Competing phases in spin ladders with ring exchange and frustration

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The ground-state properties of spin-1/2 ladders are studied, emphasizing the role of frustration and ring exchange coupling. We present a unified field theory for ladders with general coupling constants and geometry. Rich phase diagrams can be deduced by using a renormalization group calculation for ladders with in-chain next-nearest-neighbor interactions and plaquette ring exchange coupling. In addition to established phases such as Haldane, rung singlet, and dimerized phases, we also observe a surprising instability towards an incommensurate phase for weak interchain couplings, which is characterized by an exotic coexistence of self-consistent ferromagnetic and antiferromagnetic order parameters.

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I. INTRODUCTION

Spin ladders are heavily studied prototypical models exemplifying the role of quantum fluctuations in low-dimensional quantum systems [1–28]. Already in their simplest form they show some of the most discussed quantum many-body properties, such as a spin liquid or a topological Haldane gapped state. Theoretical studies have identified a number of remarkable ground-state phases, but given the large variety of possible tuning parameters it is likely that this list is far from complete. At the same time experimental progress on novel materials [29–31] as well as advancements in the field of optical lattices [32–35] give renewed interest in spin ladders in their own right beyond the larger effort to gain a better insight into complex phases of frustrated two-dimensional (2D) models. This work now focuses on SU(2) invariant ladders in order to answer the important question regarding which phases are accessible for a general choice of tunable couplings within the framework of an effective field theory. In particular, an incommensurate phase is postulated for a certain class of ladder systems in the weak-coupling limit, which has so far received little attention.

The generalized SU(2) invariant spin-1/2 ladder is described by the Heisenberg Hamiltonian plus a ring exchange interaction,

$$H = \sum_{\langle i,j\rangle} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j + K \sum_p \left(P_p + P_p^{-1}\right), \qquad (1)$$

where J_{ij} take on values of antiferromagnetic nearest-neighbor coupling $J > 0$ and next-nearest-neighbor (NNN) coupling J_2 in the two chains as well as two diagonal couplings J_d, J_d' and a rung coupling $J_\perp$ between the chains as depicted in Fig. 1. The second term in Eq. (1) sums over the plaquettes of the system, where P_p stands for the cyclic permutation of the spins on the four sites of the pth plaquette. Such a ring exchange interaction arises from the higher-order expansion of the Hubbard model in the strongly interacting limit [36,37] and can also be written as a product of two-spin permutations $P_p = P_{p_1p_2}P_{p_1p_3}P_{p_1p_4}$, which in terms of Pauli matrices $\boldsymbol{\sigma}$ reads $P_{ij} = (1 + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)/2$ [38]. The tunable parameters in Fig. 1 provide a generalization of previously studied ladder models. It has been established that weak interchain couplings give rise to four possible gapped quantum phases, which are characterized by correlations in form of singlets [4–12]:

(i) *Rung singlet phase:* Singlet formation across the rungs for an antiferromagnetic rung coupling $J_\perp > 0$.

(ii) *Haldane phase:* Singlet formation analogous to the so-called AKLT state [39] in the spin-1 model with a topological string order parameter for $J_\perp < 0$.

(iii) *Columnar dimerized phase:* Alternating singlet formation within each chain on the same bonds in both legs for $K < 0$.

(iv) *Staggered dimerized phase:* Alternating singlet formation within each chain with a shift of one site between the two legs for $K > 0$.

The rung singlet and Haldane phases have short-range correlations, while the dimer correlations in the dimerized phases are long range and break translational invariance. The four phases with singlet formation are also dominant if one considers frustrating couplings J_d, J_d', J_2, which normally enhance quantum fluctuations. Such frustrating couplings have been extensively studied in a cross-coupled ladder (CCL) for $J_d = J_d'$ [12–18] or a diagonal ladder (DL) for $J_d > J_d' = 0$ [18–20]. Both the DL and the CCL can be tuned from a rung singlet phase to a Haldane phase with increasing $J_d/J_\perp$. Interestingly, the columnar dimerized phase also appears in the DL for intermediate $J_d/J_\perp$ [18], but such a phase is debated for the CCL since it would spontaneously break translational invariance [14–18]. A frustrating in-chain next-nearest-neighbor interaction causes dimerization in single chains for $J_2 > J_{2c} \approx 0.241167J$ [40–42] and therefore also stabilizes the corresponding dimer phases in ladder systems with next-nearest-neighbor coupling [21–25].

