Magnetic-field-induced criticality in superconducting two-leg ladders

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We study magnetic-field-induced critical singularities in the superconducting phase of the hole-doped Hubbard model of repulsively interacting electrons, defined on a two-leg ladder. We argue that, provided the low-energy spin excitations in doped ladders carry electric charge, the low-temperature thermodynamic quantities, such as the specific-heat coefficient and magnetic susceptibility, will show logarithmic singularities in the quantum critical regime. This behavior is in drastic contrast to the magnetic-field-induced criticality in undoped Mott insulator ladders, which is governed by the zero-scale-factor universality with its hallmark square-root singularities.

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I. INTRODUCTION

Ladderlike geometries are a minimal step from a purely one-dimensional (1D) structure towards two dimensions, yet electron systems on ladders are amenable to being studied by powerful analytical and numerical methods available in one dimension. In addition, these systems share some similarities with two-dimensional (2D) behavior; for example, upon doping, the Hubbard model of repulsively interacting electrons on a two-leg ladder develops pairing and dominant superconducting tendencies [1–10]. Superconductivity (that is, the leading quasi-long-range order) in hole-doped ladders involves singlet pairing with an unconventional modified d-wave nature [2–4,8–10], reminiscent of behavior observed in high-T_c superconductors [11]. The fact that superconducting instability wins in purely repulsive system of electrons, without the phonon-mediated attraction, may be suggesting the common origin of superconductivity in doped ladders and 2D high-T_c systems, with the magnetic fluctuations playing an important role [12].

Two-leg ladder materials which show superconducting properties with doping, such as $(La,Ca,Sr)_{14}Cu_{24}O_{41}$, are synthesized, both as powder and as single crystals [13,14], fueling further interest in studying ladder systems.

Even though the spin gap is maintained with (not too large) doping δ [4], it does not evolve continuously with δ [9,15,16], an effect which is beyond the mean-field approximation [2]. Two scenarios have been suggested to explain this effect. In the first picture, in addition to conventional magnons, which evolve continuously with δ, there are lower-energy spin excitations: A hole pair breaks up into two holes, each of which forms a bound state with an up-spin electron on the same rung, producing spinon-holon quasiparticles (carrying spin-1/2 and charge $|e|$) [15], as illustrated in Fig. 1(a). Another candidate for the lowest-energy spin excitation in the superconducting ladder is a bound state of a magnon and hole pair [9,16]. In this scenario the hole pair does not break up; rather, it gets dressed with the magnon [17]. It was argued that when a single hole pair is present, the singlet-triplet spin gap of the system Δ_{tr} will experience a jump $\lim_{\delta\to 0}\Delta_{tr}(\delta) = (\sqrt{3}-1)\Delta_{tr}$ due to magnon-hole-pair bound-state formation and that this gap will further diminish with increasing δ [16]. A gain in the kinetic energy of holes in the local ferromagnetic environment of magnons was suggested as an intuitive mechanism for binding the magnon to the hole pair [9]. Numerical studies indicate that, generically, a hole pair is not a tightly localized object on a rung, but rather, it is spread over a few rungs, and in particular, for isotropic hoppings, the maximum probability of the two-hole configuration (participating in the bound state) is when they are located on adjacent rungs and different legs [15,18], as depicted in Fig. 1. A magnon-hole-pair bound state (carrying spin 1 and charge $2|e|$) corresponding to this case is sketched in Fig. 1(b), which should be interpreted as more depleted rungs being more polarized in the direction of the field.

II. EFFECTS OF INPLANE MAGNETIC FIELD ON SUPERCONDUCTING LADDERS

Provided that the nature of the low-energy spin excitations changes with doping, although the ground state continues to be a spin singlet and a gap in the spin excitation spectrum is maintained, a natural question arises about what happens to the magnetic-field-induced quantum critical point with δ when a magnetic field applied parallel to the ladder plane suppresses the singlet-triplet spin gap. The purpose of this work is to unveil universal singular properties of the magnetic-field-induced quantum critical point in hole-doped ladders of repulsively interacting electrons with dominant superconducting instabilities.

