PHYSICAL REVIEW B **95**, 140504(R) (2017)

Controlling Majorana states in topologically inhomogeneous superconductors

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(Received 27 June 2016; published 24 April 2017)

Majorana bound states have been recently observed at the boundaries of one-dimensional topological superconductors. Yet, controlling the localization of the Majorana states, which is essential to the realization of any topological quantum device, is an ongoing challenge. To this end, we introduce a mechanism which can break a topologically homogeneous state via the formation of topological domains, and which can be exploited to control the position of Majorana states. We found, in fact, that in the presence of amplitude-modulated fields, contiguous magnetic domains can become topologically inequivalent and, as a consequence, Majorana states can be pinned to the domain walls of the magnetic structure. The formation of topological domains and the position of Majorana states can be externally controlled by tuning an applied field (e.g., magnetic or gate).

DOI: 10.1103/PhysRevB.95.140504

Introduction. The experimental observation of Majorana bound states (MBS) [1] in topological superconductors [2–5] marks the first milestone on the pathway toward topological quantum computation [6]. In particular, topological superconductivity has been observed in proximized nanowires with strong spin-orbit coupling (SOC) [7–12], and in ferromagnetic atomic chains on the surface of a superconductor [13–15]. In general, conventional superconductivity can be turned topological by the presence of a uniform magnetic field and intrinsic SOC [16,17], an intrinsic ferromagnetic order and SOC [18], or by a noncollinear spatially dependent magnetic field [19–22]. In particular, a very promising system is represented by chains of Yu-Shiba-Rusinov states [23–25] induced by magnetic atoms on the surface of conventional superconductors with ferromagnetic [18], antiferromagnetic [26], or helimagnetic textures [22,27–36].

Nevertheless, the implementation of a reliable braiding scheme [37–39], which is essential to the realization of topological quantum devices, is an ongoing challenge. A necessary prerequisite is the ability to control the position of MBS, which are localized at the boundaries between topologically trivial and nontrivial domains. In the proposed schemes [16,17,19–21], these boundaries are determined by the experimental setup and geometry, and can be manipulated via a gradient of the gate voltage [37] or magnetic field [40], via gate-tunable valves [41], via the magnetic flux in Josephson junctions [42], via external magnetic fields in the presence of an helimagnetic order and SOC [43], or by controlling the magnetic texture in two-dimensional electron gases [44].

In this Rapid Communication, we introduce a physical mechanism which can be exploited to locally break a homogeneous superconducting state into inhomogeneous *topological* domains and to control the position of MBS. We found, in fact, that in the presence of amplitude-modulated magnetic fields, the superconducting gap and the topological invariant strongly depends on the phase offset φ of the periodic texture. Consequently, contiguous magnetic domains can become topologically inequivalent and MBS can be pinned to the domain walls of the magnetic structure. This leads to the emergence of a topologically inhomogeneous state characterized by contiguous inequivalent domains, in an otherwise homogeneous superconducting state. The main advantage is that the formation of topological domains, and thus the position of MBS, can be externally controlled by tuning an applied *uniform* field. Differently from other proposals [37–40], this mechanism does not rely on the manipulation of gradients of the field intensity.

The model. Noninteracting s-wave superconductors with periodically amplitude-modulated magnetic (Zeeman) fields can be described by a Bogoliubov–de Gennes (BdG) tight-binding Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2}\sum_n \boldsymbol{\Psi}_n^\dagger \cdot \begin{bmatrix} 2t-\mu+\mathbf{b}_n\cdot\boldsymbol{\sigma} & \imath\sigma_y\Delta \\ (i\sigma_y\Delta)^\dagger & -(2t-\mu+\mathbf{b}_n\cdot\boldsymbol{\sigma}^*) \end{bmatrix}\cdot\boldsymbol{\Psi}_n - \frac{1}{2}\sum_n \boldsymbol{\Psi}_n^\dagger \cdot \begin{bmatrix} t\Omega & 0 \\ 0 & -t\Omega \end{bmatrix}\cdot\boldsymbol{\Psi}_{n+1} + \text{H.c.}, \tag{1}$$

