Inverse participation ratios in the XX spin chain

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We continue the study of the inverse participation ratios (IPRs) of the XXZ Heisenberg spin chain initiated by Stéphan, Furukawa, Misguich, and Pasquier (2009) and continued by Misguich, Pasquier, and Luck (2016) by focusing on the case of the XX Heisenberg spin chain. For the ground state, Stéphan *et al.* note that calculating the IPR is equivalent to Dyson's constant term ex-conjecture. We express the IPRs of excited states as an apparently new "discrete" Hall inner product. We analyze this inner product using the theory of symmetric functions (Jack polynomials, Schur polynomials, the standard Hall inner product, and $\omega_{q,t}$) to determine some exact expressions and asymptotics for IPRs. We show that IPRs can be indexed by partitions, and asymptotically the IPR of a partition is equal to that of the conjugate partition. We relate the IPRs to two other models from physics, namely, the circular symplectic ensemble of Dyson and the Dyson-Gaudin two-dimensional Coulomb lattice gas. Finally, we provide a description of the IPRs in terms of a signed count of diagonals of permutohedra.

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I. INTRODUCTION

The inverse participation ratios (IPRs) of a quantum mechanical system measure how localized a given state of the system is with respect to a preferential basis. The IPR are related to the configuration (or Renyi-Shannon) entropy with Renyi parameter $n = 2$ studied in Refs. [1] and [2]. In Ref. [1], this quantity was calculated for the ground state of the periodic XXZ chain, and a close connection with the partition function of the Dyson-Gaudin gas was pointed out. Recently in Ref. [3], Misguich, Pasquier, and Luck completed a numerical study of the IPRs in the spin-1/2 XXZ chains. The authors were mainly concerned with the sum of the IPRs and looked at the distribution of IPRs of individual eigenstates as a way to understand the behavior of the former. The general interest in understanding the IPRs stems from the fact that integrable systems generally fail to reach thermal states. Misguich *et al.* ask in particular about the IPRs in the special case of the XX spin chain with respect to the Ising basis.

We let L denote the length of the spin chain and restrict attention to the space spanned by Ising bases of M down spins. To analyze the IPRs, we will proceed as follows. First, we will obtain an expression for the eigenstates in terms of the Ising basis. We will see then that the IPRs are naturally labeled by partitions λ and are equal to

$$\mathbf{t}_\lambda = (M!L^{2M})^{-1} \sum_{\theta_1,\ldots,\theta_M \in \{0,\frac{2\pi}{L},\ldots,\frac{2\pi(L-1)}{L}\}} |s_\lambda(e^{i\theta_1},\ldots,e^{i\theta_M})|^4 \times \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^4,$$

where s_λ is the Schur polynomial (a specialization of the Jack polynomials). This will naturally lead us to the theory of symmetric functions. We will analyze these quantities in complete mathematical rigor (Sec. IV and on).

In Appendix A we provide several physical interpretations of the IPRs and discuss the ground state. The calculation of the ground state IPR ($s_\lambda = 1$) is equivalent to Dyson's constant term ex-conjecture, an important motivation for much of the development of the theory of Macdonald polynomials. For excited states, we can interpret the IPR as the expectation value of $|s_\lambda|^4$ over the Dyson-Gaudin two-dimensional Coulomb lattice gas. In the limit $L \to \infty$ the gas is no longer restricted to a lattice. In this situation, the IPR can be interpreted as the expected value of $|s_\lambda|^4$ over the circular symplectic ensemble (CSE) of Dyson. The CSE is a modification of the Gaussian symplectic ensemble (GSE), a key matrix ensemble in random matrix theory.

Next we study the IPRs using the tools of symmetric function theory, namely, Jack polynomials, Hall inner product, and $\omega_{q,t}$. We interpret the IPRs in terms of a "discrete" Hall inner product and show that assuming, roughly, $L > 2M$, the discrete Hall inner product and the traditional one coincide. This allows us to use the orthogonality of the Jack polynomials to evaluate the IPRs in terms of the transition coefficients from Schur polynomials to Jack polynomials. When these are known, we are able to determine exact expressions for the IPRs and their asymptotics. Finally, we show that curiously, if λ^* is the conjugate partition of λ, then

$$\mathbf{t}_\lambda = \mathbf{t}_{\lambda^*} \quad (1)$$

assuming, roughly, that $L > 2M \gg 1$. We provide an interpretation of this result in terms of particle-hole duality together with a proof.

