Luttinger liquids, Fermi liquids, and fractional statistics

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We discuss how one-dimensional interacting fermion systems, which in the low-energy approximation are described by Luttinger liquid theory, can be reformulated as systems of weakly interacting particles with fractional charge and statistics. Our approach is to use Landau's phenomenological approach to Fermi liquid theory, where the quasiparticles are interpreted as adiabatically dressed fermions. In an earlier publication, the local charge carried by these excitations has been shown to be fractional. We focus here on the statistics of the quasiparticles and show that by a change of momentum variables the Landau parameters of the generalized Fermi fluid can be transformed to zero. This change in interaction is compensated by a change of the entropy function, which is consistent with the interpretation of the quasiparticles as satisfying generalized exclusion statistics.

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I. INTRODUCTION

In one dimension, the perturbative approach to Fermi liquid theory fails, in the sense that particle interactions generically give rise to intractable infrared divergences. However, Luttinger liquid theory, as shown by Haldane [1], can be viewed as a well functioning replacement. In this formulation bosonic operators, related to the particle densities, are the fundamental variables, and in the low-energy approximation, with linearized dispersion, the Hamiltonian takes a free-field form with the effects of interactions absorbed in velocity parameters. Nonlinear corrections to the theory can in principle be included in the form of bosonic interaction terms.

Even if Luttinger liquid theory is well described in terms of bosonic variables, elementary charged excitations do exist in the one-dimensional Fermi system, and there has been in later years an interest in studying the properties of such excitations, both theoretically and experimentally [2–10]. The conclusion is that the excitations will generally carry a fraction of the fermionic charge. The fractionalization is linked to chiral separation of charges that are introduced in the system [11,12], so that fractions of a unit charge move to the right and the left, respectively. There has in particular been interest in the study of this effect for edge excitations in quantum Hall systems, where interactions between edge modes give rise to the charge fractionalization [7–10].

However, one should note the important difference between the charge fractionalization effect in the bulk of the quantum Hall system and at the edges. In the former case, the excitations have well defined, topologically protected charge values, whereas in the latter case this is not the case. Instead the charges will there generally depend on the initial conditions which are responsible for the creation of the excitations. A particular way to introduce these excitations is to create them in the noninteracting system, and then adiabatically turn on the interaction between the edge modes. This situation was examined in Refs. [8,9], where it was shown how an integer charged edge excitation, in a quantum Hall bar with variable width, can adiabatically evolve into a fractionally charged excitation, with a compensating charge being smoothly absorbed into the background.

In the present paper, we focus on the *statistics* of the adiabatically dressed excitations. Our results are related to those in Refs. [13–15], where connections between generalized exclusion statistics and Luttinger liquids are discussed. However, our approach, which is based on the use of Landau's phenomenological approach to Fermi liquid theory, is different. The starting point is the existence of an adiabatic mapping between the noninteracting and the interacting system of one-dimensional fermions, which makes the Fermi liquid theory valid. Since this mapping is known in the low-energy approximation, the one-particle energy and the two-particle interactions can be derived from the energy of the system by functional differentiation with respect to the particle density. We further show, by use of a functional transformation, that the interaction terms can be absorbed in such a way that the quantum statistics of the quasiparticles is effectively changed. We will use in this paper the convention $\hbar = 1$.

II. THE LUTTINGER LIQUID FORMULATION

We take as the starting point the following general expression for the Hamiltonian of a one-dimensional system of spinless fermions:

$$H = \sum_k \epsilon_0(k) c_k^\dagger c_k + \frac{1}{4L} \sum_{q,k_1,k_2} V(k_1 - k_2, q) c_{k_1}^\dagger c_{k_2}^\dagger c_{k_2-q} c_{k_1+q}. \tag{1}$$

V is allowed to depend on the momentum variable $k_1 - k_2$, in addition to q, in order to be able to include in this formulation the one-dimensional description of quantum Hall systems with interedge interactions [16]. Galilei invariance is not broken by the interaction, since it only depends on the relative variable $k_1 - k_2$. We shall, however, assume that the dependence on this variable is weak. This implies that we can disregard the effect of the k dependence for low-energy particles close to the same Fermi point, while for the interaction between particles at opposite Fermi points the effect will generally be significant. For electrons in the lowest Landau level, the condition of weak dependence means that the magnetic length is much smaller than the range of the interaction between the particles [16].

The ground state of the Hamiltonian we assume to have the form of a filled Fermi sea, with well-defined Fermi points for negative and positive momenta, $k = \pm k_F$. In the low-energy approximation, the excitations of the system are restricted to

momenta near the two Fermi points, with the following low-energy form of the Hamiltonian [1]:

$$H = \bar{v}_F \sum_{\chi,k} (\chi k - k_F) : c^\dagger_{\chi,k} c_{\chi,k} : + \frac{1}{4L} \sum_{\chi,q} [V_1(q) \rho_{\chi,q} \rho_{\chi,-q} + V_2(q) \rho_{\chi,q} \rho_{-\chi,-q}], \quad (2)$$

where $\chi = \pm 1$ is a chirality parameter, associated with the two Fermi points, and $\rho_{\chi,q} = \sum_k c^\dagger_{\chi,k+q} c_{\chi,k}$ are the Fourier components of the particle density of chirality χ. The system is assumed to be confined to an interval of length L, and with periodic boundary conditions for the fermion fields, the momentum then takes discrete values $k = 2\pi n/L$, with n as an integer. The interaction has been separated in two parts with $V_1(q) = V(0,q)$ and $V_2(q) = V(2k_F,q)$, and the density operators are normal ordered relative to the filled Fermi sea. The effective Fermi velocity $\bar{v}_F$ has the form

$$\bar{v}_F = v_F - \frac{1}{4\pi}(V_1(0) - V_2(0)) \quad (3)$$

with $v_F = \frac{\partial \epsilon_0}{\partial k}(k_F)$ as the Fermi velocity of the noninteracting Fermi system, and the interaction dependent term is a correction, created by interactions between the low-energy fermions and the Fermi sea [16].

Although the k quantum number is in the low-energy approximation restricted to small deviations from $\pm k_F$, this restriction can be lifted, since the low-energy sector of the theory is not affected by this extension. Without the restriction, the model (2) describes in effect two types of fermions, characterized by different values of χ, both types with linear dispersion.

The standard way to analyze the system described by the Hamiltonian (2) is in terms of bosonization. We briefly summarize expressions to be used in the discussion to follow. The Fourier components of the charge density operators, for $q \neq 0$, are written as boson annihilation and creation operators,

$$a_q = \sqrt{\frac{2\pi}{|q|L}} \sum_\chi \theta(\chi q)\, \rho_{\chi,q}\,, \quad a^\dagger_q = \sqrt{\frac{2\pi}{|q|L}} \sum_\chi \theta(\chi q)\, \rho_{\chi,-q} \quad (4)$$

with $\theta(q)$ as the Heaviside step function. The $q = 0$ components of the charge densities define the conserved fermion number and chiral (current) quantum number

$$N = \sum_\chi N_\chi = \sum_{k\,\chi} : c^\dagger_{\chi,k}\, c_{\chi,k} :\,, \quad J = \sum_\chi \chi N_\chi = \sum_{k\,\chi} \chi\, c^\dagger_{\chi,k}\, c_{\chi,k} \quad (5)$$

with N measuring the deviation of the particle number from its ground state value N_0. The bosonized form of the Hamiltonian is [1]

$$H = \frac{\pi}{2L}(v_N N^2 + v_J J^2) + \frac{1}{2} \sum_{q\neq 0} |q| \left[\left(\bar{v}_F + \frac{V_1(q)}{4\pi} \right)(a^\dagger_q a_q + a_q a^\dagger_q) + \frac{V_2(q)}{4\pi}(a^\dagger_q a^\dagger_{-q} + a_q a_{-q}) \right], \quad (6)$$

which relative to the Hamiltonian (2) has been modified by adding (nonrelevant) terms that are constant or linear in N. The two velocity parameters v_N and v_J are

$$v_N = \bar{v}_F + \frac{1}{4\pi}(V_1(0) + V_2(0)) = v_F + \frac{1}{2\pi} V_2(0), \quad v_J = \bar{v}_F + \frac{1}{4\pi}(V_1(0) - V_2(0)) = v_F. \quad (7)$$

