_2}) = (\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)a^{\mathbb{Z}_2} + \mathrm{w}_2(a^{\mathbb{Z}_2})^2 = \mathrm{w}_1\mathrm{w}_2 a^{\mathbb{Z}_2}$, which implies that $\mathrm{w}_3 a^{\mathbb{Z}_2} = \mathrm{w}_2(a^{\mathbb{Z}_2})^2$. (4) $\mathrm{Sq}^2[(a^{\mathbb{Z}_2})^2] = (a^{\mathbb{Z}_2})^4 = (\mathrm{w}_1^2 + \mathrm{w}_2)(a^{\mathbb{Z}_2})^2$. (5) $\mathrm{Sq}^2(\mathrm{w}_1 a^{\mathbb{Z}_2}) = \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 = (\mathrm{w}_1^2 + \mathrm{w}_2)\mathrm{w}_1 a^{\mathbb{Z}_2} = \mathrm{w}_1^3 a^{\mathbb{Z}_2}$, which is the same as (2). To summarize,

$$\mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 = \mathrm{w}_1^3 a^{\mathbb{Z}_2}, \quad (a^{\mathbb{Z}_2})^4 = \mathrm{w}_1(a^{\mathbb{Z}_2})^3,$$
$$\mathrm{w}_2(a^{\mathbb{Z}_2})^2 = \mathrm{w}_3 a^{\mathbb{Z}_2}, \quad (a^{\mathbb{Z}_2})^4 + \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 + \mathrm{w}_2(a^{\mathbb{Z}_2})^2 = 0. \tag{D15}$$

There are nine 4-cocycles that can be constructed from Stiefel-Whitney classes and a 1-cocycle $a^{\mathbb{Z}_2}$:

$$(a^{\mathbb{Z}_2})^4, \quad \mathrm{w}_1(a^{\mathbb{Z}_2})^3, \quad \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2,$$
$$\mathrm{w}_2(a^{\mathbb{Z}_2})^2, \quad \mathrm{w}_1^3 a^{\mathbb{Z}_2}, \quad \mathrm{w}_3 a^{\mathbb{Z}_2},$$
$$\mathrm{w}_1^4, \quad \mathrm{w}_2^2, \quad \mathrm{w}_4. \tag{D16}$$

Only four of them are independent:

$$\mathrm{w}_1^4,\ \mathrm{w}_2^2,\ \mathrm{w}_3 a^{\mathbb{Z}_2},\ \mathrm{w}_1^3 a^{\mathbb{Z}_2}. \tag{D17}$$

5. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 2-cocycle in four dimensions

There are two relations between the Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 2-cocycle $b^{\mathbb{Z}_2}$: (1) $\mathrm{Sq}^1(\mathrm{w}_1 b^{\mathbb{Z}_2}) =$

$w_1^2 b^{\mathbb{Z}_2} + w_1 \mathcal{B}_2 b^{\mathbb{Z}_2} = w_1^2 b^{\mathbb{Z}_2}$, which implies $w_1 \mathcal{B}_2 b^{\mathbb{Z}_2} = 0$. (2) $\text{Sq}^2(b^{\mathbb{Z}_2}) = (b^{\mathbb{Z}_2})^2 = (w_1^2 + w_2) b^{\mathbb{Z}_2}$. There are seven 4-cocycles that can be constructed from Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 2-cocycle $b^{\mathbb{Z}_2}$:

$$(b^{\mathbb{Z}_2})^2, \quad w_1 \mathcal{B}_2 b^{\mathbb{Z}_2}, \quad w_1^2 b^{\mathbb{Z}_2}, \quad w_2 b^{\mathbb{Z}_2}, \quad w_1^4, \quad w_2^2, \quad w_4. \tag{D18}$$

So, the following four 4-cocycles are independent:

$$w_1^4,\ w_2^2,\ w_2 b^{\mathbb{Z}_2},\ w_1^2 b^{\mathbb{Z}_2}. \tag{D19}$$

APPENDIX E: SPIN AND PIN STRUCTURES

Stiefel-Whitney classes can determine when a manifold can have a spin structure. The spin structure is defined only for orientable manifolds. The tangent bundle for an orientable manifold M^d is a SO(d) bundle. The group SO(d) has a central extension to the group Spin(d). Note that $\pi_1[\text{SO}(d)] = \mathbb{Z}_2$. The group Spin($d$) is the double covering of the group SO(d). A spin structure on M^d is a Spin(d) bundle, such that under the group reduction Spin(d) $\to$ SO(d), the Spin(d) bundle reduces to the SO(d) bundle. Some manifolds cannot have such a lifting from SO(d) tangent bundle to the Spin(d) spinor bundle. The manifolds that have such a lifting are called spin manifolds. A manifold is a spin manifold iff its first and second Stiefel-Whitney class vanishes $w_1 = w_2 = 0$.

For a nonorientable manifold N^d, the tangent bundle is a O(d) bundle. The nonconnected group O(d) has two nontrivial central extensions (double covers) by Z_2 with different group structures, denoted by $\text{Pin}^+(d)$ and $\text{Pin}^-(d)$. So the O(d) tangent bundle has two types of lifting to a Pin^+ bundle and a Pin^- bundle, which are called Pin^+ structure and Pin^- structure, respectively. The manifolds with such liftings are called Pin^+ manifolds or Pin^- manifolds. We see that the concept of $\text{Pin}^\pm$ structure applies to both orientable and nonorientable manifolds. A manifold is a Pin^+ manifold iff $w_2 = 0$. A manifold is a Pin^- manifold iff $w_2 + w_1^2 = 0$. If a manifold N^d does admit Pin^+ or Pin^- structures, then the set of isomorphism classes of Pin^+ structures (or Pin^- structures) can be labeled by elements in $H^1(N^d;\mathbb{Z}_2)$. For example, $\mathbb{R}P^4$ admits two Pin^+ structures and no Pin^- structures since $w_2(\mathbb{R}P^4) = 0$ and $w_2(\mathbb{R}P^4) + w_1^2(\mathbb{R}P^4) \neq 0$.