II. INVERSE PARTICIPATION RATIOS

In this section we provide the basic definition of the inverse participation ratios ([4], [3]).

Definition 1. Let the normalized eigenvectors of a Hamiltonian $\mathcal{H}$ be $\{|\psi_i\rangle\}_{i=1,\ldots,D}$ and assume they are nondegenerate. The inverse participation ratio (IPR) of an eigenstate $|\psi_k\rangle$ in a preferential basis $\{|a_i\rangle\}_{i=1,\ldots,D}$ is

$$\mathbf{t}_k := \sum_{i=1}^{D} |\langle a_i|\psi_k\rangle|^4. \quad (2)$$

The maximum value of this quantity is reached when an eigenstate coincides with a single basis state, in which case

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$t_{\max} = 1$. The minimum value is reached for eigenstates which are uniform superpositions of all the basis states, with the same modulus $|\langle a_i|\psi_k\rangle| = \frac{1}{\sqrt{D}}$. This maximally delocalized limit gives $t_{\min} = \frac{1}{D}$.

We note that if

$$|\tilde{\psi}_k\rangle = \sum_i c_i |a_i\rangle \tag{3}$$

is an unnormalized multiple of $|\psi_k\rangle$, then

$$|\langle a_i|\psi_k\rangle|^4 = \frac{|c_i|^4}{\left(\sum_j |c_j|^2\right)^2} \tag{4}$$

and

$$\mathbf{t}_k = \frac{\sum_i |c_i|^4}{\left(\sum_j |c_j|^2\right)^2}. \tag{5}$$

III. PERIODIC XX HEISENBERG MODEL

We consider the XX spin chain (XXZ with anisotropy parameter $\Delta = 0$) with periodic boundary conditions and L sites. The Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y. \tag{6}$$

We will single out the preferential basis called the Ising configuration for the IPRs. These are the eigenstates of all S_i^z:

$$|\uparrow\uparrow\downarrow\cdots\rangle, |\uparrow\downarrow\uparrow\cdots\rangle, \ldots \tag{7}$$

We restrict attention to the subspace spanned by Ising configurations having M down spins. This space has dimension $\binom{L}{M}$. The coordinate Bethe ansatz tells us that the wave numbers $k_j, j = 1,\ldots,M$ and Bethe roots $I_j, j = 1,\ldots,M$ satisfy

$$Lk_j = 2\pi I_j, \quad j = 1,2,\ldots,M. \tag{8}$$

For simplicity, we will be assuming that M is odd, so that the I_j are in $\{0,1,\ldots,L-1\} \mod L$ [5]. All statements can be adapted to the case of even M, in which case the I_j are half integers. To avoid the nullity of the wave function, the wave numbers must be distinct.

We use the shorthand notation

$$|\mathbf{x}\rangle = |x_1 \cdots x_M\rangle, \quad x_1 < x_2 < \cdots < x_M \tag{9}$$

for the Ising basis with down spins at $x_1,\ldots,x_M$ and

$$\mathbf{k} = (k_1,\ldots,k_M). \tag{10}$$

The eigenvectors of $\mathcal{H}$ are now given by

$$|\psi_{\mathbf{k}}\rangle = \sum_{\mathbf{x}} c(\mathbf{x})|\mathbf{x}\rangle = \sum_{\mathbf{x}} \sum_{\pi\in S_M} \operatorname{sgn}(\pi) e^{i\pi(\mathbf{k})\cdot\mathbf{x}}|\mathbf{x}\rangle$$
$$= \sum_{\mathbf{x}} \det(e^{ik_a x_b})_{a,b}|\mathbf{x}\rangle. \tag{11}$$

By (5), the IPRs are equal to

$$\mathbf{t}_k = \frac{\sum_{\mathbf{x}} |\det(e^{ik_a x_b})_{a,b}|^4}{\left(\sum_{\mathbf{x}} |\det(e^{ik_a x_b})_{a,b}|^2\right)^2}. \tag{12}$$

The denominator of (12) is relatively easy to evaluate. We have

$$|c(\mathbf{x})|^2 = c(\mathbf{x})\bar{c}(\mathbf{x}) \tag{13}$$
$$= \left(\sum_{P\in S_M} \operatorname{sgn}(P) e^{iP(\mathbf{k})\cdot\mathbf{x}}\right)\left(\sum_{Q\in S_M} \operatorname{sgn}(Q) e^{-iQ(\mathbf{k})\cdot\mathbf{x}}\right) \tag{14}$$
$$= \sum_{P,Q\in S_M} \operatorname{sgn}(P)\operatorname{sgn}(Q) e^{i(P(\mathbf{k})-Q(\mathbf{k}))\cdot\mathbf{x}}. \tag{15}$$