Without doping, for half filling, and for strong on-site repulsion, the Hubbard model, at energies lower than the Mott gap, reduces to the Heisenberg spin-1/2 antiferromagnetic ladder. The lowest-energy spin excitations in the absence of magnetic field are degenerate triplets of magnons at wave vector (π,π) and are separated from the singlet ground state by an energy gap of Δ_{tr}, which is roughly half of the Heisenberg exchange for the isotropic exchanges. The magnetic field splits the threefold degeneracy of the triplet bands linearly, and at the critical value of magnetic field $h = \Delta_{tr}$ a phase transition is induced, and the ground-state magnetization changes with magnetic field as $m_h \sim \sqrt{h-\Delta_{tr}}\,\Theta(h-\Delta_{tr})$ [19], where Θ is the Heaviside step function. The phase transition induced by the critical magnetic field in undoped ladders shows the zero-scale-factor universality [20], which in particular implies that both the low-temperature specific-heat coefficient and the magnetic susceptibility behave as $\sim T^{-\frac{1}{2}}$.

Perhaps the simplest nontrivial model of 1D electrons that shows gap in the spin excitation spectrum is the Hubbard model of attractively interacting electrons defined on a single

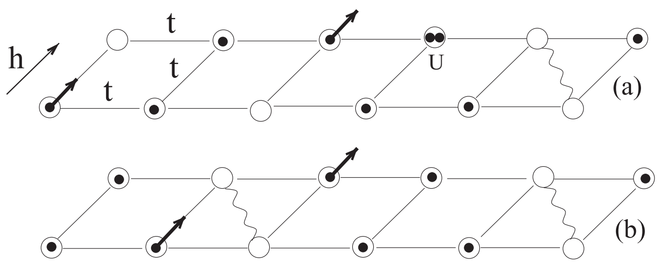

FIG. 1. Cartoon of the lowest-energy spin excitations in the doped Hubbard ladders with on-site interaction U and hopping amplitudes t. Open circles denote lattice sites, and solid circles represent electrons. In-plane magnetic field h couples only to electron spins. On the last plaquette there is a hole-pair depicted, and in (a) spinon-holons are depicted on first and third rungs, while in (b) the bound state of the magnon and hole pair is located on the second plaquette.

chain, which is integrable, and hence, one can obtain exact results, such as the critical properties of magnetic-field-induced quantum phase transition. For the attractive Hubbard model, using the Bethe ansatz method, it was shown that away from half filling the critical ground-state response of the system changes drastically compared to that of the half-filled case [21]. Similar results were obtained using a "nonlinear" bosonization approach [22], where the crucial role played by the spin-charge coupling, induced by the curvature of the Fermi surface, was unveiled. Using the Bethe ansatz basis of electron pairs and up-spin electrons, the leading low-temperature behavior of the attractive Hubbard chain at the magnetic-field-induced quantum critical point in the dilute limit was obtained in [23].

In the ladder problem that we consider, pairing originates from repulsive interelectron interactions and is hence a truly many-body effect with umklapp lattice processes playing an important role [3], as opposed to pairing in the attractive Hubbard chain. In addition, Hubbard ladders are not exactly solvable, and in the weak-coupling bosonization description one has to start with four bosonic fields [24], compared to just the spin and charge modes of the Hubbard chain.

III. EFFECTIVE DESCRIPTION

To address the nature of the magnetic-field-induced phase transition in doped superconducting ladders, we will need two assumptions for a starting point: (i) in the absence of magnetic field there is a finite energy gap in the spin excitations of these systems, while there is only one gapless mode, the charge mode corresponding to the motion of hole pairs, and (ii) the lowest-energy spin excitations are not conventional magnons, i.e., excitations involving only the spin degree of freedom (which continuously evolve from the undoped ladder case), but they involve charge degrees of freedom as well, similar to spinon-holon quasiparticles and magnon-hole-pair bound states. Consequently, such composite objects (which carry both spin and charge quantum numbers) will start to populate the ground state once magnetic field overcomes the singlet-triple spin excitation gap of the doped ladders.

The bosonization approach [16], supported by numerical simulations [9,15,18], indicates that these ingredients are indeed met in lightly doped electron ladders, which maintain the spin gap and show dominant superconducting correlations. It has been estimated numerically that the spin gap survives at least until $\delta = 0.25$ [4]. In particular, it was suggested [7,10,15,18] that lightly doped ladders at low energies realize a Luther-Emery liquid [25], and consequently, the hydrodynamic approach [26] was applied, producing an accurate low-energy description of the system, in agreement with numerical simulations [15,18].