From Eq. (D1), we see that $M\#N$ is pin^+ iff both M and N are pin^+. Similarly, $M\#N$ is pin^- iff both M and N are pin^-.

APPENDIX F: COHOMOLOGY RINGS

In this Appendix, we list some cohomology rings $H^*(M^4;\mathbb{Z}_n)$ that are used in the main text of the paper. First, let us list a few theorems:

The cohomology ring of product space (see [93], p. 216):

Let X and Y be arbitrary spaces. Assume $H^k(Y;R)$ is a free and finitely generated R module for all k. Then,

$$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R) \tag{F1}$$

is an isomorphism of graded rings. (A free R module is a module that has a basis or, equivalently, one that is isomorphic to a direct sum of copies of the ring R.)

The cohomology of connected sum:

$$H^k(M^d\#N^d,\mathbb{M}) = H^k(M^d,\mathbb{M}) \oplus H^k(N^d,\mathbb{M}),\ 0 < k < d. \tag{F2}$$

The cup product of connected sum:

$$H^k(M^d\#N^d,\mathbb{M}) \times H^l(M^d\#N^d,\mathbb{M}) \xrightarrow{\cup} H^{k+l}(M^d\#N^d,\mathbb{M}),$$
$$0 < k,l,k+l < d:$$
$$(a,a') \cup (b,b') = (a \cup b, a' \cup b'), \tag{F3}$$

where $a \in H^k(M^d,\mathbb{M})$, $b \in H^l(M^d,\mathbb{M})$, $a' \in H^k(N^d,\mathbb{M})$, and $b' \in H^l(N^d,\mathbb{M})$. The above also works for $k + l = d$ we identify

$$(\alpha v_{M^d}, \beta v_{N^d}) \sim (\alpha + \beta) v_{M^d\#N^d}, \tag{F4}$$

where v_{M^d}, v_{N^d}, and $v_{M^d\#N^d}$ are the generators in $H^d(M^d,\mathbb{M})$, $H^d(N^d,\mathbb{M})$, and $H^d(M^d\#N^d,\mathbb{M})$.

1. $H^*(T^4,\mathbb{Z}_n)$

For $M^4 = S^1 \times S^1 \times S^1 \times S^1 = T^4$, we have

$$H^*(T^4,\mathbb{Z}_n) = \frac{\mathbb{Z}_n[a_1,a_2,a_3,a_4]}{(a_1^2,a_2^2,a_3^2,a_4^2)}, \tag{F5}$$

where $a_i \in H^1(T^4,\mathbb{Z}_n)$ generate the ring. The Bockstein homomorphism all vanishes:

$$\mathcal{B}_n a_i = 0, \quad i = 1,2,3,4. \tag{F6}$$

2. $H^*(T^2 \times S^2,\mathbb{Z}_n)$

For $M^4 = T^2 \times S^2$ (where $T^2 = S^1 \times S^1$), we have

$$H^*(T^2 \times S^2,\mathbb{Z}_n) = \frac{\mathbb{Z}_n[a_1,a_2,b]}{(a_1^2,a_2^2,b^2)}, \tag{F7}$$

where $a_i \in H^1(T^2 \times S^2,\mathbb{Z}_n)$ and $b \in H^2(T^2 \times S^2,\mathbb{Z}_n)$ generate the ring. We also have

$$\mathcal{B}_n a_i = \mathcal{B}_n b = 0, \quad i = 1,2. \tag{F8}$$

3. $H^*(L^2(p);\mathbb{Z}_n)$

$L^2(p)$ space is a two-dimensional sphere with p holes removed and with the boundary of the p holes identified [see Fig. 9(a)]. It has a CW-complex decomposition as shown in Fig. 9(b). Since $\partial S = pL$, $\partial L = 0$, we can compute explicitly that

$$H_0(L^2(p),\mathbb{Z}) = \mathbb{Z}, \quad H_1(L^2(p),\mathbb{Z}) = \mathbb{Z}_p,$$
$$H_2(L^2(p),\mathbb{Z}) = 0, \tag{F9}$$

$$H^0(L^2(p),\mathbb{Z}) = \mathbb{Z}, \quad H^1(L^2(p),\mathbb{Z}) = 0,$$
$$H^2(L^2(p),\mathbb{Z}) = \mathbb{Z}_p, \tag{F10}$$

and

$$H_0(L^2(p),\mathbb{Z}_n) = \mathbb{Z}, \quad H_1(L^2(p),\mathbb{Z}_n) = \mathbb{Z}_{\langle n,p\rangle} = \{L\},$$
$$H_2(L^2(p),\mathbb{Z}_n) = \mathbb{Z}_{\langle n,p\rangle} = \left\{\frac{n}{\langle n,p\rangle} S\right\}, \tag{F11}$$

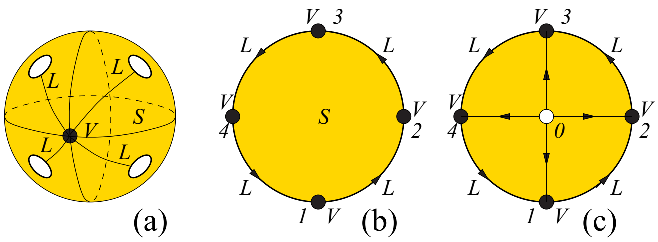

FIG. 9. (a) $L^2(p)$ (with $p=4$) space. (b) $L^2(p)$ can be described by a CW complex with a 0-cell V, a 1-cell L, and a 2-cell S. The filled dots are identified. The links (12), (23), (34), and (41) are identified.. The boundary of the 2-cell S is p copies of the 1-cell L: $\partial S = pL$ and L is a cycle $\partial L = 0$. (c) $L^2(p)$ can be described by a singular complex with two vertices (0) and V, $p+1$ links $(01),\dots,(0p)$ and L, p triangles $(012),\dots,(0,p-1,p)$. The sum of the p triangles gives us S, the whole $L^2(p)$ space.

