Spontaneous finite momentum pairing in superconductors without inversion symmetry

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We analyze the effect of magnetic fluctuations in superconductors with strong spin-orbit coupling and show that they drive a phase transition between two superconducting states: a conventional phase with zero center-of-mass momentum of Cooper pairs, and an exotic phase with nonzero pair momentum. The latter is found to exhibit persistent currents without magnetic field in doubly connected geometries such as rings. Surprisingly, the transition temperature into the superconducting state can be increased by applying a Zeeman magnetic field.

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I. INTRODUCTION

The coupling between the spin of an electron and its momentum gives rise to various phases in condensed-matter systems. In magnetic systems, spin-orbit coupling (SOC) induces numerous phases, spectacularly different from the familiar (anti)ferromagnets and with exotic low-energy excitations [1,2]. In systems with ferromagnetic tendencies, SOC leads to a different ordering pattern—helimagnetism—where the magnetic moments rotate as a function of position in a spiral structure [3,4]. An even greater variety of interesting phenomena arises from the combination of spin orbit, magnetism, and superconductivity [5–15]. Here the interplay of translations, spin rotations, and superconducting phase rotations offers many possibilities for forming ordered ground states [16]. In the presence of a Zeeman magnetic field, SOC stabilizes a condensate of Cooper pairs with finite momentum [17,18]. This is a variant of the Fulde-Ferrel-Larkin-Ovchinikov (FFLO) state [19,20] where the critical (Zeeman) magnetic field of the s-wave superconductor significantly exceeds the Pauli limit. In crucial distinction to the conventional FFLO state, SOC permits the superconducting order to exhibit a well-defined chirality even in the absence of currents. Consequently, there are no nodes in the pairing gap, and such a superconductor is robust against disorder [21,22]. An additional consequence of SOC is a finite spin susceptibility in the superconducting state—comparable to its normal-state value—down to the lowest temperatures [5,8].

The strong response of superconductors with large SOC to magnetic fields raises questions regarding the role of spin fluctuations in such systems. Superconductivity coexisting with magnetic states has been observed in several materials without inversion symmetry such as $CePt_3Si$ (Ref. [23]), $CeRhSi_3$ (Ref. [24]), and UIr (Ref. [25]). Another family of systems exhibiting superconductivity and magnetism are transition-metal-oxide heterojunctions, such as the interface between $LaAlO_3$ and $SrTiO_3$ which hosts a two-dimensional layer of high-mobility electrons [26] with strong SOC of the Rashba type [27–29]. In addition, at low temperatures the interface becomes superconducting on the background of an inhomogeneous magnetic state [30,31].

Measurements of $LaAlO_3/SrTiO_3$ heterostructures have revealed a peculiar property of superconducting films with strong SOC. A moderate magnetic field parallel to the interface increases the transition temperature T_c [32] (at high magnetic fields T_c is eventually suppressed). Such a nonmonotonic dependence of T_c on the Zeeman field is not unique to $LaAlO_3/SrTiO_3$ heterostructures; it is even more pronounced in Pb films [32]. The presence of magnetic impurities was proposed to account for this behavior [9], but has been ruled out experimentally [32]. Moreover, mere formation of a paired state at finite momentum that protects superconductors with SOC from pair-breaking effects up to high Zeeman fields does not explain the observed T_c enhancement. In fact, the conventional BCS dependence of T_c on the reduced Zeeman field H/H_c continues to hold in the presence of SOC, with the enhanced critical field H_c as described above.

Motivated by these experimental findings, we study the effect of magnetic fluctuations in phonon-mediated superconductors with strong SOC. While magnetization fluctuations have long been proposed as the pairing mechanism in many strongly correlated materials [33–35], they are known to cause breaking of Cooper pairs in weakly coupled s-wave superconductors, and hence reduce T_c. We show that spin-orbit coupled single-band superconductors can enter a finite-momentum paired state solely by short-range magnetic fluctuations. Previous works described FFLO states exclusively in systems with uniform magnetization [22] or external magnetic field [17,18]. Further studies of interplay of superconductivity and SOC considered the case where the pairing interaction involves several channels [36,37]. There, a more complex nonuniform state was found to occur only in the presence of constant magnetization and for weak SOC [37]; we are instead interested in the effect of *strong SOC* and *fluctuating magnetic order*.

Specifically, we find that magnetic fluctuations in the superconducting state can drive a second-order phase transition between two different superconducting states, one with uniform order parameter $\Delta(\mathbf{r}) = \Delta_0$ and another with $\Delta(\mathbf{r}) = \Delta_0 e^{i\mathbf{q}\cdot\mathbf{r}}$. The latter reflects the fact that Cooper pairs with center-of-mass momentum $\mathbf{q}$ are formed spontaneously without an external field. Near the transition, the superconducting phase stiffness is suppressed and T_c decreases. The quantum critical point between the two superconducting states is replaced by a smooth crossover in the presence of a Zeeman magnetic field. Consequently, T_c associated with either superconducting state initially increases with applied field.

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II. MICROSCOPIC MODEL

To study the long-wavelength properties of a superconductor with strong SOC, we start from a microscopic model $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm BCS} + \mathcal{H}_{\rm M}$. The first term describes electrons in a thin film with a Rashba term:

$$\mathcal{H}_0 = \int dr \sum_{s,s'} c^\dagger_{s,\mathbf{r}} \left[-\frac{\nabla^2}{2m}\delta_{s,s'} - i\alpha_{\rm R}\hat{z}\cdot(\nabla \times \boldsymbol{\sigma}_{ss'}) \right] c_{s',\mathbf{r}}. \quad (1)$$

Here $c^\dagger_{s,\mathbf{r}}$ creates a spin $s = \uparrow,\downarrow$ electron. $\mathcal{H}_{\rm BCS}$ describes an attractive interaction in the s-wave channel, conveniently expressed in terms of the (fluctuating) order parameter $\Delta(\mathbf{r}) = \lambda\langle c_{\downarrow,\mathbf{r}} c_{\uparrow,\mathbf{r}}\rangle$:

$$\mathcal{H}_{\rm BCS} = \int dr\{\Delta(\mathbf{r})c^\dagger_{\uparrow,\mathbf{r}}c^\dagger_{\downarrow,\mathbf{r}} + \Delta^*(\mathbf{r})c_{\downarrow,\mathbf{r}}c_{\uparrow,\mathbf{r}} + \lambda^{-1}|\Delta(\mathbf{r})|^2\}. \quad (2)$$

$\mathcal{H}_{\rm M}$ incorporates the magnetic fluctuations arising from spin exchange interactions and the coupling to an external Zeeman field:

$$\mathcal{H}_{\rm M} = -\frac{g\mu_B}{2}\int dr \sum_{s,s'} \mathbf{H}_{\rm T}(\mathbf{r})\cdot\boldsymbol{\sigma}_{ss'}c^\dagger_{s,\mathbf{r}}c_{s',\mathbf{r}} + U\int dr \mathbf{M}^2(\mathbf{r}). \quad (3)$$

Here $\mathbf{M}(\mathbf{r}) = \langle\sum_{s,s'}\boldsymbol{\sigma}_{ss'}c^\dagger_{s,\mathbf{r}}c_{s',\mathbf{r}}\rangle$, and the total magnetic field $\mathbf{H}_{\rm T}(\mathbf{r}) = \mathbf{H} + 2U\mathbf{M}(\mathbf{r})/g\mu_B$ includes the external field as well as the magnetization divided by the Bohr magneton μ_B and the Landé g factor. We focus on the effect of magnetic fluctuations in the presence/absence of an external magnetic field *parallel* to the film (with thickness smaller than the coherence length) where only Zeeman coupling is important.

