Influence of topological constraints and topological excitations: Decomposition formulas for calculating homotopy groups of symmetry-broken phases

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A symmetry broken phase of a system with internal degrees of freedom often features a complex order parameter, which generates a rich variety of topological excitations and imposes topological constraints on their interaction (topological influence); yet the very complexity of the order parameter makes it difficult to treat topological excitations and topological influence systematically. To overcome this problem, we develop a general method to calculate homotopy groups and derive decomposition formulas which express homotopy groups of the order parameter manifold G/H in terms of those of the symmetry G of a system and those of the remaining symmetry H of the state. By applying these formulas to general monopoles and three-dimensional skyrmions, we show that their textures are obtained through substitution of the corresponding $\mathfrak{su}(2)$ subalgebra for the $\mathfrak{su}(2)$ spin. We also show that a discrete symmetry of H is necessary for the presence of topological influence and find topological influence on a skyrmion characterized by a non-Abelian permutation group of three elements in the ground state of an SU(3)-Heisenberg model.

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I. INTRODUCTION

Topological excitations create nontrivial spatial structures of the order parameter that cannot be removed by continuous deformation and are characterized by a topological charge. When a system has internal degrees of freedom as in spinor Bose-Einstein condensates (BECs) [1,2], p-wave superfluids and superconductors [3,4], and multiorbital electron systems [5–9], a symmetry broken phase has a complex order parameter, accommodating a rich variety of topological excitations. Examples include fractional and non-Abelian vortices [10–12], skyrmions [13,14], Shankar skyrmions [15,16], and knot solitons [16,17]. Several different types of topological excitations have been experimentally observed in condensed matter and ultracold atomic systems: skyrmions in chiral magnets [18,19] and quantum Hall ferromagnets [20–22], half-quantum vortices in p-wave superconductors [23] and liquid ^{3}He [24], and knot solitons in liquid crystals [25]. In particular, ultracold atomic gases offer an ideal playground for the study of topological excitations due to high controllability of experimental parameters; here the controlled generations of vortices [26–28], skyrmions [29,30], monopoles [31,32], and a knot soliton [33] have been demonstrated. Yet another remarkable feature arising from internal degrees of freedom is the coexistence of different types of topological excitations, which leads to nonconservation of individual topological charges due to topological influence [34–36]. For example, the A phase of superfluid ^{3}He can simultaneously accommodate a half-quantum vortex and a monopole. When the latter makes a complete circuit of the former, the topological charge of the latter changes its sign [37].

In mathematical parlance, the set of topological charges constitutes a homotopy group of the order parameter manifold (OPM) G/H, where G and H are the symmetry of a system under consideration and the remaining symmetry of its state, respectively. Topological charges classify textures of an order parameter of topological excitations; two textures can continuously transform into each other if and only if their topological charges are the same. While the complexity of G/H leads to the richness of topological excitations, it makes the calculation of homotopy groups involved, the understanding of textures highly nontrivial, and the analysis of topological influence difficult. While topological influence of a vortex on a topological excitation is known to be described by the action of the fundamental group $\pi_1(G/H)$ on the mth homotopy group $\pi_m(G/H)$ [34–36,38], where m is the spatial dimension in which the texture of the topological excitation varies, general conditions for its presence are yet to be clarified. For topological influence on a monopole or a skyrmion, only one type is known, in which the topological influence changes the sign of the topological charge of a monopole and that of a skyrmion [34,36,37,39,40].

In the present paper, we develop a general method to calculate the homotopy group $\pi_m(G/H)$ of the order parameter manifold G/H by deriving a formula which expresses $\pi_m(G/H)$ in terms of $\pi_m(G)$ and $\pi_m(H)$. Since the homotopy groups can be determined systematically for Lie groups [41,42], $\pi_m(G/H)$ and the corresponding textures can be determined through the formula. By applying the derived formulas for $m = 2$ and 3, we show that the texture of a general monopole and that of a general three-dimensional skyrmion are obtained from that of a monopole in a ferromagnet and those of a knot soliton or a Shankar skyrmion, respectively, through substitution of an appropriate $\mathfrak{su}(2)$ subalgebra in G for the $\mathfrak{su}(2)$ spin. Consequently, their topological charges are described by a set of integers distinguished by coroots [41,42], which label different $\mathfrak{su}(2)$ subalgebras in G.

We also obtain the necessary and sufficient condition for the appearance of non-Abelian vortices and prove the absence of topological influence on a three-dimensional skyrmion. We find that possible types of topological influence on a monopole or a skyrmion can be identified with the Weyl group [41,42] of G, where only one type is shown to be allowed if G is U(1), SU(2), SO(3), or their direct product. Moreover, we find topological influence on skyrmions characterized by a non-Abelian permutation group of three elements in the ground state of an SU(3)-Heisenberg model [7,8,43,44], in which three types of skymions exchange their types through topological influence.

This paper is organized as follows. In Sec. II, we derive a decomposition formula for $\pi_m(G/H)$ for an arbitrary dimension m. In Sec. III, we derive simplified formulas for $\pi_m(G/H)$ with $m = 1,2$, and 3, and determine the texture of a general monopole and that of a general three-dimensional skyrmion. In Sec. IV, we analyze the conditions for the presence of topological influence. In Sec. V, we discuss the non-Abelian topological influence on a skyrmion. In Sec. VI, we conclude this paper. Some mathematical proofs are relegated to the appendices to avoid digressing from the main subject. Appendix A proves a lemma on the third homotopy group of a compact Lie group used in Sec. II. Appendices B and C prove formulas for $\pi_m(G/H)$ and $\pi_2(G/H)$, respectively, discussed in Sec. III. Appendix D proves a theorem concerning topological influence on a general topological excitation discussed in Sec. IV. Appendix E proves a corollary concerning topological influence on a monopole or a skyrmion discussed in Sec. IV.

II. DECOMPOSITION FORMULA FOR HOMOTOPY GROUPS OF ORDER PARAMETER MANIFOLDS

A. Homotopy groups of a Lie group

We first introduce the Cartan canonical form and the lattices of a compact Lie group, by means of which the first, second, and third homotopy groups are determined. When the parameter space of G is (not) finite, G is said to be (non)compact. If G includes translational symmetry, G is noncompact. However, for the calculation of homotopy groups, G can be replaced without loss of generality by its compact subgroup constituted from internal and rotational symmetries through the following isomorphism:

$$\pi_m(G) \simeq \pi_m(G_{\rm int} \times G_{\rm rot}) \text{ for } \forall m \geqslant 1, \tag{1}$$

where $G_{\rm int}$ and $G_{\rm rot}$ are the internal and rotational symmetries of G, respectively, and $\simeq$ denotes the group isomorphism. The relation (1) can be proved as follows. The symmetry G is, in general, constituted from an internal symmetry $G_{\rm int}$ and a space symmetry $G_{\rm space}$, i.e., $G = G_{\rm int} \times G_{\rm space}$. The former is compact, while the latter may not be. Since $\pi_m[\mathrm{SO}(d,1)] \simeq \pi_m[\mathrm{E}(d)] \simeq \pi_m[\mathrm{SO}(d)]$ and $\pi_m(\mathbb{R}^d) \simeq 0$ for any m ($\geqslant 1$) and any spatial dimension d for the Lorentz group $\mathrm{SO}(d,1)$, the Euclid group $\mathrm{E}(d)$ and the translation group $\mathbb{R}^d$, the translational part of the symmetry does not contribute to homotopy groups. Then, we have

$$\begin{aligned}\pi_m(G) &\simeq \pi_m(G_{\rm int}) \oplus \pi_m(G_{\rm space})\\ &\simeq \pi_m(G_{\rm int}) \oplus \pi_m(G_{\rm rot})\\ &\simeq \pi_m(G_{\rm int} \times G_{\rm rot}),\end{aligned} \tag{2}$$

where $\oplus$ denotes the direct sum. The first and third isomorphisms follow from the relation $\pi_m(X \times Y) \simeq \pi_m(X) \oplus \pi_m(Y)$.

1. Cartan canonical form and lattices of a compact Lie group

The Lie algebra $\mathfrak{g}$ of a compact Lie group G has a convenient basis called the Cartan canonical form [45]

$$\mathfrak{g} = \left\{\{H_j\}_{j=1}^r, \{E_{\boldsymbol\alpha}^R, E_{\boldsymbol\alpha}^I\}_{\boldsymbol\alpha\in R_+}\right\}, \tag{3}$$

where $E_{\boldsymbol\alpha}^R := (E_{\boldsymbol\alpha} + E_{\boldsymbol\alpha}^\dagger)/\sqrt{2}$ and $E_{\boldsymbol\alpha}^I := (E_{\boldsymbol\alpha} - E_{\boldsymbol\alpha}^\dagger)/(\sqrt{2}i)$ are the real and imaginary parts of the raising operator $E_{\boldsymbol\alpha}$, and r is the rank of $\mathfrak{g}$. The Cartan canonical form (3) is a generalization of the basis of the $\mathfrak{su}(2)$-Lie algebra $\{S_3,\{S_1,S_2\}\}$, and decomposes the generators of the Lie algebra into the off-diagonal matrices $\{E_{\boldsymbol\alpha}^R, E_{\boldsymbol\alpha}^I\}_{\boldsymbol\alpha\in R_+}$ and the diagonal ones (Cartan generators) $\{H_j\}_{j=1}^r$, where $\boldsymbol\alpha$ is an r-dimensional real vector known as a positive root and R_+ denotes the entire set of positive roots. The positive roots are introduced to distinguish different $\mathfrak{su}(2)$ subalgebras in $\mathfrak{g}$. It is known that any positive root can be expressed as a linear combination of the r positive roots known as simple roots, which we denote as $\{\boldsymbol\alpha_j\}_{j=1}^r$. Two matrices $E_{\boldsymbol\alpha}$ and $E_{-\boldsymbol\alpha} = E_{\boldsymbol\alpha}^\dagger$ are generalizations of the raising and lowering operators $S_+ := S_1 + iS_2$ and $S_- := S_1 - iS_2$ of the $\mathfrak{su}(2)$-spin vector $\boldsymbol{S} = (S_1,S_2,S_3)$. Physically $\boldsymbol\alpha$ describes the difference between two quantum numbers. When $E_{+\boldsymbol\alpha}$ ($E_{-\boldsymbol\alpha}$) is applied to a state, its quantum number changes by $\boldsymbol\alpha$ ($-\boldsymbol\alpha$), as S_+ (S_-) changes the magnetic quantum number of a spin state by $+1$ (-1).

Together with the Cartan generator $H_{\boldsymbol\alpha}$ defined by $H_{\boldsymbol\alpha} := \sum_{j=1}^r (\boldsymbol\alpha)_j H_j$ with $\boldsymbol\alpha = ((\boldsymbol\alpha)_1,(\boldsymbol\alpha)_2,\cdots,(\boldsymbol\alpha)_r)^T \in \mathbb{R}^r$ (T denotes the transpose of a vector), the two generators $E_{\boldsymbol\alpha}^R$ and $E_{\boldsymbol\alpha}^I$ satisfy the following commutation relations:

$$\begin{aligned}\left[E_{\boldsymbol\alpha}^R, E_{\boldsymbol\alpha}^I\right] &= i(\boldsymbol\alpha,\boldsymbol\alpha)H_{\boldsymbol\alpha},\\ \left[H_{\boldsymbol\alpha}, E_{\boldsymbol\alpha}^R\right] &= i(\boldsymbol\alpha,\boldsymbol\alpha)E_{\boldsymbol\alpha}^I,\\ \left[E_{\boldsymbol\alpha}^I, H_{\boldsymbol\alpha}\right] &= i(\boldsymbol\alpha,\boldsymbol\alpha)E_{\boldsymbol\alpha}^R.\end{aligned} \tag{4}$$

We define the coroot $\boldsymbol\alpha^c$ as a dual vector to each positive root $\boldsymbol\alpha$ and the corresponding generator $H_{\boldsymbol\alpha^c}$ as follows:

$$\boldsymbol\alpha^c := \frac{2\boldsymbol\alpha}{(\boldsymbol\alpha,\boldsymbol\alpha)}, \tag{5}$$

$$H_{\boldsymbol\alpha^c} := \sum_{j=1}^r (\boldsymbol\alpha^c)_j H_j. \tag{6}$$

One can see from Eq. (4) that a triad $\boldsymbol{S}_{\boldsymbol\alpha}$ defined by

$$\boldsymbol{S}_{\boldsymbol\alpha} := (S_{\boldsymbol\alpha,1}, S_{\boldsymbol\alpha,2}, S_{\boldsymbol\alpha,3}) := \left(\frac{E_{\boldsymbol\alpha}^R}{(\boldsymbol\alpha,\boldsymbol\alpha)}, \frac{E_{\boldsymbol\alpha}^I}{(\boldsymbol\alpha,\boldsymbol\alpha)}, \frac{H_{\boldsymbol\alpha^c}}{2}\right) \tag{7}$$

forms an $\mathfrak{su}(2)$ subalgebra satisfying the following commutation relations:

$$[S_{\boldsymbol\alpha,a}, S_{\boldsymbol\alpha,b}] = i\epsilon_{abc}S_{\boldsymbol\alpha,c} \text{ for } a,\ b,\ c = 1,\ 2,\ 3, \tag{8}$$

where ϵ_{abc} is the three-dimensional Levi-Civita symbol which is a totally antisymmetric unit tensor of rank three. We refer to $\boldsymbol{S}_{\boldsymbol\alpha}$ as a generalized $\mathfrak{su}(2)$-spin vector by analogy with the ordinary $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}$ [46]. The integral lattice L_G and the coroot lattice L_G^c of G are defined in terms of the Cartan generators of $\mathfrak{g}$ and those of the coroots as follows:

$$L_G := \{H_t \in \mathfrak{g} |\, \exp(2\pi i H_t) = e\}, \tag{9}$$

$$L_G^c := \left\{ \sum_{\alpha} n_{\alpha} H_{\alpha^c} \in \mathfrak{g} \middle| n_{\alpha} \in \mathbb{Z}, \alpha \in R_+ \right\}, \tag{10}$$

where $H_t \in \mathfrak{g}$ for t ($\in \mathbb{R}^r$) is defined by $H_t := \sum_{j=1}^r t_j H_j$ with $t = (t_1, t_2, \cdots, t_r)^T$ (T denotes the transpose of a vector). Both L_G and L_G^c form Abelian groups under the addition of matrices.

Consider an example of $\mathfrak{g} = \mathfrak{su}(3)$, which is generated by the following nine generators:

$$S_{RG,1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{RG,2} = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{RG,3} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{GB,1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_{GB,2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_{GB,3} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_{BR,1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_{BR,2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$S_{BR,3} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

The corresponding Cartan canonical form is constituted from the following three generalized $\mathfrak{su}(2)$-spin vectors

$$\begin{aligned} \boldsymbol{S}_{RG} &= (S_{RG,1}, S_{RG,2}, S_{RG,3}), \\ \boldsymbol{S}_{GB} &= (S_{GB,1}, S_{GB,2}, S_{GB,3}), \\ \boldsymbol{S}_{BR} &= (S_{BR,1}, S_{BR,2}, S_{BR,3}). \end{aligned} \tag{12}$$

Note that the three diagonal generators $S_{RG,3}$, $S_{GB,3}$, and $S_{BR,3}$ are not linearly independent because $S_{RG,3} + S_{GB,3} + S_{BR,3} = 0$. Three root vectors $\boldsymbol{\alpha}_{RG}$, $\boldsymbol{\alpha}_{GB}$, and $\boldsymbol{\alpha}_{BR}$ corresponding to generators $\boldsymbol{S}_{RG}$, $\boldsymbol{S}_{GB}$, and $\boldsymbol{S}_{BR}$ are given by

$$\boldsymbol{\alpha}_{RG} = (1, -1, 0), \quad \boldsymbol{\alpha}_{GB} = (0, 1, -1), \quad \boldsymbol{\alpha}_{BR} = (-1, 0, 1). \tag{13}$$

Since the lengths of these vectors are all $\sqrt{2}$, the corresponding coroots $\boldsymbol{\alpha}^c_{RG}, \boldsymbol{\alpha}^c_{GB}$, and $\boldsymbol{\alpha}^c_{BR}$ are given from Eq. (5) by

$$\boldsymbol{\alpha}^c_{RG} = \boldsymbol{\alpha}_{RG}, \ \boldsymbol{\alpha}^c_{GB} = \boldsymbol{\alpha}_{GB}, \ \boldsymbol{\alpha}^c_{BR} = \boldsymbol{\alpha}_{BR}. \tag{14}$$

From direct calculations using Eqs. (11), (13), and (14), one can show that the integral lattice $L_{SU(3)}$ and the coroot lattice $L^c_{SU(3)}$ coincide and that they are isomorphic to the triangular lattice (see Fig. 1):

$$\begin{aligned} L_{SU(3)} &= L^c_{SU(3)} \\ &= \left\{ \sum_{a=RG,GB,BR} m_a \boldsymbol{\alpha}^c_a \middle| m_a \in \mathbb{Z}, \sum_{a=RG,GB,BR} \boldsymbol{\alpha}^c_a = 0 \right\}. \end{aligned} \tag{15}$$

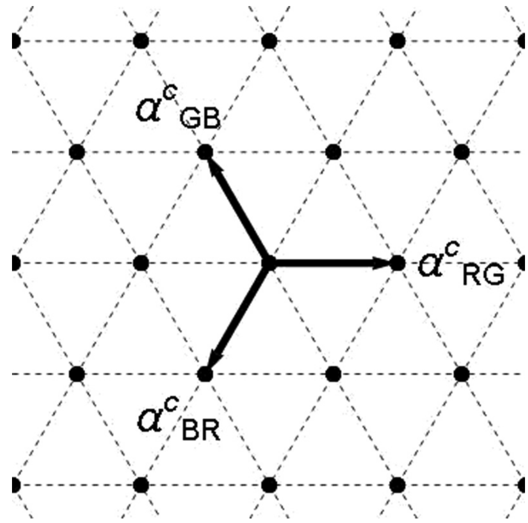

FIG. 1. Coroot lattice L^c_G of $G = SU(3)$ generated by the three coroots $\boldsymbol{\alpha}^c_{RG}, \boldsymbol{\alpha}^c_{GB}$, and $\boldsymbol{\alpha}^c_{BR}$ in Eq. (14). Since $\boldsymbol{\alpha}^c_{RG} + \boldsymbol{\alpha}^c_{GB} + \boldsymbol{\alpha}^c_{BR} = 0$, L^c_G is isomorphic to the triangular lattice. We take the system of coordinates such that $\boldsymbol{\alpha}^c_{RG} = (1,0)$, $\boldsymbol{\alpha}^c_{GB} = (-1/2, \sqrt{3}/2)$, and $\boldsymbol{\alpha}^c_{BR} = (-1/2, -\sqrt{3}/2)$.