We now want to answer the question if the enhanced entropy from competing phases may open the possibility

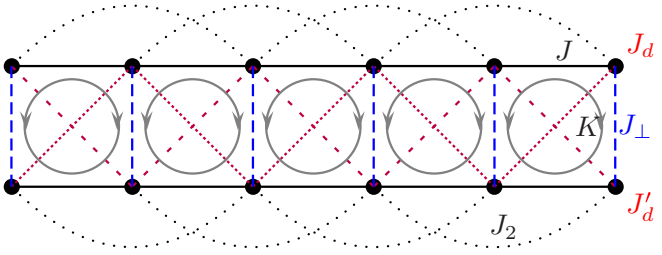

FIG. 1. Generalized spin ladder with ring exchange interaction K, NNN coupling J_2, and two diagonal couplings J_d and J_d'.

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for interesting additional phases. We therefore consider the interplay of all couplings in the full model in Fig. 1 using the framework of a renormalization group (RG) treatment. We find that for $J_d \neq J_d'$ the enlarged unit cell permits an interesting incommensurate phase, which occurs at surprisingly small coupling strengths and may serve as a guide to analogous phases in higher dimensions.

II. FIELD THEORY AND RENORMALIZATION GROUP

Starting from the established effective field-theory description of two decoupled chains, the spin operators acquire the following representation in the continuum limit [43–47]:

$$\mathbf{S}(x) \approx \mathbf{J}(x) + (-1)^x \Omega \mathbf{n}(x), \tag{2}$$

where Ω is a nonuniversal constant [48] and the lattice constant is set to unity. The uniform part of the spin operator is the sum of the chiral SU(2) currents of the Wess-Zumino-Witten (WZW) model, $\mathbf{J} = \mathbf{J}_L + \mathbf{J}_R$; the fundamental field g of the WZW model is used to express the alternating spin component $\mathbf{n} \sim \mathrm{tr}\boldsymbol{\sigma} g$ and the dimerization operator $\epsilon_j = (-1)^j \mathbf{S}_j \cdot \mathbf{S}_{j+1} \sim \mathrm{tr} g$. Without interchain couplings the field theory can be written for each leg separately in the form

$$H_0 = \frac{2\pi v}{3} \int dx \left[:\mathbf{J}_L \cdot \mathbf{J}_L : + :\mathbf{J}_R \cdot \mathbf{J}_R : + \frac{3\lambda_a}{2} \mathbf{J}_L \cdot \mathbf{J}_R \right], \tag{3}$$

where the first two normal ordered terms are conformally invariant and the last part represents a backscattering marginal operator. Without any additional couplings between the two legs of the ladder this theory describes a *spin liquid* for each chain which is generally unstable to perturbations. The velocity $v \approx \frac{\pi J}{2} - 1.65 J_2$ and the value of the marginal coupling λ_a can be tuned by the in-chain nearest-neighbor coupling J_2 [42–44].

We denote the field theories for each chain with an additional index $\eta = 1,2$. The allowed symmetries of the microscopic model dictate that up to four additional relevant or marginal operators can be generated by the coupling between the legs of the ladder

$$\delta\mathcal{H} = 2\pi v \int dx (\lambda_n O_n + \lambda_c O_c + \lambda_\epsilon O_\epsilon + \lambda_b O_b). \tag{4}$$

Here the operator $O_n = \mathbf{n}_1 \cdot \mathbf{n}_2$ of scaling dimension 1 describes the coupling between the antiferromagnetic part of the spins, which drives the system into a rung singlet phase for $\lambda_n > 0$ and into a Haldane phase for $\lambda_n < 0$. An effective coupling of the dimerization is described by the relevant operator $O_\epsilon = \epsilon_1 \epsilon_2$ of scaling dimension 1, which may drive the system into a staggered ($\lambda_\epsilon > 0$) or columnar ($\lambda_\epsilon < 0$) dimerized phase. Interestingly, for $J_d' \neq J_d$, the unit cell is enlarged, which allows another relevant operator $O_c = \mathbf{J}_1 \cdot \mathbf{n}_2 - \mathbf{J}_2 \cdot \mathbf{n}_1$ of scaling dimension 3/2, which is invariant under translation by two sites. It is not invariant under translation by one site, unless the chain index is also exchanged, in agreement with the model in Fig. 1 for $J_d' \neq J_d$. As we will see below, this operator can drive the system into yet another interesting incommensurate phase. Finally, there is another marginal coupling $O_b = \mathbf{J}_{1,L} \cdot \mathbf{J}_{2,R} + \mathbf{J}_{1,R} \cdot \mathbf{J}_{2,L}$ of scaling dimension 2, which can tip the balance of which relevant operator dominates under renormalization.