We note in particular that v_J is identical to the original Fermi velocity v_F of the noninteracting fermions, rather than to the effective Fermi velocity $\bar{v}_F$ that appears in the Luttinger Hamiltonian (2). This can be viewed as a consequence of Galilei invariance of the original Hamiltonian (1). The low-energy sector, where (6) is valid, corresponds to situations where $|q|$, as well as N/L and $|J|/L$, are effectively restricted to values much smaller than k_F.

The bosonized Hamiltonian is diagonalized by a Bogoliubov transformation of the form

$$a_q = \cosh \xi_q\, b_q + \sinh \xi_q\, b^\dagger_{-q}, \quad a^\dagger_q = \cosh \xi_q\, b^\dagger_q + \sinh \xi_q\, b_{-q}, \quad (8)$$

where ξ_q is fixed by the relation

$$\tanh 2\xi_q = -\frac{V_2(q)}{V_1(q) + 4\pi\, \bar{v}_F}. \quad (9)$$

In terms of the new bosonic operators, the Hamiltonian gets the diagonal form

$$H = \sum_{q\neq 0} \omega_q\, b^\dagger_q b_q + \frac{\pi}{2L}(v_N N^2 + v_J J^2) \quad (10)$$

with the frequency ω_q given by

$$\omega_q = \sqrt{\left(\bar{v}_F + \frac{V_1(q)}{4\pi} \right)^2 - \left(\frac{V_2(q)}{4\pi} \right)^2}\, |q|. \quad (11)$$

The bosonized form of the low-energy Hamiltonian (10) has, for given values of N and J, a free field form, which makes it straightforward to solve the many-particle problem and in particular to determine the relevant correlation functions [1]. However, for our purpose, it will be useful to reintroduce fermion variables in the expression for the Hamiltonian.

III. ADIABATICALLY DRESSED FERMIONS

The two sets of bosonic operators are unitarily equivalent,

$$b_q = U a_q U^\dagger\,, \quad b^\dagger_q = U a^\dagger_q U^\dagger \quad (12)$$

with the unitary transformation given by

$$U = \exp\left[-\sum_{q\neq 0} \frac{\xi_q}{2} \left(a^2_q - a^{\dagger 2}_q \right) \right]. \quad (13)$$

The operator U preserves the particle number of the two chiralities separately and it maps energy eigenstates of the linearized, free theory continuously into the eigenstates of the interacting theory, when the parameters ξ_q are changed. The transformation can thus be interpreted as defining an adiabatic change of the energy eigenstates during a slow turning on of the interactions V_1 and V_2.

For the fermion operators, we introduce the corresponding transformation

$$\phi(x) = U\psi(x)U^\dagger \tag{14}$$

with $\psi(x) = \frac{1}{\sqrt{L}}\sum_k c_k$ as the original fermion operator. We refer to the states created by $\phi^\dagger(x)$ as the dressed fermion states. In Ref. [8], charged excitations created by the dressed fermion operator were studied, and it was shown that the integer fermion charge introduced by $\phi^\dagger(x)$ was split in a noninteger, local contribution, equal to $\sqrt{g} = (v_J/v_N)^{1/4}$, and a compensating charge which was evenly distributed over the background. A further study of this effect in Ref. [9] confirmed this result numerically, and it showed how a slow change of the interaction between the edges of a quantum Hall bar would dynamically separate the total charge of a moving electron into a sharply defined front pulse with charge $\sqrt{g}$, followed by a long, extended tail.

The demonstration of the charge fractionalization is in fact quite easy in the bosonic representation. To show this, we focus on the Fourier components ρ_q of the fermion density operator. For $q = 0$, this is identical to the total fermion number, while the limit $q \to 0$ defines what is interpreted as the local part. The difference between these two is clearly seen in the case of a filled Fermi sea with N particles, where $\rho_0 = N$, while $\rho_q = 0$ for all $q \neq 0$. With excitations included, the charge component ρ_q (for $q \neq 0$) has, in the low-energy approximation, the following simple relation to the bosonic creation and annihilation operators:

$$\begin{aligned}\rho_q &= \sqrt{\frac{L|q|}{2\pi}}(a_q + a^\dagger_{-q}) \\ &= \sqrt{\frac{L|q|}{2\pi}}(\cosh\xi_q + \sinh\xi_q)(b_q + b^\dagger_{-q}),\end{aligned} \tag{15}$$

which gives

$$U^\dagger\rho_q U = (\cosh\xi_q + \sinh\xi_q)\rho_q. \tag{16}$$

In the limit $q \to 0$, we have the following expressions for $\cosh\xi_q$ and $\sinh\xi_q$:

$$\cosh\xi_0 = \frac{g+1}{2\sqrt{g}}, \quad \sinh\xi_0 = \frac{g-1}{2\sqrt{g}}, \tag{17}$$

which gives

$$\lim_{q\to 0} U^\dagger\rho_q U = \sqrt{g}\lim_{q\to 0}\rho_q. \tag{18}$$

Let us assume that $\Psi^\dagger$ adds a local charge $Q = 1$ to the Fermi sea of the noninteracting system, with the new (normalized) state written as

$$|\psi\rangle = \Psi^\dagger|F\rangle, \tag{19}$$

$|F\rangle$ represents the filled Fermi sea, and the assumption about the charge is

$$\lim_{q\to 0}\langle F|\Psi\rho_q\Psi^\dagger|F\rangle = \langle F|\Psi\rho_0\Psi^\dagger|F\rangle = 1. \tag{20}$$

The dressed state is $U\Psi^\dagger|F\rangle$, and the corresponding (local) dressed charge is then

$$\lim_{q\to 0}\langle F|\Psi U^\dagger\rho_q U\Psi^\dagger|F\rangle = \sqrt{g}\lim_{q\to 0}\langle F|\Psi\rho_q\Psi^\dagger|F\rangle = \sqrt{g}, \tag{21}$$

which shows the result that, with repulsive interaction ($g < 1$), only a part of the original fermion charge appears as the local charge of the dressed fermion, while the remaining part is distributed evenly as a part of the background.

However, to show the unconventional statistics of the dressed fermions is not so straightforward. The transformed field $\phi(x)$ clearly satisfies the same anticommutation relations as the electron field $\psi(x)$, and in this sense is a fermion field. However, the statistics of the particles is not necessarily apparent in the commutation relations of the fields alone, since the form of the Hamiltonian may reveal the presence of a "statistical interactions" between the particles. For this reason, we will examine more closely the form of the Hamiltonian, when it is expressed in fermionic variables.

