Universal entanglement spectra of gapped one-dimensional field theories

Gil Young Cho,[1,2] Andreas W. W. Ludwig,[3] and Shinsei Ryu[1]

[1]*Department of Physics and Institute for Condensed Matter Theory, University of Illinois, 1110 W. Green St., Urbana, Illinois 61801-3080, USA*
[2]*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea*
[3]*Department of Physics, University of California, Santa Barbara, California 93106, USA*

We discuss the entanglement spectrum of the ground state of a (1+1)-dimensional system in a *gapped* phase near a quantum phase transition. In particular, in proximity to a quantum phase transition described by a conformal field theory (CFT), the system is represented by a gapped Lorentz invariant field theory in the "scaling limit" (correlation length ξ much larger than microscopic "lattice" scale "a"), and can be thought of as a CFT perturbed by a relevant perturbation. We show that for such (1+1) gapped Lorentz invariant field theories in infinite space, the low-lying entanglement spectrum obtained by tracing out, say, left half-infinite space, is precisely equal to the physical spectrum of the unperturbed gapless, i.e., conformal field theory defined on a finite interval of length $L_\xi = \ln(\xi/a)$ with certain boundary conditions. In particular, the low-lying entanglement spectrum of the gapped theory is the finite-size spectrum of a boundary conformal field theory, and is always discrete and universal. Each relevant perturbation, and thus each gapped phase in proximity to the quantum phase transition, maps into a particular boundary condition. A similar property has been known to hold for Baxter's corner transfer matrices in a very special class of fine-tuned, namely, integrable off-critical lattice models, for the entire entanglement spectrum and independent of the scaling limit. In contrast, our result applies to completely general gapped Lorentz invariant theories in the scaling limit, without the requirement of integrability, for the low-lying entanglement spectrum. While the entanglement spectrum of the ground state of a gapped theory on a finite interval of length $2R$ with suitable boundary conditions, bipartitioned into two equal pieces, turns out to exhibit a crossover between the finite-size spectra of the same CFT with in general different boundary conditions as the system size R crosses the correlation length from the "critical regime" $R \ll \xi$ to the "gapped regime" $R \gg \xi$, the physical spectrum on a finite interval of length R with the same boundary conditions, on the other hand, is known to undergo a dramatic reorganization during the same crossover from being discrete to being continuous.

I. INTRODUCTION AND SUMMARY OF RESULTS

Considerations of quantum entanglement have provided a great deal of insight into the nature of ground [1] (and excited [2]) states of Hamiltonians of complex physical quantum systems. While the entanglement entropy is a very useful diagnostic of a quantum state, a vastly larger amount of information is contained in the spectrum of the reduced density matrix, i.e., in the spectrum of the entanglement Hamiltonian [Eq. (1) below]. For example, a particularly useful case in point is the observation that the entanglement Hamiltonian of a (2+1)-dimensional integer [3] as well fractional [4–8] quantum Hall state carries a universal fingerprint of an underlying topological phase. Indeed, this has recently become an important tool for identifying the nature of phases of a variety of microscopic gapped Hamiltonians by computing the entanglement spectrum numerically.[1]

Here we consider the entanglement spectrum of the ground state of a (1+1)-dimensional system in a gapped phase near a quantum phase transition. In particular, we consider phases in proximity to a continuous quantum phase transition with dynamical critical exponent $z = 1$, which is generally[2] described by a conformal field theory (CFT). The system near such a transition is thus represented by a gapped Lorentz

invariant field theory in the scaling limit (correlation length ξ much larger than microscopic lattice scale "a"), and can be thought of as a CFT perturbed by one or more relevant perturbations. We consider here primarily the case of a single relevant perturbation, described by a field ϕ.

In the present paper, we discuss the ground state of such a gapped (1+1)-dimensional Lorentz invariant field theory in infinite space. It is well known [9] that the entanglement Hamiltonian for the ground state of such a gapped theory, obtained by tracing out, say, left half-infinite space, is completely local (being the generator of Lorentz boosts). In this paper we will show that in general, the low-lying entanglement spectrum of such a gapped theory is the spectrum of the underlying unperturbed gapless, i.e., conformal theory on a finite interval of length $L_\xi = \ln(\xi/a)$ when $\xi/a \gg 1$, with two boundary conditions: a "free" boundary condition "F" (where the system simply ends) at the left end of the interval corresponding to the entanglement cut, and a "hard-wall" boundary condition, which we denote by "B_ϕ" at the other, right end of the interval, which corresponds to an interface of the CFT with a strongly gapped phase described by a region of the same theory where the strength of the relevant perturbation "ϕ" is by some measure large ("infinite"). We emphasize that the latter (i.e., right) boundary condition B_ϕ thus depends, as indicated, on the particular relevant perturbation "ϕ", and thus on the particular gapped phase in proximity of the transition. The entanglement Hamiltonian of the gapped theory is thus the Hamiltonian of a boundary conformal field theory (BCFT) [10], with

---

[1]See, e.g., Ref. [34–37].
[2]Under very mild assumptions [38].

these boundary conditions. (We note in passing that since the entanglement Hamiltonian is also known to be equal to the generator of Lorentz boosts [9,11,12] of the gapped relativistic theory as well as to the Hamiltonian of the same theory subject to a uniform acceleration [13] (i.e., in Rindler space-time), both the boost operator as well as the Hamiltonian in Rindler space-time also possess this boundary CFT spectrum.)

More explicitly, we show that the entanglement Hamiltonian $\hat{H}_E$ defined through the reduced density matrix in half-infinite space, region $A = \mathbb{R}_+ = (0, +\infty)$,

$$\hat{\rho}_A = \frac{1}{\mathcal{N}} \exp(-2\pi \hat{H}_E), \tag{1}$$

is of the form

$$\hat{H}_E = \frac{\pi}{L}\left(\hat{L}_0 - \frac{c}{24}\right) \tag{2}$$

with

$$L = L_\xi \equiv \ln(\xi/a). \tag{3}$$

Here, $\hat{L}_0$ is the chiral (say left-moving) Virasoro generator [see Eq. (5) below for a more explicit description] and c denotes the central charge of the unperturbed theory. The normalization factor of the reduced density matrix reads

$$\mathcal{N} = \mathrm{Tr} \, \exp(-2\pi \hat{H}_E) = \exp\left(-\frac{c}{6}L - \gamma + \dots\right), \tag{4}$$

where γ is a constant [14,15]. (The ellipsis indicates terms subleading for large L.) Equation (2) implies that the spectrum of eigenvalues ϵ of the entanglement Hamiltonian $\hat{H}_E$ is of the form

$$\epsilon - \epsilon_0 = \frac{\pi}{L}(h + n), \tag{5}$$

where ϵ_0 denotes the smallest eigenvalue. Here h runs over a subset of possible conformal weights [16] (left-moving scaling dimensions) of the CFT, which is completely determined [10] by the pair of boundary conditions "F" and "B_ϕ" at the two ends of the interval of length L, and n are non-negative integers corresponding to what are known as (conformal) descendants [16]. This is the spectrum of a boundary conformal field theory (e.g., the degeneracies of all levels are known explicitly).