where $\boldsymbol{\Psi}_n^\dagger = [c_{n\uparrow}^\dagger, c_{n\downarrow}^\dagger, c_{n\uparrow}, c_{n\downarrow}]$ is the Nambu spinor with $c_{n\uparrow\downarrow}$ and $c_{n\uparrow\downarrow}^\dagger$ the electron annihilation and creation operators, $\boldsymbol{\sigma} = [\sigma_x,\sigma_y,\sigma_z]$ the vector of Pauli matrices, μ the chemical potential, t the hopping parameter, $\Omega = \mathbb{1} + i\lambda\sigma_y$ with λ the intrinsic SOC, Δ the superconducting gap, and $\mathbf{b}_n$ the magnetic field on the site r_n. We assume for the sake of simplicity that the field modulation is commensurate with the lattice (i.e., it has spatial frequency $\theta = 2\pi p/q$ with $p,q \in \mathbb{Z}$ coprimes). Topologically nontrivial gapped phases are realized in the presence of either a collinear magnetic field (constant direction) with intrinsic SOC $\lambda \neq 0$ [16,17] or of a noncollinear field (nonconstant direction) with λ=0 [19,21,22], or both [20]. In the second case, in fact, the variation of the field direction is unitarily equivalent to an effective SOC [45]. In order to see this, one can rotate the z axis of the spin basis at each lattice site to the field direction via a unitary transformation [19,46] U_n. If the SOC vanishes, in fact, the transformed BdG Hamiltonian [19] coincides with Hamiltonian (1) after the substitutions

$$\Omega \to \Omega_n = U_n^\dagger U_{n+1}, \quad \mathbf{b}_n\cdot\boldsymbol{\sigma} \to b_n\sigma_z. \tag{2}$$

This mandates the presence of an effective collinear Zeeman field along the z axis with the same amplitude as the original

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field, a renormalized kinetic term (diagonal terms of Ω_n), and an effective SOC (off-diagonal terms of Ω_n) which is nonzero if the original field $\mathbf{b}_n$ is noncollinear [19]. In the presence of a finite magnetic field and SOC (effective or intrinsic), Hamiltonian (1) exhibits gapped phases which break time-reversal and chiral symmetries. These are the necessary ingredients which allow the realization of s-wave topological superconductors [16,17,19,21] characterized by a nontrivial $\mathbb{Z}_2$ topological invariant. This invariant coincides with the fermion parity [1,47,48] $\mathcal{P} = \prod_{k=-k} \mathrm{sgn}[F(k)]$, defined in terms of the Pfaffians $F(k) = \mathrm{Pf}[i\tau_x H(k)]$ at the time-reversal invariant momenta $k = -k$, being $H(k)$ the BdG Hamiltonian in momentum space and τ_x the first Pauli matrix in particle-hole space. In the low-energy sector, this system is equivalent to a p-wave topological superconductor [1,16,17,19].

Dependence on the phase offset. In the infinite-chain limit (i.e., neglecting finite-size effects) and for vanishing SOC ($\lambda = 0$), the Hamiltonian (1) is invariant under *global* rotations of the spin basis or, equivalently, of the Zeeman field. In order to see this, consider the case of a helical field with direction uniformly varying within the plane yz, i.e., $\mathbf{b}_n = b[\sin(n\theta + \varphi)\hat{\mathbf{y}} + \cos(n\theta + \varphi)\hat{\mathbf{z}}]$ where θ is the angular variation of the field direction between adjacent sites, and φ the phase offset describing the boundary offset of the magnetic texture. A global rotation of the spin basis corresponds to a variation of the phase offset. Moreover, in this case the transformed Hamiltonian obtained via Eq. (2) does not depend explicitly on the phase offset φ, since both the effective Zeeman and SOC terms become uniform, i.e., $b_n\hat{\mathbf{z}} = b\hat{\mathbf{z}}$ and $\Omega_n = U_n^\dagger U_{n+1} = \cos(\theta/2) + i\sigma_x \sin(\theta/2)$ (see Supplemental Material [49]). The phase offset φ is thus immaterial, since it can be absorbed by a unitary transformation. Hence, the system exhibits a global $U(1)$ gauge invariance with respect to the phase offset φ as a consequence of the more general $SO(3)$ global spin-rotational symmetry. Thus, in the case of circular helical fields without SOC, the bulk electronic spectrum does not depend on the boundary offset of the magnetic texture (except for energy contributions at the edges of the chain).