To proceed, we first make the low-energy approximation

$$V_1(q) \approx V_1(0), \quad V_2(q) \approx V_2(0), \tag{22}$$

which for the boson frequency implies

$$\omega_q \approx v_s|q|, \quad v_s = \sqrt{v_J v_N}. \tag{23}$$

For the transformation U, this gives

$$U \approx \exp\left[-\sum_{q\neq 0}\frac{\xi_0}{2}\left(a_q^2 - a_q^{\dagger 2}\right)\right], \quad \tanh\xi_0 = \frac{g-1}{g+1}, \tag{24}$$

which implies that U, in this approximation, is uniquely determined by the interaction parameter $g = \sqrt{v_J/v_N}$.

We now separate the Hamiltonian in two parts in the following way:

$$\begin{aligned}H = U v_s\left[\sum_{q\neq 0}|q|a_q^\dagger a_q + \frac{\pi}{2L}(N^2 + J^2)\right]U^\dagger \\ + v_s\frac{\pi}{2L}\left[\left(\frac{1}{g} - 1\right)N^2 + (g-1)J^2\right],\end{aligned} \tag{25}$$

where the first term can be identified as a linearized free-field Hamiltonian, with v_s as Fermi velocity, and with the field variables transformed by the operator U. This implies that the Hamiltonian can be expressed in terms of the dressed fermion field as

$$\begin{aligned}H = v_s\Bigg\{\int_0^L dx :\sum_\chi \phi_\chi^\dagger(x)(-i\chi\partial_x - k_F)\phi_\chi(x): \\ + \frac{\pi}{2L}\sum_\chi\left[\left(\frac{1}{g} + g - 2\right)N_\chi^2 + \left(\frac{1}{g} - g\right)N_\chi N_{-\chi}\right]\Bigg\},\end{aligned} \tag{26}$$

where the chiral fields are defined by

$$\psi_\chi(x) = \frac{1}{\sqrt{L}}\sum_k c_{\chi,k}e^{ikx}, \quad \phi_\chi(x) = U\psi_\chi(x)U^\dagger. \tag{27}$$

IV. FERMI LIQUID DESCRIPTION

In the low-energy approximation, we have an explicit expression (13) for the operator U that adiabatically changes the energy eigenstates of the noninteracting system into those of the interacting one. With corrections to this approximation

included, we may still assume the interacting system to be adiabatically connected to the noninteracting one, although there will be corrections to the form of the adiabatic mapping. This assumption of adiabatic connection between the noninteracting and interaction system forms the basis for Landau's Fermi liquid theory, which therefore is applicable in the present case, beyond the low-energy approximation (26).

The total energy is then given as a functional of the distribution of occupation numbers $n(k)$, associated with the noninteracting theory,

$$E = E[n(k)], \tag{28}$$

and the quasiparticle energies and interactions can be defined in terms of functional derivatives to first and second orders in the particle density [17],

$$\delta E = \sum_k \epsilon(k)\delta n(k) + \frac{1}{2}\sum_{kk'} f(k,k')\delta n(k)\delta n(k'). \tag{29}$$

The quasiparticles introduced by Landau in this way we will here identify with the dressed fermions previously discussed. For variations about the filled Fermi sea, the expressions for energy and interactions will be referred to as $\epsilon_0(k)$ and $f_0(k,k')$.

In the low-energy approximation, the energy and interaction terms can be extracted from the Hamiltonian (26). With χ related to k by $\chi = \operatorname{sgn} k$, the corresponding expression for the variation of the energy is

$$\delta E = \sum_k v_s(|k| - k_F)\delta n(k) + v_s\frac{\pi}{L}\sum_{k,k'}(\lambda_1\theta(kk') + \lambda_2\theta(-kk'))\delta n(k)\delta n(k') \tag{30}$$

with $\theta(k)$ as the Heaviside step function, and with λ_1 and λ_2 defined by

$$\lambda_1 = \frac{1}{2}\left(\frac{1}{g} + g - 2\right), \quad \lambda_2 = \frac{1}{2}\left(\frac{1}{g} - g\right). \tag{31}$$

From this follows that the single-particle energy and the interaction terms are

$$\epsilon_0(k) = v_s(|k| - k_F),$$
$$f_0(k,k') = v_s\frac{2\pi}{L}(\lambda_1\theta(kk') + \lambda_2\theta(-kk')). \tag{32}$$

When corrections to the low-energy Hamiltonian (26) are included, we interpret the above expression for the interaction to be valid at the Fermi points, written as

$$f_0(k_F,k_F) = f_0(-k_F,-k_F) = v_s\frac{2\pi}{L}\lambda_1,$$
$$f_0(k_F,-k_F) = f_0(-k_F,k_F) = v_s\frac{2\pi}{L}\lambda_2. \tag{33}$$

The symmetric and antisymmetric combinations of the interaction terms define the two Landau parameters, which after normalization with respect to the density of states are

$$F_0 = \frac{L}{2\pi v_s}(f_0(k_F,k_F) + f_0(k_F,-k_F)) = \frac{1}{g} - 1,$$
$$F_1 = \frac{L}{2\pi v_s}(f_0(k_F,k_F) - f_0(k_F,-k_F)) = g - 1. \tag{34}$$

This gives the following relation [18]:

$$1 + F_1 = \frac{1}{1+F_0} = g. \tag{35}$$

It is of interest to relate this result to the condition of Galilean invariance, as expressed in the Fermi liquid formulation. This condition is written as [17]

$$\int dk\, k\, n(k) = \int dk\, m\frac{\partial\epsilon(k)}{\partial k}n(k), \tag{36}$$

where m is the (bare) mass of the fermions and the occupation numbers are treated as a continuous function of k. We assume in the following $n(k)$ to be the particle density normalized relative to the fully occupied system, which means that it takes values in the interval $0 \leqslant n(k) \leqslant 1$. The equation above states that the total momentum is conserved when the interaction is adiabatically turned on. Variation in the particle density gives

$$\int dk\,\frac{k}{m}\delta n(k) = \int dk\frac{\partial\epsilon(k)}{\partial k}\delta n(k) + \frac{L}{2\pi}\iint dkdk'\,\frac{\partial f(k,k')}{\partial k}n(k)\delta n(k'), \tag{37}$$

where the last term is the result of treating $\epsilon(k)$ as a functional of $n(k)$. Assuming this is valid for arbitrary variations $\delta n(k)$, we get the following relation:

$$\frac{k}{m} = \frac{\partial\epsilon(k)}{\partial k} - \frac{L}{2\pi}\int dk'\, f(k,k')\frac{\partial n(k')}{\partial k'}. \tag{38}$$

For a filled Fermi sea, the derivative of the particle density is

$$\frac{\partial n_0(k')}{\partial k'} = \delta(k' + k_F) - \delta(k' - k_F) \tag{39}$$

and with $k = k_F$, Eq. (38) gets the form

$$\frac{k_F}{m} = \left.\frac{\partial\epsilon_0(k)}{\partial k}\right|_{k_F} + \frac{L}{2\pi}(f_0(k_F,k_F) - f_0(k_F,-k_F)). \tag{40}$$

We make the following identifications:

$$v_F = \frac{k_F}{m}, \quad v_s = \left.\frac{\partial\epsilon_0(k)}{\partial k}\right|_{k_F} \equiv \frac{k_F}{m^*} \tag{41}$$

with v_F as the Fermi velocity of the noninteracting fermions, v_s as the Fermi quasiparticle velocity, and m^* as the effective mass of the quasiparticles. This gives

$$\frac{m^*}{m} = \frac{v_F}{v_s} = 1 + F_1 = g. \tag{42}$$

If we further apply the identities

$$v_s = \sqrt{v_J v_N}, \quad g = \sqrt{v_J/v_N}, \tag{43}$$

we find

$$v_J = v_F, \tag{44}$$

which is consistent with the earlier result (7). Here it follows as consequence of Galilei invariance in Landau's Fermi liquid formulation, whereas the result in (7) is a consequence of the corresponding symmetry of the two-particle interaction $V(k_1 - k_2, q)$. The equality between $1 + F_1$ and $(1 + F_0)^{-1}$ in Eq. (35) can be seen as a consequence of the equality in the one

dimensional system between the velocity of the quasiparticle excitations and the velocity of sound v_s.