It is only the low-lying entanglement spectrum, describing the largest contributions in the Schmidt decomposition of the reduced density matrix, that is in general universal and described by the spectrum of the BCFT discussed above. The higher-lying spectrum depends in general on nonuniversal details. At the end of Sec. II, we provide a rough estimate of the excitation energy $(\epsilon^* - \epsilon_0)$ beyond which the conformal spectrum given in (5) is expected to be no longer applicable, which is found to be roughly $(\epsilon^* - \epsilon_0) \approx 2\pi y$. Here, $y > 0$ is the renormalization group (RG) eigenvalue of the relevant perturbation ϕ, a number of order unity. Since in view of (5) the level spacing of the low-lying spectrum is π/L, the number of levels belonging to the low-lying part of the spectrum increases with L. In Sec. V, we present numerical results for the entanglement spectrum of a system of gapped noninteracting fermions, illustrating our general analytical results. We also note that a reasonably large number of low-lying levels of the entanglement Hamiltonian is within the range of today's numerical tools even for fully interacting systems as seen,

e.g., from the numerical entanglement spectra obtained for interacting gapless (conformal) field theories in Ref. [17].[3]

A similar property as that derived in the present paper for the low-lying entanglement spectrum of a general gapped (1+1)-dimensional relativistic field theory in the vicinity of the CFT, has been known to hold (for many years) for an extremely special and fine-tuned class of theories, namely for gapped "Yang-Baxter" integrable lattice models of 2D classical statistical mechanics. Specifically, in these systems Baxter's so-called corner transfer matrix [18] (CTM) can be viewed as a lattice analog of the reduced density matrix in half space, $\hat{\rho}_A$ from Eq. (1), when suitably translated into entanglement language.[4] The surprising observation [19,20] was then made for a vast number such integrable lattice systems (see, e.g., Refs. [21,22]), that the entire spectrum of (minus) the logarithm of the CTM, which turns out to play a role analogous [9] to the entanglement Hamiltonian $\hat{H}_E$ in Eq. (1) of the field theory, equals the spectrum of a (gapless) CFT in finite size L, with the exact replacement $L \to \ln(\xi/a)$, where a and ξ are the lattice spacing and the correlation length, respectively, of the integrable lattice model.[5] Due to the fine-tuning arising from integrability this turns out to hold for all eigenvalues of (minus the logarithm of) the CTM, and moreover holds true for all, even small values of ξ/a, not only in the scaling limit.[6] The methods that have been used to demonstrate this fact for these integrable systems rely on the very special properties of integrable lattice models, such as the Yang-Baxter equation. Clearly, there is no reason for such a miraculous property to hold without the strong fine-tuning provided by the infinite number of conservation laws present in these integrable systems. However, what we show in the present paper is that in the scaling limit the low-lying entanglement spectrum is generically equal to that of the underlying gapless theory in finite size, and that this is a property completely independent of the requirement

---

[3]Recently, numerical results for entanglement spectra of a different set of (1+1)-dimensional systems, albeit for somewhat smaller system sizes than in Ref. [17], came to our attention, see Ref. [39].

[4]Specifically, in the scaling limit of the integrable lattice model (which depends on a single parameter that can be adjusted so that the system is at criticality) where the correlation length ξ of the lattice model becomes much larger than the lattice spacing a and where the lattice model turns out to be represented by a very special, namely, integrable relativistic quantum field theory, the CTM $\hat{\rho}_{\mathrm{CTM}}$ becomes equal to $[\hat{\rho}_A]^{1/4} = (1/\mathcal{N})^{1/4} \exp[-(\pi/2)\hat{H}_E]$, defined in Eq. (1). The logarithm of the CTM generalizes the entanglement Hamiltonian (times $\pi/2$) to classical 2D statistical mechanics systems defined on a lattice, i.e., where Lorentz invariance is absent.

[5]A relation between the CTM of certain integrable lattice models and the continuum Lorentz-boost operator was also discussed from a different perspective in Ref. [40].

[6]The corner transfer matrix was used by Baxter [18] to provide, e.g., an explicit, relatively simple and compact expression for the local zero-field magnetization in the 2D Ising model for all temperatures below the critical temperature. The reason why such a relatively simple and compact expression appears originates [19,20] from this relationship of the corner transfer matrix (Baxter's tool to computer the expectation value) with a CFT in finite size.

of integrability. The identity of these two spectra is thus not a property of the very restricted and special class of integrable systems, but is a completely general property of the entanglement Hamiltonian of gapped (1+1)-dimensional relativistic field theories.

In general, the high-lying excitation spectrum of the entanglement Hamiltonian contains no robust information because it is completely governed by details of the theory on distance scales comparable to the microscopic length a, which vary from case to case. On the other hand, an integrable system is known to be very special in this regard, in that even the short distance properties are completely fixed by the infinite number of conservation laws. One way of expressing this fact is to think of the integrable theory as a fixed point of the renormalization group (RG), here a CFT, perturbed by an infinite sum of terms that are ever more irrelevant (in the RG sense), with coefficients that are completely fixed by integrability. This notion has been implemented in practice in the work of Ref. [23]. At the end of Sec. II, and in particular in Appendix, we suggest that by thinking this way one may view the known results for the CTM of the integrable systems within the context given in the present paper.

Another related focus of attention in the existing literature on the entanglement spectrum of gapped (1+1)-dimensional theories has been the distribution of eigenvalues of the entanglement Hamiltonian in the regime where the eigenvalues become dense so that the distribution is described by a continuous curve. Reference [24] numerically observed a universal form of the distribution of entanglement eigenvalues. Later, it was argued in Ref. [25] that this distribution has a universal form characterized only by the central charge. This was supported by numerical work in that same paper as well as in Ref. [26]. While these interesting results are related to the discussion in the present paper, they do not focus on the resolution of the detailed structure of the entanglement spectrum on the scale of the individual levels, including their degeneracies (which are nontrivial), discussed in the present paper. All this detailed structure on the scale of the level spacing represents a rich amount of universal information contained in the entanglement spectrum of the gapped relativistic field theory.

II. DERIVATION OF THE ENTANGLEMENT SPECTRUM OF THE GAPPED FIELD THEORY OF HALF-SPACE

We now proceed to provide an explicit derivation of the entanglement Hamiltonian. We write the spatial coordinate denoted by x and the imaginary (Euclidean) time coordinate denoted by y in terms of $z = x + iy$ and $\bar{z} = x - iy$.[7] We perform a conformal transformation to a new spatial coordinate u and a new imaginary (Euclidean) time coordinate v, via the conformal transformation $z \to w(z)$, where $w = u + iv$ is given by[8]

$$z = (x + iy) = \exp(w) = \exp(u + iv), \qquad (6)$$

_____

[7] A characteristic velocity is set to unity for convenience throughout this paper.
[8] The complex conjugate relationship holds between $\bar{z}$ and $\bar{w} = u - iv$.