2. *First, second, and third homotopy groups of a compact Lie group*

It is known that L^c_G is an Abelian subgroup of L_G and that the quotient group L_G/L^c_G is isomorphic to $\pi_1(G)$ [41,42]:

$$\pi_1(G) \simeq L_G/L^c_G. \tag{16}$$

While L_G describes all loops on G, L^c_G describes only those loops on G that can continuously transform into a trivial one, so the quotient space naturally gives $\pi_1(G)$. To be concrete, let us consider an element H_t of L_G corresponding to a loop defined by

$$g_{1,n}(\phi) := \exp(i\phi H_t) \text{ for } \phi \in [0, 2\pi]. \tag{17}$$

The map $g_{1,n}$ indeed describes a loop on G, since $g_{1,n}(0) = g_{1,n}(2\pi) = e$ from Eq. (9). The triviality of loops in L^c_G can be checked by considering one of its generator $H_{\boldsymbol{\alpha}^c}$ and the corresponding loop $g_{1,d}(\phi) := \exp(i\phi H_{\boldsymbol{\alpha}^c})$. This loop can continuously transform into a trivial one through $g^{(2)}_{\boldsymbol{\alpha}}(\theta, \phi)$ defined by

$$g^{(2)}_{\boldsymbol{\alpha}}(\theta, \phi) := \mathrm{e}^{-i\theta S_{\boldsymbol{\alpha},2}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} \mathrm{e}^{i\theta S_{\boldsymbol{\alpha},2}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}}, \tag{18}$$

where θ ($\in [0, \pi]$) is the parameter of the deformation. In fact, we have

$$g^{(2)}_{\boldsymbol{\alpha}}(\theta = 0, \phi) = \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} = g_{1,d}(\phi), \tag{19}$$

$$\begin{aligned} g^{(2)}_{\boldsymbol{\alpha}}(\theta = \pi, \phi) &= \mathrm{e}^{-i\pi S_{\boldsymbol{\alpha},2}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} \mathrm{e}^{i\pi S_{\boldsymbol{\alpha},2}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} \\ &= \mathrm{e}^{-i\phi S_{\boldsymbol{\alpha},3}} \mathrm{e}^{i\phi S_{\boldsymbol{\alpha},3}} = e, \end{aligned} \tag{20}$$

where the last equality in Eq. (19) follows from the definition (7) of $S_{\boldsymbol{\alpha},3}$, and the second line in Eq. (20) is derived from $\mathrm{e}^{-i\pi S_{\boldsymbol{\alpha},2}} S_{\boldsymbol{\alpha},3} \mathrm{e}^{i\pi S_{\boldsymbol{\alpha},2}} = -S_{\boldsymbol{\alpha},3}$. It is worthwhile to mention that $\pi_1(G)$ is Abelian, which follows from the fact that L_G is Abelian and the fact that a quotient group of an Abelian group is Abelian [47].

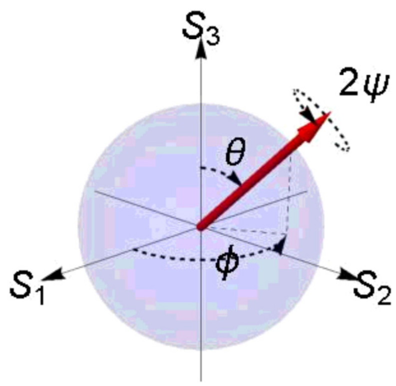

FIG. 2. Schematic illustration of the symmetry transformation $g^{(3)}_{\boldsymbol{\alpha}}(\psi,\theta,\phi)$ defined in Eq. (23). The red arrow indicates the generalized $\mathfrak{su}(2)$-spin vector parallel to the unit vector $\hat{\boldsymbol{r}}(\theta,\phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $g^{(3)}_{\boldsymbol{\alpha}}(\psi,\theta,\phi)$ describes the spin rotation about $\hat{\boldsymbol{r}}(\theta,\phi)$ through angle 2ψ.

The second homotopy group of a compact Lie group is known to vanish identically [41,42,48]:

$$\pi_2(G) \simeq 0. \tag{21}$$

We now discuss the third homotopy group. It is known that the Lie algebra $\mathfrak{g}$ of a compact Lie group G can be decomposed into the direct sum of one-dimensional Lie algebras $\mathfrak{u}(1)$ and a set of compact simple Lie algebras $\{\mathfrak{g}_i\}_{i=1}^{a}$ [42]:

$$\mathfrak{g} = \mathfrak{u}(1)^{a'} \oplus \bigoplus_{i=1}^{a} \mathfrak{g}_i, \tag{22}$$

where a and a' are the integers which are uniquely determined from $\mathfrak{g}$, and $\mathfrak{u}(1)$ is the Lie algebra of U(1), the unitary group of degree one. Let $\boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i^c$, and $\boldsymbol{S}_{\boldsymbol{\alpha}_i}$ be one of the root vectors in $\mathfrak{g}_i$ with the longest length, the corresponding coroot, and the corresponding generalized $\mathfrak{su}(2)$-spin vector defined in Eq. (7), respectively. We define $g^{(3)}_{\boldsymbol{\alpha}_i} : S^3 \to G$ for $\boldsymbol{S}_{\boldsymbol{\alpha}_i}$ by

$$g^{(3)}_{\boldsymbol{\alpha}_i}(\psi,\theta,\phi) := \exp\left[2i\psi \boldsymbol{S}_{\boldsymbol{\alpha}_i} \cdot \hat{\boldsymbol{r}}(\theta,\phi)\right], \tag{23}$$

where $\hat{\boldsymbol{r}}(\theta,\phi)$ is a unit vector on S^2 defined by $\hat{\boldsymbol{r}}(\theta,\phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, with ψ,θ, and ϕ being the polar coordinates of the three-dimensional sphere S^3:

$$S^3 = \{(\sin\psi\sin\theta\cos\phi, \sin\psi\sin\theta\sin\phi, \sin\psi\cos\theta, \cos\psi) \\ |\psi \in [0,\pi],\theta \in [0,\pi],\phi \in [0,2\pi]\}. \tag{24}$$

Since $\boldsymbol{S}_{\boldsymbol{\alpha}_i} \cdot \hat{\boldsymbol{r}}(\theta,\phi)$ is the projection of the generalized $\mathfrak{su}(2)$-spin vector in the direction of $\hat{\boldsymbol{r}}(\theta,\phi)$, $g^{(3)}_{\boldsymbol{\alpha}_i}(\psi,\theta,\phi)$ describes the rotation of $\boldsymbol{S}_{\boldsymbol{\alpha}_i}$ about $\hat{\boldsymbol{r}}(\theta,\phi)$ through angle 2ψ (see Fig. 2). For example, $g^{(3)}_{\boldsymbol{\alpha}_i}(\psi,\theta,\phi)$ for $G = \mathrm{SO}(3)$ describes a rotation in three dimensions about a vector $\hat{\boldsymbol{r}}(\theta,\phi)$ through angle 2ψ. Let $[f]_M$ be the homotopy class of f on a manifold M, which is the set of those maps to M that can continuously transform into f, where f is referred to as a representative element. Then, the following lemma holds.

Lemma 1. The third homotopy group of G is generated by the set $\{[g^{(3)}_{\boldsymbol{\alpha}_i}]_G\}_{i=1}^{a}$:

$$\pi_3(G) \simeq \left\{ \sum_{i=1}^{a} m_i \left[g^{(3)}_{\boldsymbol{\alpha}_i}\right]_G \middle| m_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^a, \tag{25}$$

where we denote the product on $\pi_3(G)$ as the sum since $\pi_3(G)$ is Abelian.

The proof of Lemma 1 is given in Appendix A. We note that the homotopy class $[g^{(3)}_{\boldsymbol{\alpha}_i}]_G$ does not depend on the choice of the root vector $\boldsymbol{\alpha}_i$ since $g^{(3)}_{\boldsymbol{\alpha}_i}$ and $g^{(3)}_{\boldsymbol{\alpha}'_i}$ can continuously transform into each other if the corresponding root vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}'_i$ in $\mathfrak{g}_i$ both have the longest length [49].

B. Homotopy groups of the order parameter manifold G/H

1. Two types of textures on G/H

Let D^m be the surface and the inner region of an m-dimensional sphere S^m with radius π:

$$D^m := \{\boldsymbol{x} \in \mathbb{R}^m | \|\boldsymbol{x}\| \leqslant \pi\}, \tag{26}$$

and consider a topological excitation without a defect characterized by $\pi_m(G/H)$, such as two-dimensional ($m = 2$) and three-dimensional ($m = 3$) skyrmions. Assuming that it is localized in D^m, we may regard its texture $O(\boldsymbol{x})$ as a map from D^m to the G/H subject to the boundary condition

$$O(\boldsymbol{x}) = O_0 \text{ for } \|\boldsymbol{x}\| = \pi, \tag{27}$$

where O_0 is a fixed value of the order parameter called the reference order parameter and $\|\boldsymbol{x}\|$ denotes the modulus of $\boldsymbol{x}$. We note that a map $O : D^m \to G/H$ subject to the boundary condition (27) can represent a texture of a topological excitation with a defect through the replacement of D^m by an m-dimensional sphere S^m enclosing the defect. A crucial point for obtaining $\pi_m(G/H)$ is to express the texture $O(\boldsymbol{x})$ in terms of a symmetry transformation $g(\boldsymbol{x})$ depending on the space coordinate $\boldsymbol{x}$ as follows:

$$O(\boldsymbol{x}) = g(\boldsymbol{x})O_0, \tag{28}$$

where gO_0 for g ($\in G$) denotes the action of g on O_0. The expression (28) relates a texture $O(\boldsymbol{x})$ on G/H to a texture $g(\boldsymbol{x})$ on G and $\pi_m(G/H)$ to $\pi_m(G)$ and $\pi_m(H)$. Although $O(\boldsymbol{x})$ is continuous on D^m, $g(\boldsymbol{x})$ may not be continuous because $g(\boldsymbol{x})$ and $g(\boldsymbol{x})h(\boldsymbol{x})$ for any discontinuous function h with $h(\boldsymbol{x}) \in H$ give the same texture $O(\boldsymbol{x})$ in Eq. (28). As we will see below, two cases arise depending on whether or not $g(\boldsymbol{x})$ is continuous on the entire region of D^m.

Given a subgroup H of G, we can define the inclusion map $i : H \to G$ by $i(h) := h$ [see Fig. 3(a)]. Then, a texture on H, i.e., a map g from D^m to H, can also be regarded as a texture on G, and we define a map $i_{*m} : \pi_m(H) \to \pi_m(G)$ between the homotopy groups as $i_{*m}([f]_H) = [i \circ f]_G$, where $\circ$ denotes the composition of two maps. We construct textures G/H in two ways from two groups Coker i_{*m} and Ker i_{*m-1} which are defined as follows:

$$\begin{aligned}\mathrm{Coker}\, i_{*m} &:= \mathrm{Coker}\{i_{*m} : \pi_m(H) \to \pi_m(G)\} \\ &:= \frac{\pi_m(G)}{\mathrm{Im}\{i_{*m} : \pi_m(H) \to \pi_m(G)\}},\end{aligned} \tag{29}$$

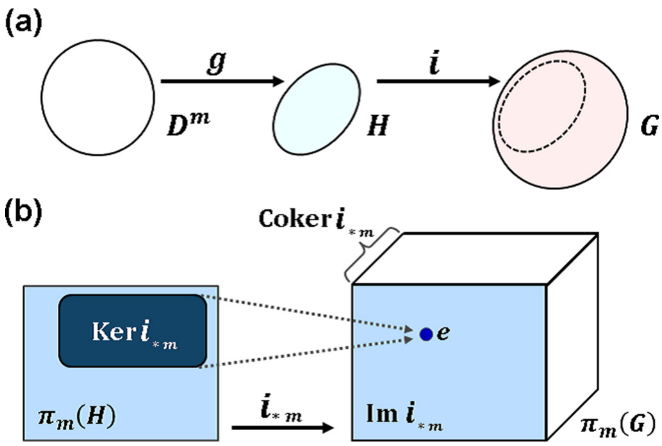

FIG. 3. (a) Schematic illustration of an inclusion map $i : H \to G$. Given a map g from D^m to H, the composition $i \circ g$ gives a map from D^m to G, where D^m is an m-dimensional disk defined in (26). (b) Schematic illustration of the cokernel Coker i_{*m}, the kernel Ker i_{*m}, and the image Im i_{*m} of $i_{*,m}$. The kernel Ker i_{*m-1} is a subgroup of $\pi_{m-1}(H)$, whose elements are mapped to the identity element e of $\pi_{m-1}(G)$. The image Im i_{*m} is a subgroup of $\pi_m(G)$, whose elements are obtained through i_{*m}. The cokernel Coker i_{*m} is the quotient group $\pi_m(G)/\mathrm{Im}\, i_{*m}$, representing the elements in $\pi_m(G)$ that cannot be obtained through i_{*m}.

$$\begin{aligned}\mathrm{Ker}\, i_{*m-1} &:= \mathrm{Ker}\{i_{*m-1} : \pi_{m-1}(H) \to \pi_{m-1}(G)\} \\ &:= \{O \in \pi_{m-1}(H) | i_{*m-1}(O) = e\},\end{aligned} \tag{30}$$

where Im $F := \{F(g)|g \in G\}$ and Coker F for $F : G \to G'$ is defined by Coker $F := G'/\mathrm{Im}\, F$ [see Fig. 3(b)]. An element of Ker i_{*m-1} represents a nontrivial texture on H that is trivial as a texture on G. While an element of Im i_{*m} represents a nontrivial texture on G that can be represented as a texture on H, that of Coker i_{*m} represents a nontrivial texture on G that cannot be represented as a texture on H. We denote the element of Coker i_{*m} corresponding to $a \in \pi_m(G)$ by $[a]$ and call a the representative element of $[a]$.

Let us construct the texture $O^{[a]}$ on G/H from $[a] \in$ Coker i_{*m}. Since a is a texture of G, we can define the texture $O^{[a]}$ through the action of a on O_0:

$$O^{[a]}(\boldsymbol{x}) := a(\boldsymbol{x})O_0 \text{ for } \boldsymbol{x} \in D^m. \tag{31}$$

Equation (31) implies that a nontrivial texture on G/H can be obtained from a nontrivial texture on G [see Fig. 4(a)]. From the boundary condition for a, i.e.,

$$a(\boldsymbol{x}) = e \text{ for } \|\boldsymbol{x}\| = \pi, \tag{32}$$

we see that $O^{[a]}(\boldsymbol{x})$ satisfies the boundary condition (27). It is worth mentioning two things. First, Coker i_{*m} is Abelian for $m \geqslant 1$ because the numerator on the right-hand side of Eq. (29), i.e., $\pi_m(G)$, is Abelian. This follows from the fact that $\pi_1(G)$ is Abelian and from the commutativity of higher-dimensional homotopy groups [50]. Second, we must consider the quotient space Coker i_{*m} instead of $\pi_m(G)$, which is the numerator on the right-hand side of Eq. (29), because the denominator Im i_{*m} gives a uniform texture through Eq. (31).