V. QUANTUM STATISTICS

A central element in the Fermi liquid theory is the assumption that the elementary excitations (quasiparticles) of the theory obey Fermi-Dirac statistics. This means that the entropy function has the same form as for the noninteracting (bare) particles,

$$S = -\sum_k [n(k)\ln n(k) + (1-n(k))\ln(1-n(k))]. \quad (45)$$

In the case discussed in the previous sections, this follows since the dressed particle field $\phi(x)$ is related to the original fermion field $\psi(x)$ by a unitary transformation. However, a further change of variable will now be introduced, which changes this relation. This is not done in the form of a transformation of the field operators, but rather by introducing a new momentum variable, with a stronger repulsion between neighboring values than demanded by the Pauli exclusion.

To this end, assume $k_i = 2\pi n_i/L$ to be the coordinates in k-space of a set of particles, with n_i as integers that increase monotonically with i. The transformation to the new momentum coordinates κ_i is then defined by

$$\kappa_i = k_i + \lambda\frac{\pi}{L}\sum_{j\neq i}\mathrm{sgn}(k_i - k_j), \quad i = 1,2,\ldots, \quad (46)$$

where λ is a new, real parameter. Transformations of this form, known as Bethe ansatz equations, are used to solve certain types of one-dimensional many-body problems. Here we focus, however, on the transformation's relation to generalization of the Pauli exclusion between fermions. Thus the equation leads to the following effective repulsion between the κ values:

$$\kappa_{i+1} = \kappa_i + \frac{2\pi}{L}(\Delta n_i + \lambda) \quad (47)$$

with Δn_i as a positive integer. The equation can be interpreted as expressing that each new particle introduced in the system will occupy a one-dimensional volume $2\pi(1+\lambda)/L$, as compared to $2\pi/L$ for fermions. This can be expressed more directly by the formula

$$\Delta d = -(1+\lambda)\Delta N, \quad (48)$$

where Δd is the change in the number of available single-particle states within a fixed, finite interval in κ space, when ΔN particles are introduced in the interval. This formulation corresponds to Haldane's defining relation of generalized exclusion statistics [19], with d interpreted as the dimension of the Hilbert space, which is available for a new particle that is added to the system, and with $1+\lambda$ as the exclusion statistics parameter.

In the thermodynamic limit, the relation (46) between k_i and κ_i defines a mapping between the corresponding continuous variables k and κ, which depend on the particle density $n(k)$ in the following way:

$$\kappa = k + \frac{1}{2}\lambda\int dk'\, n(k')\,\mathrm{sgn}(k-k'). \quad (49)$$

The density $\nu(\kappa)$ corresponding to the new variable κ is defined by

$$\nu(\kappa)\,d\kappa = n(k)\,dk, \quad (50)$$

which simply states that the number of occupied states is conserved (locally) under the mapping $k \to \kappa$. It follows directly that the two densities are related by

$$\nu(\kappa) = \frac{n(k)}{1+\lambda n(k)}, \quad n(k) = \frac{\nu(\kappa)}{1-\lambda\nu(\kappa)} \quad (51)$$

with k and κ related as shown in (49). With $n(k)$ limited by $0 \leqslant n(k) \leqslant 1$, the corresponding restriction on $\nu(\kappa)$ is $0 \leqslant \nu(\kappa) \leqslant 1/(1+\lambda)$.

The fermion entropy (45), in the continuum form is

$$S = -\frac{L}{2\pi}\int_{-\infty}^{\infty} dk[n(k)\ln n(k) + (1-n(k))\ln(1-n(k))] \quad (52)$$

and as follows from (50) and (51), it takes the following form when expressed in terms of the new variables,

$$S = -\frac{L}{2\pi}\int_{-\infty}^{\infty} d\kappa[\nu(\kappa)\ln\nu(\kappa) - (1-\lambda\nu(\kappa))\ln(1-\lambda\nu(\kappa)) + (1-(1+\lambda)\nu(\kappa))\ln(1-(1+\lambda)\nu(\kappa))]. \quad (53)$$

This expression agrees with expressions earlier found in Refs. [20,21] for the entropy of exclusion statistics particles.

However, one should note that at in our case the transformation introduced above is at this point only a change of variables. We cannot make any conclusion about the true quantum statistics of the particles without considering what the transformation makes to the energy functional of the system. The point to show is that by choosing a particular value for the parameter λ, the leading part of the quasiparticle interaction, defined in the previous section by the Landau parameters F_0 and F_1, is transformed to zero. This implies that the statistics defined by the new form of the entropy is not modified by a statistical interaction term.

In order to show this, we consider (49) as introducing a change of variables for the total energy, $E[n(k)] \to E[\nu(\kappa)]$. This redefines the quasiparticle energy and interaction

$$\tilde{\epsilon}(\kappa) = \frac{2\pi}{L}\frac{\delta E}{\delta\nu(\kappa)}, \quad \tilde{f}(\kappa,\kappa') = \frac{4\pi^2}{L^2}\frac{\delta^2 E}{\delta\nu(\kappa)\delta\nu(\kappa')}, \quad (54)$$

and the idea is to express these in terms of the previous functionals $\epsilon(k)$ and $f(k,k')$. For the single-particle energy the transformation gives

$$\tilde{\epsilon}(\kappa') = \frac{2\pi}{L}\int dk\,\frac{\delta n(k)}{\delta\nu(\kappa')}\frac{\delta E}{\delta n(k)} = \int dk\,\frac{\delta n(k)}{\delta\nu(\kappa')}\epsilon(k) \quad (55)$$

and for the interaction

$$\begin{aligned}\tilde{f}(\kappa'',\kappa') &= \frac{4\pi^2}{L^2}\frac{\delta}{\delta\nu(\kappa'')}\int dk\,\frac{\delta n(k)}{\delta\nu(\kappa')}\frac{\delta E}{\delta n(k)} \\ &= \frac{2\pi}{L}\int dk\,\frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')}\epsilon(k) \\ &\quad + \iint d\bar{k}\,dk\,\frac{\delta n(\bar{k})}{\delta\nu(\kappa'')}\frac{\delta n(k)}{\delta\nu(\kappa')}f(\bar{k},k). \quad (56)\end{aligned}$$

After some tedious manipulations (see the Appendix), the following rather simple expressions are found for the transformation matrices:

$$\frac{\delta n(k)}{\delta v(\kappa')} = \frac{1}{2}\frac{d}{dk}[(1+\lambda n(k))\mathrm{sgn}(k-k')],$$
$$\frac{\delta^2 n(k)}{\delta v(\kappa'')\delta v(\kappa')} = \frac{1}{4}\lambda\frac{d^2}{dk^2}[(1+\lambda n(k))\mathrm{sgn}(k-k')\mathrm{sgn}(k-k'')], \tag{57}$$

where the pairs of variables k',κ' and k'',κ'' are related by the transformation (49).