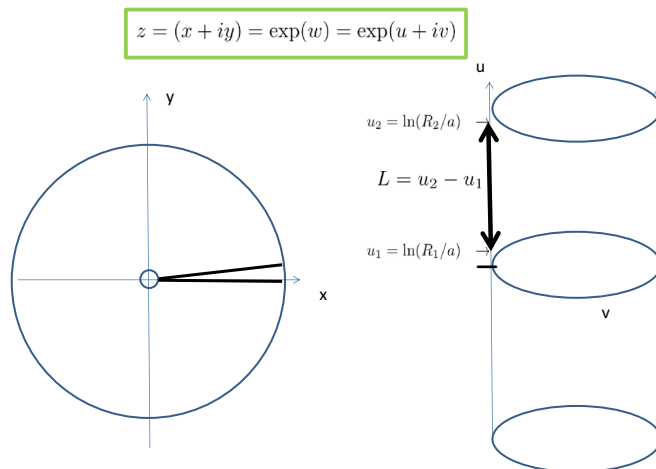$z = (x + iy) = \exp(w) = \exp(u + iv)$

FIG. 1. Conformal map.

mapping the complex z plane into a cylinder, see Fig. 1.

As it is well known, there are two equivalent ways of thinking about this transformation: (i) as angular quantization where the angular variable v is treated as the imaginary (Euclidean) time variable or (ii) as the study of the quantum field theory in Rindler space-time which describes the original quantum field theory subject to a constant acceleration (here set to unity in suitable units).

Consider the annulus $R_1/a < |z| < R_2/a$, in the complex z plane (where a is a short distance scale), which is mapped (see Fig. 1) under the conformal transformation (6) into a piece $u_1 < u < u_2$ of a cylinder (the coordinate v is periodic with period 2π) of length

$$L = (u_2 - u_1) = \ln(R_2/R_1), \qquad (7)$$

where

$$R_1/a = \exp(u_1), \quad R_2/a = \exp(u_2). \qquad (8)$$

Now consider, as discussed in Introduction, the imaginary (Euclidean) time action of a CFT in the (x, y) coordinate system, perturbed by a primary [16] field $\phi(z, \bar{z})$ of conformal weight $(h, \bar{h})$, which is relevant in the RG sense,

$$S_{z,\bar{z}} = S_* + g \int d^2 z\, \phi(z, \bar{z}), \quad \text{where } \bar{h} = h < 1. \quad (9)$$

Here, S_* denotes the action of the CFT itself. $S_{z,\bar{z}}$ in Eq. (9) defines the gapped relativistic field theory in infinite space, described by coordinates (x, y) or $(z, \bar{z})$. In order to obtain its entanglement Hamiltonian in half-space, we need to express this action in (u, v), or $(w, \bar{w})$ coordinates, describing angular quantization or, equivalently, Rindler space-time coordinates. To this end, we use the transformation properties of the primary field $\phi(z, \bar{z})$, which transforms [16] in the new coordinates to the new field $\Phi(w, \bar{w})$ defined by

$$\phi(z, \bar{z}) = \Phi(w, \bar{w}) \left(\frac{dz}{dw} \right)^{-h} \left(\frac{d\bar{z}}{d\bar{w}} \right)^{-\bar{h}}. \qquad (10)$$

Using the explicit form (6) of the map, we obtain

$$\left(\frac{dz}{dw} \right)\left(\frac{d\bar{z}}{d\bar{w}} \right) = \exp(w + \bar{w}) = \exp(2u),$$

which leads to the following form of the action from (9) when expressed in the $(w, \bar{w})$ coordinates:

$$
\begin{aligned}
S_{w, \bar{w}} &= S_* + \delta S \\
&= S_* + g \int_{u_1}^{\infty} du \int_0^{2\pi} dv \; e^{yu} \; \Phi(w, \bar{w}) \\
&= S_* + \int_{u_1}^{\infty} du \int_0^{2\pi} dv \; e^{y[u - \ln(\kappa \xi/a)]} \; \Phi(w, \bar{w}), \quad (11)
\end{aligned}
$$

where we made use of the invariance of S_* under conformal transformations. Here, $y = 2(1 - h) > 0$ is the renormalization-tion group eigenvalue of the relevant coupling constant g, inducing[9] a finite (dimensionless) correlation length $\xi/a = \kappa^{-1} g^{-1/y}$ (where κ is a nonuniversal dimensionless constant), which in turn was used to write the coupling constant in the form $g = (\kappa \frac{\xi}{a})^{-y} = e^{-y \ln(\kappa \xi/a)}$. We have considered the limit $R_2/a \to \infty$ [implying $u_2 \to \infty$, due to Eq. (8)], and u_1 was defined in Eq. (8).

Note that the second term δS in (11) arises from the presence of the relevant perturbation in (9), which leads to the lack of invariance of the total action S under the conformal transformation. In the $(w, \bar{w})$ coordinates, the term δS describes a "potential" which grows exponentially with the spatial coordinate u and describes an interface between the gapless theory ($g = 0$), and a gapped theory in which the coupling g is not small, and the dimensionless correlation length ξ/a is not large. The term δS therefore confines the theory to a finite spatial interval,

$$
u_1 < u < L, \qquad L = L_\xi = \ln(\xi/a). \quad (12)
$$

We thus see from Eqs. (11) and (12) that the action $S_{w, \bar{w}}$ in the (u, v) coordinates of Fig. 1 describes the gapless theory but now on a space of finite size $L = L_\xi$, with certain boundary conditions imposed at the two ends which will be discussed below. [Since the imaginary (Euclidean) time coordinate v is periodic with period 2π, this action describes the gapless theory at inverse temperature $\beta = 2\pi$.] Therefore the Hamiltonian of the theory in the (u, v) coordinates, which by construction is precisely the entanglement Hamiltonian $\hat{H}_E$, is the Hamiltonian of the underlying gapless theory, i.e., of the theory where the relevant perturbation is switched off, $g \equiv 0$, but on the finite interval (12) of length $L = L_\xi$. The boundary condition on the right end $u = L = L_\xi$ of the interval corresponds, as seen from (11), to an interface between the gapless theory (where $g = 0$) and the fully gapped theory emerging when g is not small. This interface is sharp[10] when L is large. As mentioned in Sec. I, since the corresponding gapped theory, appearing when g does not vanish, clearly depends on the relevant perturbation ϕ, so does the resulting boundary condition, denoted by B_ϕ. On the other hand, the boundary condition at the entanglement cut on the left side $u = u_1$ of the interval is independent of

the relevant perturbation ϕ, and it is typically[11] just a free boundary condition (where the system "simply ends").

In summary, we have shown that the low-lying spectrum of the entanglement Hamiltonian $\hat{H}_E$ of the gapped relativistic field theory is simply the finite size spectrum of the corresponding gapless (conformal) theory with boundary conditions "F" and "B_ϕ" discussed above. This is the spectrum of the corresponding boundary conformal field theory, as displayed in (2), (3), and (5). The corresponding eigenstates in the u coordinates are localized within the finite range specified in (12), corresponding in the original x coordinates, upon using (6), as expected to a finite region around the entanglement cut (at $x = 0$) of spatial extent of the order of the correlation length ξ. The limitation to the low-lying entanglement spectrum arises from the replacement of the exponentially increasing potential in (11) by a boundary condition representing a sharp interface. This replacement is certainly asymptotically valid for the low-energy, long-wavelength part of the spectrum when $L = \ln(\xi/a)$ is large. More precisely, one expects this replacement to stop being valid for eigenstates of the entanglement Hamiltonian varying on wavelengths of order $1/y$, the scale on which the potential rises exponentially.[12] This replacement is therefore expected to certainly cease to be valid for wave vectors $k_n = n(\pi/L)$ with integer n where $n \gtrsim 2Ly$, which roughly corresponds, using (5), to excitation energies $\epsilon - \epsilon_0 \gtrsim \epsilon_* - \epsilon_0 = 2\pi y$. Since the level spacing is π/L, the number of energy levels belonging to the so-defined low-lying spectrum increases with L, and can in practice be large in numerical work (see, e.g., Ref. [17], which we mentioned already before; note this reference chose to address only the entanglement spectrum of gapless theories). A brief comment on how one may view, within the context of the present paper, the result observed for the logarithm of Baxter's corner transfer matrix in gapped integrable lattice models, which is known to reproduce exactly the entire spectrum of the boundary conformal field theory (i.e., $\epsilon_* \to \infty$ in the above equation), is provided in Appendix.