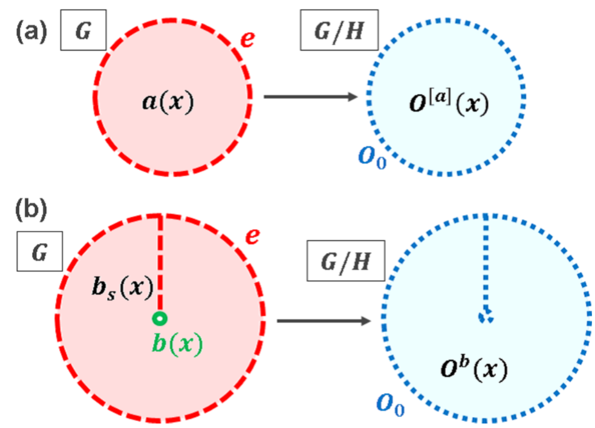

FIG. 4. (a) Construction of the texture $O^{[a]}$ from the map a defined in Eq. (31) for $m = 2$, where a is a map from a two-dimensional disk D^2 to G that maps the boundary (red dashed line) of D^2 to the identity element e. The texture $O^{[a]}$ is a map from D^2 to G/H that maps the boundary (blue dotted line) of D^2 to the reference order parameter O_0. (b) Construction of the texture O^b from the map b defined in Eq. (37) for $m = 2$. Here b is a map from a circle S^1 to H and b_s is a continuous deformation from b to the uniform texture e subject to the condition (36), where the points on the red dashed line are mapped to e. The texture O^b is a map from D^2 to G/H which maps the points on the blue dotted line to O_0.

Indeed, for $a_h := i_{*m}(a_H) \in \mathrm{Im}\, i_{*m}$, we have

$$\begin{aligned}O^{[a_h]}(\boldsymbol{x}) &:= a_h(\boldsymbol{x})O_0 = [i(a_H)](\boldsymbol{x})O_0 \\ &= a_H(\boldsymbol{x})O_0 = O_0 \text{ for } \forall \boldsymbol{x} \in D^m,\end{aligned} \tag{33}$$

where we use the invariance of O_0 under the transformation in H in obtaining the last equality. The simplest example of the construction (31) is an integer-quantum vortex in a scalar BEC. Let Ψ be the mean-field wave function of the condensate. Then, the texture $\Psi(\phi)$ around a vortex with a unit winding number is given by

$$\Psi(\phi) = \exp(i\phi)\Psi_0, \tag{34}$$

where ϕ and Ψ_0 are the azimuth angle around the vortex and the value of the mean-field wave function at $\phi = 0$. Thus the nontrivial texture $\Psi(\phi)$ of the vortex is expressed in terms of the nontrivial winding $\exp(i\phi)$ on the symmetry group $G = \mathrm{U}(1)$. Another example of the construction (31) is a three-dimensional skyrmion in a ferromagnet called a knot soliton [51], whose texture $\boldsymbol{M}(r,\theta,\phi)$ of the spin is written as

$$\boldsymbol{M}(r,\theta,\phi) := \exp\left[2i\psi(r)\boldsymbol{S}\cdot\hat{\boldsymbol{r}}(\theta,m\phi)\right]\boldsymbol{M}_0, \tag{35}$$

where (r,θ,ϕ) is the three-dimensional polar coordinates and $m \in \mathbb{Z}$ denotes the topological charge of the knot soliton. Here, $\boldsymbol{M}_0 := (0,0,1)$ and $\psi(r)$ is a function that satisfies $\psi(0) = 0$ and $\psi(\infty) = \pi$. One can see from Eq. (23) that the nontrivial texture (35) is expressed in terms of a nontrivial winding $\exp\left[2i\psi(r)\boldsymbol{S}\cdot\hat{\boldsymbol{r}}(\theta,\phi)\right]$ on a symmetry group SO(3) of spin rotation.

To construct the texture O^b on G/H from $b \in \mathrm{Ker}\, i_{*m-1}$, we regard b as a map from S^{m-1}. Since b is a trivial texture on G from its definition (30), there exists a continuous deformation b_s from $b_{s=0} = b$ to the uniform texture $b_{s=\pi} = e$ subject to the boundary condition

$$b_s(\hat{\boldsymbol{x}}_0) = O_0 \text{ for } \forall s \in [0,\pi], \tag{36}$$

where $\hat{\boldsymbol{x}}_0$ is a point on S^{m-1} and s is the parameter of the deformation. Hence we define O^b as

$$O^b(\boldsymbol{x}) := b_{s=\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}})O_0 \text{ for } \boldsymbol{x} \in D^m, \tag{37}$$

where $\hat{\boldsymbol{x}}$ is the unit vector parallel to $\boldsymbol{x}$ [see Fig. 4(b)]. We note that $b_{s=\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}})$ in the construction (37) is not continuous at the origin $\boldsymbol{x} = \boldsymbol{0}$. From the comparison of Eq. (28) with Eq. (31) [Eq. (37)], a texture on G/H is expressed by a texture on G that is (not) continuous on D^m, and is described by an element of Coker i_{*m} (Ker i_{*m-1}). Examples of the construction (37) include a half-vortex in a uniaxial nematic liquid crystal and a monopole in a ferromagnet. The order parameter of a uniaxial nematic liquid crystal is the orientation $\boldsymbol{d}$ of molecules. The texture $\boldsymbol{d}(\phi)$ around a half-vortex is given by

$$\boldsymbol{d}(\phi) := \exp(\phi L_2/2)\boldsymbol{d}_0, \tag{38}$$

where $\boldsymbol{d}_0 := (0,0,1)$, and ϕ and L_2 are the azimuth angle around the vortex and a generator of rotation about the y axis, respectively. The nontrivial texture $\boldsymbol{d}(\phi)$ is expressed not by a loop on $G = \mathrm{SO}(3)$ but by a path from e to $\exp(\pi L_2)$ on SO(3). Due to the discrete π-rotational symmetry $\boldsymbol{d} \to -\boldsymbol{d}$, the start point $\boldsymbol{d}_0$ and the end point $\exp(\pi L_2)\boldsymbol{d}_0 = -\boldsymbol{d}_0$ should be identified, where the texture (38) is continuous at $\phi = 0$ ($\phi = 2\pi$). The texture $\boldsymbol{M}(\theta,\phi)$ of a monopole in a ferromagnet is described by a hedgehog configuration of the spin, i.e., $\boldsymbol{M}(\theta,\phi) = \hat{\boldsymbol{r}}(\theta,\phi)$, which can be rewritten in terms of the $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}$ as follows:

$$\boldsymbol{M}(\theta,\phi) := \exp(i\phi S_3)\exp(i\theta S_2)\boldsymbol{M}_0, \tag{39}$$

where θ and ϕ denote the polar angle and the azimuth angle around the monopole, respectively, $\boldsymbol{M}_0 := (0,0,1)$, and S_2 (S_3) is a generator of the rotation about the y axis (z axis). As shown in Fig. 5, the hedgehog texture is obtained by the successive applications of spin rotation $\exp(i\theta S_2)$ about the y axis followed by spin rotation $\exp(i\phi S_3)$ about the z axis. Under continuous deformation $\boldsymbol{M}_u(\theta,\phi) = \exp(-iu\theta S_2)\exp(i\phi S_3)\exp(i\theta S_2)\boldsymbol{M}_0$, with u ($\in [0,1]$) being the parameter of the deformation, $\boldsymbol{M}_{u=0}(\theta,\phi) = \boldsymbol{M}(\theta,\phi)$ transforms into

$$\begin{aligned}\boldsymbol{M}_{u=1}(\theta,\phi) &= \exp(-i\theta S_2)\exp(i\phi S_3)\exp(i\theta S_2)\boldsymbol{M}_0\\ &= \exp(-i\theta S_2)\exp(i\phi S_3)\exp(i\theta S_2)\exp(i\phi S_3)\boldsymbol{M}_0\\ &= g^{(2)}(\theta,\phi)\boldsymbol{M}_0,\end{aligned} \tag{40}$$

where in the second equality we use the invariance of $\boldsymbol{M}_0$ under the rotation about the z axis, and $g^{(2)}(\theta,\phi) := \exp(-i\theta S_2)\exp(i\phi S_3)\exp(i\theta S_2)\exp(i\phi S_3)$ is the map (18) with $\boldsymbol{S}_{\boldsymbol{\alpha}}$ replaced by $\boldsymbol{S}$. The expression $g^{(2)}(\theta,\phi)\boldsymbol{M}_0$ gives the texture of a monopole in the form of Eq. (28). Indeed, since

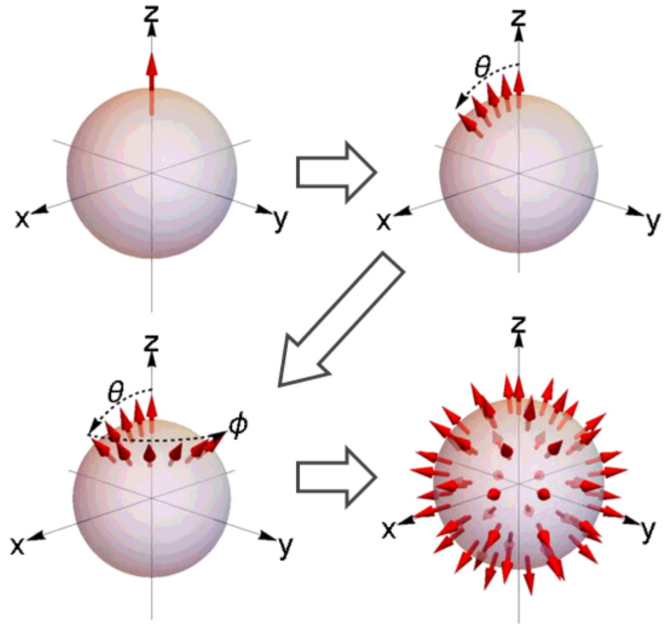

FIG. 5. Schematic illustration of a monopole in a ferromagnet with topological charge $m = +1$. Each arrow indicates the direction of the spin. The texture of the monopole is described by a hedgehog configuration of the spin, which is obtained from successive applications of spin rotation $\exp(i\theta S_2)$ about the y axis through angle θ followed by spin rotation $\exp(i\phi S_3)$ about the z axis through angle ϕ, where θ and ϕ are the polar angle and the azimuth angle, respectively.

we have

$$g^{(2)}(\theta = 0,\phi) = \exp(2i\phi S_3), \tag{41}$$

$$\begin{aligned}g^{(2)}(\theta = \pi,\phi) &= \mathrm{e}^{-i\pi S_2}\mathrm{e}^{i\phi S_3}\mathrm{e}^{i\pi S_2}\mathrm{e}^{i\phi S_3}\\ &= \mathrm{e}^{-i\phi S_3}\mathrm{e}^{i\phi S_3} = e,\end{aligned} \tag{42}$$

$$g^{(2)}(\theta,\phi = 0) = e, \tag{43}$$

where $g^{(2)}(\theta,\phi)$ is a continuous deformation from the loop $g^{(2)}(\theta = 0,\phi) = \exp(2i\phi S_3)$ on $H = \mathrm{SO}(2)$ to the trivial loop $g^{(2)}(\theta = \pi,\phi) = e$ subject to the boundary condition (36).

Finally, we define the map

$$f : \mathrm{Ker}\, i_{*m-1} \times \mathrm{Ker}\, i_{*m-1} \to \mathrm{Coker}\, i_{*m}, \tag{44}$$

which is used in Theorem 1 below. As shown in Sec. III, f vanishes for $m = 2$ and 3, so long as we focus on the first, second, and third homotopy groups. We therefore first define f for $m = 1$ and then define it for an arbitrary dimension m. For $m = 1$, we have the following isomorphism:

$$\mathrm{Ker}\, i_{0*} \simeq \pi_0(H \cap G_0), \tag{45}$$

where G_0 is a Lie group constituted from the connected component in G. This follows from the fact that $[h] \in \mathrm{Ker}\, i_{0*}$ implies that h, a representative element of $[h]$, is connected to e by a path on G, which implies $h \in G_0$, and from the fact that $[h] \in \pi_0(H \cap G_0)$ describes a common element of $\pi_0(H)$ and $\pi_0(G)$. Let $[\sigma]$ and $[\tau]$ be elements of $\pi_0(H \cap G_0)$ with representative elements σ ($\in H \cap G_0$) and τ ($\in H \cap G_0$), respectively. Since they are connected with the identity element e, there exist paths $\gamma_\sigma, \gamma_\tau$, and $\gamma_{\sigma\tau}$ from σ, τ, and

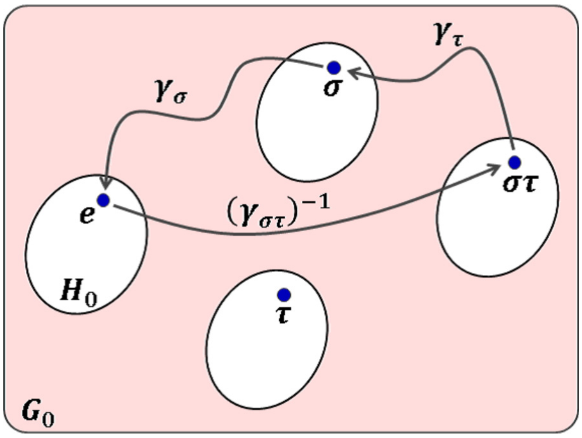

FIG. 6. Schematic illustration of the map f defined in Eq. (46). The entire region shows the connected component G_0 of G and the white regions represent the connected components of H, where H_0 shows the connected component including the identity element e, and σ, τ, and $\sigma\tau$ denote the elements of G_0 in different connected components in H. The map f is obtained from the composition $\gamma_\sigma\gamma_\tau(\gamma_{\sigma\tau})^{-1}$ of the three paths γ_σ, γ_τ, and $\gamma_{\sigma\tau}$.

$\sigma\tau$, respectively, to e. Then, the composition $\gamma_\sigma\gamma_\tau(\gamma_{\sigma\tau})^{-1}$ is a loop from e to itself (see Fig. 6). We define f as

$$f([\sigma],[\tau]) := \gamma_\sigma\gamma_\tau(\gamma_{\sigma\tau})^{-1}O_0. \tag{46}$$

Let b and b' be elements of Ker i_{*m-1}. Then, there exist continuous deformations b_s, b'_s, and $(bb')_s$ from b, b', and bb', respectively, to a trivial map subject to the boundary condition (36). We fix the parameter s, consider b_s and b'_s to be elements of $\pi_{m-1}(G)$, and denote their composition as $b_s \circ b'_s$. Then, we define $f(b,b')$ by

$$[f(b,b')](\boldsymbol{x}) := [\tilde{f}(b,b')](\boldsymbol{x})O_0, \tag{47}$$

$$[\tilde{f}(b,b')](\boldsymbol{x}) := \begin{cases} (bb')_{s=\pi-2\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}}) & \text{for } 0 \leqslant \|\boldsymbol{x}\| \leqslant \frac{\pi}{2}; \\ (b_{s=2\|\boldsymbol{x}\|-\pi} \circ b'_{s=2\|\boldsymbol{x}\|-\pi})(\hat{\boldsymbol{x}}) & \text{for } \frac{\pi}{2} \leqslant \|\boldsymbol{x}\| \leqslant \pi. \end{cases} \tag{48}$$

Since $\tilde{f}$ satisfies the boundary condition (32), which gives $\tilde{f} \in \pi_m(G)$, f is indeed a map to Coker i_{*m}. The map f does not depend on the choices of the representative elements of b and b'. Let B and $\bar{B}$ be two representative elements of b, and B_s ($\bar{B}_s$) be the continuous deformations from B ($\bar{B}$) to the trivial homotopy class. Since B and $\bar{B}$ transform into each other through continuous deformation on H, so do B_s and $\bar{B}_s$. Therefore the map f in Eq. (48) defined from B and B_s and that defined from $\bar{B}$ and $\bar{B}_s$ transform into each other continuously.

2. A decomposition formula for $\pi_m(G/H)$

Let us define the product $\times_f$ on the product set Coker i_{*m} × Ker i_{*m-1} by

$$([a],b) \times_f ([a'],b') := ([a]+[a']+f(b,b'),bb'), \tag{49}$$

where f : Ker i_{*m-1} × Ker $i_{*m-1} \to$ Coker i_{*m} is the map defined in Eq. (48) and we denote the product in Coker i_{*m} as the sum since Coker i_{*m} is Abelian. Then the following theorem holds.

Theorem 1. Under the product defined in Eq. (49), Coker i_{*m} × Ker i_{*m-1} becomes a group. This group denoted by Coker $i_{*m} \times_f$ Ker i_{*m-1} is isomorphic to the mth homotopy group of G/H:

$$\pi_m(G/H) \simeq \text{Coker } i_{*m} \times_f \text{Ker } i_{*m-1}. \tag{50}$$

Any topological charge $([a],b)$ in $\pi_m(G/H)$ can be uniquely decomposed into the product of an element of Coker i_{*m} and that of Ker i_{*m-1}:

$$([a],b) = ([a],e) \times_f (e,b). \tag{51}$$

Furthermore, the texture $O^{[a]}$ (O^b) of a topological excitation with topological charge $([a],e)$ ((e,b)) is given by Eq. (31) [Eq. (37)].