Nevertheless, if the SOC vanishes, an amplitude-modulated field can break the phase-offset invariance even without breaking the full spin-rotational symmetry. Consider, e.g., a noncollinear amplitude/modulated field realized via an elliptical helical field $\mathbf{b}_n = b_y \sin(n\theta + \varphi)\hat{\mathbf{y}} + b_z \cos(n\theta + \varphi)\hat{\mathbf{z}}$, with $b_y \neq b_z$, shown in Fig. 1(a), or via a circular helical field superimposed with a coplanar uniform field $\mathbf{b}_n = b_{yz}[\sin(n\theta + \varphi)\hat{\mathbf{y}} + \cos(n\theta + \varphi)\hat{\mathbf{z}}] + \mathbf{b}_0$ with $\mathbf{b}_0$ in the yz plane, shown in Fig. 1(b). As one can verify, in these cases the field amplitude is not uniform, but periodically modulated along the chain as $b_n = \sqrt{\langle b^2\rangle + \delta b^2 \cos[j(n\theta + \varphi)]}$ with $j = 2$ and 1, respectively, in the case of the elliptical and circular helical field. Hence the boundary phase offset φ cannot be absorbed by unitary rotations, since the phase offset affects not only the direction, but also the amplitude modulation of the field. Analogously, one can consider a collinear and amplitude/modulated magnetic field, e.g., $\mathbf{b}_n = [\langle b\rangle + \delta b \cos(n\theta + \varphi)]\hat{\mathbf{z}}$, shown in Fig. 1(c), with a finite SOC $\Omega = \mathbb{1} + \lambda i\sigma_y$. Notice that on a discrete lattice, the field texture is not invariant under translations $n \to n - \varphi/\theta$ unless φ/θ is an integer. Hence, the phase offset cannot be absorbed by spatial translations in the general case.

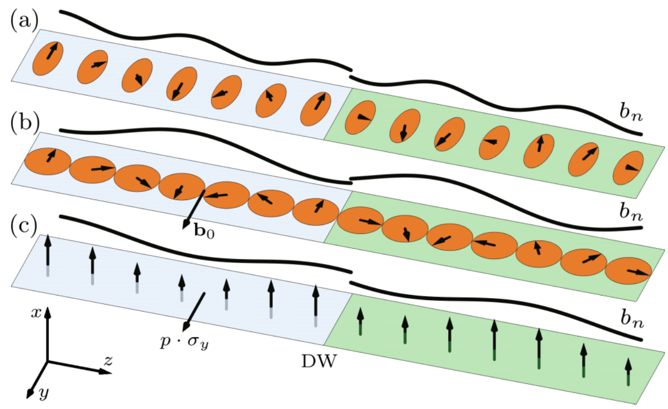

FIG. 1. Topologically inhomogeneous superconductors can be realized in superconducting systems with amplitude-modulated magnetic fields, e.g., (a) elliptical helical field, (b) circular helical field superimposed with an externally applied field $\mathbf{b}_0$, or (c) collinear amplitude-modulated field with intrinsic SOC. The field amplitude b_n is not constant, but periodically modulated along the chain. These systems can exhibit domain walls (DW) which break the spatial periodicity of the field.