III. CROSSOVER OF ENTANGLEMENT SPECTRUM OF HALF-SPACE

It is illuminating to compare the finite-size spectra of the entanglement and the physical Hamiltonian.

---

[9]See, e.g., Ref. [41].

[10]The "potential" in (11) rises by a factor e when u increases by $1/y$ (a number of order unity), which is steep as compared to the length L of the interval, when the latter is large.

[11]The ground state typically does not possess any constraints between the degrees of freedom immediately on the left (B) and the right (A) of the entanglement cut. Therefore, upon employing the Schmidt decomposition of the ground state for a bipartition $A \bigcup B$ of space and performing the trace over, say, part B, there is no constraint on the leftmost degree of freedom of part A; this therefore implies a "free" boundary condition. This is also borne out in recent numerical work on the entanglement spectrum of gapless theories, see Ref. [17]. The boundary condition at the entanglement cut can be modified if the ground state contains a specific constraint on the above-discussed degrees of freedom adjacent to the entanglement cut.

[12]One can check such estimates using the exactly solvable example of a quantum mechanical particle in an exponential potential, known from quantum Liouville theory [42].

A. Finite-size crossover of entanglement spectrum

Let us first define the finite-size entanglement spectrum for open boundary conditions (also often used in numerical work). Specifically, we consider the gapped theory on a finite interval $-R < x < +R$, choosing some, for simplicity identical boundary conditions B_0 at the two ends which we assume here to yield a unique ground state on the interval. When tracing over the negative half $-R < x < 0$ of the interval, we obtain the density matrix whose spectrum we are interested in.

In the critical regime where the correlation length is much larger than the entire interval, $R \ll \xi$, the entanglement spectrum is that of the gapless theory ($g = 0$), which is known [27,28] to be of the form of Eq. (5), where $L = L_R = \ln(R/a)$.[13] This is the spectrum of the CFT ($g = 0$) on an interval of length $L = L_R$ with typically[14] a free boundary condition "F" on the left end (arising from the entanglement cut), and the same boundary condition "B_0" that was imposed in physical space on the right end. As we increase R, the level spacing of the entanglement spectrum initially decreases as $\pi/\ln(R/a)$, and ultimately saturates at $\pi/\ln(\xi/a)$ when we reach the gapped regime:

In the gapped regime where the correlation length is much smaller than the length of the interval, $\xi \ll R$, the entanglement spectrum is precisely the one studied in Sec. II. This is the spectrum of the same CFT ($g = 0$) on an interval of length $L = L_\xi = \ln(\xi/a)$, with again typically a free boundary condition F on the left end (arising from the entanglement cut), and the boundary condition B_ϕ arising from the relevant perturbation (discussed in Sec. II) imposed on the right end.

We thus see that upon increasing the size of the interval from $R \ll \xi$ to $\xi \ll R$, the entanglement spectrum evolves from the spectrum of the CFT on an interval of length $L = L_R = \ln(R/a)$ and boundary conditions (F, B_0), to the spectrum of the same CFT on an interval of length $L = L_\xi = \ln(\xi/a)$ and boundary conditions (F, B_ϕ). This describes the evolution of a boundary renormalization group (RG) flow, while the bulk theory describing the entanglement Hamiltonian always remains gapless. A simple example is provided by the transverse field quantum Ising model perturbed by a bulk magnetic field described by the operator $\phi = \sigma$ (spin field) and a free Ising-spin boundary condition B_0. Upon the above-described crossover of the entanglement spectrum, the bulk magnetic field induces a boundary magnetic field at the free spin boundary condition B_0 that flows under the RG to the new fixed-spin boundary condition B_ϕ. In Sec. V A 1, we discuss an example where the boundary conditions B_0 and B_ϕ are in fact the same, and only the level spacing changes upon the crossover. This is confirmed numerically in Fig. 3.

B. Finite-size crossover of physical spectrum

In contrast to the boundary RG flow of the entanglement spectrum discussed above, the physical spectrum is known to evolve completely differently under the analogous crossover.

In particular, consider the physical spectrum of the gapped theory defined on a finite interval of length R. In order to be able to make a direct comparison, we choose the same boundary conditions as those for the entanglement spectrum, namely a free boundary condition F on the left end and the boundary condition B_0 on the right end of the interval.

In the critical limit where the correlation length is much larger than the entire interval, $R \ll \xi$, the physical spectrum is identical to that of entanglement spectrum, Eq. (5); namely, it is the spectrum of the CFT with the same boundary conditions (F, B_0), upon making the replacement $L_R = \ln(R/a) \to R$. Upon crossover to the corresponding gapped limit where the correlation length is much smaller than the system size, $\xi \ll R$, however, the physical spectrum of the gapped theory undergoes a dramatic, highly nontrivial reorganization from the boundary CFT spectrum with a finite level spacing to the continuous spectrum in infinite space describing continuous single- and multi-particle states of the gapped field theory. In cases where the relevant perturbation of the CFT defining the gapped theory is integrable, this reorganization of the physical finite-size spectrum has been extensively studied in great detail[15] by means of the so-called truncated space conformal field theory approach and the thermodynamic Bethe ansatz.[16] In summary, the finite-size entanglement spectrum and the finite-size physical spectrum exhibit entirely different behavior upon crossover from the critical to the gapped regime.

IV. ENTANGLEMENT SPECTRUM OF A FINITE INTERVAL

For the same gapped field theory, we now discuss the entanglement spectrum of a spatial interval $A = (-R + a, +R - a)$ where $0 < a \ll R$ (a is again a short distance scale), depicted on the real axis with coordinate ζ_1 in the top panel of Fig. 2. We use the conformal map $w(\zeta) = \ln(\frac{R+\zeta}{R-\zeta})$, with inverse $\zeta(w) = R \tanh(w/2)$, to map from the complex $\zeta = (\zeta_1 + i\zeta_2)$-plane into a finite cylinder parametrized by $w = u + iv$ with $-u_R \leqslant u \leqslant +u_R$ where $u_R = \ln(2R/a)$, as also shown in Fig. 2.