The proof of Theorem 1 is given in Appendix B. Equation (50) implies that there are two distinct types of topological excitations expressed by either Coker i_{*m} or Ker i_{*m-1}. One can see from Eq. (51) that any topological excitation can be written as a composition of these two types. The presence of f in Eq. (49) implies that these two types of topological excitations are, in general, not independent: the composition of two topological excitations described by Ker i_{*m-1} can produce a topological excitation described by Coker i_{*m}, because we have

$$\begin{aligned}(e,b) \times_f (e,b') &= (f(b,b'),bb') \\ &= (f(b,b'),e) \times_f (e,bb'),\end{aligned} \tag{52}$$

and $f(b,b') \neq e$ in general. The group Coker $i_{*m} \times_f$ Ker i_{*m-1} is referred to as the group extension of Coker i_{*m} by Ker i_{*m-1} with the factor set f [47] (see Appendix B 1 or Ref. [47] for detail).

III. FORMULAS FOR HOMOTOPY GROUPS FOR LOW-DIMENSIONAL TOPOLOGICAL EXCITATIONS

A. First homotopy group: vortices

Since Coker i_1^* is a quotient group of $\pi_1(G)$ from Eq. (29) and $\pi_1(G)$ is a quotient group of L_G from Eq. (9), we write an element of Coker i_1^* as $[H_t]$ where $H_t \in L_G$. Let us define the product $\times_f$ on Coker $i_1^* \times \pi_0(H\cap G_0)$ by

$$\begin{aligned}&([H_t],[\sigma]) \times_f ([H_s],[\tau]) \\ &\quad := ([H_t]+[H_s]+f([\sigma],[\tau]),[\sigma][\tau]),\end{aligned} \tag{53}$$

where f is the map defined in Eq. (46). Then the following corollary holds.

Corollary 1. Under the product defined in Eq. (53), Coker $i_1^* \times_f \pi_0(H\cap G_0)$ is isomorphic to the first homotopy group of G/H:

$$\pi_1(G/H) \simeq \text{Coker } i_1^* \times_f \pi_0(H\cap G_0). \tag{54}$$

Any topological charge $([H_t],[\sigma])$ in $\pi_1(G/H)$ can be uniquely decomposed into the product of an element of Coker i_1^* and that of $\pi_0(H\cap G_0)$:

$$([H_t],[\sigma]) = ([H_t],e) \times_f (e,[\sigma]). \tag{55}$$

Let ϕ and O_0 be the azimuth angle around a vortex and the reference order parameter, respectively. The texture $O^{([H_t],e)}(\phi)$

of a vortex with topological charge $([H_t],e)$ is given by

$$O^{([H_t],e)}(\phi) = \exp(i\phi H_t)O_0, \tag{56}$$

while the texture $O^{(e,[\sigma])}(\phi)$ of a vortex with topological charge $(e,[\sigma])$ is given by

$$O^{(e,[\sigma])}(\phi) = \gamma_\sigma(\phi)O_0, \tag{57}$$

where $\gamma_\sigma(\phi)$ is a path from σ $(\in H \cap G_0)$ to the identity element e.

Proof. Due to the isomorphism $\mathrm{Ker}\ i_{0*} \simeq \pi_0(H \cap G_0)$ in Eq. (45), we have Eqs. (54) and (55) from Eqs. (50) and (51), respectively. Then, it is sufficient to show Eqs. (56) and (57) to prove Corollary 1. From Eq. (31), the texture of a vortex with topological charge $([H_t],e)$ is given by $O^{([H_t],e)}(\phi) = a(\phi)O_0$, where $a(\phi)$ is a loop on G. From Eqs. (16) and (17), we have $a(\phi) = \exp(i\phi H_t)$ and hence Eq. (56). For a vortex with topological charge $(e,[\sigma])$, Eq. (37) can be expressed as $O^{(e,[\sigma])}(\phi) = b_\phi O_0$, where b is a path from $b_{s=0} = \sigma$ to $b_{s=\pi} = e$. Defining a path γ_σ by $\gamma_\sigma(\phi) := b_{s=\phi/2}$, we obtain Eq. (57), which completes the proof of Corollary 1.

Equation (54) implies that there are two types of vortices expressed by either $\mathrm{Coker}\ i_1^*$ or $\pi_0(H \cap G_0)$, and it follows from Eq. (55) that any vortex can be written as their composition. Examples of the former include an integer-quantum vortex (34), which is obtained from Eq. (57) through substitution of $\exp(i\phi H_t)$ and $O^{([H_t],e)}(\phi)$ with $\exp(i\phi)$ and $\Psi(\phi)$, respectively. The latter term $\pi_0(H \cap G_0)$ describes a vortex associated with a discrete symmetry of the state such as a half-vortex (38) in a uniaxial nematic liquid crystal, where the discrete symmetry is the π-rotational symmetry of the orientation $\boldsymbol{d}$. One can reproduce from Eq. (54) the formula $\pi_1(G/H) \simeq \pi_0(\widetilde{H})$ based on the lift method [38], where G and H are lifted to a simply connected group $\widetilde{G}$ and the corresponding subgroup $\widetilde{H}$, respectively. Since $\pi_0(\widetilde{G}) \simeq 0$ and $\pi_1(\widetilde{G}) \simeq 0$, we have $\mathrm{Coker}\ i_1^* \simeq 0$ and $\pi_0(\widetilde{H} \cap \widetilde{G}_0) \simeq \pi_0(\widetilde{H})$. We thus obtain Eq. (54). In contrast to the lift method, we find the distinction between vortices represented by $\mathrm{Coker}\ i_1^*$ and $\pi_0(H \cap G_0)$. In Sec. IV A, this distinction is shown to be crucial since only the latter can be a cause of nontrivial topological influence.

B. Second homotopy group: monopoles and skyrmions

We generalize the texture (39) of a monopole in a ferromagnet through the replacement of $\boldsymbol{S}$ by a generalized $\mathfrak{su}(2)$-spin vector. Provided that $S_{\boldsymbol{\alpha},3}$ is an unbroken generator, we define the mapping $O^{\boldsymbol{\alpha}^c} : S^2 \to G/H$ for a coroot $\boldsymbol{\alpha}^c$ by

$$O^{\boldsymbol{\alpha}^c}(\theta,\phi) := g^{(2)}_{\boldsymbol{\alpha}}(\theta,\phi)O_0, \tag{58}$$

$$g^{(2)}_{\boldsymbol{\alpha}}(\theta,\phi) := \exp(i\phi S_{\boldsymbol{\alpha},3})\exp(i\theta S_{\boldsymbol{\alpha},2}). \tag{59}$$

Comparing Eqs. (58) and (59) with Eq. (39), we find that Eq. (58) describes the hedgehog configuration of the generalized $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}_{\boldsymbol{\alpha}}$. More precisely, $O^{\boldsymbol{\alpha}^c}(\theta,\phi)$ is invariant under spin rotations generated by the generalized $\mathfrak{su}(2)$-spin vector parallel to $\hat{\boldsymbol{r}}(\theta,\phi)$:

$$\exp[i\psi \boldsymbol{S}_{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{r}}(\theta,\phi)]O^{\boldsymbol{\alpha}^c}(\theta,\phi) = O^{\boldsymbol{\alpha}^c}(\theta,\phi) \text{ for } \forall\psi \in \mathbb{R}. \tag{60}$$

This follows from the assumption that O_0 is invariant under unitary transformations generated by $S_{\boldsymbol{\alpha},3}$ and from the decomposition $\exp[i\psi \boldsymbol{S}_{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{r}}(\theta,\phi)] = g^{(2)}_{\boldsymbol{\alpha}}(\theta,\phi)e^{i\psi S_{\boldsymbol{\alpha},3}}[g^{(2)}_{\boldsymbol{\alpha}}(\theta,\phi)]^\dagger$, which is derived directly from the commutation relations (8).

The following Corollary 2 shows that the topological charge and the texture of a general monopole are described by coroots and the hedgehog configuration of the generalized $\mathfrak{su}(2)$-spin vector, respectively. Reflecting the fact that a general compact Lie algebra includes more than one $\mathfrak{su}(2)$-Lie algebra in contrast to $\mathfrak{su}(2)$, the topological charge should be described by a set of coroots. The connection with the coroots and the generalized $\mathfrak{su}(2)$-spin vectors are pointed out in Refs. [52,53], where non-Abelian gauge theories are considered and H is assumed to include a maximal Abelian subgroup of G [41,42]. We here generalize their results to arbitrary systems with arbitrary patterns of symmetry breaking.

Corollary 2. Let L_H and L^c_H (L^c_G) be the integral lattice of H and the coroot lattice of H (G), respectively. Then, L^c_H is an Abelian subgroup of $L_H \cap L^c_G$, and the quotient space of $L_H \cap L^c_G$ by L^c_H is isomorphic to $\pi_2(G/H)$:

$$\pi_2(G/H) \simeq \left(L_H \cap L^c_G\right)/L^c_H. \tag{61}$$

Therefore the topological charge n of a monopole can be expressed in terms of the coroots corresponding to the simple roots as

$$n = \sum_{j=1}^{r} m_j \boldsymbol{\alpha}^c_j, \tag{62}$$

where $\{m_j\}_{j=1}^r$ is the set of integers, and its texture $O(\theta,\phi)$ is given by

$$O(\theta,\phi) = g^{(2)}_{\boldsymbol{\alpha}_1}(\theta,m_1\phi)g^{(2)}_{\boldsymbol{\alpha}_2}(\theta,m_2\phi)\cdots g^{(2)}_{\boldsymbol{\alpha}_r}(\theta,m_r\phi)O_0. \tag{63}$$

The proof of Corollary 2 is given in Appendix C. One can reproduce from Eq. (61) the formula $\pi_2(G/H) \simeq \pi_1(\widetilde{H})$ based on the lift method [38]. Indeed, we have $\pi_1(\widetilde{G}) \simeq 0$ and hence $L_{\widetilde{G}} \simeq L^c_{\widetilde{G}}$. Then, we obtain $\pi_2(G/H) \simeq \pi_1(\widetilde{H})$. However, the texture of each topological excitation is described by a deformable loop on $\widetilde{G}$; in Ref. [38] the existence of the texture is shown but no explicit form is given. We here explicitly determine the texture as shown in Eq. (63).

C. Third homotopy group: three-dimensional skyrmions

Two prototypical examples of three-dimensional skyrmions are a Shankar skyrmion and a knot soliton, which are characterized by the homotopy groups $\pi_3(S^3) \simeq \mathbb{Z}$ and $\pi_3(S^2) \simeq \mathbb{Z}$, respectively. Both of their textures are expressed in terms of the $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}$ as

$$O(\psi,\theta,\phi) = \exp[2i\psi \boldsymbol{S} \cdot \hat{\boldsymbol{r}}(\theta,m\phi)]O_0, \tag{64}$$

where (ψ,θ,ϕ) is the polar coordinates (24) on S^3 and $m \in \mathbb{Z}$ denotes the topological charge of the three-dimensional

TABLE I. Two types of topological excitations and their examples. Coker i_{*m} and Ker i_{*m-1} are the cokernel of i_{*m} and the kernel of i_{*m-1} defined in Eqs. (29) and (30), respectively. The entry "absent" means the absence of examples.

	Coker i_{*m}	Ker i_{*m-1}
$m=1$	integer-quantum vortex	half-vortex
$m=2$	absent	monopole
$m=3$	knot soliton Shanker skyrmion	absent

skyrmion. This unified description is based on the isomorphism $\pi_3(S^3)\simeq\pi_3(S^2)$ derived from the Hopf fibration [41,50,54]. When all of the generators in $\boldsymbol{S}$ are broken, Eq. (64) describes a Shankar skyrmion [2,55]; otherwise it describes a knot soliton [33,56]. We generalize the texture (64) through the replacement of $\boldsymbol{S}$ by $\boldsymbol{S}_{\boldsymbol{\alpha}}$, and define the mapping $O^{\boldsymbol{\alpha}^c}: S^3\to G/H$ for coroot $\boldsymbol{\alpha}^c$ by

$$O^{\boldsymbol{\alpha}^c}(\psi,\theta,\phi):=g^{(3)}_{\boldsymbol{\alpha}}(\psi,\theta,\phi)O_0, \tag{65}$$

$$g^{(3)}_{\boldsymbol{\alpha}}(\psi,\theta,\phi):=\exp\left[2i\psi\boldsymbol{S}_{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{r}}(\theta,\phi)\right], \tag{66}$$

where $g^{(3)}_{\boldsymbol{\alpha}}$ is defined in Eq. (23).

The following two corollaries show that a general three-dimensional skyrmion may be regarded as the composition of several different types of three-dimensional skyrmions whose topological charges and textures are described by coroots and the corresponding textures (65), respectively.

Corollary 3. The third homotopy group $\pi_3(G/H)$ is given as follows:

$$\pi_3(G/H)\simeq\text{Coker}\{i_3^*:\pi_3(H)\to\pi_3(G)\}. \tag{67}$$

Proof. Since any subgroup H of a compact Lie group G is compact, $\pi_2(H)$ vanishes. Therefore we obtain Ker $i_2^*\simeq 0$ and hence Eq. (67) from Theorem 1, which completes the proof of Corollary 3.

We next analyze a topological charge and a texture. Let $\boldsymbol{\alpha}_i^c$ be a coroot of the Lie algebra $\mathfrak{g}_i$ defined in Eq. (22). Since the numerator $\pi_3(G)$ of Eq. (67) is generated by $\{[g^{(3)}_{\boldsymbol{\alpha}_i^c}]\}_{i=1}^a$ from Lemma 1, the quotient space $\pi_3(G/H)$ is generated by $\{[O^{\boldsymbol{\alpha}_i^c}]_{G/H}\}_{i=1}^{\bar{a}}$ for a suitable choice of the subset $\{\boldsymbol{\alpha}_k\}_{k=1}^{\bar{a}}$ of $\{\boldsymbol{\alpha}_i\}_{i=1}^a$. Thus we obtain the following corollary.

Corollary 4. The topological charge n of a three-dimensional skyrmion can be written in terms of coroots as

$$n=\sum_{k=1}^{\bar{a}}m_k\left[\boldsymbol{\alpha}_k^c\right], \tag{68}$$

where $\{m_k\}_{k=1}^r$ is a set of integers and $[\boldsymbol{\alpha}^c]$ represents the topological charge of the texture $O^{\boldsymbol{\alpha}^c}$ defined in Eq. (65). The results of this section are summarized in Table I.

IV. GENERAL CONDITIONS FOR THE PRESENCE OF TOPOLOGICAL INFLUENCE

A. Topological influence on a general topological excitation

When a topological excitation with topological charge $n\in\pi_m(G/H)$ makes a complete circuit of a vortex with topological charge $l\in\pi_1(G/H)$, the resulting topological charge $\lambda_m^l(n)$ is given by the action of l on n, where the corresponding texture $O^{\lambda_m^l(n)}(\boldsymbol{x})$ is defined as follows [36,57]:

$$O^{\lambda_m^l(n)}(\boldsymbol{x}):=\begin{cases}O^n(2\boldsymbol{x}) & \text{for } 0\leqslant\|\boldsymbol{x}\|\leqslant\frac{\pi}{2};\\ O^l(4\|\boldsymbol{x}\|-2\pi) & \text{for } \frac{\pi}{2}\leqslant\|\boldsymbol{x}\|\leqslant\pi,\end{cases} \tag{69}$$

where $\boldsymbol{x}\in D^m$ and $O^n: D^m\to G/H$ ($O^l:[0,2\pi]\to G/H$) is the texture of a topological excitation (vortex) with topological charge n (l). We can express the topological charges n and l as $n=([a],b)$ and $l=([H_t],[\sigma])$ from Theorem 1 and Corollary 1, respectively, Then, the following theorem holds.

Theorem 2. The topological charge $\lambda_m^l(n)$ is given by

$$\lambda_m^l(n)=([a],\sigma^{-1}b\sigma), \tag{70}$$

where the homotopy class $\sigma^{-1}b\sigma$ is defined as $[\sigma^{-1}b\sigma](\boldsymbol{x}):=\sigma^{-1}b(\boldsymbol{x})\sigma O_0$ for $\boldsymbol{x}\in D^{m-1}$.

The proof of Theorem 2 is given in Appendix D. The result of Theorem 2 is summarized in Table II. As shown in Table II, only vortices characterized by discrete symmetries can have nontrivial topological influence. To understand this, let us consider a situation in which a topological excitation with texture $O(\boldsymbol{x})$ makes a complete circuit of a vortex with texture $O^{(e,[\sigma])}(\phi)=\gamma_\sigma(\phi)O_0$, where $\gamma_\sigma(\phi)\sigma^{-1}$ describes a path from e to σ^{-1}. When the former goes around the latter by angle ϕ, it undergoes a nontrivial texture produced by the latter, changing its texture from $O(\boldsymbol{x})$ to $\gamma_\sigma(\phi)\sigma^{-1}O(\boldsymbol{x})$. The final texture is given by $\sigma^{-1}O(\boldsymbol{x})$. A crucial observation here is that the final texture $\sigma^{-1}O(\boldsymbol{x})$, in general, does not coincide with the initial one $O(\boldsymbol{x})$. On the other hand, when the topological excitation goes around a vortex characterized by Coker i_1^* by angle ϕ, its texture changes from $O(\boldsymbol{x})$ to $\exp(i\phi H_t)O(\boldsymbol{x})$. Therefore the initial and final textures coincide because we have $\exp(2\pi iH_t)=e$ from Eq. (9).