We adopt such a hydrodynamic description of hole pairs in doped spin ladders, and as a new ingredient we add magnetic field via the Zeeman coupling, which is close to the critical value equal to singlet-triplet spin gap $h \simeq \Delta_{tr}(\delta)$. The magnetic field does not couple to hole pairs since hole pairs do not carry spin. Rather, the magnetic field acts as a chemical potential for the particles that represent low-energy magnetic excitations that we will describe in the effective continuous model by field ψ. As our aim is to describe the leading low-temperature behavior of the superconducting ladder in the vicinity of the critical magnetic field, for the effective model we may retain only those modes which are gapless (hole pairs) and which will become gapless after the magnetic field exceeds the critical value (ψ particles). With these ingredients the effective theory governing the low-energy properties of the superconducting ladder in near-critical magnetic field reads

$$\mathcal{H} = \frac{v_p}{2}\int dx\left(\frac{[n_p(x)-\bar{n}_p]^2}{K_p\pi} + K_p\pi(\partial_x\theta_p(x))^2\right) + \int dx\,\psi^\dagger(x)\left(-\frac{\hbar^2\partial_x^2}{2M_\psi} - \left(h-h_c^0\right)s_\psi\right)\psi(x) + \int dx\{g\psi^\dagger(x)\psi(x)[n_p(x)-\bar{n}] + g_\psi|\psi^\dagger(x)\psi(x)|^2\}. \quad (1)$$

We have fixed units $\hbar = 1$ and introduced notation for the spin value carried by the lowest-energy spin excitation $s_\psi = 1$ or $1/2$ depending on whether ψ describes spin-1 magnon-hole pairs or spin-1/2 spinon-holons. Hole pairs are described by hydrodynamic variables, conjugate Gaussian fields (n_p,θ_p) corresponding to density and phase fluctuations, respectively, with the canonical commutation relation $[n_p(x),\theta(y)] = i\delta(x-y)$. The Luttinger liquid constant K_p and the sound velocity v_p can be estimated numerically [15,18]. In particular, the Luttinger liquid parameter of pairs assumes the universal value $K_p = 1$ [10,15,18] for the lightly doped case, meaning that hole pairs behave as hard-core bosons and $v_p \sim J\delta$, where $J = 4t^2/U$ is the spin-exchange constant in the Hubbard ladder. In the following we will set $K_p = 1$ for simplicity.

The low-energy spin excitations, which are either spinon-holon fermionic quasiparticles or bosonic magnon-hole-pair particles, experience vacuum to a finite-density transition when magnetic field is swept across the critical value and are described by either fermionic or bosonic field ψ. For the latter case, the last term in Eq. (1) is important, with $g_\psi > 0$ [27]. Since the dilute limit is a strong-coupling limit in one dimension, governed by the Tonks gas fixed point, in the vicinity of the critical value of magnetic field, effectively, $g_\psi \to \infty$, and hence we can treat magnon-hole pairs as hard-core bosons.

The density-density interaction ($\sim g$) between the low-energy modes in Eq. (1) can shift the critical value of magnetic field $h_c^0 \to \Delta_{tr}(\delta)$; however, it cannot influence the nature of the underlying quantum critical point [28], and in particular $g \to 0$ under renormalization. Equation (1) includes all terms up to quartic in fields allowed by the symmetry of the microscopic problem. This is so in particular due to the $U(1)$ rotation symmetry in the spin space around the axes set by the magnetic field, and hence there cannot be terms that contain an odd number in the ψ field. For the very same reason, we retained only terms which contain an equal number of ψ and $\psi^\dagger$ operators in the effective Hamiltonian (1), hence disregarding the so-called pair-hopping processes between ψ particles and hole pairs, which can strongly influence the nature of phase transition [29]. Here a note is in order. In the spin-ladder model that we consider, which is derived from the Hubbard model, there is an exact $SU(2)$ symmetry in spin space in the absence of magnetic field and an exact $U(1)$ symmetry when magnetic field is present. Hence magnetization is a conserved quantity in our model. In reality, however, in condensed-matter systems, magnetization is controlled by the external magnetic field since there are inevitably processes which break conservation of magnetization [30]. Despite the fact that these processes are crucial to relax system magnetization, we assume they are much weaker than the terms retained in Eq. (1), an assumption which will break down for $\delta \to 0$ when $v_p \to 0$. However, even down to dopings of 5%, $v_p \sim 0.1J$, and hence terms in Hamiltonian (1) are expected to be much larger than those inducing relaxation of magnetization (e.g., the typical scale of magnetic dipolar interactions between electrons, assuming the average distance between them, ∼5 Å, is $\sim 10^{-2}$ K, whereas the spin exchange energy scale ranges from ∼10 K for organic ladders to $\sim 10^3$ K for nonorganic ladders).