The expression obtained for the energy is then the following:

$$\begin{aligned}\tilde{\epsilon}(\kappa') &= \frac{1}{2}\int dk\,\epsilon(k)\frac{d}{dk}[(1+\lambda n(k))\mathrm{sgn}(k-k')] \\ &= (1+\lambda n(k'))\epsilon(k') + \frac{1}{2}\lambda\int dk\,\epsilon(k)n'(k)\mathrm{sgn}(k-k')\end{aligned} \tag{58}$$

with $n'(k) = dn/dk$.

In the case of a filled Fermi sea, the particle density and its derivative are

$$n_0(k) = \tfrac{1}{2}(\mathrm{sgn}(k+k_F) - \mathrm{sgn}(k-k_F)),$$
$$n_0'(k) = \delta(k+k_F) - \delta(k-k_F). \tag{59}$$

This gives for the pseudomomentum

$$\begin{aligned}\kappa &= k + \frac{1}{2}\lambda\int_{-k_F}^{k_F} d\bar{k}\,\mathrm{sgn}(k-\bar{k}) \\ &= \begin{cases} k+\lambda k_F & k > k_F \\ k(1+\lambda) & -k_F < k < k_F, \\ k-\lambda k_F & k < -k_F \end{cases}\end{aligned} \tag{60}$$

and in particular $\kappa_F = (1+\lambda)k_F$. The transformed particle density then is

$$v_0(\kappa) = \frac{1}{2(1+\lambda)}(\mathrm{sgn}(\kappa+\kappa_F) - \mathrm{sgn}(\kappa-\kappa_F)). \tag{61}$$

Introducing this in the expression for the quasiparticle energy gives

$$\tilde{\epsilon}_0(\kappa) = \begin{cases} \epsilon_0\left(\kappa - \frac{\lambda}{1+\lambda}\kappa_F\right) & \kappa > \kappa_F \\ (1+\lambda)\epsilon_0\left(\frac{\kappa}{1+\lambda}\right) - \lambda\epsilon_F & |\kappa| < \kappa_F \\ \epsilon_0\left(\kappa + \frac{\lambda}{1+\lambda}\kappa_F\right) & \kappa < -\kappa_F \end{cases} \tag{62}$$

with $\epsilon_F \equiv \epsilon_0(k_F) = \tilde{\epsilon}_0(\kappa_F)$.

For the interaction, we cite here only the results for variations around the filled Fermi sea, and refer to Appendix for more details,

$$\begin{aligned}\tilde{f}_0(\kappa'',\kappa') = &-\frac{\pi}{L}\lambda[\epsilon_0'(k')(1+\lambda n_0(k')) - \epsilon_0'(k'')(1+\lambda n_0(k''))]\mathrm{sgn}(k'-k'') \\ &+\frac{\pi}{2L}\lambda^2\epsilon_0'(k_F)[\mathrm{sgn}(k''+k_F)\mathrm{sgn}(k'+k_F) + \mathrm{sgn}(k''-k_F)\mathrm{sgn}(k'-k_F)] \\ &+(1+\lambda n_0(k'))(1+\lambda n_0(k''))f_0(k'',k') \\ &-\frac{1}{2}\lambda(1+\lambda n_0(k'))[f_0(-k_F,k')\mathrm{sgn}(k''+k_F) - f_0(k_F,k')\mathrm{sgn}(k''-k_F)] \\ &-\frac{1}{2}\lambda(1+\lambda n_0(k''))[f_0(k'',-k_F)\mathrm{sgn}(k'+k_F) - f_0(k'',k_F)\mathrm{sgn}(k'-k_F)] \\ &+\frac{1}{4}\lambda^2[(\mathrm{sgn}(k''+k_F)\mathrm{sgn}(k'+k_F) + \mathrm{sgn}(k''-k_F)\mathrm{sgn}(k'-k_F)f_0(k_F,k_F)) \\ &-(\mathrm{sgn}(k''+k_F)\mathrm{sgn}(k'-k_F) + \mathrm{sgn}(k''-k_F) + \mathrm{sgn}(k'-k_F))f_0(k_F,-k_F)].\end{aligned} \tag{63}$$

In spite of the discontinuous behavior of the particle density $n(k)$ at the Fermi points, the function $\tilde{f}_0(\kappa'',\kappa')$ is continuous at these points, as one can check from the above expression. The values at the Fermi points are

$$\begin{aligned}\tilde{f}_0(\kappa_F,\kappa_F) &= f_0(k_F,k_F) + \left(\lambda+\frac{1}{2}\lambda^2\right)(f_0(k_F,k_F) - f_0(k_F,-k_F)) + \frac{\pi}{L}\lambda^2 v_s, \\ \tilde{f}_0(\kappa_F,-\kappa_F) &= f_0(k_F,-k_F) - \left(\lambda+\frac{1}{2}\lambda^2\right)\left(f_0(k_F,k_F) - f_0(k_F,-k_F) + \frac{2\pi}{L}v_s\right).\end{aligned} \tag{64}$$

If the new Landau parameters $\tilde{F}_0$ and $\tilde{F}_1$ are normalized as in (34), this gives the following relation between these and the original Landau parameters:

$$\begin{aligned}\widetilde{F}_0 &= F_0 - \lambda, \\ \widetilde{F}_1 &= (1+\lambda)^2\left(F_1 + \frac{\lambda}{1+\lambda}\right).\end{aligned} \tag{65}$$

Furthermore, the relation (35) between F_0 and F_1 gives the following relation between $\tilde{F}_0$ and $\tilde{F}_1$:

$$\left(1+\frac{\tilde{F}_0}{1+\lambda}\right) = \left(1+\frac{\tilde{F}_1}{1+\lambda}\right)^{-1}. \tag{66}$$

Assuming now that the value of the parameter λ is specified as

$$\lambda = \lambda_1 + \lambda_2 = \frac{1}{g} - 1 \tag{67}$$

and using the values earlier found for F_0 and F_1 in (35), we find that both the new Landau parameters vanish,

$$\widetilde{F}_0 = \widetilde{F}_1 = 0. \tag{68}$$

This means that both $\tilde{f}_0(\kappa_F,\kappa_F)$ and $\tilde{f}_0(\kappa_F, -\kappa_F)$ vanish, and therefore that the interaction in the low-energy regime is weak, in the sense

$$\lim_{|\kappa''|\to k_F} \lim_{|\kappa'|\to k_F} \tilde{f}_0(\kappa'',\kappa') = 0. \tag{69}$$

The conclusion is thus that the interactions of the one-dimensional fermion system effectively change the particle statistics, and make the system appear as a weakly interacting system of (quasi)particles with generalized statistics. The modified exclusion parameter is given by $1 + \lambda = 1/g$, which agrees with the value of the statistics parameter in Refs. [13–15].

VI. THE LOW-ENERGY LIMIT

As a consistency check we briefly return to the linearized low-energy approximation, with the Hamiltonian described by (10). In this approximation, the interaction terms $\tilde{f}(\kappa'',\kappa')$ are negligible, and the energy function $\tilde{\epsilon}(\kappa)$ is essentially independent of the particle density and can be approximated by $\tilde{\epsilon}_0(\kappa)$. The point to check is whether the energies $E = \int d\kappa\, \nu(\kappa)\, \tilde{\epsilon}_0(\kappa)$, with the particle distributions specified by (46), coincide with the eigenvalues of the Hamiltonian (10).