TABLE II. Topological influence in four combinations of topological excitations and vortices. Here $\pi_m(G/H)$ and $\pi_0(H\cap G_0)$ are the mth and zeroth homotopy groups of the order parameter manifold G/H and $H\cap G_0$, respectively; $i_{*m}:\pi_m(H)\to\pi_m(G)$ is a homomorphism induced by the inclusion map $i:H\to G$; Coker i_{*m} and Ker i_{*m-1} are the cokernel of i_{*m} and the kernel of i_{*m-1} defined in Eqs. (29) and (30), respectively. The entry "may appear" ("absent") means that topological influence may exist (does not exist).

		topological excitation $\pi_m(G/H)$	
		Coker i_{*m}	Ker i_{*m-1}
vortex	Coker i_1^*	absent	absent
$\pi_1(G/H)$	$\pi_0(H\cap G_0)$	absent	may appear

B. Topological influence on low-dimensional topological excitations

1. Topological influence on a vortex

The necessary and sufficient condition for the presence of topological influence on a vortex is the non-Abelianness of the first homotopy group [35]. It is known that non-Abelian vortices behave differently from Abelian ones in the collision dynamics [58–60], quantum turbulance [61], and the coarsening dynamics [62–64] due to the tangling between vortices. However, the conditions for their appearances are yet to be understood from a unified point of view. The following corollary shows that their presence is solely determined by discrete symmetries, where the non-Abelian property is shown to emerge only between pairs of vortices characterized by $\pi_0(H \cap G_0)$.

Corollary 4. The first homotopy group $\pi_1(G/H)$ is Abelian if and only if $\pi_0(H \cap G_0)$ is Abelian and f defined in Eq. (46) satisfies

$$f([\sigma],[\tau]) = f([\tau],[\sigma]) \text{ for } \forall[\sigma],[\tau] \in \pi_0(H \cap G_0). \quad (71)$$

Proof. Comparing the following two equations:

$$([a],[\sigma]) \times_f ([b],[\tau]) = ([a] + [b] + f([\sigma],[\tau]),[\sigma][\tau]),$$
$$([a],[\tau]) \times_f ([a],[\sigma]) = ([a] + [b] + f([\tau],[\sigma]),[\tau][\sigma]), \quad (72)$$

we find that $\pi_1(G/H)$ is Abelian if and only if

$$\left.\begin{array}{l}[\sigma][\tau] = [\tau][\sigma];\\ f([\sigma],[\tau]) = f([\tau],[\sigma])\end{array}\right. \quad \text{for } \forall[\sigma],[\tau] \in \pi_0(H \cap G_0). \quad (73)$$

The first equation in Eq. (73) implies that $\pi_0(H \cap G_0)$ is Abelian and we have Eq. (71) from the second equation of Eq. (73), which completes the proof of Corollary 5.

2. *Topological influence on a monopole, a skyrmion, and a three-dimensional skyrmion*

Since one topological charge changes into another due to topological influence, λ_2^l is an automorphism on $\pi_2(G/H)$ [38,57], i.e., a one-to-one map from $\pi_2(G/H)$ to itself satisfying the homomorphic relation $\lambda_2^l(nn') = \lambda_2^l(n)\lambda_2^l(n')$. Therefore topological influence is characterized by the action of the automorphism group $\mathcal{G}_2$ on $\pi_2(G/H)$ defined by

$$\mathcal{G}_2 := \{\lambda_2^l | l \in \pi_1(G/H)\}. \quad (74)$$

From Corollary 2, $\pi_2(G/H)$ is described by a coroot lattice L_G^c. Let us define the Weyl reflection $w_{\boldsymbol{\alpha}} : L_G^c \to L_G^c$ for $\boldsymbol{\alpha} \in R_+$ by

$$w_{\boldsymbol{\alpha}}(H_{\boldsymbol{t}}) := H_{\boldsymbol{t}'}, \ \boldsymbol{t}' := \boldsymbol{t} - \frac{2(\boldsymbol{\alpha},\boldsymbol{t})}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\boldsymbol{\alpha}, \quad (75)$$

where $w_{\boldsymbol{\alpha}}$ describes the reflection across the plane perpendicular to $\boldsymbol{\alpha}$. It is known that $w_{\boldsymbol{\alpha}}$ is an automorphism of L_G^c [41,42]. The Weyl group W_G of G is defined as the automorphism group of L_G^c generated by the Weyl reflections:

$$W_G := \text{Gen}\{w_{\boldsymbol{\alpha}} | \boldsymbol{\alpha} \in R_+\}, \quad (76)$$

where Gen S for a set S is defined as the group generated by the elements of S. It is instructive to consider an example of $\mathfrak{g} = \mathfrak{su}(2)$. Since $\mathfrak{g}$ is constituted from only one $\mathfrak{su}(2)$-subalgebra, its coroot lattice is a one-dimensional lattice $L_G^c = \{mH_{\boldsymbol{\alpha}^c} | m \in \mathbb{Z}\}$. The Weyl reflection acts on L_G^c as its inversion: $w_{\boldsymbol{\alpha}}(mH_{\boldsymbol{\alpha}^c}) = -mH_{\boldsymbol{\alpha}^c}$; thus, $W_G \simeq \mathbb{Z}_2$. More generally, the Weyl group for $\mathfrak{g} = \mathfrak{su}(N)$ is given by $W_G \simeq S_N$, where S_N denotes the permutation group of N elements [41,42]. For $\mathfrak{g} = \mathfrak{u}(1)$, $W_G \simeq 0$, since $\mathfrak{u}(1)$ does not have a coroot. From Corollary 2, we write an element of $\pi_2(G/H) \simeq (L_H \cap L_G^c)/L_H$ by $H_{\boldsymbol{t}} + L_H^c$. The following corollary shows that topological influence is described by a Weyl reflection (75) and that possible forms of the automorphism group (74) are restricted from the Weyl group W_G.

Corollary 6. For each discrete symmetry $[\sigma] \in \pi_0(H \cap G_0)$, there exists a Weyl reflection $w_\sigma \in W_G$ that satisfies

$$\lambda_2^{(e,[\sigma])}(H_{\boldsymbol{t}} + L_H^c) = w_{[\sigma]}(H_{\boldsymbol{t}}) + L_H^c, \quad (77)$$

and the automorphism group $\mathcal{G}_2$ is a subgroup of W_G.

The proof of Corollary 6 is given in Appendix B. For all the examples studied so far, $\mathfrak{g}$ is $\mathfrak{u}(1)$,$\mathfrak{su}(2)$,$\mathfrak{so}(3)$, or their direct sum [34,36,37,39,40]. Therefore it follows from Corollary 6 that $\mathcal{G}_2$ is either trivial or a direct sum of $\mathbb{Z}_2$, where a possible form of nontrivial topological influence is essentially the sign change of a topological charge. Since a larger group, in general, has a larger Weyl group, it is natural to ask whether other forms appear when we consider a group larger than $G =$ SU(2) or SO(3). We answer this affirmatively in Sec. V by showing an example of $\mathcal{G}_2 \simeq S_3$. Finally, there is no topological influence for the case of a three-dimensional skyrmion as stated in the following corollary.

Corollary 7. The topological influence on a three-dimensional skyrmion is trivial.

Proof. From Corollary 3, we have $\pi_3(G/H) \simeq$ Coker i_3^*. Then, Corollary 7 follows directly from Theorem 2.

V. NON-ABELIAN TOPOLOGICAL INFLUENCE ON A SKYRMION IN AN SU(3)-HEISENBERG MODEL

Since topological influence on a monopole and that on a skyrmion are the same in that they are characterized by the action (70) of $\pi_1(G/H)$ on $\pi_2(G/H)$ [36], we consider topological influence on a skyrmion in the two-dimensional space. We include in G and H the space symmetry and the lattice symmetry, respectively, because dislocations and disclinations, which result from the breaking of the space symmetry, play a vital role in the topological influence analyzed below.

A. Vortices and skyrmions in a 3-CDW state

1. *SU(3)-Heisenberg model and its ground state*

The Hamiltonian of the SU(3)-Heisenberg model on a triangular lattice L is given by

$$H = J \sum_{\langle i,j \rangle} \sum_{a=1}^{8} T_{a,i} T_{a,j}, \quad (78)$$

where $\langle i,j \rangle$ denotes a pair of nearest-neighbor sites i and j, and $\{T_{a,i}\}_{a=1}^{8}$ is a set of the generators of $\mathfrak{su}(3)$ at site i. On each site, there are three degenerate states, which we refer to as red, green, and blue, and write them as

$$|R\rangle = (1,0,0)^T, \ |G\rangle = (0,1,0)^T, \ |B\rangle = (0,0,1)^T. \quad (79)$$

We call these internal degrees of freedom as color. This model is expected to be realized in an ultracold atomic gas of alkaline-earth atoms in an optical lattice [65–68] and can be regarded as a spin-1 bilinear-biquadratic model with equal

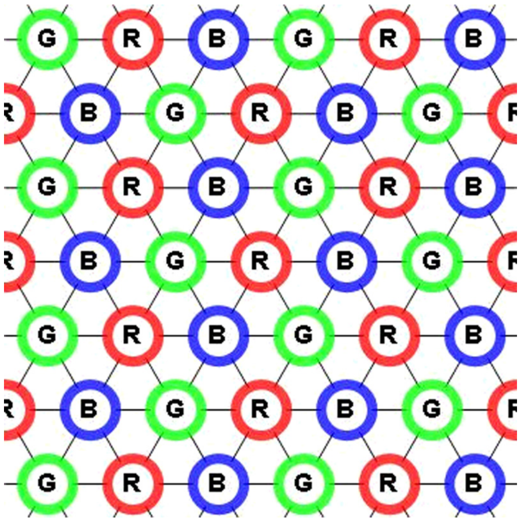

FIG. 7. Schematic illustration of the three-color density-wave state on a triangular lattice. The red (R), green (G), and blue (B) disks show the internal states $|R\rangle, |G\rangle$, and $|B\rangle$ defined in Eq. (79), respectively. The sites in states $|R\rangle$, $|G\rangle$, and $|B\rangle$ constitute the sublattices L_R, L_G, and L_B, respectively.

bilinear and biquadratic couplings [8,69]. For the case of an antiferromagnetic interaction ($J < 0$), the ground state $|\Psi\rangle_{\rm GS}$ is described by the three-sublattice ordering with a periodic alignment of three colors [7,8,43], and known as the three-color density-wave state (3-CDW state) [43,44] (see Fig. 7):

$$|\Psi\rangle_{\rm GS} = \bigotimes_{i\in L_R} |R\rangle_i \otimes \bigotimes_{i\in L_G} |G\rangle_i \otimes \bigotimes_{i\in L_B} |B\rangle_i, \tag{80}$$

where L_R, L_G, and L_B denote the three sublattices. For the triangular lattice, they are given by

$$L_R := \{(m_1 - m_2)\boldsymbol{a}_1 + (m_1 + 2m_2)\boldsymbol{a}_2 | m_1, m_2 \in \mathbb{Z}\}, \tag{81}$$

$$L_G := \{\boldsymbol{x} + \boldsymbol{a}_2 | \boldsymbol{x} \in L_R\}, \tag{82}$$

$$L_B := \{\boldsymbol{x} + \boldsymbol{a}_1 | \boldsymbol{x} \in L_R\}, \tag{83}$$

where $\boldsymbol{a}_1 = (1,0)^T$ and $\boldsymbol{a}_2 = (1/2, \sqrt{3}/2)^T$ are the primitive vectors of the triangular lattice in units of the lattice constant $a = 1$. The 3-CDW state appears as the ground state of the SU(3)-Heisenberg model on various lattices including triangular, square, and cubic lattices [7,8,43,70–72].

2. Symmetries of the system and the state

When we include the space symmetry, the symmetry of the system is given by

$$G = \mathrm{SU}(3) \times \mathrm{E}(2), \tag{84}$$

where $\mathrm{E}(2) := \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ is the two-dimensional Euclidian group generated by the two-dimensional translation group $\mathbb{R}^2$ and the two-dimensional rotational group SO(2), where the semidirect product on $H \rtimes N$ is defined by $(h,n) \rtimes (h',n') := (hnh'n^{-1}, nn')$. The ground state $|\Psi\rangle_{\rm GS}$ has the continuous symmetry H_0 generated by diagonal matrices:

$$H_0 = \left\{ \exp\left(i \sum_{a=\mathrm{RG,GB,BR}} c_a H_{\boldsymbol{\alpha}_a^c} \right) \middle| c_a \in \mathbb{R} \right\}. \tag{85}$$

Also, $|\Psi\rangle_{\rm GS}$ has the discrete symmetries that exchange the three colors R, G, and B and three sublattices L_R, L_G, and L_B simultaneously. The permutations of the colors are described by the symmetry group S_3:

$$\begin{aligned} S_3 &\simeq \mathrm{Gen}\{\sigma_{RG}, \sigma_{GB}, \sigma_{BR}\} \\ &\simeq \{I_3, \sigma_{RG}, \sigma_{GB}, \sigma_{BR}, \sigma_{RG}\sigma_{GB}, \sigma_{GB}\sigma_{BR}\}, \end{aligned} \tag{86}$$

where I_3 is the identity matrix with size three and the generators $\sigma_{RG} := \mathrm{e}^{i\pi S_{RG,1}}, \sigma_{GB} := \mathrm{e}^{i\pi S_{GB,1}}$, and $\sigma_{BR} := \mathrm{e}^{i\pi S_{BR,1}}$ are given by

$$\sigma_{RG} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_{GB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_{BR} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{87}$$

The permutations of the sublattices are described by the symmetry $H_{\rm lat}$ of the lattice:

$$\begin{aligned} H_{\rm lat} = \{(m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2, R(n\pi/3))| \\ m_1, m_2 \in \mathbb{Z}, n = 0,1,\cdots 5\}, \end{aligned} \tag{88}$$

where $m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2$ and $R(\phi)$ describe translation and rotation, respectively. The discrete symmetry is isomorphic to the lattice symmetry: $\pi_0(H \cap G_0) \simeq H_{\rm lat}$. Since h ($\in H_{\rm lat}$) induces a color exchange, we define σ_h as the corresponding matrix in S_3. From the above discussion, H is generated by the continuous symmetry H_0 and the discrete symmetry $H_{\rm lat}$:

$$H \simeq H_0 \rtimes H_{\rm lat}, \tag{89}$$

where $H_0 \rtimes H_{\rm lat}$ is the semidirect product defined by $(h_0,h) \rtimes (h_0',h') := (h_0\, \sigma_h h_0' (\sigma_h)^{-1}, hh')$.

3. Vortices in a 3-CDW state

Vortices and skyrmions in the 3-CDW state are determined in Ref. [73], where E(2) and its symmetry breaking are not considered. Here we show that the vortices characterized by S_3 indeed emerge. From Eq. (54), we have $\pi_1(G) \simeq \mathbb{Z}$, and $\pi_0(H \cap G_0) \simeq H_{\rm lat}$. Therefore we obtain Im $i_1^* \simeq 0$ and Coker $i_1^* \simeq \mathbb{Z}$ and hence

$$\pi_1(G/H) \simeq \mathbb{Z} \times_f H_{\rm lat}. \tag{90}$$

For a topological charge (m,h), we refer to σ_h defined above as a spin topological charge. For the analysis of topological influence, only a spin topological charge is necessary for the two reasons. First, vortices described by $\mathbb{Z}$ cannot have nontrivial topological influence from Theorem 2. Second, since skyrmions are shown to be described by an SU(3)-spin texture [see Eqs. (97), (98), and (99)], topological influence is solely determined by the color exchange σ_h. When we focus on spin topological charges, vortices are characterized by S_3:

$$\{\sigma_h \in S_3 | (m,h) \in \pi_1(G/H)\} \simeq S_3. \tag{91}$$

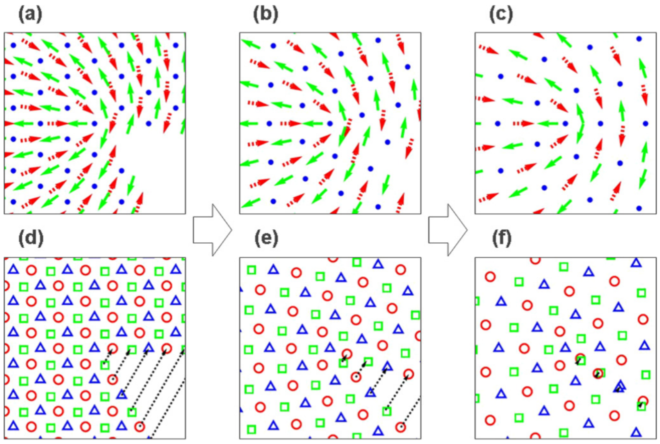

FIG. 8. Schematic illustrations of an RG vortex in the 3-CDW state for a triangular lattice. It is a disclination with the Frank angle $\pi/3$ around a site belonging to the sublattice L_B. Here, the Frank angle describes the angle over which the lattice sites are missing. The texture of the RG-vortex described in Eqs. (93), (94), and (95) is shown in (c), and (a) and (b) illustrate how the RG-spin vortex is obtained as the Frank angle vanishes. In (a)–(c), each arrow indicates the expectation value of $(S_{RG,2},S_{RG,3})$, where the red dashed (green) arrows correspond to the sublattice L_R (L_G). In (d)–(f), the red circle, green square, and blue triangule at each site indicate the sublattices L_R, L_G, and L_B, respectively. Across the disclination, the sublattices L_R and L_G are exchanged.