In all these systems, the boundary phase offset cannot be gauged away by any unitary transformation, if the amplitude modulation δb is finite. Thus, the energy spectrum will depend explicitly on the offset of the magnetic texture at the boundary. This dependence is not due to a change of the average field per unit cell, which in fact does not depend on the phase offset φ in the cases considered here (harmonic modulation), as shown in the Supplemental Material [49]. Nevertheless, changes of the phase offset $\Delta\varphi = m\theta$ which are integer multiples of the angle θ, are equivalent to a discrete lattice translation $n \to n - \Delta\varphi/\theta$. This mandates a periodicity of the Hamiltonian (up to lattice translations) in the phase offset φ with period $\Delta\varphi = 2\pi/q$ (see also Ref. [50]). Moreover, the system is also periodic in the momentum with the same period (see Supplemental Material [49]), which results in a reduced Brillouin zone given by $[0,2\pi/q]$. The global $U(1)$ continuous symmetry (arbitrary variations $\Delta\varphi$ of the phase offset) is broken down into a discrete symmetry (periodicity in the phase offset $\Delta\varphi = m\theta$ integer). This is analogous to the case of a crystal lattice which breaks the continuous translational symmetry of free space. Notice that the boundary dependence can be obtained not only via magnetic fields, but more generally via any amplitude-modulated field as, e.g., electric fields [50–52] (charge-density waves) or strain-induced SOC fields [53–57].

Figure 2 shows the dependence on the boundary phase offset φ of the density of states (DOS) at low-energy spectra of a one-dimensional superconductor with an amplitude-modulated magnetic field, calculated from Hamiltonian (1). In particular, we considered a circular helical field with spatial frequency $\theta = 2\pi/3$ superimposed with an applied uniform field [see Fig. 1(b)], for two choices of the applied field. The bulk energy spectrum depends periodically on the phase offset φ with period $\Delta\varphi = \theta$.

Closing the particle-hole gap. As we have shown, the energy spectrum of a superconductor in the presence of amplitude-modulated magnetic fields can depend explicitly

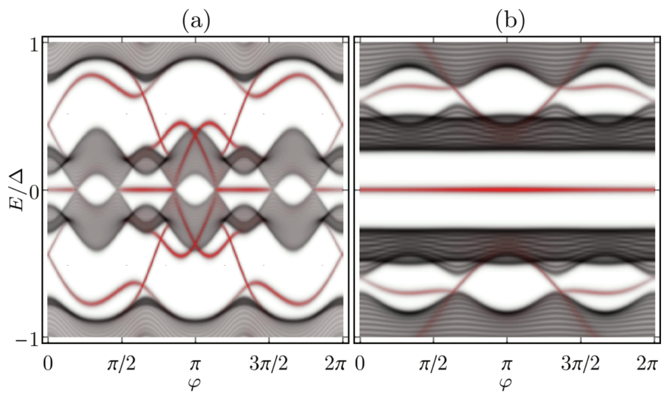

FIG. 2. DOS in the bulk (dark) and at the edges (color) of the system described by Hamiltonian (1) in the low-energy range as a function of the boundary phase offset φ in the case of a circular helical field with spatial frequency $\theta = 2\pi/3$ and amplitude b_{yz} superimposed with an applied field $\mathbf{b}_0$ [as in Fig. 1(b)]. (a) Nontrivial gaps with MBS alternating with trivial gaps as a function of the phase offset φ obtained for $b_0/b_{yz} = 0.75$ ($\delta b^2/\langle b^2\rangle \approx 1$). (b) Nontrivial gap with MBS in the whole range $\varphi \in [0,2\pi]$ for $b_0/b_{yz} = 0.3$ ($\delta b^2/\langle b^2\rangle \approx 0.5$). Notice that in both cases the bulk energy levels are periodic in the phase offset with period $\Delta\varphi = \theta = 2\pi/3$. We assume $\Delta = t/2$, $\mu = 3t$, $b_{yz} = 1.5t$, and $\mathbf{b}_0 \parallel \hat{\mathbf{z}}$.