The corresponding RG equations of the bare coupling constants are determined up to second order according to the operator product expansion [49,50]

$$\frac{d\lambda_k}{dl} = (2 - d_k)\lambda_k - \frac{\pi}{v} \sum_{i,j} C_{ijk} \lambda_i \lambda_j, \tag{5}$$

with d_k the scaling dimension of each operator and the coefficients C_{ijk} are obtained by the operator product expansion [47,51] as discussed in Appendix 1. For the operator content in Eq. (4), we arrive at the following RG flow ($\dot{\lambda} = d\lambda/dl$):

$$\dot{\lambda_a} = {\lambda_a}^2 + \tfrac{1}{2}{\lambda_\epsilon}^2 - \tfrac{1}{2}{\lambda_n}^2, \tag{6a}$$

$$\dot{\lambda_b} = {\lambda_b}^2 - \lambda_\epsilon \lambda_n + {\lambda_n}^2, \tag{6b}$$

$$\dot{\lambda_\epsilon} = \lambda_\epsilon + \tfrac{3}{2}\lambda_a \lambda_\epsilon - \tfrac{3}{2}\lambda_b \lambda_n - \tfrac{3}{2}{\lambda_c}^2, \tag{6c}$$

$$\dot{\lambda_n} = \lambda_n - \tfrac{1}{2}\lambda_a \lambda_n - \tfrac{1}{2}\lambda_b \lambda_\epsilon + \lambda_b \lambda_n - {\lambda_c}^2, \tag{6d}$$

$$\dot{\lambda_c} = \tfrac{1}{2}\lambda_c - \tfrac{1}{4}\lambda_a \lambda_c + \lambda_b \lambda_c + \tfrac{1}{2}\lambda_c \lambda_\epsilon - \lambda_c \lambda_n. \tag{6e}$$

The bare coupling constants can be determined from the microscopic model using Eq. (2) (see also the Appendix)

$$\begin{aligned} \lambda_n &= \Omega^2 \frac{J_\perp - \tilde{J}_d}{2\pi v}, \lambda_\epsilon = 36 \frac{\Omega^2}{\pi^2} \frac{K}{2\pi v}, \lambda_c = \Omega \frac{J_d - J_d'}{2\pi v}, \\ \lambda_a &= 1.723(J_2 - J_{2c}) + \left[2 - \Omega^2 + \frac{3\Omega^2}{\pi^2}(1 + \Omega^2) \right] \frac{K}{2\pi v}, \\ \lambda_b &= \frac{J_\perp + \tilde{J}_d}{2\pi v} + 4 \left[1 - \frac{(1 + \Omega^2)^2}{\pi^2} \right] \frac{K}{2\pi v}, \end{aligned} \tag{7}$$

where we have used $\tilde{J}_d = J_d + J_d'$. As expected, O_c is forbidden by symmetry for $J_d' = J_d$ and stays zero under renormalization in Eqs. (6) in this case. The system remains approximately scale invariant above an energy scale Λ as long as the coupling constants are small. However, as the cutoff $\Lambda(l) = \Lambda_0 e^{-l}$ is lowered, typically one of the coupling constants effectively becomes of order unity under renormalization, which determines the dominant ground-state correlations. In turn, the value of $\Lambda(l)$ at this breakdown point defines a new intrinsic energy scale, below which scale invariance is lost and no further renormalization of all couplings is possible. This means that there generally is an implied assumption that subdominant couplings remain small so they do not turn around the RG flow in higher order. Good agreement with numerically determined phase transitions has been achieved when using $\Omega = 1$ and integrating the RG equations up to a breakdown point where one coupling reaches $\lambda_* = 1$, which is the procedure we use in the following.