We now use the fact that the determinant is zero whenever any two of x_i are equal. This allows us to remove the restriction of the x_i being distinct. In addition, we must divide by $M!$ to account for the order. We also note that

$$\sum_{x_1,\ldots,x_M=1}^{L} \prod_{j=1}^{M} e^{ix_j m_j} = \prod_{j=1}^{M} \sum_{x_j=1}^{L} e^{ix_j m_j}. \tag{16}$$

We have

$$\sum_{\mathbf{x}} |c(\mathbf{x})|^2$$
$$= \frac{1}{M!} \sum_{x_1,\ldots,x_M=1}^{L} |c(\mathbf{x})|^2$$
$$= \frac{1}{M!} \sum_{x_1,\ldots,x_M=1}^{L} \sum_{P,Q\in S_M} \operatorname{sgn}(P)\operatorname{sgn}(Q) \prod_{j=1}^{M} e^{ix_j(k_{P(j)}-k_{Q(j)})}$$
$$= \frac{1}{M!} \sum_{P,Q\in S_M} \operatorname{sgn}(P)\operatorname{sgn}(Q) \sum_{x_1,\ldots,x_M=1}^{L} \prod_{j=1}^{M} e^{ix_j(k_{P(j)}-k_{Q(j)})}$$
$$= \frac{1}{M!} \sum_{P,Q\in S_M} \operatorname{sgn}(P)\operatorname{sgn}(Q) \prod_{j=1}^{M} \sum_{x_j=1}^{L} e^{ix_j(k_{P(j)}-k_{Q(j)})}. \tag{17}$$

Now $k_j = \frac{2\pi I_j}{L}$ and $0 < |I_j - I_i| < L$. Consequently,

$$\sum_{x_j=1}^{L} e^{ix_j(k_{P(j)}-k_{Q(j)})} = L\delta(k_{P(j)} = k_{Q(j)}). \tag{18}$$

Substituting yields the result

$$\sum_{\mathbf{x}} |c(\mathbf{x})|^2 = L^M. \tag{19}$$

Therefore

$$\mathbf{t}_k = \frac{\sum_{\mathbf{x}} |\det(e^{ik_a x_b})_{a,b}|^4}{L^{2M}}. \tag{20}$$

(An anonymous referee kindly pointed out that one could have obtained this result using a standard second-quantization property: The state $c_{k_1}^\dagger \cdots c_{k_M}^\dagger|0\rangle$ is normalized if the fermion creation operators $c_{k_j}^\dagger$ satisfy the canonical fermionic anticommutation relations. This condition is met if we add a factor $1/\sqrt{L}$ in each single-particle wave function, and we thus get L^M for the norm squared of the many-body state.)

Next, set $\theta_j := \frac{2\pi}{L} x_j$, so that

$$e^{ik_a x_b} = e^{iI_a\theta_b}. \tag{21}$$

We set

$$c(\boldsymbol{\theta}) = \det(e^{iI_a\theta_b})_{a,b} \tag{22}$$

and introduce the notation $\sum_{\boldsymbol{\theta}}$ to denote summation over distinct values of $\theta_1,\ldots,\theta_M$ in $\frac{2\pi}{L}\{0,\ldots,L-1\}$.

Example 1. IPR of the ground state. For the ground state,

$$\{I_1,I_2,\ldots,I_M\} = \left\{\frac{M-1}{2},\frac{M-1}{2}-1,\ldots,-\frac{M-1}{2}\right\}. \tag{23}$$

Therefore

$$c(\boldsymbol{\theta}) = \begin{vmatrix} e^{i(-\frac{M-1}{2})\theta_1} & e^{i(-\frac{M-1}{2})\theta_2} & \cdots & e^{i(-\frac{M-1}{2})\theta_M} \\ e^{i(-\frac{M-1}{2}+1)\theta_1} & e^{i(-\frac{M-1}{2}+1)\theta_2} & \cdots & e^{i(-\frac{M-1}{2}+1)\theta_M} \\ \vdots & \vdots & \cdots & \vdots \\ e^{i(\frac{M-1}{2})\theta_1} & e^{i(\frac{M-1}{2})\theta_2} & \cdots & e^{i(\frac{M-1}{2})\theta_M} \end{vmatrix}. \tag{24}$$