$$H^0(L^2(p),\mathbb{Z}_n) = \mathbb{Z},\quad H^1(L^2(p),\mathbb{Z}_n) = \mathbb{Z}_{\langle n,p\rangle} = \{a\},$$
$$H^2(L^2(p),\mathbb{Z}_n) = \mathbb{Z}_{\langle n,p\rangle} = \{b\}, \tag{F12}$$

where we have listed the generators of $H_*(L^2(p),\mathbb{Z}_n)$ and $H^*(L^2(p),\mathbb{Z}_n)$.

Using the CW complex of $L^2(p)$, we can compute the Bockstein homomorphism for $\mathbb{Z}_n$ coefficient. Let $\tilde a \in Z^1(L^2(p);\mathbb{Z})$ to be a generator of $H^1(L^2(p);\mathbb{Z}_p)$, and $\tilde b \in C^2(L^2(p);\mathbb{Z})$ to be a generator of $H^2(L^2(p);\mathbb{Z})$:

$$\langle \tilde a, L\rangle = 1,\quad \langle \tilde b, S\rangle = 1. \tag{F13}$$

We see that $p = \langle p\tilde a, L\rangle = \langle \tilde a, pL\rangle = \langle \tilde a, \partial S\rangle = \langle d\tilde a, S\rangle$. Thus, $d\tilde a = 0 \mod p$, confirming that $\tilde a$ is a cocycle in $H^1(L^3(p,q);\mathbb{Z}_p)$, but $\tilde a$ is not a cocycle in $H^1(L^3(p,q);\mathbb{Z})$. From the above calculation, we also see that $d\tilde a = p\tilde b$ or $\frac{1}{p} d\tilde a = \tilde b$. Therefore, $\frac{1}{n}\frac{n}{\langle p,n\rangle} d\tilde a = \frac{p}{\langle p,n\rangle}\tilde b$ or $\frac{1}{n} d(\frac{n}{\langle p,n\rangle}\tilde a) = \mathcal{B}_n(\frac{n}{\langle p,n\rangle}\tilde a) = \frac{p}{\langle p,n\rangle}\tilde b$. We note that $\frac{n}{\langle p,n\rangle}\tilde a$ is an integer-valued cochain that satisfies $d\frac{n}{\langle p,n\rangle}\tilde a = 0 \mod n$. Thus, $a = \frac{n}{\langle p,n\rangle}\tilde a$ is a cocycle and a generator in $H^1(L^3(p,q);\mathbb{Z}_n)$. Also, $b=\tilde b$ is a cocycle and a generator in $H^2(L^3(p,q);\mathbb{Z}_n)$. The Bockstein homomorphism can be written as

$$\mathcal{B}_n a = \frac{p}{\langle p,n\rangle} b. \tag{F14}$$

We can calculate the cohomology ring $H^*(L^2(p);\mathbb{Z}_n)$ by decomposing $L^2(p)$ into a singular complex characterized by the vertices $0,1,2,\dots,p$ [see Fig. 9(c)]. Note that $1,2,\dots,p$ corresponds to the same vertex. First a,b [the generators of $H^1(L^2(p);\mathbb{Z}_n)$ and $H^2(L^2(p);\mathbb{Z}_n)$] are given by

$$\langle a,(m,m+1)\rangle = \frac{n}{\langle p,n\rangle},$$
$$\langle a,(0m)\rangle = \frac{(m-1)n}{\langle p,n\rangle},$$
$$\langle b,(012)\rangle = \langle b,(00'2)\rangle = 1,\quad \langle b,\text{others}\rangle = 0,$$
$$m = 1,\dots,p. \tag{F15}$$

We see that

$$\langle a,L\rangle = \frac{n}{\langle p,n\rangle},\quad \left\langle b,\frac{n}{\langle p,n\rangle}S\right\rangle = \frac{n}{\langle p,n\rangle}, \tag{F16}$$

where $\frac{n}{\langle p,n\rangle}S$ is a 2-cycle $\partial\frac{n}{\langle p,n\rangle}S = \frac{np}{\langle p,n\rangle}L = 0 \mod n$.

Now, we can calculate the cup product

$$\langle a^2,(0,m,m+1)\rangle = \langle a,(0,m)\rangle\langle a,(m,m+1)\rangle$$
$$= \frac{(m-1)n}{\langle p,n\rangle}\frac{n}{\langle p,n\rangle} \tag{F17}$$

or

$$\left\langle a^2,\frac{n}{\langle p,n\rangle}S\right\rangle = \sum_{m=1}^{p}\frac{(m-1)n}{\langle p,n\rangle}\frac{n^2}{\langle p,n\rangle^2} = \frac{n^3 p(p-1)}{2\langle p,n\rangle^3}$$
$$\overset{n}{=} \begin{cases}\frac{n}{2} & \text{if } p_2>1,\ \frac{n}{2^{p_2}} = \text{odd},\\ 0 & \text{otherwise},\end{cases} \tag{F18}$$

where p_2 is the number of prime factor 2 in p. The above implies that

$$a^2 = \frac{n^2 p(p-1)}{2\langle p,n\rangle^2} b$$
$$= \begin{cases}\frac{\langle n,p\rangle}{2} b & \text{if } p_2>1,\ \frac{n}{2^{p_2}} = \text{odd},\\ 0 & \text{otherwise}.\end{cases} \tag{F19}$$

The ring $H^*(L^2(p);\mathbb{Z}_n)$ is determined by Eqs. (F12) and (F19).