The operators with the smaller scaling dimension, i.e., O_n and O_ϵ, renormalize faster and will drive the systems into one of the four known phases as discussed above. However, there are regions of the parameter space where the outcome of the renormalization procedure is less trivial to predict due to the competition of the frustrating couplings. In particular, if the bare couplings $\lambda_\epsilon, \lambda_n$ are very small, it is possible that the relevant coupling λ_c may drive the phase into an incommensurate phase as discussed below. Nonetheless, λ_ϵ and λ_n will always be generated by higher-order terms in the

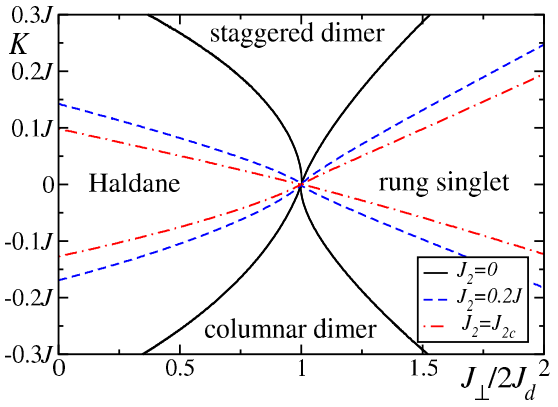

FIG. 2. Phase diagram of the cross coupled ladder as a function of K, and the ratio $J_\perp/2J_d$, for $J_d = J_d' = 0.2J$, and three values of $J_2 = 0$, $0.2J$, and J_{2c}, denoted by black solid, blue dashed, and red dot-dashed lines, respectively.

renormalization procedure [14] and may still dominate the low-energy physics depending also on the marginal couplings.

III. PHASE DIAGRAMS

To illustrate the interplay of different coupling constants we first consider the CCL with $J_d = J_d'$ in the presence of in-chain frustration J_2 and ring exchange coupling K. As shown in Fig. 2 the four known phases dominate the phase diagram since $\lambda_c = 0$ by symmetry in this case. For $K = 0$ there is a direct phase transition from rung singlet to Haldane phase at $J_\perp/2J_d = 1$ with no intermediate dimerized phase. It has been argued before that by including higher-order irrelevant operators such an intermediate dimerized phase can be generated in a narrow region for $K = 0$ [14,15], so the possibility of such a phase obviously depends on higher orders and cannot easily be settled using RG. Numerically, an indirect signature was seen in Ref. [24], but other works have found no evidence [16–18]. A next-nearest-neighbor coupling $J_2 < J_{2c}$ does not change the basic topology of the phase diagram but has a large effect on the range of the dimer phases. This is due to the fact that the in-chain frustration J_2 leads to a reduced starting value of λ_a and therefore makes the dimerization operator effectively more relevant.

For comparison the phase diagram of the DL with $J_d' = 0$ is shown in Fig. 3 for $K = 0$ and three values of the NNN interaction, $J_2 = 0, 0.2J, J_{2c}$ as function of the two interchain couplings $J_\perp$ and J_d. The symmetry properties of this model fundamentally differ from the CCL, because the larger unit cell permits the presence of the operator O_c, which plays an important role in generating the dimer operator O_ϵ under the renormalization in Eq. (6). Interestingly, in contrast to the CCL, therefore an intermediate columnar dimerized phase exists even *without* ring exchange, where the generated operator O_ϵ dominates the correlations. This phase again becomes larger in the presence of in-chain frustration J_2 for reasons mentioned above.

One interesting aspect of the phase diagram in Fig. 3 is the fact that there is no phase which is generated directly by the relevant operator O_c even in cases where its initial bare coupling constant λ_c may be largest. This invites interesting

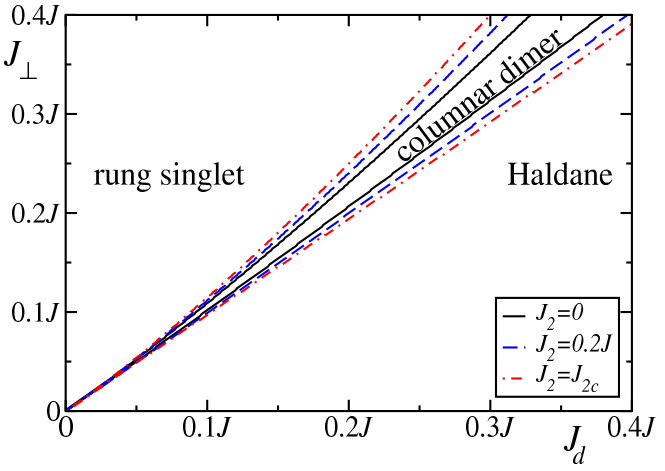

FIG. 3. Phase diagram of the diagonal ladder ($J_d' = 0$) as a function of J_d and $J_\perp$ for $K = 0$ and three values of $J_2 = 0$, $0.2J$, and J_{2c}, denoted by black solid, blue dashed, and red dot-dashed lines respectively.

questions: First, is it ever possible to generate phases which are characterized by a dominant coupling λ_c in the extended parameter space? And, second, if such a phase exist, what dominant correlation would be expected?