We notice that (24) is equal to

$$e^{i(-\frac{M-1}{2})(\theta_1+\ldots+\theta_M)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_M} \\ \vdots & \vdots & \cdots & \vdots \\ e^{i(M-1)\theta_1} & e^{i(M-1)\theta_2} & \cdots & e^{i(M-1)\theta_M} \end{vmatrix}, \tag{25}$$

the determinant being a Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_M} \\ \vdots & \vdots & \cdots & \vdots \\ e^{i(M-1)\theta_1} & e^{i(M-1)\theta_2} & \cdots & e^{i(M-1)\theta_M} \end{vmatrix} = \prod_{j<k}(e^{i\theta_k} - e^{i\theta_j}). \tag{26}$$

Therefore the IPR of the ground state is equal to

$$\mathbf{t}_0 = \frac{1}{L^{2M}}\sum_{\boldsymbol{\theta}}\prod_{j<k}|e^{i\theta_k} - e^{i\theta_j}|^4. \tag{27}$$

We will complete the evaluation in (A10).

Notice that $|\det(e^{iI_a\theta_b})_{a,b}|$ is invariant under shifts of the Bethe roots I, since

$$\det(e^{i(I_a+n)\theta_b})_{a,b} = e^{in(\theta_1+\ldots+\theta_M)}\det(e^{iI_a\theta_b})_{a,b}. \tag{28}$$

Therefore IPRs corresponding to two excited states whose Bethe roots are related by a shift will be equal.

Assume without loss of generality that $I_1 > I_2 > \ldots > I_M$. Set

$$\lambda_j = I_j - \frac{M+1}{2} + j \tag{29}$$

so that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_M$ yields a partition $\lambda = [\lambda_1,\ldots,\lambda_M]$. In this notation,

$$|\det(e^{iI_a\theta_b})_{a,b}| = |s_\lambda(e^{i\theta_1},\ldots,e^{i\theta_M})V(e^{i\theta_1},\ldots,e^{i\theta_M})|, \tag{30}$$

where s is the Schur polynomial (discussed in Appendix B) and $V(x_1,\ldots,x_M)$ is the Vandermonde determinant:

$$V(x_1,\ldots,x_M) = \prod_{i<j}(x_j - x_i). \tag{31}$$

We then have

$$\mathbf{t}_\lambda = \frac{\sum_{\boldsymbol{\theta}}|s_\lambda(e^{i\theta_1},\ldots,e^{i\theta_M})|^4\prod_{i<j}|e^{i\theta_i} - e^{i\theta_j}|^4}{L^{2M}}, \tag{32}$$

where we have introduced a labeling by partitions λ.

IV. CALCULATING THE IPRS OF XX

Throughout this section, we refer the reader to Appendix B for notation and necessary definitions from symmetric function theory. Set

$$\Delta(x;\beta) = \prod_{i\neq j}\left(1 - x_i x_j^{-1}\right)^{\beta/2}. \tag{33}$$

Then

$$\Delta(x;\beta) = \prod_{i<j}\left[(x_j - x_i)\left(x_j^{-1} - x_i^{-1}\right)\right]^{\beta/2}. \tag{34}$$

In particular,

$$\Delta(e^{i\theta};\beta) = \prod_{k<l}|e^{i\theta_l} - e^{i\theta_k}|^\beta. \tag{35}$$

Define T_L to be a discrete torus:

$$T_L := \left\{e^{i\theta} \in \mathbb{C} : \theta \in \frac{2\pi\mathbb{Z}}{L}\ \forall j\right\}. \tag{36}$$

Define a scalar product on symmetric polynomials by

$$\langle f,g\rangle_{L;\beta} := \frac{1}{L^M M!}\sum_{T_L^M} f(z)\overline{g}(z)\Delta(z;\beta). \tag{37}$$

In terms of this "discrete" Hall inner product,

$$\mathbf{t}_\lambda = \frac{\left\langle s_\lambda^2, s_\lambda^2\right\rangle_{L;4}}{L^M}. \tag{38}$$

Note that

$$\sum_{z_1,\ldots,z_M\in T_L} z_1^{a_1}z_2^{a_2}\cdots z_M^{a_M} = \begin{cases} L^M & \text{if } a_1 \equiv \cdots \equiv a_M \equiv 0 \pmod{L} \\ 0 & \text{otherwise.}\end{cases} \tag{39}$$