4. $H^*[L^3(p,q)\times S^1,\mathbb{Z}_n]$

We know that S^3 can be described by two complex numbers z_1,z_2 satisfying $|z_1|^2+|z_2|^2=1$. Let p and q be coprime integers. We can see that the action $(z_1,z_2)\to(e^{i\frac{2\pi}{p}}z_1, e^{i\frac{2\pi q}{p}}z_2)$ is a free action on S^3. Quotienting out such a free action, the resulting space is the lens space $L^3(p,q)$. We see that $L^3(2,1) = \mathbb{R}P^3$. $L^3(p,q_1)$ and $L^3(p,q_2)$ are homotopically equivalent if and only if $q_1q_2 = \pm m^2 \mod p$ for an integer m.

$L^3(p,q)$ is described by the CW complex in Fig. 10 for $(p,q)=(4,1)$, which has a 0-cell V (the four vertices 1,2,3,4 are identified and correspond to V), a 1-cell L [the four links

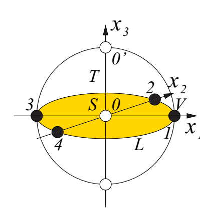

FIG. 10. The S^3 is parametrized by $(x_1,x_2,x_3) = \frac{(\text{Re}z_1,\text{Im}z_1,\text{Re}z_2)}{1+\text{Im}z_2}$ which is the whole $\mathbb{R}^3$. The open dots are the points $(z_1,z_2) = (0,e^{i2\pi m/p})$, $m=0,\dots,p-1$. The shaded disk is B_0^2. The north and the south hemispheres are $B_{\pm1}^2$. The volume between B_0^2 and B_1^2 is the lens space $L^3(p,q)|_{(p,q)=(4,1)}$. The lens space $L^3(p,q)$ is described by a CW complex with a 0-cell V, a 1-cell L, a 2-cell S, and a 3-cell T. The filled dots are identified under the quotient map and correspond to the 0-cell V. The shaded disk S is the 2-cell. The boundary of the 2-cell S is p copies of the 1-cell L: $\partial S = pL$ and L is a cycle $\partial L = 0$. The 3-cell T is the half-ball above the shaded disk.

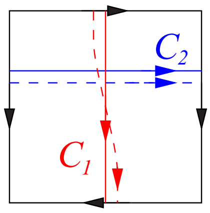

FIG. 11. The Klein bottle: the top and bottom boundaries are identified with a twist and the left and right boundaries are identified without twist. $H_1(K;\mathbb{Z}_2)$ is generated by C_1 and C_2 cycles.

(12),(23),(34),(41) are identified and correspond to L], a 2-cell S [which is the union of (012),(023),(034),(041)], and a 3-cell T [which is the union of (00′12),(00′23),(00′34),(00′41)]. To describe the lens space, let us first consider p points $(z_1,z_2)=(0,e^{i2\pi m/p})$, $m=0,\dots,p-1$ (which become one point after the quotient). The 2-cell B^2_m is formed by the points $(z_1,z_2)=\cos\theta(0,e^{i2\pi m/p})+\sin\theta(z_1,0)$, $|z_1|=1$ (where B^2_m and $B^2_{m'}$ are identified by the quotient map). The volume between B^2_m and B^2_{m+1} is the lens space $L^3(p,q)$ which is also the 3-cell T. The 0-cell is given by $(z_1,z_2)=(e^{i2\pi m/p},0)$, $m=0,\dots,p-1$ (which becomes one point after the quotient).

Since $\partial S=pL$, $\partial L=\partial T=0$, we see that

$$\begin{aligned} &H_0(L^3(p,q),\mathbb{Z})=\mathbb{Z},\quad H_1(L^3(p,q),\mathbb{Z})=\mathbb{Z}_p,\\ &H_2(L^3(p,q),\mathbb{Z})=0,\quad H_3(L^3(p,q),\mathbb{Z})=\mathbb{Z},\end{aligned}\tag{F20}$$

and, by Poincaré duality,

$$\begin{aligned} &H^0(L^3(p,q),\mathbb{Z})=\mathbb{Z},\quad H^1(L^3(p,q),\mathbb{Z})=0,\\ &H^2(L^3(p,q),\mathbb{Z})=\mathbb{Z}_p,\quad H^3(L^3(p,q),\mathbb{Z})=\mathbb{Z}.\end{aligned}\tag{F21}$$

Then, using the universal coefficient theorem (A11) and Eq. (A8), we find that

$$\begin{aligned} &H_0(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_n,\quad H_1(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_{\langle n,p\rangle},\\ &H_2(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_{\langle n,p\rangle},\quad H_3(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_n,\end{aligned}\tag{F22}$$

$$\begin{aligned} &H^0(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_n,\quad H^1(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_{\langle n,p\rangle},\\ &H^2(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_{\langle n,p\rangle},\quad H^3(L^3(p,q),\mathbb{Z}_n)=\mathbb{Z}_n.\end{aligned}\tag{F23}$$

$H_1(L^3(p,q),\mathbb{Z}_n)$ is generated by L and $H_2(L^3(p,q),\mathbb{Z}_n)$ is generated by $\frac{n}{\langle n,p\rangle}S$.

The cohomology rings $H^*(L^3(p,q);\mathbb{Z}_p)$ are given by (see [93], p. 251)

$$\begin{aligned} &H^*(L^3(p,q);\mathbb{Z}_p)\\ &\quad=\left\{m_0+m_1a+m_2b+m_3ab|a^2=\frac{p}{2}(\langle p,2\rangle-1)b\right\},\\ &H^*(L^3(p,q);\mathbb{Z})=\{m_0+m_2b+m_3c\}.\end{aligned}\tag{F24}$$

We also have $\mathcal{B}_pa=b$.