The first question is answered by finding suitable parameters for such a phase by taking advantage of canceling out the effect from competing dimer phases. We understand that the dimerized phase in Fig. 3 is created by O_ϵ, which grows beyond bounds after it is generated in second order by the initial bare coupling λ_c. To suppress this generation, it is possible to add a small ring exchange K, which initially pushes the value of λ_ϵ to be slightly positive in Eq. (7). The analogous reasoning applies for λ_n, where a small positive bare value can be achieved by choosing $J_\perp$ slightly larger than J_d. Accordingly, we show the phase diagram for the diagonal ladder with $J_\perp = 1.1J_d$ and $J_2 = 0.2J$ as a function of K and J_d in Fig. 4. In this case, there is indeed an extended region with this additional phase for $K > 0$, which we describe as *incommensurate* for reasons that will be explained below. This incommensurate phase separates the staggered dimer from the columnar dimer phase.

To illustrate another example of possible parameters for this phase, we consider a general ratio of the diagonal couplings

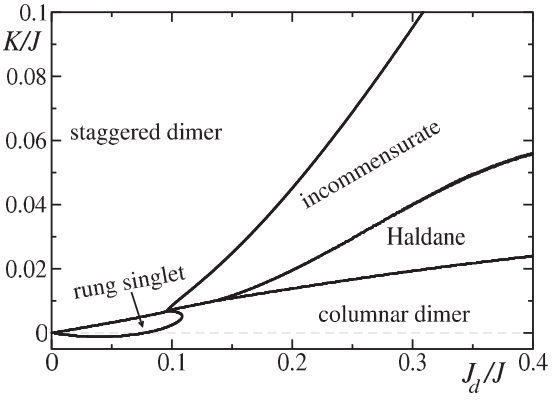

FIG. 4. Phase diagram of the diagonal ladder ($J_d' = 0$) as a function of J_d and K for $J_\perp/J_d = 1.1$ and $J_2 = 0.2J$.

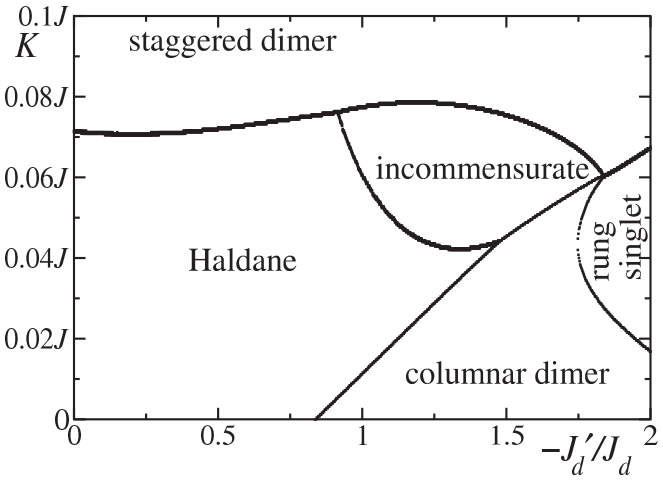

FIG. 5. Phase diagram as a function of $-J_d'/J_d$ and K, for $J_\perp = 0$, $J_2 = J_{2c}$, and $J_d = 0.2J$.

J_d'/J_d in Fig. 5 for $J_\perp = 0$, $J_2 = J_{2c}$, and $J_d = 0.2J$. According to Eq. (7) the initial value of λ_c becomes strongest for $J_d' = -J_d$, while the other initial couplings are zero for $J_\perp = K = 0$ and $J_2 = J_{2c}$. Indeed, the intermediate incommensurate phase appears for small positive K and $J_d' \approx -J_d$, again separating the staggered and columnar dimer phases.