III. EFFECTIVE THEORY

Upon integrating out the electronic degrees of freedom and assuming a finite superconducting gap $\langle\Delta\rangle \neq 0$, we obtain the effective low-energy description of the system at wavelengths much longer than the coherence length ξ. This can be expressed as a lattice free energy with lattice spacing $a \gg \xi$:

$$F = \sum_{\vec{j},\hat{\nu}} \Big\{ \alpha(T-T_c)|\Delta_{\vec{j}}|^2 + \beta|\Delta_{\vec{j}}|^4 + \frac{w}{2}|\Delta_{\vec{j}} - \Delta_{\vec{j}+\hat{\nu}}|^2 + U\mathbf{M}^2_{\vec{j}} - \frac{\chi_\perp}{2}H^2_{{\rm T}\perp,\vec{j}} - \frac{\chi_{||}}{2}\mathbf{H}^2_{{\rm T}||,\vec{j}} - i\frac{\eta}{4}(\hat{z}\times\hat{\nu})\cdot(\mathbf{H}_{{\rm T}\vec{j}} + \mathbf{H}_{{\rm T}\vec{j}+\hat{\nu}})[\Delta^*_{\vec{j}}\Delta_{\vec{j}+\hat{\nu}} - \Delta^*_{\vec{j}+\hat{\nu}}\Delta_{\vec{j}}] \Big\}. \quad (4)$$

For a detailed derivation see Appendix A. For simplicity we assume here a square lattice for which $\hat{\nu} = \hat{x},\hat{y}$ connects neighboring lattice sites denoted by $\vec{j}$. The total magnetic field is separated into its in-plane $\mathbf{H}_{{\rm T}||,\vec{j}}$ and perpendicular $H_{{\rm T}\perp,\vec{j}}$ components. The first line of Eq. (4) is the conventional Ginzburg-Landau free energy of superconductivity [38]. The second line of Eq. (4) describes the magnetic fluctuations in the itinerant electron system [39]. In the absence of external field it becomes $U\sum_\mu[1-\nu(\varepsilon_F)U\chi_\mu/\chi_N]M^2_{\mu,\vec{j}}$ where $\mu = \perp, \|$ and $\chi_N = (g\mu_B)^2\nu(\varepsilon_F)/2$ is the electron-spin susceptibility. This form is familiar, e.g., from the study of the Stoner instabilities. We are interested in the case $\nu(\varepsilon_F)U \ll 1$ (far from the magnetic instability) where magnetic fluctuations are massive. Therefore, we can neglect higher-order terms such as gradients of $M_{\vec{j}}$ and $M^4_{\vec{j}}$. The second line of Eq. (4) already contains an important consequence of SOC: the spin susceptibilities χ_μ are only weakly affected by superconductivity [5], and remain nonzero deep in the superconducting state $T \to 0$. This is in contrast to the vanishing spin susceptibility of conventional superconductors without SOC. Furthermore, SOC gives rise to an anisotropic χ in the superconducting state; the spin susceptibility normal $\chi_\perp$ and parallel $\chi_{||}$ to the plane are no longer equal. Another important manifestation of SOC is the appearance of the final term that explicitly breaks inversion symmetry. Its coefficient η is proportional to $\alpha_{\rm R}$. In the continuum limit it takes the form

$$-i\Delta^*(\mathbf{r})[\hat{z}\cdot(\mathbf{H}_{\rm T}\times\nabla)]\Delta(\mathbf{r}), \quad (5)$$

which has been derived in Ref. [7] (see also Appendix A).

To study the universal properties, it is convenient to adopt a phase-only formulation, writing $\Delta_{\vec{j}} = \Delta_0 e^{i\Phi_j}$. Under this approximation, the free energy takes the form

$$F = \sum_{\vec{j},\hat{\nu}} \Big\{ -\rho_s\cos(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \frac{\kappa}{2}(\hat{z}\times\hat{\nu})\cdot(\mathbf{H}_{{\rm T}\vec{j}} + \mathbf{H}_{{\rm T}\vec{j}+\hat{\nu}})\sin(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + U\mathbf{M}^2_{\vec{j}} - \frac{\chi_\perp}{2}H^2_{{\rm T}\perp,\vec{j}} - \frac{\chi_{||}}{2}\mathbf{H}^2_{{\rm T}||,\vec{j}} \Big\}. \quad (6)$$

Here, ρ_s is the superfluid stiffness in the ν direction, and κ grows from $\propto \alpha_R|\Delta(T)|^2/T_c^2$ near T_c to $\propto \alpha_R$ as $T \to 0$ [40].

The unique form of the free energy of superconductors with SOC in a uniform external Zeeman field has been studied in the past [7]. Here we focus on the role of magnetic fluctuations (with and without external field). Integrating out massive fluctuations of $\mathbf{M}$ (see Appendix B for details) generates short-range interactions for the pairing field, i.e.,

$$F = -\sum_{\vec{j},\hat{\nu}} \Big\{ \tilde{\rho}_{s,\nu}\cos(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}} - \Theta_{\hat{\nu}}) + \frac{\zeta}{4}[\sin(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \sin(\Phi_{\vec{j}} - \Phi_{\vec{j}-\hat{\nu}})]^2 \Big\}. \quad (7)$$

The coefficient $\zeta \equiv U\kappa^2(1 - 2\chi_{\|}U/g^2\mu_B^2)^{-1}/(g\mu_B)^2$ is positive in the paramagnetic phase; it grows as the magnetic transition is approached and the corresponding fluctuations become stronger.

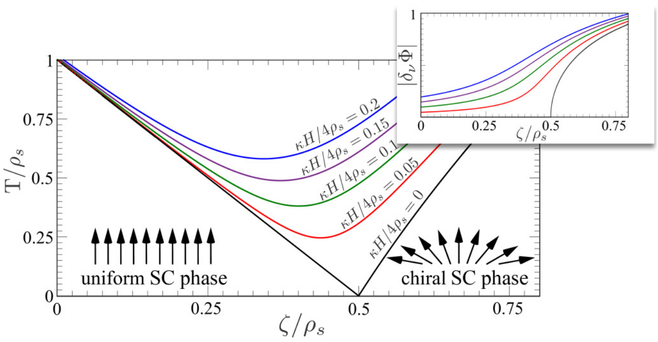

FIG. 1. The phase diagram as a function of temperature T and ζ in units of ρ_s. In the absence of magnetic field (black curve) superconductivity is suppressed to zero ($T_c = 0$) at the transition between the states with uniform phase and chiral winding of the phase. The transition temperature is shown to increase under application of an external Zeeman field. The phase difference $\delta_\nu\Phi$ as a function of ζ is plotted in the inset.

IV. PHASE DIAGRAM

We begin by exploring the phase diagram in the absence of magnetic field, where $\Theta_{\hat\nu} = 0$ and $\tilde\rho_{s,\nu} = \rho_s$ is the conventional superfluid stiffness. While the first term is minimized by configurations with uniform $\Phi_{\vec j}$, the second term favors the phase on neighboring sites to differ by $\pi/2$. The same model also describes XY spins with ferromagnetic nearest-neighbor and antiferromagnetic next-nearest-neighbor exchange [41]. Expressed in terms of effective XY spins $\mathbf{S}_{\vec j} = (\cos\Phi_j, \sin\Phi_j)$, the free energy becomes

$$F = -\sum_{\vec j,\hat\nu}[\rho_s \mathbf{S}_{\vec j}\cdot\mathbf{S}_{\vec j+\hat\nu} + \zeta(\mathbf{S}_{\vec j}\times\mathbf{S}_{\vec j+\hat\nu} + \mathbf{S}_{\vec j}\times\mathbf{S}_{\vec j-\hat\nu})^2]. \quad (8)$$

There, frustration induces a transition into a helical ferromagnetic state $C_\nu \equiv \langle \mathbf{S}_{\vec j}\times\mathbf{S}_{\vec j+\hat\nu}\rangle \neq 0$. Similarly, depending on the relative strength of the two terms in Eq. (7) the system can be in one of two states: (i) a superconductor with a uniform phase and (ii) a superconductor with $\delta_\nu\Phi \equiv \Phi_{\vec j+\hat\nu} - \Phi_{\vec j} = \text{const} \neq 0$. The latter is only possible when both SOC and magnetic fluctuations are present ($\zeta > 0$).