As before, consider the imaginary (Euclidean) time action of a CFT in the (ζ_1, ζ_2) coordinate system, perturbed by a relevant primary [16] field $\phi(\zeta, \bar{\zeta})$ of conformal weight $(h, \bar{h})$,

$$S_{\zeta, \bar{\zeta}} = S_* + g \int d^2\zeta \, \phi(\zeta, \bar{\zeta}), \quad \text{where } \bar{h} = h < 1.$$

Using

$$\frac{d\zeta}{dw} = \frac{R}{2} \frac{1}{\cosh(w/2)} = \frac{R}{1 + \cosh(w)},$$

we obtain for the action in the w coordinates

$$S_{w, \bar{w}} = S_* + \delta S, \qquad (13)$$

---

[13]This result can be obtained by setting $g = 0$, $R_2 \to R$, $R_1 \to a$ and $L = L_R = \ln(R/a)$ in Sec. II.

[14]See the corresponding footnote in Sec. II.

[15]Level crossings seen in the integrable cases as a consequence of the additional conservation laws will typically turn into avoided crossings in the generic, nonintegrable settings.
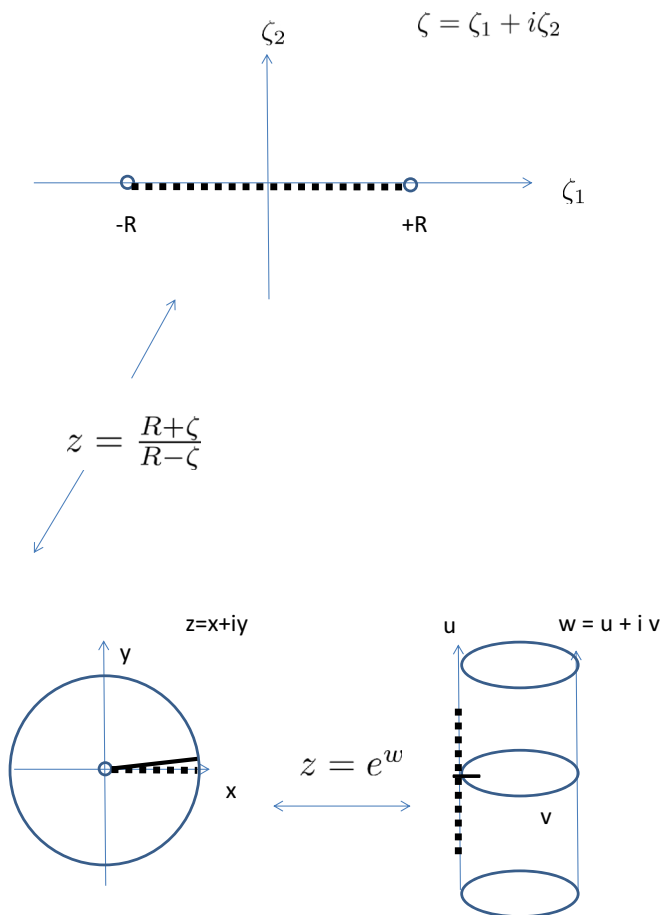
[16]See, e.g., Refs. [43,44].

$$\zeta = \zeta_1 + i\zeta_2$$

$$z = \frac{R+\zeta}{R-\zeta}$$

$$z = e^w$$

FIG. 2. Conformal map.

where

$$\delta S = g R^y \int_{-u_R}^{+u_R} du \int_0^{2\pi} dv \left| \frac{1}{1+\cosh(w)} \right|^y \Phi(w,\bar{w})$$

$$= \frac{\left(\frac{R}{a}\right)^y}{\left(\frac{\xi}{a}\right)^y} \int_{-u_R}^{+u_R} du \int_0^{2\pi} dv \left| \frac{1}{1+\cosh(w)} \right|^y \Phi(w,\bar{w})$$

$$= \int_{-u_R}^{+u_R} du \int_0^{2\pi} dv \left| \frac{e^{(u_R-L_\xi)}}{1+\cosh(w)} \right|^y \Phi(w,\bar{w}), \quad (14)$$

and $\left(\frac{R}{a}\right)^y = e^{yu_R}$.

In the critical regime $u_R \ll L_\xi$, where δS is small, we obtain the already-known [28] entanglement spectrum of the gapless theory ($g = 0$) on the finite interval.[17] In the gapped regime $u_R \gg L_\xi$, on the other hand, the "effective u-dependent coupling constant" in the last equation of (14) is never small unless $|u|$ is "close" to u_R,

$$(u_R - |u|) \ll L_\xi, \quad (\text{recall} : -u_R < u < +u_R), \quad (15)$$

in which case the expression $e^{(u_R-L_\xi)}/|1+\cosh(w)|$ appearing in Eq. (14) tends to $\to e^{-[L_\xi-(u_R-|u|)]}$, which is small when

---

[17]We see from Fig. 2 that in the critical regime the entanglement spectrum is the spectrum of the CFT on a finite interval $-u_R < u < u_R$ with boundary conditions F and B_ϕ imposed at the two ends, as in Ref. [28].

L_ξ is large. The condition in Eq. (15) for δS to be small describes two disjoint intervals, $-u_R < u \ll -(u_R - L_\xi)$ and $(u_R - L_\xi) \ll u < u_R$; these are two segments of length L_ξ ($\ll u_R$) each at the right and the left ends of the full interval $-u_R < u < u_R$ in which u is defined. Therefore the entanglement spectrum of the interval $A = (-R+a, +R-a)$ in this regime is the sum of two finite-size spectra of the corresponding gapless (conformal) theory on a space of length $L_\xi = \ln(\xi/a)$ each. The boundary condition at the ends $u = \pm u_R$ of each of these two intervals is (typically) "free" F, whereas it is B_ϕ at the other ends of the two intervals. Therefore each of these two spectra is precisely the spectrum discussed in Sec. II.

V. NUMERICAL RESULTS

In this section, we present numerical calculations of the (low-lying) entanglement spectrum of a chain of spinless fermions, which confirm the above discussion of Lorentz invariant quantum field theories. Let us consider the Su-Schrieffer-Heeger (SSH) model defined on a one-dimensional lattice:

$$H = \sum_i \Psi_i^\dagger h^0 \Psi_i + \sum_i (\Psi_{i+1}^\dagger h^x \Psi_i + \text{H.c.}), \quad (16)$$

where i labels a site on a one-dimensional chain, and Ψ is a two-component fermion annihilation operator, which includes two fermion operators c_A and c_B defined for each two-site unit cell:

$$\Psi_i = \begin{pmatrix} c_A \\ c_B \end{pmatrix}_i. \quad (17)$$

The matrix elements h^0 and h^x are given by

$$h^0 = \begin{pmatrix} \mu_s & t+\delta t \\ t+\delta t & -\mu_s \end{pmatrix}, \quad h^x = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}. \quad (18)$$

where t ("hopping"), δt ("dimerization"), and μ_s ("staggered chemical potential") are real parameters. In addition, a proper boundary condition must be specified (see below). For convenience, we will choose $t = 1$ and change δt and μ_s (which are properly normalized with respect to the unit $t = 1$). When $\mu_s = 0$, the SSH model is particle-hole symmetric, and can be thought of as a member of symmetry class D [29,30]. We will mostly set $\mu_s = 0$ in the following. Hence, when $\mu_s = 0$, the SSH model realizes two topologically distinct gapped phases which are distinguished by a $\mathbb{Z}_2$-valued topological invariant. A topologically trivial phase is realized when