To answer the question about the nature of the phase for strong $O_c = \mathbf{J}_1 \cdot \mathbf{n}_2 - \mathbf{J}_2 \cdot \mathbf{n}_1$ it is possible to invoke a self-consistency argument, analogous to a chain mean-field theory [52–54]. Clearly, the operator O_c causes an alternating magnetic order $\mathbf{n}$ in one chain to induce a collinear ferromagnetic order $\mathbf{J}$ in the other chain and vice versa. Assuming small finite-order parameters we can write for the expectation values

$$\begin{aligned} \langle \mathbf{J}_2 \rangle &= -\lambda_c \chi_0 \langle \mathbf{n}_1 \rangle; \quad \langle \mathbf{n}_1 \rangle = -\lambda_c \chi_1 \langle \mathbf{J}_2 \rangle \\ \langle \mathbf{J}_1 \rangle &= \lambda_c \chi_0 \langle \mathbf{n}_2 \rangle; \quad \langle \mathbf{n}_2 \rangle = \lambda_c \chi_1 \langle \mathbf{J}_1 \rangle, \end{aligned} \tag{8}$$

where χ_0 is the uniform susceptibility and χ_1 is the alternating susceptibility. While χ_0 is finite [55], χ_1 diverges with $1/T$ [53,54], so there will always be a low-enough energy scale at which $\lambda_c^2 \chi_0 \chi_1 = 1$ results in self-consistent finite values for both the alternating antiferromagnetic and the collinear ferromagnetic correlations. Semiclassically, $\mathbf{n}_{1(2)}$ is perpendicular to $\mathbf{J}_{1(2)}$, so therefore also $\mathbf{n}_1$ is perpendicular to $\mathbf{n}_2$ in this mean-field argument. For coupled chains in a higher-dimensional array this will result in an *incommensurate* ordered phase, characterized by a small collinear ferromagnetic part which effectively shifts the wavelength of the antiferromagnetic order [56,57]. In the one-dimensional ladder system a spontaneously broken SU(2) symmetry is not possible, so for this model the phase is characterized by short-range incommensurate correlations [58]. Note that a finite expectation value of O_c in this phase does not break translational invariance or any other symmetries spontaneaously, since O_c is part of the original Hamiltonian.

IV. CONCLUSIONS

Incommensurate behavior due to frustration has been much discussed in the literature, not only for strongly frustrated chains [58] and coupled ladder systems [22,59,60] but, more recently, also for anisotropic triangular lattices [61–64]. However, the incommensurate phase we have predicted in this paper arises for *small* frustration and interchain couplings. It is generated by a complex combination of competing instabilities, which requires only a small perturbation from the spin-liquid fixed point of decoupled chains. This opens the door for the search also in 2D systems for corresponding incommensurate phases near instabilities.

In conclusion, we have examined a general SU(2)-invariant ladder model with focus on frustration and ring exchange. For small coupling strengths the renormalization flow to four known phases can be quantitatively examined. In addition, we predict that there is an instability to an incommensurate phase in parts of the parameter space. In contrast to other models with incommensurate behavior, this phase can be observed even for very small values of frustration and interchain couplings. While frustration is an important ingredient for enhanced quantum fluctuations in order to generate the phase, the underlying instability towards incommensurate order is already present in the field theory of any weakly coupled chains with a broken translational symmetry $J_d \neq J_d'$. It therefore appears that an essential aspect for the observed incommensurate behavior is the enlarged unit cell, which in turn provides an important clue what choice of couplings may be promising to show corresponding phases in a 2D generalization of the model.

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APPENDIX

In this Appendix additional information for the derivation of the renormalization group equations is given and the bosonization of the four spin interaction is discussed.

1. Field theory and renormalization group flow

For the derivation of the bosonization formulas and the operator product expansion (OPE) it is useful to consider an interacting spinful Fermion model as the underlying physical realization where only the spin channel will be considered in the low-energy limit. For the half-filled Hubbard model, the charge channel is gapped and the Heisenberg couplings considered in the paper corresponds to the spin channel.