At critical magnetic field, $h = \Delta_{tr}(\delta)$, the low-energy dispersion of ψ particles is quadratic, $E_\psi(k) = k^2/M_\psi$, invalidating their hydrodynamic description. From now on we will set $M_\psi = 1/2$. The leading low-energy dispersion of hole pairs, however, is linear in momentum, $E_p(k) \simeq v_p||k| - k_F^p|$ for $|k| \to k_F^p$, where the Fermi wave vector of hole pairs is related to the linear density of pairs $\bar{n}_p$ by the standard 1D relation $k_F^p = \bar{n}_p\pi$ [31].

A. Ground state properties

For $h < \Delta_{tr}(\delta)$, the number of ψ particles in the ground state is zero. Once magnetic field exceeds $\Delta_{tr}(\delta)$, ψ particles start to populate the ground state by breaking up hole pairs or dressing hole pairs by magnons. Hence, even though the magnetic field does not couple directly to hole pairs, it alters the quantum numbers of the ground state; in particular the mean ground-state density of hole pairs changes with magnetic field for $h > \Delta_{tr}(\delta)$ [32] as

$$\bar{n}_p = \delta/2 - {}_0\langle h|\psi^\dagger\psi|h\rangle_0\, s_\psi, \tag{2}$$

where $|h\rangle_0$ is the ground state for a given value of magnetic field h, which is a global spin-singlet state and independent of field for $h < \Delta_{tr}(\delta)$. For $s_\psi = 1$ one hole pair is converted to one magnon-hole pair, whereas for $s = 1/2$ when one hole pair breaks up, two spinon-holon quasiparticles are produced. In both cases Eq. (2) can be written in a unified manner,

$$\bar{n}_p = \delta/2 - m_h. \tag{3}$$

By minimizing the ground-state energy, the expectation value of the effective Hamiltonian (1) with respect to $\bar{n}_p$, and expressing $\bar{n}_p$ with the help of Eq. (3), we obtain for the ground-state magnetization,

$$m_h = \frac{\sqrt{4[h - \Delta_{tr}(\delta)]/\pi^2 + v_p^2} - v_p}{2}\Theta[h - \Delta_{tr}(\delta)]. \tag{4}$$

In particular, for $0 < h - \Delta_{tr}(\delta) \ll v_p$ we obtain from Eq. (4) $m_h = \chi_{cr}[h - \Delta_{tr}(\delta)] + O\{[h - \Delta_{tr}(\delta)]^2/v_p^2\}$, where $\chi_{cr} = 1/(\pi^2 v_p) \sim 1/\delta$ is the ground-state critical magnetic susceptibility. Note that for $\delta = 0$ ($v_p = 0$) we find that χ_{cr} diverges and we recover from Eq. (4) the conventional square-root behavior $m_h \sim \sqrt{h - \Delta_{tr}}$.

B. Leading low-temperature properties

Even though at zero temperature, for $h \leqslant \Delta_{tr}(\delta)$, the ground-state density of ψ particles is zero, temperature fluctuations create spin excitations from different sources. For very low temperatures, only gapless excitation sources, which stem from the hole pairs, will be important. Hence thermally induced magnetization, for $h \leqslant \Delta_{tr}(\delta)$, is

$$m_h(T) = \delta/2 - \bar{n}_p(T), \tag{5}$$

where $\bar{n}_p(T)$ is the mean density of hole pairs at temperature T. As already mentioned, hole pairs behave as hard-core bosons, at least in the lightly doped case. Similarly, ψ particles, in the case when they are bosonic, behave as hard-core bosons at the onset of magnetization due to $g_\psi > 0$ and the inherent strong-coupling nature of the critical point. In one dimension, the thermodynamic properties of hard-core bosons are identical to those of fermions; hence we can use Fermi-Dirac statistics to describe the finite-temperature distribution of both types of particles: hole pairs and ψ particles. We can rewrite Eq. (5) with the help of the Lagrange multiplier chemical potential $\mu_h(T)$, which is a solution of the following equation:

$$\int_0^\infty \frac{dk/\pi}{e^{\frac{E_\psi(k) - h + \Delta_{tr}(\delta) - \mu_h(T)}{k_B T}} + 1} + \int_0^\infty \frac{dk/\pi}{e^{\frac{E_p(k) - \mu_h(T)}{k_B T}} + 1} = \frac{\delta}{2}. \tag{6}$$

The solution for $\mu_h(T)$ from Eq. (6) at $T \to 0$ is given in terms of the Lambert W function,

$$\frac{\mu_h(T)}{k_B T} = -W\left(\frac{v_p e^{\frac{h - \Delta_{tr}(\delta)}{k_B T}}}{2\sqrt{k_B T/\pi}}\right). \tag{7}$$

Once $\mu_h(T)$ is known, within the grand-canonical formalism, we easily obtain all interesting thermodynamic quantities of the system.