We refer to vortices with spin topological charges σ_{RG}, σ_{GB}, and σ_{BR} as RG-, GB-, and BR-vortices, respectively, according to their exchanges of the colors and sublattices. An example of an RG-vortex is the disclination with the Frank angle $\pi/3$ around a site belonging to the sublattice L_B, around which colors R and G and sublattices L_R and L_G are exchanged simultaneously. Here the Frank angle is the angle over which the lattice sites are missing [see Figs. 8(a) and 8(d)]. Let ϕ be the azimuth angle around the vortex. Due to the exchange of the sublattices, there is an ambiguity in the correspondence between a site and the sublattice it belongs to. We therefore fix the range of ϕ to $[0,2\pi)$ to assign one sublattice to each site. We take the reference order parameter as the expectation value of $\boldsymbol{S}_{\rm RG}$ with respect to $|\Psi\rangle_{\rm GS}$ in Eq. (80):

$$\langle \boldsymbol{S}_{\rm RG}\rangle_{R,0} = (0,0,1),\ \langle \boldsymbol{S}_{\rm RG}\rangle_{G,0} = (0,0,-1),$$
$$\langle \boldsymbol{S}_{\rm RG}\rangle_{B,0} = (0,0,0), \tag{92}$$

where $\langle A\rangle_X$ stands for the expectation value of A over sites of sublattice L_X and the subscript 0 indicates the expectation value with respect to $|\Psi\rangle_{\rm GS}$. Then, the texture of a vortex is obtained by operating $\exp(i\phi S_{{\rm RG},1}/2)$ on the reference order parameter (92):

$$\langle \boldsymbol{S}_{\rm RG}\rangle_R(\phi) = \left\langle {\rm e}^{-i\frac{\phi}{2}S_{{\rm RG},1}}\boldsymbol{S}_{\rm RG}{\rm e}^{i\frac{\phi}{2}S_{{\rm RG},1}}\right\rangle_{R,0} = \left[0,\sin\left(\frac{\phi}{2}\right),\cos\left(\frac{\phi}{2}\right)\right], \tag{93}$$

$$\langle \boldsymbol{S}_{\rm RG}\rangle_G(\phi) = \left\langle {\rm e}^{-i\frac{\phi}{2}S_{{\rm RG},1}}\boldsymbol{S}_{\rm RG}{\rm e}^{i\frac{\phi}{2}S_{{\rm RG},1}}\right\rangle_{G,0} = -\left[0,\sin\left(\frac{\phi}{2}\right),\cos\left(\frac{\phi}{2}\right)\right], \tag{94}$$

$$\langle \boldsymbol{S}_{\rm RG}\rangle_B(\phi) = \left\langle {\rm e}^{-i\frac{\phi}{2}S_{{\rm RG},1}}\boldsymbol{S}_{\rm RG}{\rm e}^{i\frac{\phi}{2}S_{{\rm RG},1}}\right\rangle_{B,0} = (0,0,0). \tag{95}$$

One can see from Eqs. (93) and (94) that $\boldsymbol{S}_{\rm RG}$ rotates by angle π around the vortex. We note that Eqs. (93) and (94) indeed give a continuous map to the OPM because the sublattices L_R and L_G are exchanged at $\phi = 0$ and 2π.

4. Skyrmions in a 3-CDW state

Since H does not include $\mathfrak{su}(2)$-subalgebras from Eq. (85), L_H^c vanishes. Hence, from Corollary 2, the second homotopy group is isomorphic to the triangular lattice, which, in turn, is isomorphic to the coroot lattice of SU(3):

$$\pi_2(G/H) \simeq \left(L_H\cap L_G^c\right)/L_H^c \simeq L_H\cap L_G^c \simeq \left\{\sum_{a={\rm RG,GB,BR}} m_a\boldsymbol{\alpha}_a^c \,\middle|\, m_a\in\mathbb{Z},\ \sum_{a={\rm RG,GB,BR}}\boldsymbol{\alpha}_a^c = 0\right\} \simeq L^c_{\rm SU(3)}. \tag{96}$$

Reflecting the triangular geometry of $L^c_{\rm SU(3)}$ (see Fig. 1), the 3-CDW state has three types of skyrmions [see Fig. 10(c)]. Let (r,ϕ) be the polar coordinates in $\mathbb{R}^2$, and $\theta(r)$ be a real function that satisfies $\theta(0) = 0$ and $\theta(\infty) = \pi$. From Corollary 2, the texture of a skyrmion with topological charge $\boldsymbol{\alpha}_{RG}^c$ is obtained by operating $g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)$ on the reference order parameter in Eq. (92):

$$\langle \boldsymbol{S}_{\rm RG}\rangle_R(\theta,\phi) = \left\langle \left[g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right]^\dagger \boldsymbol{S}_{\rm RG}\, g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right\rangle_{R,0} = \hat{\boldsymbol{r}}(\theta(r),\phi), \tag{97}$$

$$\langle \boldsymbol{S}_{\rm RG}\rangle_G(\theta,\phi) = \left\langle \left[g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right]^\dagger \boldsymbol{S}_{\rm RG}\, g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right\rangle_{G,0} = -\hat{\boldsymbol{r}}(\theta(r),\phi), \tag{98}$$

$$\langle \boldsymbol{S}_{\rm RG}\rangle_B(\theta,\phi) = \left\langle \left[g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right]^\dagger \boldsymbol{S}_{\rm RG}\, g^{(2)}_{\boldsymbol{\alpha}_{RG}}(\theta(r),\phi)\right\rangle_{B,0} = 0. \tag{99}$$

Thus this skyrmion is described by a hedgehog configuration of $\boldsymbol{S}_{\rm RG}$ with winding number $+1$ (-1) on L_R (L_G) as shown in Fig. 9. We refer to this skyrmion as an RG skyrmion. Similarly, the texture of a skyrmion with topological charge $\boldsymbol{\alpha}_{GB}^c$ is described by a hedgehog configuration of $\boldsymbol{S}_{\rm GB}$ with winding number $+1$ (-1) on L_G (L_B) and we refer to it as a GB skyrmion [see Fig. 10(b)]. Also, there exists a skyrmion with topological charge $\boldsymbol{\alpha}_{BR}^c$ described by a hedgehog configuration of $\boldsymbol{S}_{\rm BR}$ with winding number $+1$ (-1) on the sublattice L_B (L_R), and we refer to it as a BR skyrmion [see Fig. 10 (c)]. It follows from the relation $\boldsymbol{\alpha}_{RG}^c + \boldsymbol{\alpha}_{GB}^c + \boldsymbol{\alpha}_{BR}^c = 0$ (see Fig. 1) that these three skyrmions are not independent; the composition of all of them results in a trivial texture.

B. Topological influence in a 3-CDW state

From Eq. (96) and Fig. 1, $\pi_2(G/H)$ is isomorphic to the triangular lattice and $\pi_1(G/H)$ is isomorphic to S_3 as far as the spin topological charge is considered. We will see below that $\mathcal{G}_2$ defined in Eq. (74) is isomorphic to S_3, where three skyrmions with $\boldsymbol{\alpha}_{RG}$, $\boldsymbol{\alpha}_{GB}$, and $\boldsymbol{\alpha}_{BR}$ together

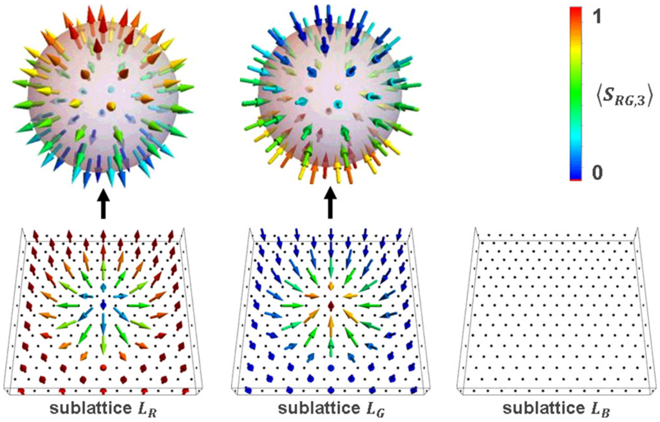

FIG. 9. Schematic illustrations of an RG skyrmion. Each arrow shows the expectation value of the generalized $\mathfrak{su}(2)$-spin vector $\langle \boldsymbol{S}_{\mathrm{RG}}\rangle(\theta,\phi)$, where the color represents the third component $\langle S_{\mathrm{RG},3}\rangle(\theta,\phi)$. The absence of an arrow on a site indicates that the expectation value vanishes there. When the two-dimensional plane is compactified into a sphere, in which the points at infinity are mapped onto the north pole, the texture on the sublattice L_R (L_G) describes a hedgehog texture of $\langle \boldsymbol{S}_{\mathrm{RG}}\rangle$ with winding number $+1$ (-1).

with their antiskyrmions with $-\boldsymbol{\alpha}_{RG}$, $-\boldsymbol{\alpha}_{GB}$, and $-\boldsymbol{\alpha}_{BR}$ are exchanged through topological influence, reflecting the S_3 symmetry of the triangular lattice. From Theorem 2, the topological influence of a vortex with spin topological charge σ on a skyrmion with topological charge n is described by the conjugation by σ:

$$\lambda_2^{\sigma}(n) := \sigma^{-1} n \sigma. \tag{100}$$

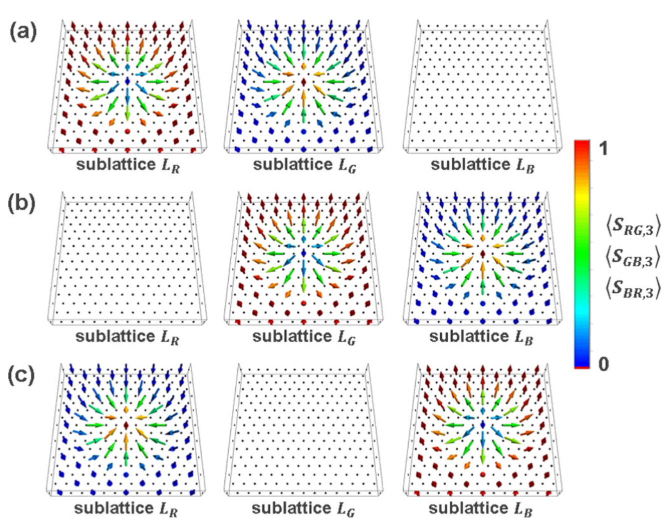

FIG. 10. Three types of skyrmions in a 3-CDW state. (a) An RG skyrmion has topological charge $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c_{RG}$ and a hedgehog texture of $\boldsymbol{S}_{RG}$ with winding number $+1$ (-1) on the sublattice L_R (L_G). (b) A GB skyrmion has topological charge $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c_{GB}$ and a hedgehog texture of $\boldsymbol{S}_{GB}$ with winding number $+1$ (-1) on the sublattice L_G (L_B). (c) A BR skyrmion has topological charge $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c_{BR}$ and a hedgehog texture of $\boldsymbol{S}_{BR}$ with winding number $+1$ (-1) on the sublattice L_B (L_R).

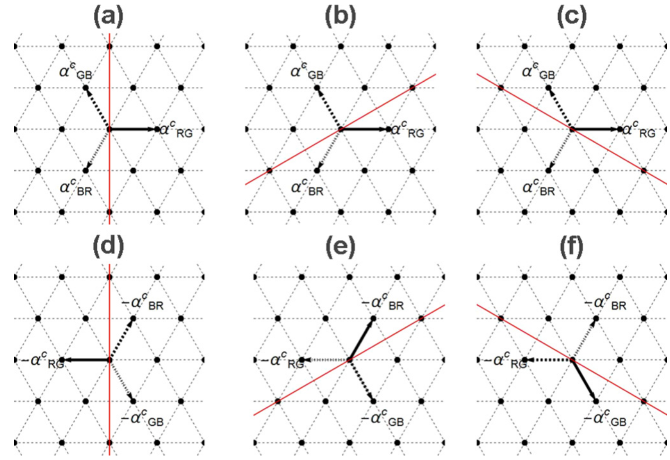

FIG. 11. Topological charges of skyrmions (a)–(c) before and (d)–(f) after making a complete circuit of the RG, GB, and BR vortices, respectively. The RG, GB, and GR vortices act on skyrmions by inverting them about the line perpendicular to $\boldsymbol{\alpha}_{RG}$, $\boldsymbol{\alpha}_{GB}$, and $\boldsymbol{\alpha}_{BR}$, respectively.

For example, for $\sigma = \sigma_{RG}$ and $n = \boldsymbol{\alpha}_{RG}, \boldsymbol{\alpha}_{GB}$, and $\boldsymbol{\alpha}_{BR}$. Direct calculations of the matrices in Eqs. (11) and (87) give

$$\lambda_2^{\sigma}(\boldsymbol{\alpha}_{RG}) = -\boldsymbol{\alpha}_{RG},\ \lambda_2^{\sigma}(\boldsymbol{\alpha}_{GB}) = -\boldsymbol{\alpha}_{BR},$$
$$\lambda_2^{\sigma}(\boldsymbol{\alpha}_{BR}) = -\boldsymbol{\alpha}_{GB}. \tag{101}$$

A crucial observation here is that this vortex acts on the triangular lattice, inverting it about the line perpendicular to $\boldsymbol{\alpha}_{RG}$ [see Figs. 11(a) and 11(d)]. Similarly, a vortex with topological charge σ_{GB} (σ_{BR}) acts on the triangular lattice inverting it about the line perpendicular to $\boldsymbol{\alpha}_{GB}$ ($\boldsymbol{\alpha}_{RG}$) as shown in Figs. 11(b) and 11(e) [Figs. 11(c) and 11(f)]. Since S_3 is generated by σ_{RG}, σ_{GB}, and σ_{BR}, we have $\mathcal{G}_2 \simeq S_3$.

The non-Abelian property of $\mathcal{G}_2$ emerges when we consider topological influence of two vortices. Let σ and τ be the topological charges of the vortices and n be that of a skyrmion. Suppose that the skyrmion goes around the vortex with σ clockwise, goes around the vortex with τ clockwise, goes around the vortex with σ anticlockwise, and finally goes around the vortex with τ anticlockwise (see Fig. 12). Since the third (fourth) process is the inverse process of the first (second) one, the change in the topological charge is given by

$$\lambda_2^{\tau^{-1}}\left(\lambda_2^{\sigma^{-1}}\left\{\lambda_2^{\tau}\left[\lambda_2^{\sigma}(n)\right]\right\}\right) = \lambda_2^{\rho}(n) \text{ for } \rho = \tau^{-1}\sigma^{-1}\tau\sigma. \tag{102}$$

While the final topological charge coincides with the initial one for an Abelian $\mathcal{G}_2$ because we have $\rho = e$ for any pair of vortices, it does not for a non-Abelian $\mathcal{G}_2$ because $\rho \neq e$ in general. For the case of $\sigma = \sigma_{RG}, \tau = \sigma_{GB}$, and $n = \boldsymbol{\alpha}_{RG}$, we have $\rho = \sigma_{BR}\sigma_{GB}$ and $\lambda_2^{\rho}(\boldsymbol{\alpha}_{RG}) = \boldsymbol{\alpha}_{GB} \neq \boldsymbol{\alpha}_{RG}$.

VI. CONCLUSION AND DISCUSSION

In the present paper, we have developed a general method to determine the homotopy group $\pi_m(G/H)$ of the order parameter manifold G/H by deriving the formula (50) which expresses $\pi_m(G/H)$ in terms of $\pi_m(G)$ and $\pi_m(H)$. Since the homotopy group of a Lie group and each texture on it can be calculated systematically by means of the Cartan canonical

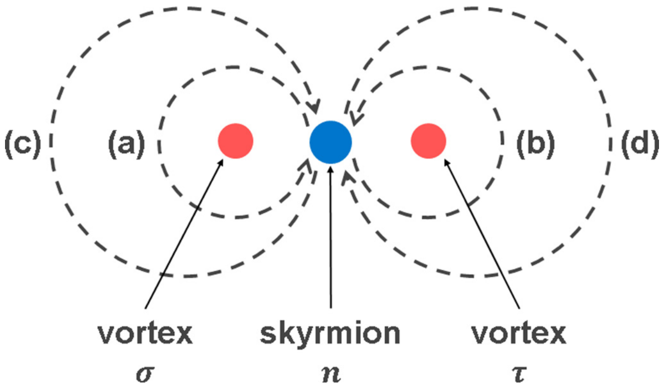

FIG. 12. Topological influence of two vortices with spin topological charges σ and τ on a skyrmion with topological charge n. The skyrmion goes around the vortex with σ clockwise (a), then around the vortex with τ clockwise (b), then around the vortex with σ anticlockwise (c), and finally around the vortex with τ anticlockwise (d). Through these processes, the topological charge of a skyrmion changes from n to $\lambda_2^\rho(n)$ with $\rho = \tau^{-1}\sigma^{-1}\tau\sigma$.

forms (3) and the lattices defined in Eqs. (9) and (10), the obtained formulas allow us to calculate $\pi_m(G/H)$ and the texture O of each topological excitation systematically. We find that the textures of a monopole and that of a three-dimensional skyrmion are obtained by the replacement of the $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}$ by the generalized $\mathfrak{su}(2)$-spin vector $\boldsymbol{S}_{\boldsymbol{\alpha}}$ defined in Eq. (7), and that their topological charges are described by a set of coroots, reflecting the fact that a Lie algebra $\mathfrak{g}$, in general, includes multiple $\mathfrak{su}(2)$ subalgebras. We have also shown the necessity of a discrete symmetry $\pi_0(H \cap G_0)$ for the presence of nontrivial topological influence. Moreover, we derive the necessary and sufficient condition for the presence of non-Abelian vortices and prove the absence of topological influence on a three-dimensional skyrmion. As for topological influence on a monopole or a skyrmion, we prove that the automorphism group $\mathcal{G}_2$ of topological influence is a subgroup of the Weyl group W_G, clarifying why only one type of topological influence is known so far. Seeking for other types, we find that topological influence characterized by a non-Abelian group S_3 emerges in the three-color density-wave state of the SU(3)-Heigenberg model, where three types of skyrmions and vortices characterized by S_3 appear. These skyrmions change their types through the topological influence, giving $\mathcal{G}_2 = S_3$.