Consequently, the inner product can be expressed as an extraction of coefficients:

$$\langle f,g\rangle_{L;\beta} = \frac{1}{M!}\sum_{i_1,\ldots,i_M\in\mathbb{Z}}\left[x_1^{Li_1}\cdots x_M^{Li_M}\right] \times f(x_1,\ldots,x_M)g\left(x_1^{-1},\ldots,x_M^{-1}\right)\Delta(x;\beta). \tag{40}$$

As we show next, for fixed f,g,M and all sufficiently large L,

$$\langle f,g\rangle_{L;\beta} = \frac{1}{M!}\mathrm{CT}[f(x)g(x^{-1})\Delta(x;\beta)] = \frac{1}{M!}\int_T f(z)\overline{g(z)}\Delta(z;\beta), \tag{41}$$

the Hall inner product of (B9).

Proposition 1. Let f,g be symmetric polynomials and suppose that $\beta/2 \in \mathbb{N}$. Let $\deg_i(f)$ denote the degree of x_i

in f. Set

$$p_1 := \max_i \deg_i(f), \quad q_1 := \max_i \deg_i(g). \tag{42}$$

$$p_2 := \min_i \deg_i(f), \quad q_2 := \min_i \deg_i(g). \tag{43}$$

Then

$$|\max_i \deg_i(f\bar{g}\Delta(x;\beta))| \leqslant \max\{|p_1-q_2|,|p_2-q_1|\} + (\beta/2)(M-1). \tag{44}$$

Proof. Let $\delta = (0,1,\ldots,M-1)$. The product $f\bar{g}\Delta(x;\beta)$ is a sum of monomials of the following form: with $\sigma^{(i_1)},\sigma^{(i_2)}$ permutations, $x_1^{f_1}\cdots x_M^{f_M}$ monomials coming from f and $x_1^{g_1}\cdots x_M^{g_M}$ monomials from g,

$$x_1^{f_1}\cdots x_M^{f_M} x_1^{-g_1}\cdots x_M^{-g_M}\prod_{i=1}^{\beta/2}\left(x^{\sigma^{(i_1)}\delta}x^{-\sigma^{(i_2)}\delta}\right). \tag{45}$$

The degree of x_i is equal to

$$f_i - g_i + \sum_{i=1}^{\beta/2}\sigma^{(i_1)}(\delta) - \sigma^{(i_2)}(\delta). \tag{46}$$

It satisfies

$$\left|f_i - g_i + \sum_{i=1}^{\beta/2}\sigma^{(i_1)}(\delta) - \sigma^{(i_2)}(\delta)\right| \leqslant \max\{|p_1-q_2|,|p_2-q_1|\} + (\beta/2)(M-1). \tag{47}$$

■

Example 2. Proposition 1 is tight. Taking $M=4,\lambda=[2,1,1]$ and $\beta=2$, we have

$$p_1 = q_1 = 2, \quad p_2 = q_2 = 0. \tag{48}$$

The proposition guarantees that the maximum degree of a variable in $f\bar{g}\Delta(x;2)$ does not exceed $M+1=5$. This means that when $L>5$, the proposition guarantees that the discrete Hall inner product will be equal to the Hall inner product. Computing on the monomial symmetric functions,

$$\langle m_\lambda,m_\lambda\rangle_{4;2} = 16 \tag{49}$$

$$\langle m_\lambda,m_\lambda\rangle_{5;2} = 9 \tag{50}$$

$$\langle m_\lambda,m_\lambda\rangle_{L;2} = \frac{1}{M!}\mathrm{CT}[m_\lambda\overline{m}_\mu\Delta(x)] = \langle m_\lambda,m_\lambda\rangle_2 = 2 \quad \forall L \geqslant 6. \tag{51}$$

We see that the proposition is tight in the sense that no smaller L would work.

Proposition 1 clarifies when we can truncate the sum in (40). Namely, each i should satisfy

$$Li \leqslant \max\{|p_1-q_2|,|p_2-q_1|\} + (\beta/2)(M-1). \tag{52}$$

The case $\beta=4$ is of particular interest to us. According to Proposition 1, if M,L,λ,μ satisfy $L-2M > \max\{\lambda_1,\mu_1\}-2$, then

$$\left\langle J_\lambda^{(1/2)},J_\mu^{(1/2)}\right\rangle_{L;4} = \left\langle J_\lambda^{(1/2)},J_\mu^{(1/2)}\right\rangle_4. \tag{53}$$

In particular, under these conditions these Jack polynomials are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle_{L;4}$.