$$\delta t > 0, \quad (19)$$

while a topologically nontrivial phase is realized when

$$\delta t < 0. \quad (20)$$

There is a quantum phase transition separating these phases when

$$\delta t = 0. \quad (21)$$

In the following, we compute the entanglement spectrum of these phases and at the critical point between them numerically.
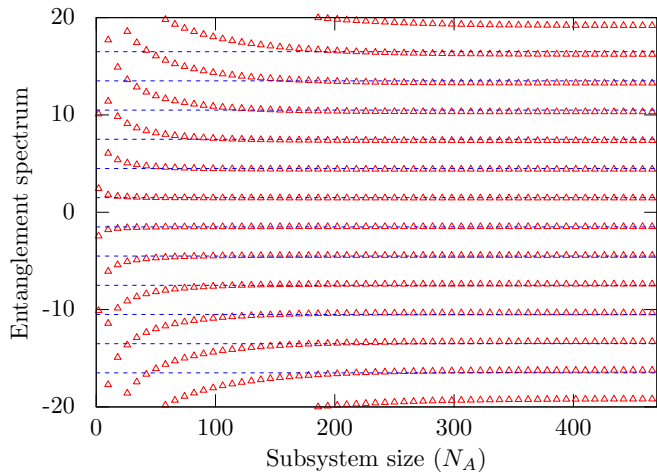
FIG. 3. The single-particle entanglement spectrum of the SSH model with OBC in its topologically trivial phase ($\delta t = 0.01$). The blue dotted lines are guide for eyes.
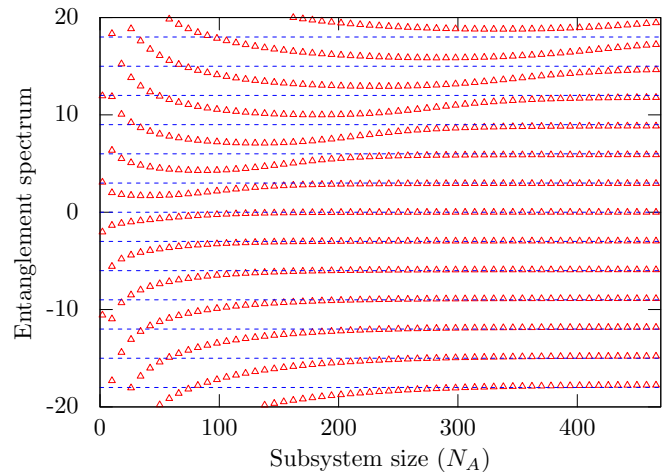

FIG. 4. The entanglement spectrum of the SSH model with OBC in its topologically nontrivial phase ($\delta t = -0.01$). The blue dotted lines are guide for eyes.

A. Open boundary conditions (OBC)

We first consider the SSH model defined on a finite lattice consisting of N unit cells, and with open boundary condition imposed on both ends. We will take $N \in 2\mathbb{Z}$ for convenience.

1. Topologically trivial phase

Let us start with the topologically trivial phase, $\delta t > 0$. In this phase, the SSH model with open boundary condition has a unique ground state, $|\Psi\rangle$. We then consider a (pure) density matrix $\rho = |\Psi\rangle\langle\Psi|$ made out of the ground state, and trace out the left $N/2$ sites to define the reduced density matrix for the remaining subsystem (subsystem A). The number of unit cells in subsystem A is denoted by $N_A(= N/2)$. The computed single-particle entanglement spectrum is presented in Fig. 3. As we make the system size (and hence the subsystem size) bigger, the spectrum approaches the prediction from BCFT. That is, the levels of the single-particle entanglement spectrum are all equally spaced, and the level spacing does not scale with N_A. This spectrum is the spectrum of the free chiral fermion conformal field theory with antiperiodic spatial boundary condition, the so-called Neveu-Schwartz (NS) spectrum.

2. Topologically nontrivial phase

Next, let us consider the nontrivial topological phase, $\delta t < 0$. In this case, the SSH model of finite length has near double-degenerate ground states when open boundary conditions (OBC) are imposed, due to near zero-energy single particle modes localized near the ends.[18] of the degeneracy, we have several options to define the density matrix and hence the entanglement spectrum. One option would be to take a proper linear combination of the two degenerate ground states. For example, one can make the linear combination such that the

(near) zero-energy single particle eigenstate is localized at a given end. One motivation for taking such a linear combination is that the so-constructed state may well be compared with the ground state defined for a semi-infinite system—the geometry that we considered in the bulk of the paper for the entanglement Hamiltonian.

In practice, such a ground state can be constructed by turning on a small μ_s (near a boundary, say). One should, however, note that such a procedure breaks particle-hole symmetry. In fact, for any finite system size and finite correlation length, if we take a linear combination of near zero-energy modes to respect particle-hole symmetry, they are not localized at a given end. (Note, however, that there is one exception for this: the "zero correlation length limit" that we can take in the SSH model.) Only in the semi-infinite limit, (one of) the localized zero energy mode is an exact particle-hole symmetry eigenstate of the Hamiltonian.

In Fig. 4, the entanglement spectrum with this construction of the ground state (i.e., the unique ground state selected by turning on a finite μ_s) is shown. As we make the subsystem size bigger, the entanglement spectrum "crosses over". In particular, while for small N_A, the single-particle entanglement spectrum does not have a zero mode, as we make the system size bigger, one level approaches zero from below, and asymptotically the single-particle entanglement spectrum has one exact zero mode. We call this entanglement spectrum the Ramond (R) spectrum (the spectrum of the free chiral fermion conformal field theory with periodic spatial boundary condition). The R spectrum, whose many-body spectrum displays a double degeneracy, is what is predicted from BCFT (in the "gapped regime").

Let now us discuss the "crossover" in more detail. In fact, one could discuss two kinds of features separately; one in terms of the scaling of the level spacing, and the other in terms of the structure of the levels, or more precisely the presence or absence of the zero mode.

From the former perspective, if a well-defined crossover region ever appears, the spectrum should follow the critical

---

[18]There is a small splitting, exponential in the system size divided by the correlation length, due to the interaction (tunneling) between the zero modes at the two ends of the finite system.

scaling, $\sim H_L / \ln N_A$, for N_A much smaller than the correlation length ξ ("critical regime"). On the other hand, for N_A much larger than the correlation length, the entanglement spectrum should scale as $\sim H_L / \ln \xi$ ("gapped regime"). However, it is not entirely clear how to identify such a crossover region in Fig. 4.