To study the phase diagram as a function of ζ and T, we initially consider only smooth variations of the phase. We approximate T_c by the effective phase stiffness, i.e., the coefficient of $(\nabla\Phi)^2$ in the expansion of the free energy. At the transition between the two superconducting (SC) phases the stiffness vanishes, as shown in Fig. 1. Microscopically, the parameter ζ can be modified by changing the strength of either the SOC or the magnetic fluctuations. The former is conceptually simplest: within the single band model it leaves ρ_s unaffected. In contrast, increasing the magnetic fluctuations suppresses Δ_0, and consequently, also modifies ρ_s. Nevertheless, the phase diagram (Fig. 1) equally applies to both cases provided these dependencies are taken into account. We note that suppression of T_c by magnetic fluctuations is well known from mean-field treatments of ordinary superconductors. Our result shows that in the presence of SOC, magnetic fluctuations can have a much more pronounced impact on T_c due to frustration effects. Similar behavior occurs in certain magnetic systems, such as the frustrated spin system described above, at the Lifshitz point [42].

Within mean-field theory, a second-order transition between superconducting states with $\delta_\nu\Phi = 0$ and $\delta_\nu\Phi \neq 0$ occurs at $\zeta_c = \rho_s/2$. On a square lattice and for $\zeta > \zeta_c$ the nearest-neighbors phase difference takes one of four values: $\delta_x\Phi = \pm\delta_y\Phi = \pm\cos^{-1}(\rho_s/2\zeta)$. These states are characterized by the superconducting order parameter Δ and a two-component chiral order parameter,

$$C_{\nu=x,y} = \langle\sin(\Phi_{\vec j} - \Phi_{\vec j+\hat\nu})\rangle = \begin{cases} 0 & \zeta < \frac{\rho_s}{2} \\ \pm\sqrt{1-\frac{\rho_s^2}{4\zeta^2}} & \zeta > \frac{\rho_s}{2} \end{cases}. \quad (9)$$

The phase transition into a superconducting state with finite pair momentum is a key result of our work.

The nature of the chiral transition becomes transparent when the free energy [Eq. (7)] is expressed in terms of C_ν. Note that under $\pi/2$ rotations around the z axis the order parameter transforms as $C_{x/y} \to \pm C_{y/x}$. The generic form of the free energy with this symmetry is $F = r(C_x^2 + C_y^2) + u(C_x^2 + C_y^2)^2 + vC_x^2C_y^2$, where expanding Eq. (7) yields $r = \rho_s/2 - \zeta$, and $u = -v/2 = \rho_s/8$. This model features a continuous phase transition in the XY universality class, while the dangerously irrelevant fourfold anisotropy $\sim v$ determines the nature of the ordered phase.

Above we have made several assumptions which are valid in the ordered states, but must be revisited near the Lifshitz point: (i) We neglected amplitude modulations of the order parameter; based on Eq. (4), it is straightforward to show (see Appendix C) that a single harmonics of the order parameter $\Delta(\mathbf{r}) = \Delta_0 e^{i\mathbf{q}\cdot\mathbf{r}}$ is energetically favorable. (ii) The previous analysis excluded vortex excitations. Earlier studies of the Kosterlitz-Thouless transition in closely related magnetic systems showed that near the critical point, helical order may survive even when vortex proliferation destroys magnetic order [43–46]. Vortex physics, however, becomes important only at finite temperature, thus it does not qualitatively change the phase diagram (Fig. 1).

V. MAGNETIC FIELD EFFECTS

In the presence of a constant external magnetic field $\mathbf{H}$, the phase transition is replaced by a smooth crossover, and the phase stiffness remains finite for all ζ; see Fig. 1. This follows from the free energy in Eq. (7), where

$$\begin{aligned} \tilde\rho_{s,\nu} &= \sqrt{(\kappa H_\nu^\parallel/4)^2 + \rho_s^2}, \\ \Theta_{\hat\nu} &= [\pi/2 - \cos^{-1}(\kappa H_\nu^\parallel/4\tilde\rho_{s,\nu})]\text{sgn}(H_\nu^\parallel), \\ H_\nu^\parallel &= \frac{(\hat z\times\hat\nu)\cdot\mathbf{H}}{1-2\chi_\parallel U/g^2\mu_B^2}. \end{aligned} \quad (10)$$

The phase $\Theta_{\hat\nu}$ changes from $\Theta_{\hat\nu} = \kappa H_\nu^\parallel/4\tilde\rho_s$ at low Zeeman field to $\pm\pi/2$ at very high field. Within mean-field theory, the phase difference $\delta_\nu\Phi$ in the direction perpendicular to the field is nonzero for all ζ. For $\zeta \ll \tilde\rho_s/2$ the phase difference equals $\Theta_{\hat\nu}$ and tracks the Zeeman field, while at large ζ it acquires a field-independent contribution (see Fig. 1). Consequently, the Zeeman field enhances T_c as illustrated in Fig. 2. This result applies provided the magnitude of the order parameter $|\Delta_{\vec j}|$ is

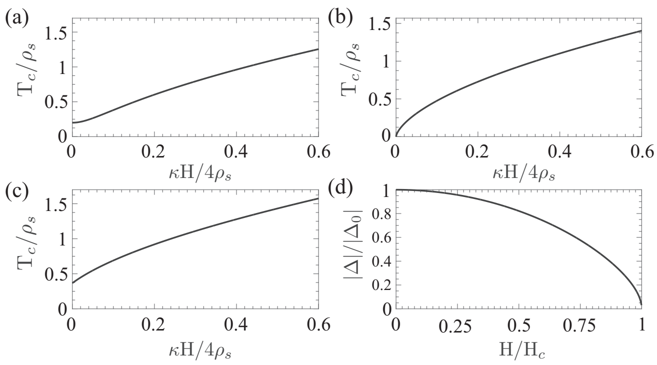

FIG. 2. The transition temperature as a function of an applied Zeeman field H for (a) $\zeta = 0.4\rho_s$, (b) $\zeta = 0.5\rho_s$, and (c) $\zeta = 0.6\rho_s$. The transition temperature is found from the phase-only model assuming the magnitude of the order parameter Δ has a weak dependence on H. Panel (d) shows the range of fields for which the phase only approximation holds.

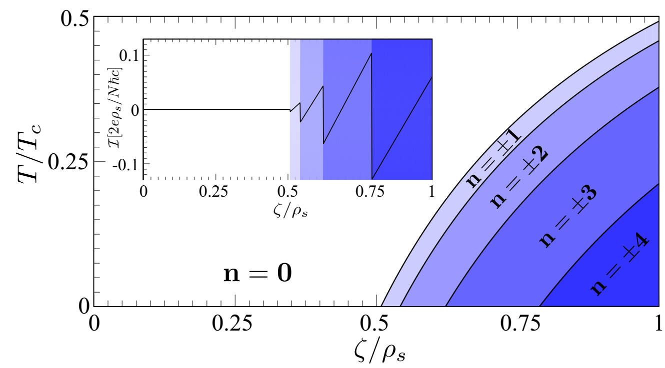

FIG. 3. The phase diagram in a ring geometry. While at low temperature the transition lines are functions of T/T_c and ζ/ρ_s, at higher temperature they also depend on ρ_s explicitly (see Appendix D). The corresponding persistent currents at $T \to 0$ are shown in the inset. The current is plotted assuming $n \geqslant 0$, for negative n the sign of the current is inverted.

field independent. Since SOC protects superconductivity from pair-breaking effects up to Zeeman fields well above the Pauli limit, the phase-only model captures the main effect at small H. At higher magnetic fields suppression of $|\Delta|$ is expected to be dominant. The dependence of $|\Delta|$ on the magnetic field in disordered superconductors [22] is illustrated in Fig. 2(d).