Theorem 1. Suppose that L and M are given. Let λ be a fixed partition with $\lambda_1 < (1/2)(L-2M+2)$. Let $s_\lambda, J_\mu^{(1/2)}$ be the Schur and Jack polynomials, respectively. Write

$$s_\lambda^2 = \sum_\nu r_\lambda^\nu J_\nu^{(1/2)}. \tag{54}$$

The IPR for partition λ is equal to

$$\mathbf{t}_\lambda = \frac{(2M)!}{M!(2L)^M}\sum_\nu \mathcal{N}_\nu^{(1/2)}(M)\mathcal{C}_\nu(1/2)\left(r_\lambda^\nu\right)^2. \tag{55}$$

Proof. Under these conditions on λ,

$$\mathbf{t}_\lambda = \frac{\left\langle s_\lambda^2,s_\lambda^2\right\rangle_{L;4}}{L^M} = \frac{\left\langle s_\lambda^2,s_\lambda^2\right\rangle_4}{L^M}. \tag{56}$$

We have

$$\left\langle s_\lambda^2,s_\lambda^2\right\rangle_4 = \left\langle \sum_\nu r_\lambda^\nu J_\nu^{(1/2)},\sum_\nu r_\lambda^\nu J_\nu^{(1/2)}\right\rangle_4 = \sum_\nu \left(r_\lambda^\nu\right)^2\left\langle J_\nu^{(1/2)},J_\nu^{(1/2)}\right\rangle_4. \tag{57}$$

Applying (B8) for $\beta=4$ yields the result. ■

The transition coefficients for Jack polynomials with a parameter α_1 to Jack polynomials of another parameter α_2 are not well understood and are closely related to several open problems, such as finding a combinatorial description of the Littlewood-Richardson coefficients of Jack polynomials. Our particular case when one of the types of Jack polynomials is a Schur polynomial offers hope, though we do not have explicit formulas. One can calculate these coefficients indirectly, however, by transitioning first to the symmetric monomial basis: $J^{\alpha_1}\to m\to J^{\alpha_2}$ (see Appendix B 4 and the Supplemental Material [6] for computer code in SageMath.).

Example 3. The simplest excited state has $\lambda=[1]$. Computing,

$$s_{[1]}^2 = s_{[1,1]} + s_{[2]} \tag{58}$$

and

$$s_{[1,1]} = (1/2)J_{[1,1]}, \quad s_{[2]} = (2/3)J_{[2]} - (1/6)J_{[1,1]}, \tag{59}$$

so that

$$s_{[1]}^2 = (2/3)J_{[2]} + (1/3)J_{[1,1]}. \tag{60}$$

The IPR is

$$\mathbf{t}_{[1]} = (1/L^M)\left\langle s_{[1]}^2,s_{[1]}^2\right\rangle_4 = (1/L^M)\langle(2/3)J_{[2]}+(1/3)J_{[1,1]},(2/3)J_{[2]} + (1/3)J_{[1,1]}\rangle_L. \tag{61}$$

We also compute

$$C_{[1,1]} = 3/2, \quad C_{[2]} = 3/4. \tag{62}$$

$$\mathcal{N}_{[1,1]} = \frac{M}{M-1/2}\frac{M-1}{M-3/2}, \quad \mathcal{N}_{[2]} = \frac{M}{M-1/2}\frac{M+1/2}{M}. \tag{63}$$

TABLE I. Table of IPRs, assuming $L - 2M > 2\lambda_1$ and $\mathcal{N} \approx 1$, the latter occurring when M is large.

λ	$\mathbf{t}_\lambda$
[]	$\mathbf{t}_0$
[1]	$(1/2)\mathbf{t}_0$
[2]	$(11/32)\mathbf{t}_0$
[1,1]	$(11/32)\mathbf{t}_0$
[3]	$(17/64)\mathbf{t}_0$
[2,1]	$(1/4)\mathbf{t}_0$
[1,1,1]	$(17/64)\mathbf{t}_0$
[4]	$(1787/8192)\mathbf{t}_0$
[3,1]	$(1451/8192)\mathbf{t}_0$
[2,2]	$(99/512)\mathbf{t}_0$
[2,1,1]	$(1451/8192)\mathbf{t}_0$
[1,1,1,1]	$(1787/8192)\mathbf{t}_0$

Consequently,

$$L^M \mathbf{t}_{[1]} = (4/9)\langle J_{[2]}, J_{[2]}\rangle_4 + (4/9)\langle J_{[2]}, J_{[1,1]}\rangle_4 + (1/9)\langle J_{[1,1]}, J_{[1,1]}\rangle_4. \tag{64}$$