on the offset φ of the magnetic texture. In fact, if the energy variation ΔE of the lowest-energy level is comparable with the superconducting pairing Δ, the particle-hole gap may close and reopen at specific values of the boundary offset $\varphi = \varphi^*$. This is indeed the case shown in Fig. 2(a), where the bulk particle-hole gap closes for certain values of the phase offset φ. Hence, the bulk properties can be deeply modified by the phase-boundary dependence: A conventional nodeless s-wave superconductor becomes an unconventional nodal superconductor for specific values of the phase offset. In general, if the lowest-energy level closes and reopens the gap linearly in the phase offset ($E \propto \varphi - \varphi^*$) at any of the two time-reversal invariant momenta $k = 0,\pi/q$ in the reduced Brillouin zone $[0,2\pi/q]$, and if no additional degeneracies are present, the Pfaffian $F(k)$ will change its sign and the topological invariant shall change at $\varphi = \varphi^*$ (see also Ref. [58]). Fermion parity transitions do indeed occur if the amplitude modulation δb is large compared with the average magnetic field. This can be verified by expanding the Pfaffian as a Fourier sum in the phase offset. For collinear amplitude-modulated fields $b_n = \langle b\rangle + \delta b \cos(n\theta + \varphi)$ [Fig. 1(c)] with finite SOC, one obtains (neglecting higher harmonics)

$$F(k) \approx F_0(k) + C\delta b^q \cos(q\varphi) \quad \text{for } k = 0,\pi/q, \tag{3}$$

where $F_0(k) = \pm\sqrt{\prod_{m=0}^{q-1} \det[h(k+m\theta)]}$, being $h(k)$ the Hamiltonian in momentum space with constant magnetic field ($\delta b = 0$), and where C is a constant prefactor (see Supplemental Material [49]). Similar equations can be obtained in the case of noncollinear amplitude-modulated fields. Thus, if the particle-hole gap remains open for any value of the phase offset φ, the fermion parity $\mathcal{P} = \operatorname{sgn}[F(0)F(\pi/q)]$ will change its sign at $\varphi \equiv \varphi_m^* = \{\arccos[(\delta b_c/\delta b)^q] + 2\pi m\}/q$ for $m \in \mathbb{Z}$ if the amplitude modulation is larger than a critical value $\delta b > \delta b_c \equiv \sqrt[q]{\min[|F_0(0)|,|F_0(\pi/q)|]/|C|}$.

Hence, for commensurate spatial frequencies $\theta = 2\pi p/q$ (p,q coprimes) with q odd, a change of the phase offset can drive a transition between trivial and nontrivial states if the amplitude modulation δb is larger than a critical value δb_c. A remarkable consequence is that MBS can be created or annihilated by changing the phase offset at the boundary. This is indeed the key result of this Rapid Communication and the main property of topological superconductors in the presence of amplitude-modulated fields: The particle-hole gap can be closed and reopened with a concurrent transition between trivial and nontrivial states by tuning only the phase offset φ of the magnetic texture, i.e., without changing the field average intensity. In particular, Fig. 2(a) shows the case where $\delta b > \delta b_c$, with nontrivial gaps ($\mathcal{P} = -1$) with MBS alternating to trivial gaps ($\mathcal{P} = 1$) for different values of the phase offset $\varphi \in [0,2\pi]$. Alternatively, a nontrivial gap can span the whole interval $[0,2\pi]$, as shown in Fig. 2(b) in the case that $\delta b < \delta b_c$, where MBS are present for any possible boundary configuration ($\mathcal{P} = -1$ for any choice of the phase offset φ).