VI. EXPERIMENTAL SIGNATURES

The mechanism described here provides a possible explanation for the enhancement of T_c with magnetic field measured in Pb films and $LaAlO_3/SrTiO_3$ heterostructures [32]. Both systems feature strong SOC, and evidence of inhomogeneous magnetism has been observed in the oxide interface. In Pb films the strength of SOC can be enhanced by reducing the film thickness. Interestingly, over a range of film thicknesses, the field-induced T_c enhancement in Pb was found to be larger for thinner samples, consistent with our theory for $\zeta < \rho_s/2$. In the $LaAlO_3/SrTiO_3$ interface such measurements were performed for a particular value of SOC only. However, in oxide heterostructures SOC can be gate tuned [27], making them ideal candidate systems for exploring the entire phase diagram. Direct observation of the chiral superconducting state is more challenging; the absence of nodes in the pairing gap rules out probes that are frequently used in search of the FFLO state, such as specific-heat measurements. Instead, phase sensitive techniques must be employed, e.g., measuring the critical current in a Josephson junction. Alternatively, numerous striking signatures arise in a ring geometry due to the sensitivity of the superconducting phase to boundary conditions.

To study the superconducting state on a ring of radius $R \gg \xi$ at low temperature, we consider Eq. (7) with periodic boundary condition in the x direction, $\Phi_{\vec{j}} = \Phi_{\vec{j}+N\hat{x}} + 2\pi n$, where N is the number of sites in the x direction, and n is an integer. This boundary condition only permits ground states with $\delta_x\Phi = 2\pi n/N$. When the ring thickness is smaller than ξ, modulations along the y direction are suppressed. The free energy of such states is obtained by setting $\Phi_{\vec{j}+\hat{x}} - \Phi_{\vec{j}} = 2\pi n/N$ and $\Phi_{\vec{j}+\hat{y}} - \Phi_{\vec{j}} = 0$ in Eq. (7), i.e.,

$$F_{\text{ring}} = -\rho_s \cos\left(\frac{2\pi n}{N}\right) - \zeta \sin^2\left(\frac{2\pi n}{N}\right). \tag{11}$$

Similar to the planar geometry, for large ζ the free energy is minimized by $n_{\min} \neq 0$. Here $n_{\min}$ changes discretely whenever $F_{\text{ring}}(n_{\min}) = F_{\text{ring}}(n_{\min}+1)$, and in general the current $\mathcal{I}_x = (2e/hc)\partial F_{\text{ring}}(n)/\partial n|_{n=n_{\min}}$ is nonzero. This is in contrast to the planar geometry where $\delta_\nu\Phi$ takes continuous values and satisfies $\partial F/\partial\delta_\nu\Phi = 0$. Consequently, a ground state with $n_{\min} \neq 0$ exhibits persistent current [47],

$$\mathcal{I}_x = \frac{2e}{N\hbar c}\sin\left(\frac{2\pi n_{\min}}{N}\right)\left[\rho_s - 2\zeta\cos\left(\frac{2\pi n_{\min}}{N}\right)\right]. \tag{12}$$

The continuous transition that occurs in the planar geometry at ζ_c is replaced by a sequence of first-order transitions as ζ increases. As illustrated in Fig. 3, these states are characterized by a persistent current even in the absence of an external magnetic field. The current changes abruptly at the transition points (in realistic systems these sharp changes would be smeared). To extend the analysis to higher temperatures, modulations of $|\Delta_{\vec{j}}|$ as well as $\Phi_{\vec{j}}$ must be considered. The complete phase diagram is presented in Fig. 3, with a detailed derivation in Appendix C.

Another signature of SOC can be obtained from measurements of the persistent current [48] induced by a magnetic field along the y direction. Here the orbital component as well as the Zeeman contribution analyzed in Eq. (7) are important. The boundary conditions are modified to $\Phi_{\vec{j}} = \Phi_{\vec{j}+N\hat{x}} + 2\pi n + \varphi$, where $\varphi = 2e\pi R^2 H/hc$ is the magnetic flux threading the ring in units of the superconducting flux quantum. Correspondingly, Eq. (11) is modified as $F_{\text{ring}} = -\tilde{\rho}_{s,x}\cos\left[\frac{2\pi(n-\varphi)}{N} - \Theta_x\right] - \zeta\sin^2\left[\frac{2\pi(n-\varphi)}{N}\right]$, and a persistent current flows in the system as a function of magnetic field even for small ζ. The current

$$\mathcal{I}_x \sim \tilde{\rho}_{s,x}\sin\left[\frac{2\pi(n_{\min}-\varphi)}{N} - \Theta_x\right] - \zeta\sin\left[\frac{4\pi(n_{\min}-\varphi)}{N}\right] \tag{13}$$

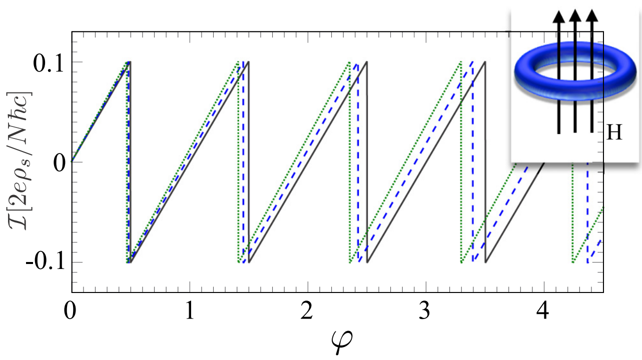

FIG. 4. The persistent currents as a function of external magnetic field for $\zeta = 0.1\rho_s < \zeta_c$. The magnetic field is expressed in terms of the flux threading the ring in units of the superconducting flux quantum φ. The change in periodicity as a function of κ is illustrated for $\kappa/\rho_s = 0$ (black), $\kappa/\rho_s = 0.4T^{-1}$ (blue), and $\kappa/\rho_s = 0.8T^{-1}$ (green). These values of κ are chosen to give the correct order of magnitude for a metallic system with a Fermi energy of 1 eV, a SOC of 10 meV, and a transition temperature of 0.1 meV. In addition, we set $N = 25$ and $R = 0.1$ μm.

is plotted in Fig. 4 as a function of φ. For weak fields $\Theta_x \propto \kappa H/\rho_s$ and the periodicity of the persistent currents with respect to φ changes as a function of κ. For stronger fields, Θ_x is no longer linear in the field and together with the H dependence of $\tilde{\rho}_{s,x}$ gives rise to a nonperiodic dependence on the magnetic field.

VII. CONCLUSION

We analyzed the effect of magnetic fluctuations on two-dimensional BCS superconductors with large SOC, and found that they suppress the phase stiffness, and hence, reduce T_c. Furthermore, magnetic fluctuations induce a phase transition between two superconducting states: a conventional one with uniform phase and a second one where the phase winds as a function of position. The latter corresponds to a superconducting state with finite-momentum pairing. We showed that in thin films the transition temperature increases when an in-plane magnetic field is applied—an effect that may already have been observed experimentally [32]. In addition, we predict interesting signatures of this system in a ring geometry, such as the occurrence of spontaneous persistent currents without an applied magnetic field in the finite-momentum paired state. We emphasize that although we assumed s-wave pairing in the derivation, our results apply for any singlet state. Consequently, pairing at finite momentum may occur whenever superconductivity arises in the vicinity of magnetic transitions.

APPENDIX A: DERIVATION OF THE FREE ENERGY

In this section we derive the free energy given in Eq. (4) starting from the model Hamiltonian equations (1)–(3). While the general form of the free energy is dictated by symmetry, the values of the various coefficients (for example α or η) depend on the microscopic theory. To demonstrate the derivation, we consider here a clean superconductor with SOC and magnetic fluctuations as described by Eqs. (1)–(3). Moreover, we focus on the free energy in the vicinity of the superconducting phase transition. In the end of this section we discuss the generalization of the free energy to describe systems deep in the superconducting state ($T \to 0$) and the effects of disorder.