At critical field $h = \Delta_{tr}(\delta)$, $\mu_h(T)$ picks up logarithmic dependence on temperature, which follows from Eq. (7), a dependence that extends to various thermodynamic quantities. In particular, the leading low-temperature behavior of the critical specific-heat coefficient is

$$\gamma_{cr}(T) = C_{cr}(T)/T \sim \ln^2 T/v_p, \tag{8}$$

and the leading low-temperature dependence of the critical magnetic susceptibility is

$$\chi_{cr}(T) - \chi_{cr} \sim 1/\ln T. \quad (9)$$

Hence, the critical properties of doped spin ladders in the magnetic field that is equal to the singlet-triplet spin gap differ drastically from those of the undoped case.

IV. SUMMARY

In summary, we studied magnetic-field-induced critical singularities in the superconducting phase of the hole-doped repulsive Hubbard model of electrons, defined on a two-leg ladder. The crucial role is played by the occurrence of a new type of low-energy spin excitations with doping which are composite in nature, carrying both spin and charge quantum numbers. When these objects start to populate the ground state, at the corresponding critical value of the magnetic field, the magnetic susceptibility stays finite at $T = 0$. The low-temperature critical magnetic susceptibility and the specific-heat coefficient show logarithmic singularities, as opposed to the celebrated square-root singularity of undoped ladders.

The magnetization curve (and thermodynamic properties) of the undoped ladders of organic compounds, with a relatively small spin gap, has been measured in experiments [33]. Similar studies have not been reported for the doped ladders because of the difficulties of doping the organic materials. In a doped nonorganic ladder, in the superconducting phase, a spin gap of 80 K has been reported [14], compared to the spin gap of the undoped system, which is several hundreds of kelvins. This gap is still too large to be suppressed by the constant magnetic field available at present. However, progress in achieving high magnetic fields and/or the possibility (as suggested by theory) of further reducing the spin gap of doped ladders by adjusting doping make experimental access of the character of the magnetic-field-induced quantum critical point in superconducting ladders feasible.