Pinning of MBS. The dependence of the topological properties on the offset of the magnetic texture can be exploited to control the localization of MBS in the presence of a domain wall which breaks the periodic rotation of the field direction, without changing its average intensity. Consider, in fact, the case shown in Fig. 1, where a domain wall at the lattice site r_m separates two contiguous and homogeneous magnetic domains (left and right domains), which are characterized by a different value of the phase offset, i.e., φ_L and φ_R with $\Delta\varphi_m \equiv \varphi_R - \varphi_L \neq 0$. If the field amplitude is constant and the SOC vanishes ($\lambda = 0$), the spin-rotational symmetry of Hamiltonian (1) is unbroken. In this case, neglecting boundary/termination conditions and additional symmetry-breaking effects, the phase offset can be gauged away by unitary rotations. Thus, under these assumptions, the two domains are unitarily and topologically equivalent, i.e., $\mathcal{P}(\varphi_L) = \mathcal{P}(\varphi_R)$, being either both trivial or nontrivial. On the contrary, if the field is amplitude modulated, the two domains are not unitarily equivalent anymore, since the phase offset cannot be gauged away by unitary transformations. Moreover, if the amplitude modulation is large enough ($\delta b > \delta b_c$), the topological invariant will depend on the phase offset: In this case their fermion parity may differ, i.e., $\mathcal{P}(\varphi_L) \neq \mathcal{P}(\varphi_R)$ for certain values of the phase difference $\Delta\varphi_m$. Hence, two distinct and contiguous *topological domains* emerge, separated by the domain wall of the underlying magnetic texture. Consequently, MBS localize at the boundaries of the topologically nontrivial domain, i.e., at one edge of the chain and at the domain wall. Notice that, for the sake of simplicity, the domain wall thickness is assumed to be equal to one interatomic distance. However, domain walls in magnetically ordered materials typically extend over several lattice sites: In this case the pinned MBS would be localized over a similar length scale.

To illustrate the formation of topological domains and the consequent pinning of MBS, we calculate the local density of states (LDOS) at zero energy, which corresponds to the probability to find an intragap MBS at a given position r_n along the chain. Fig. 3(a) shows the LDOS in the case of a circular helimagnetic field with spatial frequency $\theta = 2\pi/3$ superimposed with a uniform field, as a function of the ratio

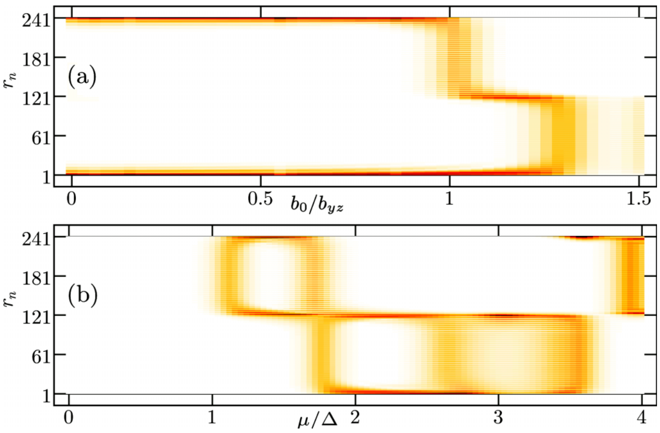

FIG. 3. LDOS at zero energy in the presence of a domain wall localized at the center $m = 121$ of the chain ($N = 241$ sites), as a function of the lattice site r_n (a) of the ratio between the superimposed uniform field b_0 and the helical field b_{yz}, and (b) of the chemical potential (gate field). The zero-energy peaks signal the presence of MBS localized alternatively at the edges, or at one edge and at the domain wall. We assume a circular helimagnetic field with spatial frequency $\theta = 2\pi/3$ superimposed with a uniform field $\mathbf{b}_0 \parallel \hat{\mathbf{z}}$, and with $\Delta = t/2$, $b_{yz} = 1.5t$, $\Delta\varphi_m = \pi/3$, $\mu = 3t$ [panel (a)], and $b_0/b_{yz} = 0.75$ ($\delta b^2/\langle b^2\rangle \approx 1$) [panel (b)].

between the superimposed uniform field b_0 and the helical field b, with a fixed phase difference $\Delta\varphi_m = \pi/3$ at the domain wall. At $b_0 = 0$, the field amplitude is constant ($\delta b = 0$) and thus the left and right domains are topologically equivalent [$\mathcal{P}(\varphi_L) = \mathcal{P}(\varphi_R) = -1$]. In this case MBS are necessarily localized at the two edges of the chain. Nevertheless, as the magnitude of the applied field b_0 increases such that $\delta b > \delta b_c$, one of the MBS becomes pinned to the domain wall, leaving one side of the chain topologically trivial [$\mathcal{P}(\varphi_L) \neq \mathcal{P}(\varphi_R)$].