The expression for the free energy can be obtained from the quantum partition function, i.e., the Euclidean path integral $\mathcal{Z} = \int \mathcal{D}\bar{c}\mathcal{D}c e^{-S[\bar{c},c]}$ where S is the imaginary time action for the mean-field Hamiltonian in Eqs. (1)–(3) and c ($\bar{c}$) is the Grassmann field corresponding to the fermion operator c ($c^\dagger$). Integrating out the electronic degrees of freedoms we get [39]

$$F = \int d\mathbf{r}\left[\frac{|\Delta(\mathbf{r})|^2}{\lambda} + U\mathbf{M}^2(\mathbf{r}) + T\sum_n \ln G^{-1}(\mathbf{r}-\mathbf{r}',\epsilon_n)\right]. \tag{A1}$$

Here the free energy is written in terms of the Matsubara single-particle Green's function calculated with respect to $\mathcal{H}_0 + \mathcal{H}_{\text{BCS}} + \mathcal{H}_M$, and $\epsilon_n = 2\pi T(n+1/2)$. Note that we are studying the mean-field Hamiltonian. An equivalent starting point is given by interacting electrons, where the interaction is decoupled in terms of two Hubbard-Stratonovich fields [39] $\mathbf{M}$ and Δ.

Near the superconducting transition temperature T_c and in the paramagnetic state, it is appropriate to expand the logarithm in powers of Δ and its gradients as well as H_{T}. In the presence of a weak external field it is sufficient to consider terms up to second order in the magnetic field $\mathbf{H}_{\text{T}}(\mathbf{r}) = \mathbf{H}(\mathbf{r}) + 2U\mathbf{M}(\mathbf{r})/g\mu_B$. The expansion with respect to the superconducting order parameter, however, must include also quartic terms to allow for a nonzero order parameter. Thus the free energy takes the following form:

$$\begin{aligned} F \approx \int d\mathbf{r}\bigg\{&\alpha|\Delta(\mathbf{r})|^2 + \beta|\Delta(\mathbf{r})|^4 + \frac{\tilde{w}}{2}|\nabla\Delta(\mathbf{r})|^2 \\ &+ \frac{\tilde{\eta}^{\nu,\nu'}}{2}[\Delta^*(\mathbf{r})\nabla_\nu\Delta(\mathbf{r}) - \Delta(\mathbf{r})\nabla_\nu\Delta^*(\mathbf{r})]H_{\text{T}}^{\nu'}(\mathbf{r}) + UM_\nu^2 \\ &- \chi^0_{\nu,\nu'}H_{\text{T}}^\nu(\mathbf{r})H_{\text{T}}^{\nu'}(\mathbf{r}) + \delta\chi_{\nu,\nu'}H_{\text{T}}^\nu(\mathbf{r})H_{\text{T}}^{\nu'}(\mathbf{r})|\Delta(\mathbf{r})|^2\bigg\}, \end{aligned} \tag{A2}$$

where repeated indices ν and ν', which denote the vector components, are summed over. Note that since the superconducting order parameter carries electric charge, odd powers of Δ itself are prohibited. In contrast, as we show below there are odd terms in $\mathbf{H}_{\text{T}}$ and the gradient of Δ in the presence of SOC.

The contribution to the free energy proportional to $H_{\text{T}}^\nu H_{\text{T}}^{\nu'}$ has been calculated in Ref. [5] to all orders in Δ; it was shown there that these terms can be written as $\chi_\parallel(H_{\text{T}}^x)^2/2 + \chi_\parallel(H_{\text{T}}^y)^2/2 + \chi_\perp(H_{\text{T}}^z)^2/2$, where $\chi_\perp$ remains equal to the spin susceptibility of the normal state χ_N for all temperatures while $\chi_\parallel$ drops from its normal value χ_N near T_c to $\chi_N/2$ as $T \to 0$. Consequently, we will focus here on the derivation of the remaining terms, starting with those proportional to $|\Delta|^2$. These terms are given by [39]

$$F_\Delta^{(2)} = \int d\mathbf{r}\left[\lambda^{-1}|\Delta(\mathbf{r})|^2 - \int d\mathbf{r}'\Pi(\mathbf{r}-\mathbf{r}')\Delta^*(\mathbf{r})\Delta(\mathbf{r}')\right]. \tag{A3}$$

Here, $\Pi(\mathbf{r}-\mathbf{r}')$ is the s-wave, static pair-correlation function

$$\Pi(\mathbf{r}-\mathbf{r}') = \int dt \langle c^{\dagger}_{\mathbf{r},\uparrow}(t) c^{\dagger}_{\mathbf{r},\downarrow}(t) c_{\mathbf{r}',\downarrow}(0) c_{\mathbf{r}',\uparrow}(0)\rangle. \tag{A4}$$

The thermal average $\langle\ldots\rangle$ is performed with respect to the Hamiltonian in Eqs. (1) and (3). In other words, the contributions to $F^{(2)}_{\Delta}$ include the effect of magnetic fluctuations but are calculated in the absence of superconducting order $\Delta = 0$. Furthermore, the pair-correlation function is calculated up to second order in the gradients and first order in the total field:

$$\Pi(\mathbf{q}) \approx \Pi(\mathbf{q}=0,\mathbf{H}_{\mathrm{T}}=0) + q_\nu q_{\nu'} \left.\frac{\partial^2 \Pi(\mathbf{q},\mathbf{H}_{\mathrm{T}})}{\partial q_\nu \partial q'_\nu}\right|_{q,H_{\mathrm{T}}=0} + 2 q_\nu H^{\nu'}_{\mathrm{T}} \left.\frac{\partial^2 \Pi(\mathbf{q},\mathbf{H}_{\mathrm{T}})}{\partial q_\nu \partial H_{\nu'}}\right|_{q,H_{\mathrm{T}}=0}, \tag{A5}$$

where $\Pi(\mathbf{q})$ is the Fourier transform of Eq. (A4). The perturbative calculation of $\Pi(\mathbf{q})$ is determined by the unperturbed single-particle Green's function corresponding to $\mathcal{H}_0$ in Eq. (1). In the presence of SOC it has both diagonal and off-diagonal spin components:

$$\hat{g}_{\mathbf{k},\epsilon_n} = [i\epsilon_n - \hat{\mathcal{H}}_0 + \mu]^{-1} = \frac{\left(i\epsilon_n - \frac{k^2}{2m} + \mu\right)\sigma_0 + \alpha_R \hat{z}\cdot(\mathbf{k}\times\boldsymbol{\sigma})}{\left(i\epsilon_n - \frac{k^2}{2m} + \mu\right)^2 - \alpha_R^2 k^2}, \tag{A6}$$

where $\boldsymbol{\sigma}$ is a vector of Pauli matrices, and σ_0 is the identity matrix.

We turn now to the derivation of the various terms in Eq. (A5). The first term is simply

$$\Pi(\mathbf{q}=0,\mathbf{H}_{\mathrm{T}}=0) = T\sum_n \int \frac{d\mathbf{k}}{(2\pi)^2}\left[g^{\uparrow\uparrow}_{\mathbf{k},\epsilon_n} g^{\downarrow\downarrow}_{-\mathbf{k},-\epsilon_n} - g^{\uparrow\downarrow}_{\mathbf{k},\epsilon_n} g^{\downarrow\uparrow}_{-\mathbf{k},-\epsilon_n}\right] = T\sum_{n,\gamma=\pm}\int_{-\infty}^{\infty} d\xi_\gamma \frac{\mathcal{N}_\gamma}{\epsilon_n^2+\xi_\gamma^2}, \tag{A7}$$

where $\xi_\pm = k^2/2m \pm \alpha_R|k| - \mu$ is the energy spectrum which contains two Rashba bands labeled by $\gamma\pm$. The sum of the density of states ($\mathcal{N}_\pm$) of these bands is equal to the density of states in the absence of SOC. Therefore, the coefficient of the $|\Delta|^2$ term in the free energy is not modified by the SOC.