The spin currents are therefore conveniently expressed in terms of left- and right-moving Fermion operators

$$J_\kappa^a(z_\kappa) =: \psi_{\kappa\eta}^\dagger \frac{\sigma^a_{\eta\eta'}}{2} \psi_{\kappa\eta'} : (z_\kappa), \tag{A1}$$

where σ^a are the Pauli matrices with a summed over spin-index $\eta = \uparrow, \downarrow$, and $\kappa = R, L$ denotes the chirality. The chiral complex coordinates are $z_{L/R} = \pm ix + v\tau$. The dimerization

and staggered magnetization operators are given by [51]

$$\epsilon(z) \sim \frac{i}{2}[:\psi^\dagger_{R\eta}\psi_{L\eta}:(z) - :\psi^\dagger_{L\eta}\psi_{R\eta}:(z)],$$
$$n^a(z) \sim \frac{1}{2}\sigma^a_{\eta\eta'}[:\psi^\dagger_{R\eta}\psi_{L\eta'}:(z) + :\psi^\dagger_{L\eta}\psi_{R\eta'}:(z)], \tag{A2}$$

where z implies a dependence on both chiral variables z_R, z_L in this case.

The OPEs among J^a_κ, ϵ, and n^b can be calculated using Wick's theorem [47] and the two-point correlation function

$$\langle\langle \psi_{\kappa\eta}(z_\kappa)\psi^\dagger_{\kappa'\eta'}(w_{\kappa'})\rangle\rangle = \delta_{\kappa\kappa'}\delta_{\eta\eta'}\frac{\gamma}{z_\kappa - w_\kappa}, \tag{A3}$$

where we choose a normalization of $\gamma = 1/2\pi$. Keeping all relevant terms, the important OPEs are [51]

$$J^a_\kappa(z_\kappa)J^b_{\kappa'}(w_{\kappa'}) = \delta_{\kappa\kappa'}\left[\frac{(\gamma^2/2)\delta_{ab}}{(z_\kappa - w_\kappa)^2} + i\epsilon_{abc}\gamma\frac{J^c_\kappa(w_\kappa)}{z_\kappa - w_\kappa}\right]$$
$$J^a_\kappa(z_\kappa)\epsilon(w) = i\kappa\frac{\gamma/2}{z_\kappa - w_\kappa}n^a(w)$$
$$J^a_\kappa(z_\kappa)n^b(w) = i\frac{\gamma/2}{z_\kappa - w_\kappa}[\epsilon_{abc}n^c(w) - \kappa\delta_{ab}\epsilon(w)]$$
$$\epsilon(z)\epsilon(w) = \frac{\gamma^2}{|z-w|} - |z-w|\mathbf{J}_R\cdot\mathbf{J}_L(w)$$
$$n^a(z)\epsilon(w) = -i\gamma|z-w|\left[\frac{J^a_R(w_R)}{z_L - w_L} - \frac{J^a_L(w_L)}{z_R - w_R}\right]$$
$$n^a(z)n^b(w) = |z-w|\left[\frac{\gamma^2\delta_{ab}}{|z-w|^2} + i\epsilon_{abc}\gamma \times\left[\frac{J^c_R(w_R)}{z_L - w_L} + \frac{J^c_L(w_L)}{z_R - w_R}\right] + \hat{Q}^{ab}(w)\right], \tag{A4}$$

where δ_{ab} is the Kronecker δ function and ϵ_{abc} the Levi Civita symbol. Here

$$\hat{Q}^{ab} = \tfrac{1}{2}\sigma^a_{\eta\eta'}\sigma^b_{\tau\tau'}\psi^\dagger_{R\eta}\psi_{L\eta'}\psi^\dagger_{L\tau}\psi_{R\tau'}, \tag{A5}$$

denotes the zeroth-order contraction between the fermionic fields. After freezing out the gapped charge degrees of freedom, only the trace of this operator is relevant for the calculation of the renormalization group flow, which reads

$$\hat{Q}^{aa} = \mathbf{J}_R\cdot\mathbf{J}_L. \tag{A6}$$

The evolution of the bare couplings is determined from the OPEs of the perturbing operators using [49]

$$\frac{d\lambda_k}{dl} = (2 - d_k)\lambda_k - \frac{\pi}{v}\sum_{i,j}C_{ijk}\lambda_i\lambda_j, \tag{A7}$$

where d_k is the scaling dimension of the operator, v is the velocity, and C_{ijk} is the coefficient extracted from the OPE $O_i(z)O_j(w) \sim O_k(w)$. Using Eqs. (A4) and (A7), we arrive at the renormalization group equations in the main text.