Finally, we raise three problems for future study. First, the dynamical stability of the textures of topological excitations derived in Sec. II needs to be clarified. These textures and their variations have widely been used as candidates for dynamically stable textures of topological excitations [1,50,54,74–76]. In fact, the dynamically stability has been demonstrated in a number of examples [54,73,77–81]. The texture $O(\theta,\phi) := g_{\boldsymbol{\alpha}}^{(2)}(\theta,\phi)O_0$ and its variation $O(r,\phi) := g_{\boldsymbol{\alpha}}^{(2)}[\theta(r),\phi]O_0$ are widely used as candidates for the textures of a monopole and that of a skyrmion, respectively [1,50,54,74–76], where $\theta(r)$ is a function subject to the boundary conditions $\theta(0) = 0$ and $\theta(\infty) = \pi$. Moreover, they indeed give stable textures [54,73,77–82] for an appropriate choice of $\theta(r)$. It merits further study to clarify their dynamical stability. Second, we represent in Corollary 5 the necessary and sufficient condition for non-Abelian vortices in terms of the map f defined in Eq. (46). However, its physical implication is yet to be clarified. Considering the growing interest in the dynamics of non-Abelian vortices [59,61–64], it is of interest to understand whether we can simplify the condition (73). Third, analogous concepts of topological influence in topological insulators and superconductors have recently been discussed in specific examples [83–85], where the domain S^m of $\pi_m(G/H)$ and the order parameter manifold G/H are replaced by a Brillouin zone and a space of Hamiltonians, respectively. When a lower-dimensional topological invariant is nontrivial, one can change a higher-dimensional topological invariant under continuous deformation of a Hamiltonian. It is worthwhile to analyze general conditions for nontrivial topological influence in topological insulators and superconductors and clarify its difference from topological influence in topological excitations by using the general formulas developed in the present paper.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF LEMMA 1

Lemma 1 follows from the following theorem on the third homotopy group of a simple compact Lie group [49].

Theorem 3. Let G and $\boldsymbol{S}_{\boldsymbol{\alpha}}$ be a simple compact Lie group and the generalized $\mathfrak{su}(2)$-spin vector for the coroot $\boldsymbol{\alpha}^c$ with shortest length in G, respectively. Then, we have

$$\pi_3(G) \simeq \left\{ m\left[g_{\boldsymbol{\alpha}}^{(3)}\right]_G \middle| m \in \mathbb{Z} \right\}, \tag{A1}$$

where $[g_{\boldsymbol{\alpha}}^{(3)}]_G$ denotes the homotopy class of G with representative element $g_{\boldsymbol{\alpha}}^{(3)}$ defined in Eq. (23). The isomorphism in Eq. (A1) is given by $i_{*3} : \pi_3(H) \to \pi_3(G/H)$, where $H' = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ is the subgroup of G generated by $\boldsymbol{S}_{\boldsymbol{\alpha}}$. The right-hand side of Eq. (A1) does not depend of the choice of the coroot since $[g_{\boldsymbol{\alpha}_1}^{(3)}]_G = [g_{\boldsymbol{\alpha}_2}^{(3)}]_G$ for two coroots $\boldsymbol{\alpha}_1^c$ and $\boldsymbol{\alpha}_2^c$ with the shortest length.

Proof of Lemma 1. Let Eq. (22) and $\boldsymbol{\alpha}_i^c$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G and a coroot in $\mathfrak{g}_i$, respectively. If we denote the universal covering group of G' by $\widetilde{G}'$, $\widetilde{G}$ is given by $\mathbb{R}^{a'} \times \widetilde{G}_1 \times \cdots \times \widetilde{G}_a$ and hence we obtain

$$\begin{aligned}\pi_3(G) &\simeq \pi_3(\mathbb{R}^{a'} \times \widetilde{G}_1 \times \cdots \times \widetilde{G}_a) \simeq \bigoplus_{i=1}^{a} \pi_3(\widetilde{G}_k)\\ &\simeq \left\{ \sum_{i=1}^{a} m_i\left[g_{\boldsymbol{\alpha}_i}^{(3)}\right] \middle| m_i \in \mathbb{Z} \right\},\end{aligned} \tag{A2}$$

where the first isomorphism follows from the relation $\pi_m(G') \simeq \pi_m(\widetilde{G}')$ for $\forall m \geqslant 2$, the second one from the relation $\pi_m(X \times Y) \simeq \pi_m(X) \oplus \pi_m(Y)$, and the third one from Theorem 3, which completes the proof of Lemma 1.

APPENDIX B: PROOF OF THEOREM 1

Theorem 1 is proved by applying the theory of a group extension with an Abelian kernel [47].

1. Group extension with an Abelian kernel

Definition 1: Group extension. (a) Let Q and N be two groups. Then G is a group extension of Q by N if N is a normal subgroup of G and Q is a quotient group of G by N, i.e., $Q = G/N$. In particular, if N is Abelian, G is referred to as a group extension of Q by an Abelian kernel N. Let G and G' be group extensions of Q by N. We denote the projection from G (G') to Q as T (T'). If there exists an isomorphism $F: G \to G'$ such that $T' \circ F = T$, the two group extensions G and G' are regarded as equivalent.

Let G, N, and Q be a group, a normal subgroup of G, and the quotient group of G by N, respectively, and consider a situation in which we know N and Q but do not know G. The problem of constructing G from N and Q is referred to as an group extension problem. As we will see below, there is a general theory to solve the group extension problem if N is Abelian.

Let T be the projection from G to Q. A map $s: Q \to G$ that satisfies $T \circ s = \mathrm{id}_Q$ is referred to as a section of T, where id_Q denotes the identity map on Q. We assume that N is Abelian and that a section s of T is given. We define the map $f: Q \times Q \to N$ referred to as the factor set of G associated with s by

$$f(q,q') := s(q)s(q')s(qq')^{-1}. \tag{B1}$$

Since $T[f(q,q')] = e$ and hence $f(q,q') \in N$, f is indeed a map to N. Since N is a normal subgroup of G, N is invariant under the inner isomorphism $g' \mapsto gg'g^{-1}$. Moreover, the inner isomorphisms of N acts on N trivially because N is Abelian. Therefore the inner isomorphism of $g \in G$ depends only on the quotient element $q = T(g)$. We define the map $\theta_q: N \to N$ for q by $\theta_q(n) := gng^{-1}$, where $g \in G$ satisfies $q = T(g)$. Then, the following theorem holds [47].

Theorem 4. (a) Let the product $\times_f$ on the product set $N \times Q$ be defined by

$$(n,q) \times_f (n',q') := (n + \theta_q(n') + f(q,q'), qq'), \tag{B2}$$

where we write the product on N by the sum because N is Abelian. Then, $N \times Q$ becomes a group, where the identity element is given by $(f(e,e)^{-1}, e)$ and the inverse of (n,q) is given by $(n^{-1} + f(q,q^{-1})^{-1}, q^{-1})$. This group denoted by $N \times_f Q$ is a group extension of Q by N and isomorphic to G under this product. (b) Let $\bar{s}$ and $\bar{f}$ be another section of T and the factor set associated with $\bar{s}$, respectively. If there exists a map $\alpha: Q \to N$ that satisfies $\bar{s}(q) = \alpha(q)s(q)$, the two group extensions $N \times_f Q$ and $N \times_{\bar{f}} Q$ are equivalent.

2. Proof of Theorem 1

We start from the relation derived in Ref. [46]:

$$\frac{\pi_m(G/H)}{\mathrm{Coker}\ i_{*m}} \simeq \mathrm{Ker}\ i_{*m-1}, \tag{B3}$$

which follows from the homotopy exact sequence [46,86]. From the homotopy lifting theorem [86], any homotopy class O of G/H can be written as

$$O(\boldsymbol{x}) = g(\boldsymbol{x})O_0 \text{ for } \forall \boldsymbol{x} \in D^m, \tag{B4}$$

where g is a map from D^m to G subject to the boundary condition

$$g(\boldsymbol{x}) = e \text{ for } \|\boldsymbol{x}\| = \pi. \tag{B5}$$

Then, the projection map $T: \pi_m(G/H) \to \mathrm{Ker}\ i_{*m-1}$ in Eq. (B3) is given by

$$[T(O)](\hat{\boldsymbol{x}}) := \lim_{r\to 0} g^O(r\hat{\boldsymbol{x}}) \text{ for } \hat{\boldsymbol{x}} \in S^{m-1}, \tag{B6}$$

where $T(O)$ is a map from S^{m-1}.

We first prove that the inner isomorphism of $\pi_m(G/H)$ on $\mathrm{Coker}\ i_{*m}$ is trivial:

$$n[a]n^{-1} = [a] \text{ for } \forall n \in \pi_m(G/H), \forall [a] \in \mathrm{Coker}\ i_{*m}. \tag{B7}$$

For $m \geqslant 2$, Eq. (B7) follows from the commutativity of higher-dimensional homotopy groups. For $m = 1$, from Eq. (B4), we can express the loop l^n ($l^{[a]}$) corresponding n ($[a]$) as $l^n(\phi) = \gamma_n(\phi)O_0$ ($l^{[a]}(\phi) = a(\phi)O_0$), where γ_n is a path from $\gamma_n(0) = \sigma_n$ to $\gamma_n(2\pi) = e$ and a is a loop on G. Then, defining $a_s(\phi)$ for $s,\phi \in [0,2\pi]$ by

$$a_s(\phi) := \begin{cases} \gamma_n(2\pi - 3\phi)O_0 & 0 \leqslant \phi \leqslant \frac{s}{3}; \\ a\left(\frac{\pi(3\phi - s)}{3\pi - s}\right)\gamma_n(2\pi - s)O_0 & \frac{s}{3} \leqslant \phi \leqslant 2\pi - \frac{s}{3}; \\ \gamma_n(3\phi - 4\pi)O_0 & 2\pi - \frac{s}{3} \leqslant s \leqslant 2\pi, \end{cases} \tag{B8}$$

we find that a_s is a continuous deformation from $a_{s=0} = [a]$ to $a_{s=2\pi} = n[a]n^{-1}$, which completes the proof of Eq. (B7).

We next apply Theorem 4 to derive Eqs. (49)–(51). Let b and $[O^b]_{G/H}$ be an element of $\mathrm{Ker}\ i_{*m-1}$ and that of $\pi_m(G/H)$ defined in Eq. (37), respectively. When we define $S: \mathrm{Ker}\, i_{*m-1} \to \pi_m(G/H)$ by $S(b) := [O^b]_{G/H}$, it is a section of T since

$$\begin{aligned}\{[T \circ S](b)\}(\boldsymbol{x}) &= [T([O^b]_{G/H})](\boldsymbol{x}) = b_{s=0}(\hat{\boldsymbol{x}}) \\ &= b(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in S^{m-1}.\end{aligned} \tag{B9}$$

We define $f: \mathrm{Ker}\, i_{*m-1} \times \mathrm{Ker}\, i_{*m-1} \to \mathrm{Coker}\, i_{*m}$ by Eq. (B1) with substitution of S for s. Since $\mathrm{Coker}\ i_{*m}$ is Abelian for any m, Theorem 4 gives the isomorphism

$$\pi_m(G/H) \simeq \mathrm{Coker}\ i_{*m} \times_f \mathrm{Ker}\ i_{*m-1}, \tag{B10}$$

where the product on the right-hand side of Eq. (B10) is defined by

$$([a],b) \times_f ([a'],b) := \big([a] + \theta_b([a']) + f(b,b'), bb'\big). \tag{B11}$$

From Eq. (B7), the inner isomorphism of $\pi_m(G/H)$ on $\mathrm{Coker}\ i_{*m}$ is trivial: $\theta_b([a']) := b[a']b^{-1} = [a']$, which gives

Eqs. (49) and (50). Since we take the section of the identity element as $S(e) := [O^e]_{G/H} = e$, we obtain $f(e,b) = S(e)S(b)[S(eb)]^{-1} = e$ and hence Eq. (51).

We finally prove that any choice of the section gives an equivalent group extension. Let $[O^b]_{G/H}$ be another element of $\pi_m(G/H)$ corresponding to another deformation $\bar{b}_s$ of $b \in \mathrm{Ker}\, i_{*m-1}$ to the trivial homotopy class. Then, the map $\bar{S} : b \mapsto [\bar{O}^b]_{G/H}$ provides another section. From the relation

$$[(O^b)^{-1}\bar{O}^b](\boldsymbol{x}) = \begin{cases} b_{s=\pi-2\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}})O_0 & \text{for } 0 \leqslant \|\boldsymbol{x}\| \leqslant \frac{\pi}{2}; \\ \bar{b}_{s=2\|\boldsymbol{x}\|-\pi}(\hat{\boldsymbol{x}})O_0 & \text{for } \frac{\pi}{2} \leqslant \|\boldsymbol{x}\| \leqslant \pi, \end{cases} \tag{B12}$$

we have

$$\bar{b}_{s=2\|\boldsymbol{x}\|-\pi}(\hat{\boldsymbol{x}}) = e \text{ for } \|\boldsymbol{x}\| = \pi, \tag{B13}$$

$$\bar{b}_{s=\pi-2\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}}) = e \text{ for } \|\boldsymbol{x}\| = 0, \tag{B14}$$

and hence $([O^b]_{G/H})^{-1}[\bar{O}^b]_{G/H} \in \mathrm{Coker}\ i_{*m}$. Therefore, defining the map $\alpha : \mathrm{Ker}\ i_{*m-1} \to \mathrm{Coker}\ i_{*m}$ by $\alpha(b) := [S(b)]^{-1}\bar{S}(b)$, we find from Theorem 4(b) that the two sections S and $\bar{S}$ give the equivalent group extensions, which completes the proof of Theorem 1.