From the perspective of the structure of the spectrum, the entanglement spectrum for small N_A looks like the NS spectrum (i.e., there is no zero mode in the single particle entanglement spectrum), which crosses over to the R spectrum. (However, this "NS-like" spectrum should be distinguished from the NS-spectrum that appears in the topologically trivial case, because of the different scaling of the level spacing.) It is here crucial to recall that, by our construction of the unique ground state, the entanglement spectrum breaks particle-hole symmetry. In fact, what would look like a crossover in numerics is possible because of the particle-hole symmetry breaking. This should be compared with what we expect for the ideal, semi-infinite limit. In the semi-infinite limit, the ground state is unique and respects particle-hole symmetry, and so is the reduced density matrix. If so, the entanglement spectrum should be particle-hole symmetric. As a corollary, the spectrum cannot cross over from the NS spectrum to R spectrum in the presence of particle-hole symmetry. In short, the crossover from the NS spectrum to R spectrum in numerics is due to particle-hole symmetry breaking, which we can think of as a finite-size artifact. On the other hand, as we make the system size bigger, particle-hole symmetry breaking eventually goes away, and hence, in this ideal limit, the entanglement spectrum will be particle-hole symmetric. There should in fact be no crossover from an actual NS to the R spectrum.[19]

B. Periodic boundary conditions (PBC)

Let us now discuss the entanglement spectrum for the case of periodic boundary conditions. With periodic boundary conditions, the ground state is always unique as far as there is a spectral gap, irrespective of the sign of δt. That is, even in the topological case, one does not have to choose between ground states. The numerically computed entanglement spectrum is shown in Fig. 5. As compared to the case of open boundary conditions, there is no crossover from the NS to the R spectrum. (As mentioned above, such a crossover is not to be expected

---

[19]We should remark that the crossover (from small to large size N_A) observed in the numerical (single-particle) entanglement spectrum displayed in Fig. 4 is not quite covered by the cases discussed analytically in Sec. III. This is because in Sec. III for simplicity the technical assumption was made that the finite interval, to be bipartitioned into two pieces, had identical boundary conditions (called B_0 in that section) imposed at the two ends. (This permitted the simple analytical prediction of the crossover of the entanglement spectrum detailed in Sec. III.) For the case where different boundary conditions are imposed at the two ends, the corresponding entanglement spectrum has not yet been studied using analytical means. The numerical study presented in Fig. 4 of such a case may help the development of a future analytical description of this type of crossover of the entanglement spectrum.
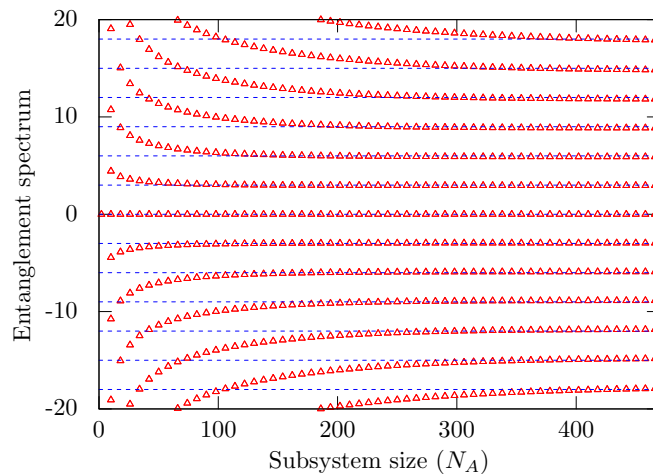

FIG. 5. The single-particle entanglement spectrum of the SSH model with PBC in its topologically nontrivial phase ($\delta t = -0.01$). The blue dotted lines are guide for eyes. Each level is doubly degenerate.

in the ideal semi-infinite limit.) One important feature for the case of periodic boundary conditions, which we expect from our discussion in the preceding section, is that, due to the presence of two entangling boundaries, the entanglement spectrum consists of two identical copies of a BCFT spectrum. That is, each level in the single-particle entanglement spectrum is doubly degenerate. The double-degeneracy is indeed confirmed in numerics. In this special case of equal bipartition, $N_B = N - N_A = N_A$, this double degeneracy turns out to be exact. This is the consequence of an accidental chiral symmetry that exists when $N_A = N_B$ and when there is an symmetry relating regions A and B (reflection/inversion symmetry in our case) [31–33]. Combined with the sublattice symmetry that exists when μ_s is zero, the accidental symmetry commutes with the entanglement Hamiltonian, resulting in the double degeneracy. This degeneracy is lifted by turning on the finite
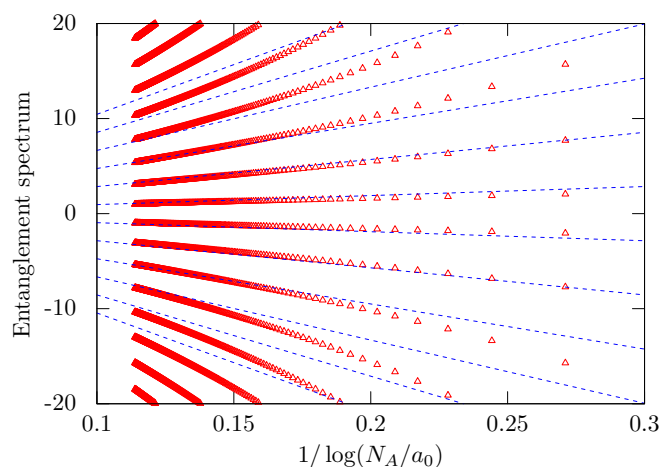

FIG. 6. The single-particle entanglement spectrum of the SSH model with periodic boundary conditions at its critical point ($\delta t = 0$). The blue dotted lines are guide for eyes. To compare with the expected result, we chose the "scale" a_0 to be $a_0 = 0.1$.

staggered chemical potential μ_s or taking $N_A \neq N_B$. This can be understood as a "tunneling" (coupling) between the two copies of the BCFT spectrum.

C. Critical scaling

Finally, as an aside, let us take a look at the entanglement spectrum at the critical point $\delta t = 0$. From the field theory considerations, we expect that the spectrum scales as $\sim 1/\ln N_A$ instead of $\sim 1/\ln \xi$. In Fig. 6, we fit the entanglement spectrum at the critical point, which roughly follows what we expect; one observes critical scaling $H_L/\ln N_A$ when N_A is large enough.

In gapless systems (critical points) in general, since there is no spectral gap, there are many (near) degenerate states. In our numerics, we let our computers choose the ground state. This procedure in principle may be tricky; for example, the behavior of the entanglement spectrum may depend severely on the choice of N_A and N. (There may be an even-odd like effect.) In Fig. 6, we choose open boundary conditions, and as before, fix N_A as $N_A = N/2$.

VI. CONCLUSIONS

In conclusion, we have considered in this paper (1+1)-dimensional gapped relativistic field theories in the scaling limit which can be viewed as describing gapped phases in the vicinity of a quantum phase transition described by a CFT. We have shown that the low-lying entanglement spectrum of such a gapped field theory is the spectrum of the underlying CFT on a finite interval of size $L_\xi = \ln(\xi/a)$ with a free boundary condition F and a boundary B_ϕ determined by the relevant perturbation of the CFT defining the gapped theory. We have also calculated the entanglement spectrum of the gapped field theory on a finite interval. This result provides, at the same time, the structure of the entanglement spectrum of the theory with periodic boundary conditions, i.e., on a circle of circumference $2R$, bipartitioned into two half-circles of length R each. In the limit where the size R is larger than the correlation length ξ, in analogy with the case of an interval, the entanglement spectrum is the sum of two spectra arising from the two ends of the semicircle. We would like to mention that interesting features arise for entanglement spectra of (1+1) dimensional symmetry protected topological (SPT) phases, and these will be discussed in a companion paper by the authors which will appear very shortly.