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[1] E. Dagotto, J. Riera, and D. Scalapino, Phys. Rev. B **45**, 5744(R) (1992).
[2] M. Sigrist, T. M. Rice, and F. C. Zhang, Phys. Rev. B **49**, 12058 (1994).
[3] D. V. Khveshchenko and T. M. Rice, Phys. Rev. B **50**, 252 (1994); D. V. Khveshchenko, *ibid.* **50**, 380 (1994).
[4] R. M. Noack, S. R. White, and D. J. Scalapino, Phys. Rev. Lett. **73**, 882 (1994).
[5] C. A. Hayward, D. Poilblanc, R. M. Noack, D. J. Scalapino, and W. Hanke, Phys. Rev. Lett. **75**, 926 (1995).
[6] E. Dagotto and T. M. Rice, Science **271**, 618 (1996).
[7] L. Balents and M. P. A. Fisher, Phys. Rev. B **53**, 12133 (1996).
[8] D. Poilblanc, D. J. Scalapino, and S. Capponi, Phys. Rev. Lett. **91**, 137203 (2003).
[9] G. Roux, S. R. White, S. Capponi, and D. Poilblanc, Phys. Rev. Lett. **97**, 087207 (2006); G. Roux, E. Orignac, P. Pujol, and D. Poilblanc, Phys. Rev. B **75**, 245119 (2007).
[10] M. Dolfi, B. Bauer, S. Keller, and M. Troyer, Phys. Rev. B **92**, 195139 (2015).
[11] C. C. Tsuei and J. R. Kirtley, Rev. Mod. Phys. **72**, 969 (2000); C. C. Tsuei, J. R. Kirtley, G. Hammerl, J. Mannhart, H. Raffy, and Z. Z. Li, Phys. Rev. Lett. **93**, 187004 (2004).
[12] P. W. Anderson, Science **235**, 1196 (1987).
[13] M. Uehara, T. Nagata, J. Akimitsu, H. Takahashi, N. Mori, and K. Kinoshita, J. Phys. Soc. Jpn. **65**, 2764 (1996); T. Nagata, M. Uehara, J. Goto, J. Akimitsu, N. Motoyama, H. Eisaki, S. Uchida, H. Takahashi, T. Nakanishi, and N. Mori, Phys. Rev. Lett. **81**, 1090 (1998).
[14] Y. Piskunov, D. Jérome, P. Auban-Senzier, P. Wzietek, C. Bourbonnais, U. Ammerhal, G. Dhalenne, and A. Revcolevschi, Eur. Phys. J. B **24**, 443 (2001).
[15] M. Troyer, H. Tsunetsugu, and T. M. Rice, Phys. Rev. B **53**, 251 (1996).
[16] H.-H. Lin, L. Balents, and M. P. A. Fisher, Phys. Rev. B **58**, 1794 (1998).
[17] One can view the magnon-hole-pair bound state sketched in Fig. 1(b) as a bound state of two spinon-holons depicted in Fig. 1(a). Thus the difference between the two scenarios is whether the low-energy spin excitation is carried by two-spinon-holon scattering or the bound state.
[18] T. Siller, M. Troyer, T. M. Rice, and S. R. White, Phys. Rev. B **63**, 195106 (2001).
[19] G. I. Japaridze and A. A. Nersesyan, JETP Lett. **27**, 334 (1978); V. L. Pokrovsky and A. L. Talapov, Phys. Rev. Lett. **42**, 65 (1979); H.-J. Schulz, Phys. Rev. B **22**, 5274 (1980).
[20] S. Sachdev, T. Senthil, and R. Shankar, Phys. Rev. B **50**, 258 (1994); S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1998).
[21] T. B. Bahder and F. Woynarovich, Phys. Rev. B **33**, 2114 (1986); F. Woynarovich, *ibid.* **43**, 11448 (1991); F. Woynarovich and K. Penc, Z. Phys. B **85**, 269 (1991).
[22] T. Vekua, S. I. Matveenko, and G. V. Shlyapnikov, JETP Lett. **90**, 289 (2009).
[23] T. Vekua, Phys. Rev. B **89**, 121112(R) (2014); **80**, 201402(R) (2009).
[24] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, 1999).
[25] A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).
[26] K. B. Efetov and A. I. Larkin, Zh. Eksp. Theor. Fiz. **69**, 764 (1975) [Sov. Phys. JETP **42**, 390 (1976)].
[27] From the site-resolved magnetization (and density) profile [9] in lightly doped ladders it is clear that when the magnetic field slightly exceeds the critical value, magnon-hole pairs repel each other, as they try to be distributed as far as possible from each other.

[28] U. Ledermann and K. Le Hur, Phys. Rev. B **61**, 2497 (2000); L. Balents, *ibid.* **61**, 4429 (2000).

[29] J. S. Meyer, K. A. Matveev, and A. I. Larkin, Phys. Rev. Lett. **98**, 126404 (2007); M. Sitte, A. Rosch, J. S. Meyer, K. A. Matveev, and M. Garst, *ibid.* **102**, 176404 (2009); G. Sun and T. Vekua, Phys. Rev. B **93**, 205137 (2016).

[30] Within our model such processes would induce terms like $\psi^\dagger\psi_p + \text{H.c.}$ and $\psi^\dagger\psi^\dagger\psi_p\psi_p + \text{H.c.}$, where ψ_p is the hole-pair annihilation operator.

[31] Since in one dimension the spectrum of hard-core bosons is identical to that of free fermions, for hole pairs we adopt the low-energy dispersion of free fermions, in particular to be consistent when describing thermodynamic properties of hole pairs using the Fermi-Dirac distribution.

[32] Note that even in the picture where hole pairs get dressed by magnons, we still have to assume formally that hole-pair density drops for $h > h_c$ to avoid double counting, as ψ particles include in them hole pairs (bound to magnons).

[33] G. Chaboussant, P. A. Crowell, L. P. Lévy, O. Piovesana, A. Madouri, and D. Mailly, Phys. Rev. B **55**, 3046 (1997); M. Ikeda, Z. Honda, M. Sakai, S. Kimura, K. Okunishi, and M. Hagiwara, J. Low Temp. Phys. **170**, 296 (2013).