Similarly, the leading contributions to the second term in Eq. (A5), i.e., the superfluid stiffness, as well as the quartic term in Δ [which appears in Eq. (A2)] are also not modified by SOC. The third term is the first nontrivial contribution which is unique to systems without inversion symmetry. To demonstrate its derivation we first consider the case where $\mathbf{H}_{\mathrm{T}}$ points along the x direction:

$$\begin{aligned}
&2q_\nu H^x_{\mathrm{T}} \left.\frac{\partial^2\Pi(\mathbf{q},\mathbf{H}_{\mathrm{T}})}{\partial q_\nu \partial H_x}\right|_{q,H_{\mathrm{T}}=0}\\
&= -q_\nu g\mu_B H^x_{\mathrm{T}} \frac{\partial}{\partial q_\nu} T\sum_n \int \frac{d\mathbf{k}d\mathbf{k}'d\mathbf{k}''}{(2\pi)^6}\langle c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\uparrow} c^{\dagger}_{-\mathbf{k}+\mathbf{q}/2,\downarrow}[c^{\dagger}_{\mathbf{k}',\uparrow}c_{\mathbf{k}',\downarrow} + c^{\dagger}_{\mathbf{k}',\downarrow}c_{\mathbf{k}',\uparrow}] c_{-\mathbf{k}''+\mathbf{q}/2,\downarrow} c_{\mathbf{k}''+\mathbf{q}/2,\uparrow}\rangle\\
&= -q_\nu g\mu_B H^x_{\mathrm{T}} \frac{\partial}{\partial q_\nu} T\sum_n \int \frac{d\mathbf{k}}{(2\pi)^2}\big\{\big[g^{\uparrow\uparrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n} g^{\downarrow\downarrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n} + g^{\uparrow\uparrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n} g^{\downarrow\downarrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n}\big]\big[g^{\uparrow\downarrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n} + g^{\downarrow\uparrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n}\big]\\
&\quad - g^{\uparrow\uparrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n} g^{\downarrow\downarrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n}\big[g^{\downarrow\uparrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n} + g^{\uparrow\downarrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n}\big] - \big(g^{\downarrow\uparrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n}\big)^2 g^{\uparrow\downarrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n} - \big(g^{\uparrow\downarrow}_{\mathbf{k}+\mathbf{q}/2,\epsilon_n}\big)^2 g^{\downarrow\uparrow}_{-\mathbf{k}+\mathbf{q}/2,-\epsilon_n}\big\}.
\end{aligned} \tag{A8}$$

Using the expression in Eq. (A4), we find that the leading contributions are obtained when the derivative acts on the denominators of the various components of the Green's functions. Correspondingly, Eq. (A6) can be further simplified to give

$$\begin{aligned}
&2q_\nu H^x_{\mathrm{T}} \left.\frac{\partial^2\Pi(\mathbf{q},\mathbf{H}_{\mathrm{T}})}{\partial q_\nu \partial H_x}\right|_{q,H_{\mathrm{T}}=0}\\
&= \delta_{\nu,y} q_\nu g\mu_B H^x_{\mathrm{T}} \frac{\partial}{\partial q_\nu} T\sum_{n,\gamma=\pm}\gamma\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{k_y}{|k|}\frac{1}{[i\epsilon_n - \xi_\gamma(\mathbf{k}+\mathbf{q}/2)]^2[-i\epsilon_n - \xi_\gamma(-\mathbf{k}+\mathbf{q}/2)]}\\
&= q_y g\mu_B H^x_{\mathrm{T}} \frac{\partial}{\partial q_y} T\sum_{n,\gamma=\pm}\gamma\int_0^{2\pi}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty} d\xi_\gamma \mathcal{N}_\gamma \frac{\sin\theta}{(i\epsilon_n - \xi_\gamma - \mathbf{v}_F\cdot\mathbf{q}/2)^2(-i\epsilon_n - \xi_\gamma + \mathbf{v}_F\cdot\mathbf{q}/2)} = C(\mathcal{N}_+ - \mathcal{N}_-)\frac{g\mu_B H^x_{\mathrm{T}} v_F q_y}{T^2}.
\end{aligned} \tag{A9}$$

Here, $\mathbf{v}_F$ is the velocity at the Fermi energy which is identical for both Rashba bands, and $C \approx -0.03$. Note that the omitted contributions where the derivative acts on the numerators are smaller by T/ε_F which is several orders of magnitude below unity in BCS superconductors. The term in Eq. (A9) is proportional to the difference in the density of states of the two Rashba bands, and thus vanishes in the absence of SOC. The same calculation for the y component of the magnetic field yields a result proportional to q_x with the same magnitude but opposite sign. Therefore the full expression in the free energy can be written as $\propto \hat{z} \cdot (\mathbf{q} \times \mathbf{H}_T)$. Moreover, it is easy to show that there is no term that is linear in the out-of-plane component of the field $\mathbf{H}_{\mathrm{T}} \cdot \hat{z}$.

Preforming a Fourier transform on Eq. (A9) and collecting all contributions to $F^{(2)}$ we get

$$F \approx \int d\mathbf{r} \bigg\{ \alpha(T - T_c)|\Delta(\mathbf{r})|^2 + \beta|\Delta(\mathbf{r})|^4 + \frac{\tilde{w}}{2}|\nabla\Delta(\mathbf{r})|^2 + U\mathbf{M}^2(\mathbf{r}) - \frac{\chi_\perp}{2}H^2_{\mathrm{T}\perp}(\mathbf{r}) - \frac{\chi_{||}}{2}\mathbf{H}^2_{\mathrm{T}||}(\mathbf{r}) - i\frac{\tilde{\eta}}{2}[\hat{z} \times \mathbf{H}_{\mathrm{T}}(\mathbf{r})] \cdot [\Delta^*(\mathbf{r})\nabla\Delta(\mathbf{r}) - \Delta(\mathbf{r})\nabla\Delta^*(\mathbf{r})] \bigg\}. \tag{A10}$$

Here α, $\tilde{w}$, and β are given by their value in the absence of SOC [as demonstrated in Eq. (A6)]. The spin susceptibilities χ_ν remain finite deep in the superconducting state due to the SOC as shown in Ref. [5] and discussed below Eq. (A2). Finally, the parameter η was computed in Eqs. (A8) and (A9).

The derivation above was performed in the continuum, while our analysis is performed most conveniently on a lattice model. As explained in the main text, this does not refer to the crystalline lattice, but an effective description of the properties at wavelength longer than the superconducting coherence length ξ. As a final step we thus perform the following substitutions: (i) In purely local terms (without derivatives) we replace the continuous coordinate $\mathbf{r}$ by the discrete site label $\vec{j}$. (ii) In terms quadratic in the derivatives of the order parameter we replace $\nabla\Delta(\mathbf{r})$ by $[\Delta_{\vec{j}+\hat{\nu}} - \Delta_{\vec{j}}]/a$ where $\hat{\nu}$ is a vector connecting two nearest neighbors and a is the lattice spacing. (iii) For the final term, it is convenient (though not essential) to associate the magnetic field with bonds. Consequently, this term takes the form $(\hat{z} \times \hat{\nu}) \cdot (\mathbf{H}_{\mathrm{T}\vec{j}} + \mathbf{H}_{\mathrm{T}\vec{j}+\hat{\nu}})[\Delta^*_{\vec{j}}\Delta_{\vec{j}+\hat{\nu}} - \Delta^*_{\vec{j}+\hat{\nu}}\Delta_{\vec{j}}]$. Following this prescription we obtain Eq. (4) with $w = \tilde{w}/a^2$ and $\eta = \tilde{\eta}/a$.