APPENDIX : COMMENTS ON THE RELATIONSHIP WITH THE CORNER TRANSFER MATRIX SPECTRUM OF YANG-BAXTER INTEGRABLE 2D CLASSICAL STATISTICAL MECHANICS MODELS

In this appendix, we briefly suggest a way in which the observation made (many years ago) in the literature about the spectrum of the logarithm of the CTM of integrable lattice models could be viewed within the context of the notions used in the present paper about the low-lying entanglement spectrum of gapped relativistic (1+1)-dimensional field theories. This will also provide some intuition about how the highly special constraints of integrability manage to generate the exact BCFT spectrum at arbitrarily high excitation "energies" of the entanglement Hamiltonian, even far off the scaling limit, i.e., for values of the correlation length ξ down to distances of the lattice scale a. First, recall that all the many off-critical integrable lattice models in which the behavior in question of the CTM was observed, are in fact one-parameter families of integrable models, which have the property that the system is critical ($\xi/a = \infty$) for one special value λ_* of the parameter λ, where they represent lattice realizations of a certain set of CFTs. Moreover, the deviation $\delta\lambda = (\lambda - \lambda_*)$ of the parameter from this special value is a relevant perturbation, and couples to a particular relevant field ϕ of the CFT. (This particular field is also special in the context of the CFT, in that even in the CFT perturbed by it an infinite subset of the conservation laws of the CFT survive. Only a few very special relevant perturbations of the CFT have this property.) Second, recall that (i) one can represent the critical lattice theory at $\lambda = \lambda_*$ as the CFT perturbed by an infinite number of irrelevant perturbations. Some of these irrelevant perturbations include powers of the energy-momentum tensor of the CFT which lead to nonlinear contributions to the energy-momentum relationship, thus representing the breaking of Lorentz-invariance present in the lattice theory. Moreover, (ii) the off-critical lattice model at $\delta\lambda \neq 0$ may be represented by perturbing the so-represented critical lattice theory at $\lambda = \lambda_*$ by yet another infinite set of perturbations of the CFT, the most relevant of which is the field ϕ discussed above. It is the constraints arising from the integrability of the lattice model that fix exactly the infinite number of expansion coefficients. A practical implementation of this general principle of representing an integrable lattice theory in terms of such perturbations of a CFT, can be found, e.g., in Ref. [23]. Now it is clear that the presence of all these perturbations (in principle infinite in number) can be treated in precisely the same way the single relevant perturbation ϕ is treated in Sec. II of the present paper. In that section, it is shown that a single relevant perturbation ϕ, when added to the CFT, leads to a "domain wall potential" in the coordinate u parametrizing the coordinate space on which the entanglement Hamiltonian $\hat{H}_E$ acts (angular quantization or Rindler space-time). All the additional perturbations that have to be added to the CFT to represent the lattice theory exactly will simply modify the shape of that domain wall potential in some way. Since the spectrum of the logarithm of the CTM of the gapped integrable models is known to be exactly the entire spectrum of the boundary CFT with the boundary conditions mentioned in the present paper, it ought to be the case that the exact domain wall potential being generated in this way in

the integrable theory, exactly describes a conformal boundary condition, all the way up to arbitrarily high (entanglement) energies. It may be possible to study explicitly, in the spirit of Ref. [23] mentioned above, approximations to the entire entanglement spectrum of the integrable system in terms of that of a CFT with a finite, but increasing number of perturbations.

[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).

[2] B. Bauer and C. Nayak, J. Stat. Mech. (2014) P09005.

[3] S. Ryu and Y. Hatsugai, Phys. Rev. B **73**, 245115 (2006).

[4] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).

[5] A. Chandran, M. Hermanns, N. Regnault, and B. A. Bernevig, Phys. Rev. B **84**, 205136 (2011).

[6] X.-L. Qi, H. Katsura, and A. W. W. Ludwig, Phys. Rev. Lett. **108**, 196402 (2012).

[7] J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B **86**, 245310 (2012).

[8] B. Swingle and T. Senthil, Phys. Rev. B **86**, 045117 (2012).

[9] D. Kabat and M. Strassler, Phys. Lett. B **329**, 46 (1994).

[10] J. Cardy, Nucl. Phys. B **324**, 581 (1989).

[11] J. J. Bisognano and E. H. Wichmann, J. Math. Phys. **16**, 985 (1975).

[12] J. J. Bisognano and E. H. Wichmann, J. Math. Phys. **17**, 303 (1976).

[13] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, New York, 1984).

[14] P. Calabrese and J. Cardy, J. Stat. Mech. (2002) P06002.

[15] I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991).

[16] A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).

[17] A. Laeuchli, arXiv-1303.0741.

[18] R. J. Baxter, *Exactly Solved Models in Statistical Mechancis* (Academic, New York, 1982).

[19] J. Cardy, in *Fields, Strings and Critical Phenomena*, edited by E. Brézin and J. Zinn-Justin, Les Houches Summer School, XLIX, 1988 (North-Holland, Amsterdam, 1990), p. 169.

[20] J. L. Cardy, Adv. Stud. Pure Math. **19**, 127 (1989).

[21] E. Date, M. Jimbo, T. Miwa, and M. Okado, Phys. Rev. B **35**, 2105(R) (1987).

[22] H. Saleur and M. Bauer, Nucl. Phys. B **320**, 591 (1989).

[23] S. Lukyanov and V. Terras, Nucl. Phys. B **654**, 323 (2003).

[24] K. Okunishi, Y. Hieida, and Y. Akutsu, Phys. Rev. E **59**, R6227 (1999).

[25] P. Calabrese and A. Lefevre, Phys. Rev. A **78**, 032329 (2008).

[26] F. Pollmann and J. E. Moore, New J. Phys. **12**, 025006 (2010).

[27] I. Peschel and T. T. Truong, Z. Phys. B **69**, 385 (1987).

[28] J. Cardy, Talk, KITP, June 2015: On the (unmetaphorical) entanglement gap in CFTs (and related topics).

[29] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).

[30] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).

[31] A. M. Turner, Y. Zhang, and A. Vishwanath, Phys. Rev. B. **82**, 241102(R) (2010).

[32] T. L. Hughes, E. Prodan, and B. A. Bernevig, Phys. Rev. B **83**, 245132 (2011).

[33] P.-Y. Chang, C. Mudry, and S. Ryu, J. Stat. Mech. (2014) P09014.

[34] L. Cincio and G. Vidal, Phys. Rev. Lett. **110**, 067208 (2013).

[35] B. Bauer, L. Cincio, B. Keller, M. Dolfi, G. Vidal, S. Trebst, and A. Ludwig, Nat. Commun. **5**, 5137 (2014).

[36] W. Zhu, S. S. Gong, and D. N. Sheng, Phys. Rev. B **92**, 014424 (2015).

[37] W. Zhu, S. S. Gong, F. D. M. Haldane, and D. N. Sheng, Phys. Rev. B **92**, 165106 (2015).

[38] J. Polchinski, Nucl. Phys. B **303**, 226 (1988).

[39] P. Kim, H. Katsura, N. Trivedi, and J. Han, Phys. Rev. B **94**, 195110 (2016).

[40] H. B. Thacker, Physica D **18**, 348 (1986).

[41] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, New York, 1992).

[42] P. Ginsparg and G. Moore, arXiv:hep-th/9304011.

[43] V. P. Yurov and A. B. Zamolodchikov, Int. J. Mod. Phys. A **05**, 3221 (1990).

[44] P. Dorey, M. Pillin, A. Pocklington, I. Runkel, R. Tateo, and G. Watts, arXiv:hep-th/0010278.