We first assume the particle distribution to be without holes, where all momentum states k_i are filled between a minimum value $k_{\min}$, close to the Fermi point $-k_F$, and a maximum value $k_{\max}$, close to k_F. $k_{\min}$ and $k_{\max}$ are then related to the particle numbers N and J in the following way:

$$k_{\max} = k_F + \frac{\pi}{L}(N + J), \quad k_{\min} = -k_F - \frac{\pi}{L}(N - J), \tag{70}$$

and the relation (49) between κ and k now simplifies to

$$\kappa = (1 + \lambda)k - \frac{\pi}{L}\lambda J. \tag{71}$$

The integrated energy of the system is

$$\begin{aligned} E &= \frac{L}{2\pi}\int_{\kappa_{\min}}^{\kappa_{\max}} \tilde{\epsilon}_0(\kappa)\nu_0(\kappa)d\kappa = \frac{L}{2\pi}\frac{1}{1+\lambda}\left(\int_{-\kappa_F}^{\kappa_F} \tilde{\epsilon}_0(\kappa)d\kappa + \int_{\kappa_{\min}}^{-\kappa_F} \tilde{\epsilon}_0(\kappa)d\kappa + \int_{\kappa_F}^{\kappa_{\max}} \tilde{\epsilon}_0(\kappa)d\kappa\right) \\ &= E_0 + \frac{L}{2\pi}\frac{1}{1+\lambda}\left[\int_{\kappa_{\min}}^{-\kappa_F} (\epsilon_F - v_s(\kappa + \kappa_F))d\kappa + \int_{\kappa_F}^{\kappa_{\max}} (\epsilon_F + v_s(\kappa - \kappa_F))d\kappa\right] \\ &= E_0 + \epsilon_F N + \frac{\pi}{2L}v_s\left((1+\lambda)N^2 + \frac{1}{1+\lambda}J^2\right), \end{aligned} \tag{72}$$

where E_0 is the energy of the filled Fermi sea, and where $\tilde{\epsilon}_0(\kappa)$ has been expanded to linear order in κ in the intervals close to the Fermi points. The quadratic terms in N and J agree with those of the Hamiltonian (10), while the two first terms in (72) can be absorbed in a redefinition of the zero point of the energy and by introducing a chemical potential, which effectively makes the Fermi sea the ground state of the system. The bosonic excitation term in (10) corresponds to the particle-hole excitations of the fermionic system, which so far has not been included.

In Fig. 1, the situation is illustrated for a small system with quadratic dispersion, $\tilde{\epsilon}_0 = \frac{1}{2}a\kappa^2$, $a = v_s/\kappa_F$, and with statistics parameter $\lambda = 0.5$. The single-particle energies $\tilde{\epsilon}_0$ are shown as functions of k in two cases, the first one with a filled Fermi sea, with $N_0 = 23$ particles and $N = J = 0$. In the second case, two particles have been added in the lowest available states close to the Fermi point k_F, so that $N = J = 2$. The figure demonstrates how the effects of the modified Pauli exclusion are transmitted through the Fermi sea. This produces effectively a shift of the energy curve along the k axis. Thus the insertion of new particles near the Fermi point k_F will affect the values of κ, and thus of $\tilde{\epsilon}_0$, not only near the Fermi point k_F, but also near $-k_F$. It is straightforward to to show that this will effectively move a part of any new fermion charge inserted at k_F, so that the fraction $(1 + g)/2$ of the charge remains at k_F, while the fraction $(1 - g)/2$ is moved to $-k_F$.

Particle-hole excitations can be introduced by changing the (discrete) momenta k_i of the occupied states in the following way:

$$\begin{aligned} k_i &= k_i^0 + \Delta k_i = (2\pi/L)(i + n_i), \\ i &= i_{\min}, i_{\min} + 1, \ldots, i_{\max}, \end{aligned} \tag{73}$$

where k_i^0 gives the momentum values of the occupied states without holes, and n_i are integers, which introduce holes in the distribution. We assume $n_{i+1} \geqslant n_i$ to make the ordering of k_i with respect to i to be unchanged when introducing the holes. For the pseudomomenta κ_i, we get a similar change in the values. This is a direct consequence of the transformation formula (46)

$$\begin{aligned} \kappa_i &= k_i + \lambda\frac{\pi}{L}\sum_{j=i_{\min}}^{i_{\max}} \mathrm{sgn}(k_i - k_j) \\ &= (2\pi/L)\left(i + n_i + \frac{1}{2}\lambda\sum_{j=i_{\min}}^{i_{\max}} \mathrm{sgn}(i - j)\right) \\ &\equiv \kappa_i^0 + \Delta k_i \end{aligned} \tag{74}$$

with $\Delta k_i = (2\pi/L)n_i$. The shifts are thus the same as for the momenta k_i, which means that they are independent of the statistics parameter λ.

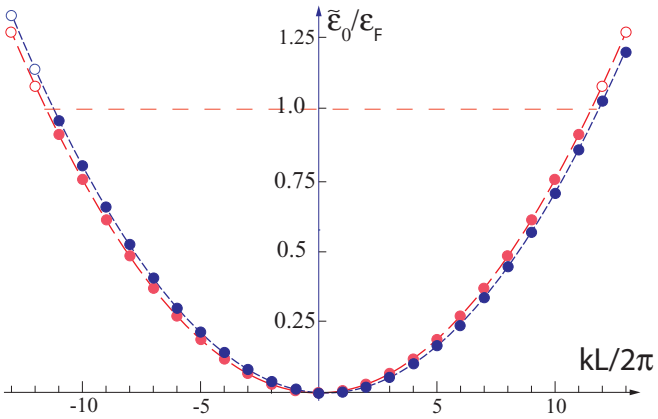

FIG. 1. The single-particle energy $\tilde{\epsilon}_0$, shown as function of the momentum variable k, with quadratic dispersion and with statistics parameter $\lambda = 0.5$. Filled circles correspond to occupied states and open circles to unoccupied states. Two cases are shown. The first case (red circles, long dashed curve) corresponds to a filled Fermi sea, representing the ground state with $N_0 = 23$ particles. In the second case (blue circles, short dashed curve), two particles are added in the lowest available momentum states close to the Fermi point k_F. This corresponds to the case $N = J = 2$. The generalized Pauli exclusion between the particles gives rise to a relative shift of the two curves along the k axis.

With the excitations restricted to the neighborhoods of the Fermi points, linearization of the energy as function of momentum can be made, which gives

$$\sum_i \tilde{\epsilon}_0(\kappa_i) = \sum_i \tilde{\epsilon}_0(\kappa_i^0) + v_s \sum_i |\Delta k_i|. \tag{75}$$

The excitation term is the same as for free fermions, although with v_s as the effective Fermi velocity. It corresponds precisely to the bosonic excitation term in (10), when the boson frequency is linearized in q, $\omega_q \approx v_s|q|$. Thus the expression for the full energy, given as a sum over single-particle energies $\tilde{\epsilon}_0(\kappa_i)$ reproduces precisely the energy eigenvalues of the Hamiltonian (10) within this approximation.