In the following, we will only consider $L^3(p,1)\equiv L^3(p)$. We like to calculate the cohomology ring $H^*(L^3(p);\mathbb{Z}_n)$ by decomposing the lens space $L^3(p)$ into a simplicial complex characterized by the vertices $0,0',1,2,\dots,p$ (see Fig. 10). Note that 0 and 0′ correspond to the same vertex and $1,2,\dots,p$ correspond to the same vertex. Also note that, for example, the 2-simplices (012) and (0′23) are identified. First, a,b [the generators of $H^1(L^3(p);\mathbb{Z}_n)$ and $H^2(L^3(p);\mathbb{Z}_n)$] are given by

$$\begin{aligned} &\langle a,(00')\rangle=\langle a,(m,m+1)\rangle=\frac{n}{\langle p,n\rangle},\\ &\langle a,(0m)\rangle=\frac{(m-1)n}{\langle p,n\rangle},\\ &\langle b,(012)\rangle=\langle b,(00'2)\rangle=1,\ \ \langle b,\text{others}\rangle=0,\\ &\qquad m=1,\dots,p.\end{aligned}\tag{F25}$$

We see that

$$\langle a,L\rangle=\frac{n}{\langle p,n\rangle},\quad \left\langle b,\frac{n}{\langle p,n\rangle}S\right\rangle=\frac{n}{\langle p,n\rangle},\tag{F26}$$

where $\frac{n}{\langle p,n\rangle}S$ is a 2-cycle $\partial\frac{n}{\langle p,n\rangle}S=\frac{np}{\langle p,n\rangle}L=0 \bmod n$.

Now, we can calculate the cup product

$$\begin{aligned} \langle a^2,(0,m,m+1)\rangle&=\langle a,(0,m)\rangle\langle a,(m,m+1)\rangle\\ &=\frac{(m-1)n}{\langle p,n\rangle}\frac{n}{\langle p,n\rangle}\end{aligned}\tag{F27}$$

or

$$\begin{aligned} \left\langle a^2,\frac{n}{\langle p,n\rangle}S\right\rangle&=\sum_{m=1}^{p}\frac{(m-1)n}{\langle p,n\rangle}\frac{n^2}{\langle p,n\rangle^2}=\frac{n^3p(p-1)}{2\langle p,n\rangle^3}\\ &\overset{n}{=}\begin{cases}\frac{n}{2} & \text{if } p_2>1,\ \frac{n}{2^{p_2}}=\text{odd},\\ 0 & \text{otherwise},\end{cases}\end{aligned}\tag{F28}$$

where p_2 is the number of prime factor 2 in p. The above implies that

$$\begin{aligned} a^2&=\frac{n^2p(p-1)}{2\langle p,n\rangle^2}b\\ &=\begin{cases}\frac{\langle n,p\rangle}{2}b & \text{if } p_2>1,\ \frac{n}{2^{p_2}}=\text{odd},\\ 0 & \text{otherwise}.\end{cases}\end{aligned}\tag{F29}$$

We also note that

$$\langle ab,T\rangle=-\langle a,(0'0)\rangle\langle b,(012)\rangle=\frac{n}{\langle p,n\rangle},\tag{F30}$$

which implies that

$$ab=\frac{n}{\langle p,n\rangle}c.\tag{F31}$$

Thus, the cohomology ring $H^*(L^3(p);\mathbb{Z}_n)$ is given by

$$\begin{aligned} &H^*(L^3(p);\mathbb{Z}_n)=\{\zeta+\alpha a+\beta b+\gamma c\},\\ &\quad\text{with}\ \ a^2=\begin{cases}\frac{\langle n,p\rangle}{2}b & \text{if } p_2>1,\ \frac{n}{2^{p_2}}=\text{odd},\\ 0 & \text{otherwise},\end{cases}\\ &\qquad\quad ab=\frac{n}{\langle p,n\rangle}c,\end{aligned}\tag{F32}$$

where $\zeta,\gamma\in\mathbb{Z}_n$ and $\alpha,\beta\in\mathbb{Z}_{\langle p,n\rangle}$. We also have $\mathcal{B}_na=\frac{p}{\langle p,n\rangle}b$.

Notice that

$$H^*(S^1;\mathbb{Z}_n)=\frac{\mathbb{Z}_n[a_1]}{(a_1^2)}\tag{F33}$$

is a free $\mathbb{Z}_n$ module. This allows us to compute the cohomology ring $H^*(S^1\times L^3(p);\mathbb{Z}_n)$.

5. $H^*(F^4;\mathbb{Z}_n)$

In order for the volume-independent partition function $Z^{\text{top}}(M^4)$ on an orientable space-time M^4 to be a topological invariant, we require the Euler number and the Pontryagin number of M^4 to vanish: $\chi(M^4)=P_1(M^4)=0$. We also like M^4 to be complicated enough so that its second Stiefel-Whitney class w_2 is nonzero. How to construct such an four-dimensional manifold?

First, let us introduce intersection form Q_{M^4}: $H^2(M^4;\mathbb{Z})\times H^2(M^4;\mathbb{Z})\to\mathbb{Z}$ defined by

$$Q_{M^4}(a,b)=\langle a\cup b,[M^4]\rangle=\int_{M^4}ab. \tag{F34}$$

The intersection form has the following properties:

(1) Under connected sum,

$$Q_{M^4\#N^4}=Q_{M^4}\oplus Q_{N^4}. \tag{F35}$$

(2) Poincaré duality implies that the intersection form Q_{M^4} is unimodular.

(3) If M^4 is spin, then $Q_{M^4}(a,a)=$ even for all $a\in H^2(M^4;\mathbb{Z})$. If M^4 is orientable and Q_{M^4} is even, then M^4 is spin.

(4) The signature of Q_{M^4} is one third of the Pontryagin number: $\sigma(M^4)=\frac{1}{3}P_1(M^4)$.