To conclude this section, we note that although we derived the free energy near T_c, the same form holds deep in the superconducting state. This again follows directly from the symmetries of the problem. Alternatively, the expansion of the free energy in gradients of Δ and magnetic field can be calculated via the current-current, current-spin, and spin-spin correlation functions inside the superconducting state. Such a derivation has been performed in Ref. [40] for superconductors with strong SOC in the presence of a uniform magnetic field. It was found there that all parameters in Eq. (A10) remain nonzero near $T \approx 0$. Furthermore, from the treatment in Ref. [22] it follows that the free energy maintains its structure in the presence of disorder as long as the mean scattering rate is smaller than the SOC energy.

APPENDIX B: INTEGRATING OUT MAGNETIC FLUCTUATIONS

In this Appendix we provide technical details of integrating out the massive magnetic fluctuations from the free energy in Eq. (6). Since the free energy is quadratic in the magnetization, the corresponding partition function can be brought into the form $\mathcal{Z} = \int d\Phi_{\vec{j}} d\mathbf{M}_{\vec{j}} \exp[-a_{\vec{j}}(\mathbf{M}_{\vec{j}} - \mathbf{b}_{\vec{j}})^2 - F_{\mathrm{eff}}(\Phi_{\vec{j}})] \sim \int d\Phi_{\vec{j}} \exp[-F_{\mathrm{eff}}(\Phi_{\vec{j}})]$. Here F_{eff} is the effective free energy obtained by integrating out $\mathbf{M}$. To determine F_{eff} it is convenient to change the integration variable from $\mathbf{M}$ to $\mathbf{H}_{\mathrm{T}} = \mathbf{H} + 2U/g\mu_B\mathbf{M}$. Correspondingly, we write the free energy in Eq. (6) as

$$F = \sum_{\vec{j},\hat{\nu}} \bigg\{ -\rho_s \cos(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \frac{g^2\mu_B^2}{4U}\left(1 - \frac{2\chi_\perp U}{g^2\mu_B^2}\right)H^2_{\mathrm{T}\perp,\vec{j}} + \frac{g^2\mu_B^2}{4U}\left(1 - \frac{2\chi_{\|} U}{g^2\mu_B^2}\right)\mathbf{H}^2_{\mathrm{T}\|,\vec{j}} - \frac{g^2\mu_B^2}{4U}\mathbf{H}_{\vec{j}} \cdot \mathbf{H}_{\mathrm{T}\|,\vec{j}} + \frac{\kappa}{2}(\hat{z} \times \hat{\nu}) \cdot \mathbf{H}_{\mathrm{T}\|,\vec{j}}[\sin(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \sin(\Phi_{\vec{j}} - \Phi_{\vec{j}-\hat{\nu}})] \bigg\}. \tag{B1}$$

In deriving the above terms (and from here on), we have omitted contributions to the free energy that depend only on the external magnetic field $\mathbf{H}$ (and are independent of Φ and $\mathbf{H}_{\mathrm{T}}$). From Eq. (B1), the effective free energy can be easily read off to be

$$F_{\mathrm{eff}} = \sum_{\vec{j},\hat{\nu}} \left\{ -\rho_s \cos(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \frac{U/g^2\mu_B^2}{(1 - 2\chi_{\|}U/g^2\mu_B^2)}\left(\frac{g^2\mu^2}{2U}\mathbf{H}_{\vec{j}} - \frac{\kappa}{2}(\hat{z} \times \hat{\nu})[\sin(\Phi_{\vec{j}+\hat{\nu}} - \Phi_{\vec{j}}) + \sin(\Phi_{\vec{j}} - \Phi_{\vec{j}-\hat{\nu}})]\right)^2 \right\}. \tag{B2}$$

To bring F_{eff} into its final form [Eq. (7)], we again drop the constant term and absorb the linear terms in the external magnetic field into the cosine (first term) via the identity $a\cos x + b\sin x = \sqrt{a^2 + b^2}\cos(x + \Upsilon)$ with $\Upsilon = [\cos^{-1} b/\sqrt{a^2 + b^2} - \pi/2]\mathrm{sgn}(b)$ for $a > 0$. The explicit expressions for a, b, and Υ are given in Eq. (10).

APPENDIX C: MODULATED SUPERCONDUCTING STATE

In the main text we considered only phase modulations of the order parameter; here we examine the free energy allowing for spatial modulations of the amplitude. For this purpose, we start with Eq. (4) and integrate out the massive magnetic fluctuations in the absence of an external field:

$$F=\sum_{\vec{j},\hat{\upsilon}}\left\{\alpha(T-T_c)|\Delta_{\vec{j}}|^2+\beta|\Delta_{\vec{j}}|^4+\frac{w}{2}|\Delta_{\vec{j}}-\Delta_{\vec{j}+\hat{\upsilon}}|^2+\frac{\tilde{\eta}}{16}[\Delta^*_{\vec{j}}\Delta_{\vec{j}+\hat{\upsilon}}-\Delta^*_{\vec{j}+\hat{\upsilon}}\Delta_{\vec{j}}+\Delta^*_{\vec{j}-\hat{\upsilon}}\Delta_{\vec{j}}-\Delta^*_{\vec{j}}\Delta_{\vec{j}-\hat{\upsilon}}]^2\right\}. \tag{C1}$$

In the integration we neglected the weak dependence of the spin susceptibility on the superconducting order parameter, and introduced $\tilde{\eta}=U\eta^2(1-2\chi_\perp U/g^2\mu_B^2)^{-1}/(g\mu_B)^2$. In general, an amplitude-modulated order parameter can be written as $\Delta_{\vec{j}}=\sum_q\Delta_q e^{i\vec{j}\cdot\vec{q}}$. For simplicity, we consider here a trial order parameter with only two nonzero elements in the sum, $\Delta_{\vec{j}}=\Delta_1 e^{i\vec{j}\cdot\vec{q}_1}+\Delta_2 e^{i\vec{j}\cdot\vec{q}_2}$, where $\vec{q}_1$ and $\vec{q}_2$ are arbitrary wave vectors. The generalization to a higher number of terms is straightforward. Consequently, the free energy for the trial order parameter is

$$\begin{aligned}F=\sum_{\hat{\upsilon}}\Big\{&\alpha(T-T_c)\big[\Delta_1^2+\Delta_2^2\big]+\beta\big[\Delta_1^4+4\Delta_1^2\Delta_2^2+\Delta_2^4\big]+w\Delta_1^2[1-\cos(q_{1,\hat{\upsilon}})]+w\Delta_2^2[1-\cos(q_{2,\hat{\upsilon}})]\\&-\frac{\tilde{\eta}}{2}\{2\Delta_1^4\sin^2(q_{1,\hat{\upsilon}})+2\Delta_2^4\sin^2(q_{2,\hat{\upsilon}})+6\Delta_1^2\Delta_2^2\sin(q_{1,\hat{\upsilon}})\sin(q_{2,\hat{\upsilon}})+\Delta_1^2\Delta_2^2[\sin(q_{1,\hat{\upsilon}})+\sin(q_{2,\hat{\upsilon}})]^2\}\Big\}.\end{aligned} \tag{C2}$$

Next, we show that this free energy is minimal only when either Δ_1 or Δ_2 vanishes. The above expression can be further simplified by defining the effective parameters:

$$\begin{aligned}r_i&=\alpha(T-T_c)+w\sum_{\hat{\upsilon}}\big[1-\cos(q_{i,\hat{\upsilon}})\big];\\u_i&=\beta-\tilde{\eta}\sum_{\hat{\upsilon}}\sin^2(q_{i,\hat{\upsilon}});\\v&=4\beta-\frac{\tilde{\eta}}{2}\sum_{\hat{\upsilon}}[\sin^2(q_{1,\hat{\upsilon}})+\sin^2(q_{2,\hat{\upsilon}})+3\sin(q_{1,\hat{\upsilon}})\sin(q_{2,\hat{\upsilon}})].\end{aligned} \tag{C3}$$