APPENDIX C: PROOF OF COROLLARY 2

Since $\pi_2(G)$ vanishes from Eq. (21), Coker i_2^* vanishes. It follows from Theorem 1 that

$$\pi_2(G/H) \simeq \mathrm{Ker}\{i_1^* : \pi_1(H) \to \pi_1(G)\}. \tag{C1}$$

Since an element of L_H^c describes a trivial loop on H, it is trivial as a loop on G, which indicates $L_H^c \subset L_G^c$. Since $L_H^c \subset L_H$, L_H^c is an Abelian subgroup of $L_H \cap L_H^c$. Let us write elements of $\pi_1(H) \simeq L_H/L_H^c$ as $H_t + L_H^c$ ($H_t \in L_H$) and those of $\pi_1(G) \simeq L_G/L_G^c$ as $H_t \in L_G$ ($H_t + L_G^c$). For an element $H_t + L_H^c$ of $\pi_2(G/H) \simeq \mathrm{Ker}\, i_1^*$, we have $i_1^*(H_t + L_H^c) = e$, and hence $H_t + L_H^c \in L_G^c$. We thus obtain Eqs. (61) and (62):

$$\begin{aligned} \pi_2(G/H) &\simeq \{H_t + L_H^c | H_t \in L_H, H_t + L_H^c \in L_G^c\} \\ &\simeq \{H_t + L_H^c | H_t \in L_H \cap L_G^c\} \\ &\simeq (L_H \cap L_G^c)/L_H^c. \end{aligned} \tag{C2}$$

We next derive the texture (63) for the topological charge n given in Eq. (62), which can be done straightforwardly from the construction (37) and Theorem 1. The right-hand side of Eq. (62) describes the loop

$$g_n(\phi) = \exp\left(i\phi \sum_{j=1}^{r} m_j H_{\boldsymbol{\alpha}_j^c}\right), \tag{C3}$$

which can continuously transform into a trivial one through the continuous deformation $g(\theta,\phi)$ defined by

$$g(\theta,\phi) := \mathrm{e}^{-i\theta S_{\alpha_1,2}} g_{\boldsymbol{\alpha}_1}^{(2)}(\theta,m_1\phi)\mathrm{e}^{-i\theta S_{\alpha_2,2}} g_{\boldsymbol{\alpha}_2}^{(2)}(\theta,m_2\phi) \times \cdots \times \mathrm{e}^{-i\theta S_{\alpha_r,2}} g_{\boldsymbol{\alpha}_r}^{(2)}(\theta,m_r\phi) \exp\left(i\phi \sum_{j=1}^{r} m_j S_{\boldsymbol{\alpha}_j,3}\right), \tag{C4}$$

where θ ($\in [0,\pi]$) is the parameter of the deformation. In fact, since

$$g_{\boldsymbol{\alpha}_j,1}^{(2)}(0,m_j\phi) = \mathrm{e}^{i\phi m_j S_{\alpha_j,3}}, \tag{C5}$$

$$\begin{aligned} \mathrm{e}^{-i\pi S_{\alpha_j,2}} g_{\boldsymbol{\alpha}_j}^{(2)}(\pi,m_j\phi) &= \mathrm{e}^{-i\pi S_{\alpha_j,2}}\mathrm{e}^{i\phi m_j S_{\alpha_j,3}}\mathrm{e}^{i\pi S_{\alpha_j,2}} \\ &= \mathrm{e}^{-i\phi m_j S_{\alpha_j,3}}, \end{aligned} \tag{C6}$$

we obtain $g(\theta = 0,\phi) = g_n(\phi)$ and $g(\theta = \pi,\phi) = e$. Thus the texture $\widetilde{O}(\theta,\phi)$ is given by

$$\begin{aligned} \widetilde{O}(\theta,\phi) = {}& \mathrm{e}^{-i\theta S_{\alpha_1,2}} g_{\boldsymbol{\alpha}_1}^{(2)}(\theta,m_1\phi)\mathrm{e}^{-i\theta S_{\alpha_2,2}} g_{\boldsymbol{\alpha}_2}^{(2)}(\theta,m_2\phi) \\ & \times \cdots \times \mathrm{e}^{-i\theta S_{\alpha_r,2}} g_{\boldsymbol{\alpha}_r}^{(2)}(\theta,m_r\phi)O_0. \end{aligned} \tag{C7}$$

Let us define

$$\widetilde{O}_u(\theta,\phi) = g_{u,1}(\theta,\phi)g_{u,2}(\theta,\phi)\cdots g_{u,r}(\theta,\phi)O_0, \tag{C8}$$

where $g_{u,b}(\theta,\phi) := \mathrm{e}^{-i\theta(1-u)S_{\alpha_j,2}} g_{\boldsymbol{\alpha}_j}^{(2)}(\theta,m_j\phi)$ and $u \in [0,1]$. Equation (C8) gives a continuous deformation from $O_{u=0} = \widetilde{O}$ to $O_{u=1} = O$, because $g_{u,b}$ is a continuous deformation from $g_{u=0,b}(\theta,\phi) = \mathrm{e}^{-i\theta S_{\alpha_j,2}} g_{\boldsymbol{\alpha}_j}^{(2)}(\theta,m_j\phi)$ to $g_{u=1,b}(\theta,\phi) = g_{\boldsymbol{\alpha}_j}^{(2)}(\theta,m_j\phi)$, which completes the proof of Corollary 2.

APPENDIX D: PROOF OF THEOREM 2

It follows from Eq. (49) that the action of $([H_t],[\sigma])$ can be decomposed into the action of $([H_t],e)$ and that of $(e,[\sigma])$ as follows:

$$\lambda_m^l(n) = \lambda_m^{(e,[\sigma])}\{\lambda_m^{([H_t],e)}[([a],b)]\}. \tag{D1}$$

From Eq. (56), we can write the texture $O_1(\phi)$ of a vortex with topological charge $([H_t],e)$ as $O_1(\phi) := g_1(\phi)O_0$, where $g_1(\phi) := \exp(i\phi H_t)$ and ϕ ($\in [0,2\pi]$) is the azimuth angle around the vortex. Let $O^{([a],b)}(\boldsymbol{x})$ be the texture of a topological excitation with topological charge $([a],b)$ and let us define λ_s for $s \in [0,2\pi]$ by

$$\lambda_s(\boldsymbol{x}) := \begin{cases} g_1(s)O^{([a],b)}\left[\left(\frac{\pi}{\frac{\pi}{2}+\frac{s}{4}}\right)\boldsymbol{x}\right] & \text{for } 0 \leqslant \|\boldsymbol{x}\| \leqslant \frac{\pi}{2} + \frac{s}{4}; \\ g_1(4\|\boldsymbol{x}\| - 2\pi)O_0 & \text{for } \frac{\pi}{2} + \frac{s}{4} \leqslant \|\boldsymbol{x}\| \leqslant \pi. \end{cases} \tag{D2}$$

Since $g_1(2\pi) = e$, λ_s is a continuous deformation from $\lambda_{s=0}(\boldsymbol{x}) = \lambda_m^{([H_t],e)}([a],b)$ to

$$\lambda_{s=2\pi}(\boldsymbol{x}) = O^{([a],b)}(\boldsymbol{x}) = O^{([a],b)}(\boldsymbol{x}), \tag{D3}$$

subject to the boundary condition (27). We thus have $\lambda_m^{([H_t],e)}[([a],b)] = ([a],b)$. Then, Eq. (D1) reduces to

$$\begin{aligned} &\lambda_m^{([H_t],[\sigma])}[([a],b)] \\ &\quad = \lambda_m^{(e,[\sigma])}[([a],b)] = \lambda_m^{(e,[\sigma])}[([a],e) \times_f (e,b)] \\ &\quad = \lambda_m^{(e,[\sigma])}[([a],e)] \times_f \lambda_m^{(e,[\sigma])}[(e,b)], \end{aligned} \tag{D4}$$

where we use the homomorphic property of λ_m^l in the third equality. Let γ_σ be a path from $\gamma_\sigma(0) = \sigma$ to $\gamma_\sigma(2\pi) = e$ and we write the texture of a topological excitation with topological charge $([a],e)$ $[(e,b)]$ by $O'(\boldsymbol{x}) = g(\boldsymbol{x})O_0$ with $g(\boldsymbol{x}) = a(\boldsymbol{x})$

$[g(\boldsymbol{x}) = b_{\|\boldsymbol{x}\|}(\hat{\boldsymbol{x}})]$. Let us define λ'_s for $s \in [0,2\pi]$ by

$$\lambda'_s(\boldsymbol{x}) := \begin{cases} \gamma_\sigma(s)\sigma^{-1}O'\big[\big(\frac{\pi}{\frac{\pi}{2}+\frac{s}{4}}\big)\boldsymbol{x}\big] & \text{for } 0 \leqslant \|\boldsymbol{x}\| \leqslant \frac{\pi}{2}+\frac{s}{4}; \\ \gamma_\sigma(4\,\|\boldsymbol{x}\| - 2\pi)O_0 & \text{for } \frac{\pi}{2}+\frac{s}{4} \leqslant \|\boldsymbol{x}\| \leqslant \pi. \end{cases} \tag{D5}$$

Then, λ'_s is a continuous deformation from $\lambda'_{s=0}(\boldsymbol{x}) = \lambda_m^{(e,[\sigma])}(\boldsymbol{x})$ to

$$\lambda'_{s=2\pi}(\boldsymbol{x}) = \sigma^{-1}O'(\boldsymbol{x}) = \sigma^{-1}g(\boldsymbol{x})\sigma O_0 = [\sigma^{-1}g\sigma](\boldsymbol{x}), \tag{D6}$$

where we use $\sigma \in H \cap G_0$ and hence $\sigma O_0 = O_0$ in the second equality in Eq. (D6). When the topological charge is $([a],e)$, $[\gamma_\sigma(s)]^{-1}a\gamma_\sigma(s)$ for $s \in [0,2\pi]$ describes a continuous deformation from $\sigma^{-1}a\sigma$ to a subject to the boundary condition (27). We therefore have $\lambda_m^{(e,[\sigma])}[([a],e)] = ([a],e)$ and hence Eq. (70):

$$\begin{aligned} &\lambda_m^{(e,[\sigma])}[([a],e)] \times_f \lambda_m^{(e,[\sigma])}[(e,b)] \\ &\quad = ([a],e) \times_f (e,\sigma^{-1}b\sigma) = ([a],\sigma^{-1}b\sigma), \end{aligned} \tag{D7}$$

which completes the proof of Theorem 2.

APPENDIX E: PROOF OF COROLLARY 6

Let $\mathfrak{g}_C$ be the Cartan subalgebra of G and we define two subalgebras of $\mathfrak{g}$ by

$$\begin{aligned} \mathfrak{h}_C^\perp &:= \{H_{\boldsymbol{u}} \in \mathfrak{g}_C | (\boldsymbol{u},\boldsymbol{t}) = 0 \text{ for } \forall H_{\boldsymbol{t}} \in L_H \cap L_G^c\}, \\ \mathfrak{h}^\perp &:= \mathfrak{h}_C^\perp \oplus \mathrm{Span}\{E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I | \boldsymbol{\alpha} \in R_+, \\ &\qquad (\boldsymbol{\alpha},\boldsymbol{t}) = 0 \text{ for } \forall H_{\boldsymbol{t}} \in L_H \cap L_G^c\}, \end{aligned} \tag{E1}$$

where Span$S$ denotes the vector space spanned by the elements of S.

We first prove that $\mathfrak{h}^\perp$ is a subalgebra of $\mathfrak{g}$ that commutes with $L_H \cap L_G^c$. From the commutation relations of the Cartan canonical form, the commutators among $H_{\boldsymbol{u}} \in \mathfrak{h}_C^\perp$ and $E_{\boldsymbol{\alpha}}^{R,I} \in \mathfrak{h}^\perp$ are spanned by $H_{\boldsymbol{u}}$ and $E_{\boldsymbol{\alpha}}^{R,I}$ that satisfy $(\boldsymbol{u},\boldsymbol{t}) = (\boldsymbol{\alpha},\boldsymbol{t}) = 0$ for $\forall H_{\boldsymbol{t}} \in L_H \cap L_G^c$. Therefore $\mathfrak{h}^\perp$ forms a subalgebra of $\mathfrak{g}$. Since both $\mathfrak{h}_C^\perp$ and $L_H \cap L_G^c$ are generated by the Cartan generators, $\mathfrak{h}_C^\perp$ commutes with $L_H \cap L_G^c$. For $H_{\boldsymbol{t}} \in L_H \cap L_G^c$ and $E_{\boldsymbol{\alpha}}^R, E_{\boldsymbol{\alpha}}^I \in \mathfrak{h}^\perp$, we have $[H_{\boldsymbol{t}}, E_{\boldsymbol{\alpha}}^{R,I}] = \pm i(\boldsymbol{\alpha},\boldsymbol{t})E_{\boldsymbol{\alpha}}^{I,R} = 0$ from Eq. (E1), indicating that $L_H \cap L_G^c$ commutes with $\mathfrak{h}^\perp$.

Let σ be a representative element of $[\sigma] \in \pi_0(H \cap G_0)$. We next prove $\mathrm{Ad}(\sigma)\mathfrak{h}_C^\perp \subset \mathfrak{h}^\perp$, where $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ for $g \in G$ is defined by $[\mathrm{Ad}(g)](X) := gXg^{-1}$ for $X \in \mathfrak{g}$. For $H_{\boldsymbol{u}} \in \mathfrak{h}_C^\perp$, we expand $\mathrm{Ad}(\sigma)H_{\boldsymbol{u}}$ in terms of the Cartan canonical form as

$$\mathrm{Ad}(\sigma)H_{\boldsymbol{u}} = H_{\boldsymbol{u}'} + \sum_{\boldsymbol{\alpha} \in R_+} (c_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^R + d_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^I), \tag{E2}$$

where $c_{\boldsymbol{\alpha}}$ and $d_{\boldsymbol{\alpha}}$ are real numbers. Let $H_{\boldsymbol{t}}$ be an element of $L_H \cap L_G^c$. Since $\mathrm{Ad}(\sigma)$ is an automorphism on $(L_H \cap L_G^c)/L_H^c$ from Theorem 2, we have $\mathrm{Ad}(\sigma^{-1})H_{\boldsymbol{t}} \in L_H \cap L_G^c$ and hence it can be written as $H_{\boldsymbol{t}} = \mathrm{Ad}(\sigma)H_{\boldsymbol{t}'}$ for some $H_{\boldsymbol{t}'} \in L_H \cap L_G^c$. From Eq. (E1), we have

$$\begin{aligned} (\boldsymbol{t},\boldsymbol{u}') &= \mathrm{Tr}[H_{\boldsymbol{t}}H_{\boldsymbol{u}'}] = \mathrm{Tr}[\mathrm{Ad}(\sigma)H_{\boldsymbol{t}'}\mathrm{Ad}(\sigma)H_{\boldsymbol{u}}] \\ &= (\boldsymbol{t}',\boldsymbol{u}) = 0. \end{aligned} \tag{E3}$$

Hence we obtain $H_{\boldsymbol{u}'} \in \mathfrak{h}_C^\perp$. Also, it follows from Eq. (E1) that

$$\begin{aligned} 0 &= \mathrm{Ad}(\sigma)[H_{\boldsymbol{t}}, H_{\boldsymbol{u}}] \\ &= \left[H_{\boldsymbol{t}'}, H_{\boldsymbol{u}'} + \sum_{\boldsymbol{\alpha} \in R_+} (c_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^R + d_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^I)\right] \\ &= \sum_{\boldsymbol{\alpha} \in R_+} i(\boldsymbol{\alpha},\boldsymbol{t}')(c_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^I - d_{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}^R). \end{aligned} \tag{E4}$$

This gives $c_{\boldsymbol{\alpha}} = d_{\boldsymbol{\alpha}} = 0$ if $\boldsymbol{\alpha}$ satisfies $(\boldsymbol{\alpha},\boldsymbol{t}') \neq 0$ for some $\boldsymbol{t}' \in L_H \cap L_G^c$, resulting in $\mathrm{Ad}(\sigma)H_{\boldsymbol{u}} \in \mathfrak{h}^\perp$, which completes the proof of $\mathrm{Ad}(\sigma)\mathfrak{h}_C^\perp \subset \mathfrak{h}^\perp$.

Let T_G be a maximum Abelian group of G. Let $H^\perp$ be the connected Lie group generated by $\mathfrak{h}^\perp$ including the identity element. Since $\mathrm{Ad}(\sigma)\mathfrak{h}_C^\perp \subset \mathfrak{h}^\perp$, $\mathfrak{h}_C^\perp$ and $\mathrm{Ad}(\sigma)\mathfrak{h}_C^\perp$ are maximum Abelian subgroups of $H^\perp$. Since any two maximum Abelian subgroups are conjugate to each other [41,42], there exists an element $h_\sigma^\perp$ of $H^\perp$ such that $\mathrm{Ad}(h_\sigma^\perp)[\mathrm{Ad}(\sigma)\mathfrak{h}_C^\perp] = \mathfrak{h}_C^\perp$. On the other hand, $\mathrm{Ad}(h_\sigma^\perp)$ acts on $L_H \cap L_G^c$ trivially from the commutativity between $\mathfrak{h}^\perp$ and $L_H \cap L_G^c$. We therefore have $\mathrm{Ad}(h_\sigma^\perp)\mathrm{Ad}(\sigma)L_H \cap L_G^c = L_H \cap L_G^c$. Since $\mathfrak{g}_C$ is generated by $L_H \cap L_G^c$ and $\mathfrak{h}_C^\perp$, we have

$$h_\sigma^\perp\sigma \in N_W := \{g \in G | \mathrm{Ad}(g)X \in \mathfrak{g}_C \text{ for } \forall X \in \mathfrak{g}_C\}. \tag{E5}$$

It is known that T_G is a normal subgroup of N_W and that the quotient group N_W/T_G is isomorphic to W_G [41,42]. We define $w_{[\sigma]} \in W_G$ as the projection of $h_\sigma^\perp\sigma \in N_W$ to $W_G \simeq N_W/T_G$. From Theorem 2, the action of $([H_{\boldsymbol{t}}],[\sigma]) \in \pi_1(G/H)$ on $H_{\boldsymbol{t}} + L_H^c \in \pi_2(G/H)$ can be written as

$$\begin{aligned} \lambda_2^{([H_{\boldsymbol{t}}],[\sigma])}(H_{\boldsymbol{t}} + L_H^c) &= \lambda_2^{(e,[\sigma])}(H_{\boldsymbol{t}} + L_H^c) = \mathrm{Ad}(\sigma)(H_{\boldsymbol{t}}) + L_H^c \\ &= w_{[\sigma]}(H_{\boldsymbol{t}}) + L_H^c. \end{aligned} \tag{E6}$$

We note that the right-hand side does not depend on the choice of a representative element since $\mathrm{Ad}(\sigma)$ acts on L_H^c trivially. Thus we have

$$\begin{aligned} \mathcal{G}_2 &\simeq \{\lambda_2^{([H_{\boldsymbol{t}}],[\sigma])} | ([H_{\boldsymbol{t}}],[\sigma]) \in \pi_1(G/H)\} \\ &\simeq \{w_{[\sigma]} \in W_G | [\sigma] \in \pi_0(H \cap G_0)\}. \end{aligned} \tag{E7}$$

Since the right-hand side is a subgroup of W_G, $\mathcal{G}_2$ is also a subgroup of W_G, which completes the proof of Corollary 6.

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