Assuming $L > 2M + 2$, we can replace the discrete Hall inner products with the Hall inner product. Then

$$\langle J_{[2]}, J_{[2]}\rangle_4 = C_{[2]}\mathcal{N}_{[2]}\frac{(2M)!}{M!2^M} \tag{65}$$

$$\langle J_{[1,1]}, J_{[1,1]}\rangle_4 = C_{[1,1]}\mathcal{N}_{[1,1]}\frac{(2M)!}{M!2^M} \tag{66}$$

$$\langle J_{[1,1]}, J_{[2]}\rangle_4 = 0. \tag{67}$$

This gives us the exact value of the IPR $\mathbf{t}_{[1]}$. Assuming further that M is large, so that $\mathcal{N} \approx 1$, yields

$$\mathbf{t}_{[1]} \approx (1/2)\frac{(2M)!}{M!L^M 2^M} = (1/2)\mathbf{t}_0. \tag{68}$$

Example 4. Table of IPRs.

Table I suggests the following property:

$$\langle J^{(1)}_\lambda, J^{(1)}_\lambda\rangle'_4 = \langle J^{(1)}_{\lambda^*}, J^{(1)}_{\lambda^*}\rangle'_4. \tag{69}$$

Here, we recall, $J^{(1)}$ is a (rescaled) Schur polynomial and the inner product is the Hall inner product for parameter $\beta = 4$ (or $\alpha = 1/2$ in the symmetric polynomial literature). We emphasize that the inner product has a different parameter than the Jack polynomials being operated on. Moreover, generally even

$$\langle J^{(1/2)}_\lambda, J^{(1/2)}_\lambda\rangle'_4 \neq \langle J^{(1/2)}_{\lambda^*}, J^{(1/2)}_{\lambda^*}\rangle'_4, \tag{70}$$

where the parameters are matching. Thus the choice of Schur polynomials is special among the Jack polynomials.

Theorem 2. Let λ,μ be partitions and s_λ the Schur polynomial. For any β,

$$\langle s_\lambda, s_\mu\rangle'_\beta = \langle s_{\lambda^*}, s_{\mu^*}\rangle'_\beta. \tag{71}$$

Proof. Following [7, VI §2], consider the symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$ and define an inner product on the basis of power sum polynomials p_μ:

$$z_\mu(q,t) := z_\mu \prod_{i=1}^{l(\mu)} \frac{1-q^{\mu_i}}{1-t^{\mu_i}}, \quad \langle p_\mu, p_\kappa\rangle'_{q,t} := \delta_{\mu,\kappa} z_\mu(q,t). \tag{72}$$

This inner product is a q-analogue of the inner product $\langle\cdot,\cdot\rangle'_\beta$. Indeed, denote the limit $(q,t) \to (1,1)$ with $q = t^{2/\beta}$ by $(q,t) \xrightarrow[2/\beta]{} (1,1)$. Then

$$\lim_{(q,t)\xrightarrow[2/\beta]{}(1,1)} \langle\cdot,\cdot\rangle'_{(q,t)} = \langle\cdot,\cdot\rangle'_\beta. \tag{73}$$

Let $\omega_{t,q}$ be the standard automorphism on symmetric functions with coefficients in $\mathbb{Q}(q,t)$:

$$\omega_{q,t}(p_\lambda) = (-1)^{|\lambda|+l(\lambda)} p_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{74}$$

Let

$$\omega_\beta = \lim_{(q,t)\xrightarrow[2/\beta]{}(1,1)} \omega_{q,t}. \tag{75}$$

The automorphism $\omega_{q,t}$ satisfies

$$\omega_{q,t}^{-1} = \omega_{t,q} \tag{76}$$

$$\langle \omega_{u,v} f, g\rangle'_{q,t} = \langle f, \omega_{u,v} g\rangle'_{q,t} \tag{77}$$

$$\langle \omega_{t,q} f, g\rangle'_{q,t} = \langle \omega_2 f, g\rangle'_2 \tag{78}$$

$$\langle \omega_2 f, \omega_2 g\rangle'_2 = \langle f, g\rangle'_2. \tag{79}$$

The Schur polynomials satisfy

$$\omega_2 s_\lambda = s_{\lambda^*}. \tag{80}$$

Putting this together,

$$\langle s_\lambda, s_\mu\rangle'_\beta = \langle \omega_\beta^{-1} s_\lambda, \omega_\beta s_\mu\rangle'_\beta = \langle \omega_2 s_\lambda, \omega_\beta s_\mu\rangle'_2. \tag{81}$$