(5) A smooth compact spin 4-manifold has a signature which is a multiple of 16.

(6) A 4-manifold bounds a 5-manifold if and only if it has zero signature.

We know that $Q_{\mathbb{C}P^2}$ is 1×1 matrix: $Q_{\mathbb{C}P^2}=(1)$, while $Q_{\overline{\mathbb{C}P}^2}=(-1)$. Thus, $Q_{\mathbb{C}P^2\#\overline{\mathbb{C}P}^2}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. This means $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ is not spin and has a zero Pontryagin number.

The Euler number $\chi(M)$ has the following properties:

(1) $\chi(S^d)=1+(-)^d$.

(2) $\chi(\mathbb{R}P^d)=\frac{1+(-)^d}{2}$.

(3) $\chi(\mathbb{C}P^2)=\chi(\overline{\mathbb{C}P}^2)=3$.

(4) $\chi(M\times N)=\chi(M)\chi(N)$.

(5) $\chi(M^d\#N^d)=\chi(M^d)+\chi(N^d)-\chi(S^d)$.

Using the above result, we find that

$$F^4\equiv(S^1\times S^3)\#(S^1\times S^3)\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2 \tag{F36}$$

has

$$Q_{F^4}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \chi(F^4)=P_1(F^4)=0. \tag{F37}$$

We see that F^4 is not spin.

The cohomology classes for F^4 are

$$H^1(F^4;\mathbb{Z}_n)=\mathbb{Z}_n^{\oplus 2},\ \ H^2(F^4;\mathbb{Z}_n)=\mathbb{Z}_n^{\oplus 2},$$
$$H^3(F^4;\mathbb{Z}_n)=\mathbb{Z}_n^{\oplus 2},\ \ H^4(F^4;\mathbb{Z}_n)=\mathbb{Z}_n. \tag{F38}$$

Let a_1,a_2 be the generators of $H^1(F^4;\mathbb{Z}_n)$, b_1,b_2 the generators of $H^2(F^4;\mathbb{Z}_n)$, c_1,c_2 be the generators of $H^3(F^4;\mathbb{Z}_n)$, and v be the generator of $H^4(F^4;\mathbb{Z}_n)$:

$$H^*(F^4;\mathbb{Z}_n)=\{a_1,a_2,b_1,b_2,c_1,c_2,v\}. \tag{F39}$$

We find that the nonzero cup products are given by

$$b_1^2=-b_2^2=a_1c_1=a_2c_2=v. \tag{F40}$$

All other cup products vanish.

6. $H^*(\mathbb{R}P^d;\mathbb{Z}_2)$

Next, let us list some cohomology rings with $\mathbb{Z}_2$ coefficient for nonorientable spaces. The cohomology ring $H^*(\mathbb{R}P^d;\mathbb{Z}_2)$ is given by

$$H^*(\mathbb{R}P^d;\mathbb{Z}_2)=\frac{\mathbb{Z}_2[a]}{(a^{d+1})} \tag{F41}$$

with $a\in H^1(\mathbb{R}P^d;\mathbb{Z}_2)$. $\mathbb{R}P^d$ is nonorientable if $d=$ even. The total Stiefel-Whitney class for $\mathbb{R}P^d$ is given by

$$\mathrm{w}=(1+a)^{d+1} \tag{F42}$$

(see https://amathew.wordpress.com/2010/12/17/the-stiefel-whitney-classes-of-projective-space/). We see that for $\mathbb{R}P^4$, $\mathrm{w}_1=a$ and $\mathrm{w}_2=0$. Thus, $\mathbb{R}P^4$ is a pin$^+$ manifold, but not a pin$^-$ manifold.

7. $H^*(F^4_{\text{non}};\mathbb{Z}_2)$

We note that $\mathbb{R}P^4$ has an intersection form $Q_{\mathbb{R}P^4}=(1)$ (with $\mathbb{Z}_2$ field), $\sigma(\mathbb{R}P^4)=1$ mod 2, and $\chi(\mathbb{R}P^4)=1$. So,

$$F^4_{\text{non}}\equiv\mathbb{R}P^4\#\mathbb{C}P^2\#(S^1\times S^3) \tag{F43}$$

has $\sigma(F^4_{\text{non}})=0$ mod 2 and $\chi(F^4_{\text{non}})=0$.

The cohomology classes for F^4_{non} are

$$H^1(F^4_{\text{non}};\mathbb{Z}_2)=\mathbb{Z}_2^{\oplus 2},\ \ H^2(F^4_{\text{non}};\mathbb{Z}_2)=\mathbb{Z}_2^{\oplus 2},$$
$$H^3(F^4_{\text{non}};\mathbb{Z}_2)=\mathbb{Z}_2^{\oplus 2},\ \ H^4(F^4_{\text{non}};\mathbb{Z}_2)=\mathbb{Z}_2. \tag{F44}$$

Let $a^{\mathbb{R}P^4},a^{S^1\times S^3}$ be the generators of $H^1(F^4_{\text{non}};\mathbb{Z}_n)$, $(a^{\mathbb{R}P^4})^2,b^{\mathbb{C}P^2}$ of $H^2(F^4_{\text{non}};\mathbb{Z}_n)$, $(a^{\mathbb{R}P^4})^3,c^{S^1\times S^3}$ of $H^3(F^4_{\text{non}};\mathbb{Z}_n)$, and v the generator of $H^4(F^4_{\text{non}};\mathbb{Z}_n)$:

$$H^*(F^4_{\text{non}};\mathbb{Z}_n)=\{(a^{\mathbb{R}P^4})^{m=1,2,3},a^{S^1\times S^3},b^{\mathbb{C}P^2},c^{S^1\times S^3},v\}. \tag{F45}$$

We find that the nonzero cup products are given by

$$(a^{\mathbb{R}P^4})^4=(b^{\mathbb{C}P^2})^2=a^{S^1\times S^3}c^{S^1\times S^3}=v,$$
$$(a^{\mathbb{R}P^4})^2,\ \ (a^{\mathbb{R}P^4})^3. \tag{F46}$$

All other cup products vanish. The first Stiefel-Whitney class for F^4_{non} is given by $\mathrm{w}_1=a^{\mathbb{R}P^4}$. Since $\mathbb{R}P^4$, $\mathbb{C}P^2$, and $S^1\times S^3$ are all pin$^+$ manifolds, their connected sum F^4_{non} is also a pin$^+$ manifold. Thus, the second Stiefel-Whitney class for F^4_{non} is $\mathrm{w}_2=0$. Since $\mathrm{w}_2+\mathrm{w}_1^2=(a^{\mathbb{R}P^4})^2\neq 0$, F^4_{non} is not a pin$^-$ manifold.