Well below mean-field T_c, and as long as the gradient terms in the free energy are small ($\alpha T_c\gg w$ and $\beta\gg\tilde{\eta}$), the parameters in the free energy satisfy $r_i<0$ and $u_i,v>0$. Then, the free energy can be written as

$$F=r_1\Delta_1^2+r_2\Delta_2^2+u_1\Delta_1^4+u_2\Delta_2^4+v\Delta_1^2\Delta_2^2. \tag{C4}$$

To analyze Eq. (C4) we rescale $\Delta_1\to\tilde{\Delta}_1$ and $\Delta_2\to\tilde{\Delta}_2\sqrt{r_1/r_2}$, as well as $u_1\to\tilde{u}_1$, $u_2\to\tilde{u}_2r_2^2/r_1^2$, and $v\to\tilde{v}r_2/r_1$:

$$\begin{aligned}F=&\,r_1\big[\tilde{\Delta}_1^2+\tilde{\Delta}_2^2\big]+\frac{\tilde{u}_1+\tilde{u}_2+\tilde{v}}{4}\big[\tilde{\Delta}_1^2+\tilde{\Delta}_2^2\big]^2\\&+\frac{\tilde{u}_1+\tilde{u}_2-\tilde{v}}{4}\big[\tilde{\Delta}_1^2-\tilde{\Delta}_2^2\big]^2\\&+\frac{\tilde{u}_1-\tilde{u}_2}{2}\big[\tilde{\Delta}_1^2+\tilde{\Delta}_2^2\big]\big[\tilde{\Delta}_1^2-\tilde{\Delta}_2^2\big].\end{aligned} \tag{C5}$$

Finally, we use the parametrization $\tilde{\Delta}_1=S\cos\theta$ and $\tilde{\Delta}_2=S\sin\theta$:

$$\begin{aligned}F=&\,r_1S^2+\frac{\tilde{u}_1+\tilde{u}_2+\tilde{v}}{4}S^4+\frac{\tilde{u}_1+\tilde{u}_2-\tilde{v}}{4}S^4\cos^2 2\theta\\&+\frac{\tilde{u}_1-\tilde{u}_2}{2}S^4\cos 2\theta.\end{aligned} \tag{C6}$$

Under the conditions $\alpha T_c\gg c$ and $\beta\gg\tilde{\eta}$, the coefficient of $\cos^2 2\theta$ is negative ($\tilde{u}_1+\tilde{u}_2-\tilde{v}<0$). As a result, the extrema of the free energy occur at $\theta=\pi n/2$ with $n=0,1,\ldots$, i.e., either $\Delta_1=0$ or $\Delta_2=0$. Among these, the minima and maxima are determined by the sign of $\tilde{u}_1-\tilde{u}_2$. Thus, we explicitly show that the energy is minimal in the absence of amplitude modulations, $\Delta_{\vec{j}}=\Delta_0 e^{i\vec{q}\cdot\vec{j}}$.

APPENDIX D: PHASE DIAGRAM IN RING GEOMETRY

In the main text we studied the effect of magnetic fluctuations on the superconducting state in the presence of SOC. Here we analyze the system in a ring geometry, i.e., a one-dimensional superconductor with periodic boundary conditions. We therefore assume a narrow ring of radius R with thickness smaller than the coherence length ξ. To impose the boundary conductions on the free energy in Eq. (4), we write the order parameter in terms of angular harmonics,

$$\Delta_x=\sqrt{\frac{1}{N}}\sum_{n=-N/2}^{N/2}\Lambda_n e^{2\pi ixn/N}. \tag{D1}$$

Integrating out the magnetization, and using Eq. (D1), the free energy takes the form

$$F_{\rm ring}=\sum_{n=-N/2}^{N/2}\left\{\alpha(T-T_c)+w\left[1-\cos\left(\frac{2\pi n}{N}\right)\right]\right\}|\Lambda_n|^2$$

$$+\frac{1}{4N}\sum_{n,m,p,\ell}\delta_{n-m,\ell-p}\left\{\frac{\tilde{\eta}}{4}(e^{2\pi im/N}-e^{-2\pi in/N})(e^{2\pi i\ell/N}-e^{-2\pi ip/N})+\beta\right\}$$
$$\times[2\Lambda_n^*\Lambda_m\Lambda_p^*\Lambda_\ell+\Lambda_n^*\Lambda_m\Lambda_\ell^*\Lambda_p+\Lambda_m^*\Lambda_n\Lambda_p^*\Lambda_\ell]. \quad \text{(D2)}$$

Similar to the treatment in the previous section, we neglect the weak dependence of the spin susceptibility on the superconducting order parameter, and define $\tilde{\eta}=U\eta^2(1-2\chi_\perp U/g^2\mu_B^2)^{-1}/(g\mu_B)^2$. Our analysis of the phase diagram is performed in the limit where the lattice spacing $a=2\pi R/N$ satisfies $\xi\ll a\ll R$. Under these conditions $w\ll\alpha T_c$, as inferred from the known expression for the free energy in the continuum limit, $w\sim\alpha T_c\xi^2/a^2\ll\alpha T_c$. We restrict our analysis to $\tilde{\eta}\ll\beta$. To study the opposite limit $\tilde{\eta}\gtrsim\beta$, it is necessary to take into account terms of order $|\Lambda_n|^6$ in the free energy which is beyond the scope of this work.

Below the transition temperature T_c, the free energy is minimized by $\Lambda_n\neq 0$ for a single value of n. That is, the superconducting state has a well defined angular momentum (harmonic). Upon crossing T_c from above, a superconducting state with uniform phase $\Lambda_0\neq 0$ forms when temperature is not too low. To observe nonuniform phases with $n\neq 0$, temperature has to be lowered below the n-dependent transition temperature $T_c(n)=T_c-\frac{w}{\alpha}[1-\cos(\frac{2\pi n}{N})]$. When temperature crosses $T_c(n)$ the free energy acquires two additional minima at

$$\left|\Lambda_{\pm n}^{\min}\right|^2=-N\frac{\alpha(T-T_c)+w[1-\cos(2\pi n/N)]}{2[\beta-\tilde{\eta}\sin^2(2\pi n/N)]}. \quad \text{(D3)}$$

The corresponding state is characterized by a phase that winds around the ring. The appearance of new minima does not necessarily indicate a transition into a superconducting state with $n\neq 0$. Rather, the transition occurs only when the corresponding free energy

$$F_{\text{ring}}^{\min}(n)=-N\frac{\{\alpha(T-T_c)+w[1-\cos(2\pi n/N)]\}^2}{4[\beta-\tilde{\eta}\sin^2(2\pi n/N)]} \quad \text{(D4)}$$

becomes the global minimum. Exploring the phase diagram in the ring geometry, we obtain that for

$$\tilde{\eta}<\tilde{\eta}_c=\frac{\beta w}{\alpha T_c\cos^2(\pi/N)}\left[1-\frac{w}{\alpha T_c}\sin^2(\pi/N)\right] \quad \text{(D5)}$$

the system remains in the uniform phase for all $T<T_c$. For $N\to\infty$, this condition coincides with the critical ζ obtained in the planar geometry. When the strength of SOC or magnetic fluctuations increases and $\tilde{\eta}$ grows beyond $\tilde{\eta}_c$, the global minimum changes from $\Lambda_{\pm n}$ to $\Lambda_{\pm(n+1)}$. The transition lines as a function of $\tilde{\eta}$ and T, shown in Fig. 3, are determined by the equation $F_{\text{ring}}^{\min}(n)=F_{\text{ring}}^{\min}(n+1)$. One unique property of the superconducting state with $n\neq 0$ is that it supports persistent currents without external magnetic field as indicated in Eq. (9) and illustrated in Fig. 3. Note that by expanding Eq. (D4) at low temperature with respect to $\tilde{\eta}$ and w one recovers the phase-only free energy of Eq. (8) with $\rho_s=2w\alpha T_c/\beta$ and $\zeta=\tilde{\eta}(\alpha T_c/\beta)^2$.

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