Moreover, the formation of contiguous topological domains can be obtained in proximized nanowires also by tuning the chemical potential via an applied gate field. This is similar to the case of homogeneous topological superconductors with uniform magnetic fields [16,17], where a change in the chemical potential (or magnetic field intensity) can turn the topological state from trivial to nontrivial. For example, Fig. 3(b) shows the localization of MBS as a function of the chemical potential μ. MBS are localized alternatively at the left or right edge and at the domain wall for different values of μ.

Hence, by controlling the external magnetic field or gate voltage, MBS can be moved from the edge of the chain to the domain wall. The position of MBS can be revealed by a measure of the local differential conductance at zero bias, which is proportional to the LDOS at zero energy, obtained via scanning tunneling microscopy [7,13]. The pinning of MBS is topologically robust against magnetic and nonmagnetic disorder, as long as the perturbation is small, as verified in the Supplemental Material [49].

Experimental implementation. The implementation of the model proposed here requires the realization of amplitude-modulated magnetic textures with a periodicity comparable with the lattice parameter of the superconducting chain, in the presence of domain walls on a similar length scale.

A possible implementation is to realize a heterostructure where a proximized nanowire with strong SOC is contiguous to an antiferromagnetic material. The magnetic field of an antiferromagnet is indeed dominated by a nonzero quadrupolar contribution [59,60], which results in a periodic field with the same periodicity of the ordering vector. The mismatch between the ordering vector of the antiferromagnet and the lattice parameter of the nanowire will then result in an amplitude-modulated Zeeman field at the wire lattice sites. Domain walls in antiferromagnets have a typical thickness of hundreds of atoms [61]: The ensuing pinned MBS will be thus localized on a similar length scale. Notice also that amplitude-modulated magnetic orders occur, e.g., in multiferroics [62,63] and gadolinium compounds [64,65], and in general in materials with competing ferromagnetic and antiferromagnetic exchanges and in the presence of anisotropic distortions which favors magnetization along a specific axis [66]. In these systems, where helical orders are generally coupled with a spontaneous electric polarization, electric fields or currents may be employed to induce a sliding of the helical spin texture [67]. Other means may involve using spin-torque mechanisms or spin currents to twist the phase offset of the helical order [68,69].

Another possible implementation can be realized at larger length scales via artificial one-dimensional superlattices realized using nanoscale lithography design on ultrathin films [70] or quantum dot solids [71], in the presence of nanomagnets. This would allow a fine-tuning of the magnetic texture and domain wall properties by controlling the position and orientation of the nanomagnets.

In principle, an amplitude-modulated texture can be also achieved in topological Yu-Shiba-Rusinov chains [13,21] via an external magnetic field [43]. Notice that a realistic theoretical description of such system should take into account the effect of the external field on the magnetic texture, which would likely relax into a nonplanar magnetic configuration, and the dependence of the ordering vector on the Fermi momentum [27,29,72–74]. Moreover, a more fundamental challenge is the creation of stable domain walls in these systems via, e.g., magnetic impurities.

Conclusions. We have found that amplitude-modulated magnetic fields can induce the emergence of a topologically inhomogeneous state via the formation of topological domains. In these systems, MBS become pinned to domain walls between contiguous and topologically inequivalent domains. Such systems can be in principle realized with proximized nanowires in a magnetic field induced by an antiferromagnet, in artificial one-dimensional superlattices via nanoscale lithography, or in quantum dot solids.

Remarkably, the formation of topological domains and the localization of MBS can be externally controlled by tuning an applied and uniform magnetic or gate field, without moving the domain wall. Controlling the localization of MBS is the first step in the realization of a reliable braiding scheme in topological devices.

Acknowledgments. The authors thank Wei Chen, Paola Gentile, and Andreas Schnyder for useful discussions. P.M. thanks Peter Stano, Daniel Loss, Jelena Klinovaja, and Alex Matos-Abiague for their valuable feedback and suggestions.

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