8. $H^*(K;\mathbb{Z}_2)$

The Klein bottle K has the following cohomology class:

$$H^1(K;\mathbb{Z}_2)=\mathbb{Z}_2^{\oplus 2}=\{a_1,a_2\},\quad H^2(K;\mathbb{Z}_2)=\mathbb{Z}_2=\{b\}. \tag{F47}$$

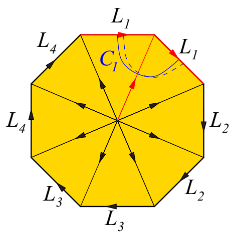

FIG. 12. A nonorientable surface Σ_g^{non} with genus $g=4$. All the corners are identified and the edges with the same label L_i are glued together along its direction. The Poincaré dual of the cycle C_1 is $a_1 \in H^1(\Sigma_g^{\text{non}};\mathbb{Z}_2)$: $\langle a_1,\text{red link}\rangle = 1$ and $\langle a_1,\text{black link}\rangle = 0$. We note that $\Sigma_1^{\text{non}} = \mathbb{R}P^2$ and $\Sigma_2^{\text{non}} =$ Klein bottle.

$H^1(K;\mathbb{Z}_2)$ is generated by a_1 and a_2 which are the Poincaré dual of C_1 and C_2 (see Fig. 11):

$$a_1 = C_1^*, \quad a_2 = C_2^*. \tag{F48}$$

We see that $a_1a_2 = b$ since C_1 and C_2 intersect once; $a_2^2 = 0$ since C_2 does not self-intersect (i.e., C_2 and its displacement does not intersect); $a_1^2 = b$ since C_1 self-intersects once (i.e., C_1 and its displacement intersect once). Therefore, $H^*(K;\mathbb{Z}_2)$ is determined by

$$a_1^2 = a_1a_2 = b, \quad a_2^2 = 0. \tag{F49}$$

9. $H^*(\Sigma_g^{\text{non}};\mathbb{Z}_2)$

The cohomology ring for nonorientable surface Σ_g^{non} (see Fig. 12), $H^*(\Sigma_g^{\text{non}};\mathbb{Z}_2)$, is given by (see [93], p. 208)

$$\begin{aligned} H^*\big(\Sigma_g^{\text{non}};\mathbb{Z}_2\big) &= \frac{\mathbb{Z}_2[a_i]}{(a_i^3, a_i^2 - a_j^2, a_ia_ja_k, a_ia_j|_{i\neq j})} \\ &= \big\{\zeta + \alpha_i a_i + \beta b | \zeta,\alpha_i,\beta \in \mathbb{Z}_2,\ a_i^2 = \delta_{ij} b\big\}, \end{aligned} \tag{F50}$$

with $a_i \in H^1(\Sigma_g^{\text{non}};\mathbb{Z}_2) = \mathbb{Z}_2^{\oplus g}$, $i = 1,2,\ldots,g$ and $b \in H^2(\Sigma_g^{\text{non}};\mathbb{Z}_2) = \mathbb{Z}_2$.

To understand the above result, we note that the cycles C_i, $i = 1,\ldots,g$, generate $H_1(\Sigma_g^{\text{non}};\mathbb{Z}_2)$ (see Fig. 12 where only C_1 is drawn). The Poincaré dual of C_i, $a_i = C_i^*$, generates $H^1(\Sigma_g^{\text{non}};\mathbb{Z}_2)$. We note that the self-intersection number for C_i is 1. Thus, $a_i^2 = b$. C_i and C_j do not intersect if $i \neq j$. Thus, $a_ia_j = 0$.

To calculate the Stiefel-Whitney class w_i, we note that the orientation reverses as we go along the loop C_i. This implies that $\oint_{C_i} \mathrm{w}_1 = 1 \bmod 2$. Since $\oint_{C_i} a_j$ is the intersection number between C_i and Poincaré dual of a_j which is C_j, we see that $\oint_{C_i} a_j = \delta_{ij}$. Therefore, $\mathrm{w}_1 = \sum_{i=1}^g a_i$. In two dimensions $\mathrm{w}_2 = \mathrm{w}_1^2 = \sum_{i=1}^g a_i^2 = [g]_2 b$. Thus, Σ_g^{non} is a pin$^+$ manifold if $g =$ even, and it is not a pin$^+$ manifold if $g =$ odd. Σ_g^{non} is always a pin$^-$ manifold.

We also note that the CW complex of Σ_g^{non} in Fig. 12 has $V = 2$ vertices, $L = 3g$ links, and $T = 2g$ triangles. Thus, the Euler number $\chi(\Sigma_g^{\text{non}}) = V - L + T = 2 - g$. The top Stiefel-Whitney class is equal to the Euler class mod 2, regardless of the $\mathbb{Z}$ orientability of the manifold. In other words, every manifold is $\mathbb{Z}_2$ orientable. So, the Euler class (with $\mathbb{Z}_2$ coefficients) coincides with the top Stiefel-Whitney class. This is another way to obtain $\mathrm{w}_2 = [g]_2 b$.