2. Four-spin interactions

In this part, we present the bosonization of the four-spin interactions. The ring exchange interaction is given by

$$H = K\sum_p\left(h_p + \frac{1}{4}\right), \tag{A8}$$

where p sums over the plaquettes of the system and the local energy operators are given by

$$h_p = \mathbf{S}_{p_1}\cdot\mathbf{S}_{p_2} + \mathbf{S}_{p_3}\cdot\mathbf{S}_{p_4} + \mathbf{S}_{p_1}\cdot\mathbf{S}_{p_4} + \mathbf{S}_{p_2}\cdot\mathbf{S}_{p_3} + \mathbf{S}_{p_1}\cdot\mathbf{S}_{p_3} + \mathbf{S}_{p_2}\cdot\mathbf{S}_{p_4} + 4(\mathbf{S}_{p_1}\cdot\mathbf{S}_{p_2})(\mathbf{S}_{p_3}\cdot\mathbf{S}_{p_4}) + 4(\mathbf{S}_{p_1}\cdot\mathbf{S}_{p_4})(\mathbf{S}_{p_2}\cdot\mathbf{S}_{p_3}) - 4(\mathbf{S}_{p_1}\cdot\mathbf{S}_{p_3})(\mathbf{S}_{p_2}\cdot\mathbf{S}_{p_4}).$$

The indices $p_1 - p_4$ count clockwise the spins of the pth plaquette. The Hamiltonian consists of products of spin operators, H_2, and products of pairs of spin operators, H_4.

The four-spin interactions are either on the same leg of the ladder or on the rungs of the ladder. The leg component H_L and the rung H_R are given by

$$H_L = 4K\sum_j(\mathbf{S}_{1,j}\cdot\mathbf{S}_{1,j+1})(\mathbf{S}_{2,j}\cdot\mathbf{S}_{2,j+1}), \tag{A9}$$
$$H_R = 4K\sum_j(\mathbf{S}_{1,j}\cdot\mathbf{S}_{2,j})(\mathbf{S}_{1,j+1}\cdot\mathbf{S}_{2,j+1}) - 4K\sum_j(\mathbf{S}_{1,j}\cdot\mathbf{S}_{2,j+1})(\mathbf{S}_{2,j}\cdot\mathbf{S}_{1,j+1}). \tag{A10}$$

Spin operators are substituted in the continuum with

$$\mathbf{S}_j/a \to \mathbf{S}(x) \approx \mathbf{J}(x) + (-1)^x\Omega\mathbf{n}(x), \tag{A11}$$

and the product of two spin operators on the same chain becomes with the help of Eq. (A4),

$$aS^a(x)S^b(x+a) \approx -(a\Omega)^2\hat{Q}^{ab}(x) + (-1)^x 2\gamma\Omega\delta_{ab}\epsilon(x) + \gamma(1 + a\Omega^2)\left[\epsilon_{abc}\left[J^c_R(x) - J^c_L(x)\right] - \frac{\gamma}{a}\delta_{ab}\right]. \tag{A12}$$

Using this equation for the products of spin operators in the leg part, we arrive at

$$H_L \approx 12aK\gamma^2\Omega^2 \times\int dx[(a + (a\Omega)^2)O_a(x) + 12O_\epsilon(x)]. \tag{A13}$$

The rung part of the four spin interactions can be written as

$$H_R \approx 4a^3K\int dx S^a_1(x)S^b_1(x+a) \times\left[S^a_2(x)S^b_2(x+a) - S^b_2(x)S^a_2(x+a)\right], \tag{A14}$$

where, due to the relative minus sign and the δ functions, relevant terms are eliminated, and only interchain marginal

terms survive,

$$H_R \approx -16aK\gamma^2(1+a\Omega^2)^2 \int dx\, O_b(x). \quad \text{(A15)}$$

Combining Eqs. (A13) and (A15) and carrying out a trivial calculation for the two spin interactions H_2,

$$H_2 \approx aK \int dx[(2-a\Omega^2)O_a(x) + 4O_b(x)], \quad \text{(A16)}$$

one arrives at the bosonized ring exchange interaction in the continuous limit

$$H \approx aK \int dx \Bigg[\left(2 - a\Omega^2 + \frac{3a\Omega^2}{\pi^2}(1+a\Omega^2)\right)O_a + 4\left(1 - \frac{(1+a\Omega^2)^2}{\pi^2}\right)O_b + 36\frac{\Omega^2}{\pi^2}O_\epsilon\Bigg], \quad \text{(A17)}$$

which is used to determine the corresponding bare coupling strengths in the main text.

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