VII. CONCLUDING REMARKS

The standard approach to study the interacting one-dimensional Fermi system is based on the use of the bosonization technique. However, to study properties of the elementary charged excitations, other methods may be more convenient. We have here applied the Fermi liquid theory of Landau to study the quantum statistics of the "dressed fermions" of the interacting theory. These particles are identified as the Landau type of quasiparticles, and they are well defined due to the adiabatic mapping which exists between the (low-energy sector of the) noninteracting and the interacting theory.

For excitations close to the Fermi points, the unitary transformation between the noninteracting and the interacting system is explicitly known and has a simple form when expressed in the bosonized variables. We have used this to rewrite the low-energy Hamiltonian in terms of dressed fermion fields, and to show that the corresponding excitations are fractionally charged. A more detailed study of this effect has been performed in a previous publication [8]. The Hamiltonian, in this form, has furthermore been used to determine the quasiparticle energy and two-body interaction, defined as functional derivatives of the full energy of the system. By a change of momentum variables in the functional derivatives, the interaction terms have been shown to vanish for momenta close to the Fermi points. This formally gives the description the form of a free theory. However, as shown by the transformed form of the entropy function, the quasiparticles obey a generalized type of Pauli exclusion.

The two-body interaction, after the momentum transformation, vanishes at the Fermi points. Away from these points, however there are nonvanishing corrections, as shown by the derived expressions. Since the main part of the interaction has been absorbed in the change of statistics, these contributions are small and can in principle be treated perturbatively.

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APPENDIX: EVALUATING THE FUNCTIONAL DERIVATIVES

In order to determine $\tilde{\epsilon}$ and $\tilde{f}$, we need to evaluate the functional derivatives

$$\frac{\delta n(k)}{\delta \nu(\kappa')}, \quad \frac{\delta^2 n(k)}{\delta \nu(\kappa'')\delta \nu(\kappa')} \tag{A1}$$

as shown in Eqs. (55) and (56). $n(k)$ can be viewed as a functional of ν (and a function of k), as specified in Eq. (51),

$$n(k) = \frac{\nu(\kappa)}{1 - \lambda\nu(\kappa)} \equiv F[\nu(\kappa)]. \tag{A2}$$

One should note that in the functional derivatives (A1), k as well as κ' and κ'' should be considered as fixed during the variation of the density ν. However, when k is replaced by κ, as in (A2), the condition that k is fixed implies that the corresponding variable κ will be a functional of ν. This is shown by the relation between the two [as earlier expressed in (49)]:

$$\kappa = k + \frac{1}{2}\lambda \int d\bar{\kappa}\, \nu(\bar{\kappa})\mathrm{sgn}(\kappa - \bar{\kappa}). \tag{A3}$$

As a consequence of this, $F[\nu(\kappa)]$ has both an explicit dependence of ν, as shown by (A2), and an implicit dependence of ν, which follows from the fact that κ depends on ν. To take care of both these effects, we write the variation of the functional F, in the following way:

$$\delta F[\nu(\kappa)]_k = F[\nu(\kappa + \delta\kappa) + \delta\nu(\kappa + \delta\kappa)] - F[\nu(\kappa)], \tag{A4}$$

where the label k indicates that k is kept fixed under the variation of ν. We are interested in determining the variation $\delta F[\nu(\kappa)]_k$ under a change in ν of the following form:

$$\delta\nu(\kappa) = \epsilon_1\delta(\kappa - \kappa') + \epsilon_2\delta(\kappa - \kappa''), \tag{A5}$$

here with κ regarded as a free variable. With δF valuated to second order in ϵ_1 and ϵ_2, the term proportional to ϵ_1 will then

give the functional derivative with respect to $\nu(\kappa')$ and the second-order term proportional to $\epsilon_1\epsilon_2$ will give the double derivative with respect to $\nu(\kappa')$ and $\nu(\kappa'')$.

We begin by evaluating (to second order in ϵ_1 and ϵ_2) the variation in κ induced by the variation in ν with k fixed,

$$\delta\kappa = \frac{1}{2}\lambda \int d\bar{\kappa}[(\nu(\bar{\kappa}) + \delta\nu(\bar{\kappa}))\mathrm{sgn}(\kappa + \delta\kappa - \bar{\kappa}) - \nu(\bar{\kappa})\mathrm{sgn}(\kappa - \bar{\kappa})]. \tag{A6}$$

We have

$$\mathrm{sgn}(\kappa + \delta\kappa - \bar{\kappa}) = \mathrm{sgn}(\kappa - \bar{\kappa}) + 2(\theta(\kappa + \delta\kappa - \bar{\kappa}) - \theta(\kappa - \bar{\kappa})) \tag{A7}$$

with $\theta(\kappa - \bar{\kappa})$ as the Heaviside step function, and using the expansion

$$\nu(\bar{\kappa}) = \nu(\kappa) + \nu'(\kappa)(\bar{\kappa} - \kappa)\ldots, \tag{A8}$$

we find the following implicit expression for $\delta\kappa$,

$$\delta\kappa = \frac{1}{2}\lambda\left[\int d\bar{\kappa}\,\delta\nu(\bar{\kappa})\mathrm{sgn}(\kappa - \bar{\kappa}) + 2(\nu(\kappa) + \delta\nu(\kappa))\delta\kappa + \nu'(\kappa)\delta\kappa^2\right] + \text{higher order}. \tag{A9}$$

From this, the first-order contribution to $\delta\kappa$ is found to be

$$\delta\kappa_1 = \frac{\lambda/2}{1 - \lambda\nu(\kappa)} \int d\bar{\kappa}\delta\nu(\bar{\kappa})\mathrm{sgn}(\kappa - \bar{\kappa}) = \frac{\lambda/2}{1 - \lambda\nu(\kappa)}(\epsilon_1\mathrm{sgn}(\kappa - \kappa') + \epsilon_2\mathrm{sgn}(\kappa - \kappa'')) \tag{A10}$$

and including explicitly only the term proportional to $\epsilon_1\epsilon_2$ we find for the second-order term:

$$\delta\kappa_2 = 2\epsilon_1\epsilon_2\left[\left(\frac{\lambda/2}{1 - \lambda\nu(\kappa)}\right)^2(\delta(\kappa - \kappa')\mathrm{sgn}(\kappa - \kappa'') + \delta(\kappa - \kappa'')\mathrm{sgn}(\kappa - \kappa')) + \left(\frac{\lambda/2}{1 - \lambda\nu(\kappa)}\right)^3 \nu'(\kappa)\mathrm{sgn}(\kappa - \kappa')\mathrm{sgn}(\kappa - \kappa'')\right] + \ldots. \tag{A11}$$

We consider now the expansion of δF in powers of ϵ_1 and ϵ_2, starting with the expression

$$\delta F[\nu(\kappa)]_k = \frac{1}{\lambda}\left[\frac{1}{1 - \lambda(\nu(\kappa + \delta\kappa) + \delta\nu(\kappa + \delta\kappa))} - \frac{1}{1 - \lambda\nu(\kappa)}\right]. \tag{A12}$$

We leave out the intermediate steps in deriving the expansion and give the expression for the coefficient proportional to ϵ_1, which determines the functional derivative of $n(k)$ with respect to $\nu(\kappa')$,

$$\frac{\delta n(k)}{\delta\nu(\kappa')} = \frac{1}{(1 - \lambda\nu(\kappa))^2}\left(\delta(\kappa - \kappa') + \frac{\lambda/2}{(1 - \lambda\nu(\kappa))}\nu'(\kappa)\mathrm{sgn}(\kappa - \kappa')\right). \tag{A13}$$