APPENDIX G: GROUP COHOMOLOGY THEORY

1. Homogeneous group cocycle

In this Appendix, we will briefly introduce group cohomology. The group cohomology class $\mathcal{H}^d(G,\mathbb{M})$ is a $\mathbb{Z}$ model constructed from a group G and a $\mathbb{Z}$ module $\mathbb{M}$ (i.e., a vector space over $\mathbb{Z}$). Each element of G also induces a mapping $\mathbb{M} \to \mathbb{M}$, which is denoted as

$$g \cdot m = m', \quad g \in G,\ m,m' \in \mathbb{M}. \tag{G1}$$

The map $g\cdot$ is a group homomorphism:

$$g \cdot (m_1 + m_2) = g \cdot m_1 + g \cdot m_2. \tag{G2}$$

The module $\mathbb{M}$ with such a G-group homomorphism is called a G module.

A homogeneous d-cochain is a function $\nu_d : G^{d+1} \to \mathbb{M}$, that satisfies

$$\nu_d(g_0,\ldots,g_d) = g \cdot \nu_d(gg_0,\ldots,gg_d), \quad g,g_i \in G. \tag{G3}$$

We denote the set of d-cochains as $\mathcal{C}^d(G,\mathbb{M})$. Clearly, $\mathcal{C}^d(G,\mathbb{M})$ is an Abelian group.

Let us define a mapping d (group homomorphism) from $\mathcal{C}^d(G,\mathbb{M})$ to $\mathcal{C}^{d+1}(G,\mathbb{M})$:

$$(d\nu_d)(g_0,\ldots,g_{d+1}) = \sum_{i=0}^{d+1} (-)^i \nu_d(g_0,\ldots,\hat{g}_i,\ldots,g_{d+1}), \tag{G4}$$

where $g_0,\ldots,\hat{g}_i,\ldots,g_{d+1}$ is the sequence $g_0,\ldots,g_i,\ldots,g_{d+1}$ with g_i removed. One can check that $d^2 = 0$. The homogeneous d-cocycles are then the homogeneous d-cochains that also satisfy the cocycle condition

$$d\nu_d = 0. \tag{G5}$$

We denote the set of d-cocycles as $\mathcal{Z}^d(G,\mathbb{M})$. Clearly, $\mathcal{Z}^d(G,\mathbb{M})$ is an Abelian subgroup of $\mathcal{C}^d(G,\mathbb{M})$.

Let us denote $\mathcal{B}^d(G,\mathbb{M})$ as the image of the map $d : \mathcal{C}^{d-1}(G,\mathbb{M}) \to \mathcal{C}^d(G,\mathbb{M})$ and $\mathcal{B}^0(G,\mathbb{M}) = \{0\}$. The elements in $\mathcal{B}^d(G,\mathbb{M})$ are called d-coboundaries. Since $d^2 = 0$, $\mathcal{B}^d(G,\mathbb{M})$ is a subgroup of $\mathcal{Z}^d(G,\mathbb{M})$:

$$\mathcal{B}^d(G,\mathbb{M}) = \{d\nu_{d-1} | \nu_{d-1} \in \mathcal{C}^{d-1}(G,\mathbb{M})\} \subset \mathcal{Z}^d(G,\mathbb{M}). \tag{G6}$$

The group cohomology class $\mathcal{H}^d(G,\mathbb{M})$ is then defined as

$$\mathcal{H}^d(G,\mathbb{M}) = \mathcal{Z}^d(G,\mathbb{M})/\mathcal{B}^d(G,\mathbb{M}). \tag{G7}$$

We note that the d operator and the cochains $\mathcal{C}^d(G,\mathbb{M})$ (for all values of d) form a so-called cochain complex

$$\cdots \xrightarrow{d} \mathcal{C}^d(G,\mathbb{M}) \xrightarrow{d} \mathcal{C}^{d+1}(G,\mathbb{M}) \xrightarrow{d} \cdots \tag{G8}$$

which is denoted as $C(G,\mathbb{M})$. So, we may also write the group cohomology $\mathcal{H}^d(G,\mathbb{M})$ as the standard cohomology of the cochain complex $H^d[C(G,\mathbb{M})]$.

2. Inhomogeneous group cocycle

The above definition of group cohomology class can be rewritten in terms of inhomogeneous group cochains/cocycles. An inhomogeneous group d-cochain is a function $\omega_d : G^d \to M$. All $\omega_d(g_1,\dots,g_d)$ form $\mathcal{C}^d(G,\mathbb{M})$. The inhomogeneous group cochains and the homogeneous group cochains are related as

$$\nu_d(g_0,g_1,\dots,g_d) = \omega_d(a_{01},\dots,a_{d-1,d}), \tag{G9}$$

with

$$g_0 = 1,\ g_1 = g_0 a_{01},\ g_2 = g_1 a_{12},\ \dots\ g_d = g_{d-1}a_{d-1,d}. \tag{G10}$$

Now, the d map has a form on ω_d:

$$\begin{aligned}(d\omega_d)(a_{01},\dots,a_{d,d+1}) &= a_{01}\cdot\omega_d(a_{12},\dots,a_{d,d+1})\\ &+ \sum_{i=1}^{d}(-)^i\omega_d(a_{01},\dots,a_{i-1,i}a_{i,i+1},\dots,a_{d,d+1})\\ &+ (-)^{d+1}\omega_d(a_{01},\dots,\tilde{a}_{d-1,d}).\end{aligned} \tag{G11}$$

This allows us to define the inhomogeneous group d-cocycles which satisfy $d\omega_d = 0$ and the inhomogeneous group d-coboundaries which have a form $\omega_d = d\mu_{d-1}$. Geometrically, we may view g_i as living on the vertex i, while a_{ij} as living on the link connecting the two vertices i to j.

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