From the term proportional to $\epsilon_1\epsilon_2$ we derive the following expression:

$$\begin{aligned}\frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')} &= \frac{1}{4}\left(\frac{\lambda^2}{(1 - \lambda\nu(\kappa))^4}\nu''(\kappa) + 3\frac{\lambda^3}{(1 - \lambda\nu(\kappa))^5}\nu(\kappa)'^2\right)\mathrm{sgn}(\kappa - \kappa')\mathrm{sgn}(\kappa - \kappa'') \\ &+ \frac{3}{2}\frac{\lambda^2}{(1 - \lambda\nu(\kappa))^4}\nu'(\kappa)(\mathrm{sgn}(\kappa - \kappa')\delta(\kappa - \kappa'') + \delta(\kappa - \kappa')\mathrm{sgn}(\kappa - \kappa'')) \\ &+ \frac{1}{2}\frac{\lambda}{(1 - \lambda\nu(\kappa))^3}(\mathrm{sgn}(\kappa - \kappa')\delta'(\kappa - \kappa'') + \delta'(\kappa - \kappa')\mathrm{sgn}(\kappa - \kappa'')) + 2\frac{\lambda}{(1 - \lambda\nu(\kappa))^3}\delta(\kappa - \kappa')\delta(\kappa - \kappa'').\end{aligned} \tag{A14}$$

Finally, we rewrite these in terms of the variable $n(k)$, by use of the identity

$$\frac{1}{1 - \lambda\nu(\kappa)} = 1 + \lambda n(k). \tag{A15}$$

Leaving out also here the intermediate steps we find the following expressions:

$$\frac{\delta n(k)}{\delta\nu(\kappa')} = (1 + \lambda n(k))^2\left[\frac{1}{1 + \lambda n(k)}\delta(k - k') + \frac{\lambda/2}{(1 + \lambda n(k))^2}n'(k)\mathrm{sgn}(k - k')\right] = \frac{1}{2}\frac{d}{dk}[(1 + \lambda n(k))\mathrm{sgn}(k - k')] \tag{A16}$$

and

$$\begin{aligned}\frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')} &= \frac{1}{4}\lambda^2 n''(k)\mathrm{sgn}(k - k')\mathrm{sgn}(k - k'') + \lambda^2 n'(k)(\mathrm{sgn}(k - k')\delta(k - k'') + \delta(k - k')\mathrm{sgn}(k - k'')) \\ &+ \frac{1}{2}\lambda(1 + \lambda n(k))(\mathrm{sgn}(k - k')\delta'(k - k'') + \delta'(k - k')\mathrm{sgn}(k - k'')) + 2\lambda(1 + \lambda n(k))\delta(k - k')\delta'(k - k'') \\ &= \frac{1}{4}\lambda\frac{d^2}{dk^2}[(1 + \lambda n(k))\mathrm{sgn}(k - k')\mathrm{sgn}(k - k'')].\end{aligned} \tag{A17}$$

The continuum form of the expression for the variation in the total energy (A18) we write as

$$\delta E = \frac{L}{2\pi}\int dk\epsilon(k)\delta n(k) + \frac{L^2}{8\pi^2}\iint dkdk' f(k,k')\delta n(k)\delta n(k'), \tag{A18}$$

which gives

$$\epsilon(k) = \frac{2\pi}{L}\frac{\delta E}{\delta n(k)}, \quad f(k,k') = \frac{4\pi^2}{L^2}\frac{\delta^2 E}{\delta n(k)\delta n(k')}. \tag{A19}$$

The energy function after the change of variables then is

$$\begin{aligned}\tilde{\epsilon}(\kappa') &= \frac{2\pi}{L}\frac{\delta E}{\delta \nu(\kappa')} = \int dk\,\frac{\delta n(k)}{\delta\nu(\kappa')}\epsilon(k)\\ &= \frac{1}{2}\int dk\epsilon(k)\frac{d}{dk}[(1+\lambda n(k))\mathrm{sgn}(k-k')]\\ &= (1+\lambda n(k'))\epsilon(k') + \frac{1}{2}\lambda\int dk\epsilon(k)n'(k)\mathrm{sgn}(k-k').\end{aligned} \tag{A20}$$

It is convenient to separate the interaction function into two parts:

$$\begin{aligned}\tilde{f}(\kappa'',\kappa) &= \frac{4\pi^2}{L^2}\frac{\delta^2 E}{\delta\nu(\kappa)\delta\nu(\kappa')}\\ &= \frac{2\pi}{L}\int dk\frac{\delta^2 n(k)}{\delta\nu(\kappa'')\delta\nu(\kappa')}\epsilon(k) + \iint d\bar{k}dk\frac{\delta n(k)}{\delta\nu(\kappa')}\frac{\delta n(k)}{\delta\nu(\kappa'')}f(\bar{k},k)\\ &= \tilde{f}_A(\kappa'',\kappa') + \tilde{f}_B(\kappa'',\kappa').\end{aligned} \tag{A21}$$

For the first term, we find

$$\begin{aligned}\tilde{f}_A(\kappa'',\kappa') &= \frac{\pi}{2L}\lambda\int dk\epsilon(k)\frac{d^2}{dk^2}[(1+\lambda n(k))\mathrm{sgn}(k-k')\mathrm{sgn}(k-k'')]\\ &= -\frac{\pi}{L}\lambda[\epsilon'(k')(1+\lambda n(k')) - \epsilon'(k'')(1+\lambda n(k''))]\mathrm{sgn}(k'-k'')\\ &\quad -\frac{\pi}{2L}\lambda^2\int dk\,\epsilon'(k)\,n'(k)\,\mathrm{sgn}(k-k')\mathrm{sgn}(k-k'')\end{aligned} \tag{A22}$$

and for the second term

$$\begin{aligned}\tilde{f}_B(\kappa'',\kappa') &= \frac{1}{4}\iint d\bar{k}\,dk\,\frac{d}{d\bar{k}}[(1+\lambda n(\bar{k}))\mathrm{sgn}(\bar{k}-k'')]\frac{d}{dk}[(1+\lambda n(k))\mathrm{sgn}(k-k')]f(\bar{k},k)\\ &= (1+\lambda n(k''))(1+\lambda n(k'))f(k'',k') + \frac{1}{2}\lambda\big(1+\lambda n(k'')\big)\int dk\,n'(k)\,\mathrm{sgn}(k-k')f(k'',k)\\ &\quad + \frac{1}{2}\lambda(1+\lambda n(k'))\int dk\,n'(k)\,\mathrm{sgn}(k-k'')f(k,k') + \frac{1}{4}\lambda^2\iint d\bar{k}dkn'(\bar{k})n'(k)\mathrm{sgn}(\bar{k}-k'')\mathrm{sgn}(k-k')f(\bar{k},k).\end{aligned} \tag{A23}$$

In the expressions given above the pair k' and κ', as well as k'' and κ'', are related by Eq. (A3).

The expressions for the energy $\tilde{\epsilon}_0(\kappa)$ and for the interaction $\tilde{f}_0(\kappa'',\kappa')$ are finally found by introducing the density $n_0(k)$ and its derivative $n_0'(k)$, as given by Eq. (59), in the above expressions (A20), (A22), and (A23). The results are displayed in Eqs. (62) and (63) in the main text.

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