By evaluating on the power sum basis, we can check that

$$\omega_{q,t}\omega_2 = \omega_2\omega_{q,t}. \tag{82}$$

Consequently,

$$\begin{aligned}\langle \omega_2 s_\lambda, \omega_\beta s_\mu\rangle'_2 &= \langle \omega_2\omega_2 s_\lambda, \omega_2\omega_\beta s_\mu\rangle'_2 \\ &= \langle \omega_2\omega_2 s_\lambda, \omega_\beta\omega_2 s_\mu\rangle'_2 = \langle \omega_2 s_{\lambda^*}, \omega_\beta s_{\mu^*}\rangle'_2,\end{aligned} \tag{83}$$

which proves the result. ■

Corollary 1. Let λ be a partition satisfying $\lambda_1 < (1/2)(L - 2M + 2)$ with $M \gg |\lambda|$ (so that $\mathcal{N}_\lambda \approx 1$). Then the IPRs satisfy the duality relation

$$\mathbf{t}_\lambda = \mathbf{t}_{\lambda^*}. \tag{84}$$

Proof. We have shown that under these hypotheses,

$$\mathbf{t}_\lambda = \frac{\langle s^2_\lambda, s^2_\lambda\rangle_4}{L^M}. \tag{85}$$

Moreover, since $\mathcal{N}_\lambda \approx 1$,

$$\frac{\langle s^2_\lambda, s^2_\lambda\rangle_4}{\langle s^2_{\lambda^*}, s^2_{\lambda^*}\rangle_4} = \frac{\langle s^2_\lambda, s^2_\lambda\rangle'_4}{\langle s^2_{\lambda^*}, s^2_{\lambda^*}\rangle'_4}. \tag{86}$$

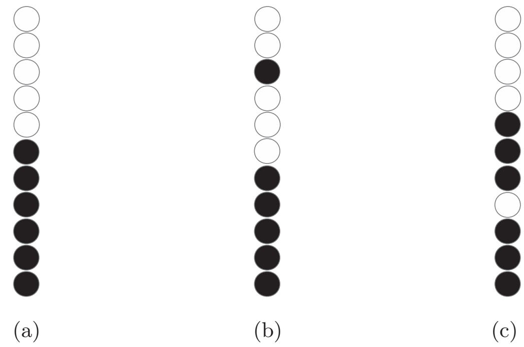

FIG. 1. (a) An infinite chain of particles below an infinite chain of holes represents the ground state. (b) The partition [3] is interpreted as taking the ground state and moving the top particle up three positions. (c) The partition [1,1,1] raises the top three particles of the ground state by one position. The map taking the configuration of a partition to that of its dual corresponds to particle-hole duality: exchanging particles with holes followed by a flip.

It will suffice to show that the latter ratio is equal to 1. Write s_λ^2 and $s_{\lambda^*}^2$ in the Schur basis:

$$s_\lambda^2 = \sum_\nu c_\nu^\lambda s_\nu, \quad s_{\lambda^*}^2 = \sum_\nu c_\nu^{\lambda^*} s_\nu. \tag{87}$$

Since ω is an automorphism, for any partitions μ,κ,

$$\omega(s_\mu s_\kappa) = s_{\mu^*} s_{\kappa^*}. \tag{88}$$

In particular,

$$s_\lambda^2 = \omega\omega\left(s_\lambda^2\right) = \omega\left(s_{\lambda^*}^2\right) \Rightarrow s_{\lambda^*}^2 = \omega\left(s_\lambda^2\right). \tag{89}$$

Comparing coefficients,

$$\sum_\nu c_\nu^{\lambda^*} s_\nu = \sum_\nu c_\nu^\lambda s_{\nu^*} = \sum_{\nu^*} c_{\nu^*}^\lambda s_\nu \Rightarrow c_\nu^{\lambda^*} = c_{\nu^*}^\lambda. \tag{90}$$

Expanding

$$\begin{aligned}\left\langle s_\lambda^2, s_\lambda^2 \right\rangle_4' &= \sum_{\nu_1,\nu_2} c_{\nu_1}^\lambda c_{\nu_2}^\lambda \left\langle s_{\nu_1}, s_{\nu_2}\right\rangle_4' \\ &= \sum_{\nu_1,\nu_2} c_{\nu_1'}^\lambda c_{\nu_2'}^\lambda \left\langle s_{\nu_1'}, s_{\nu_2'}\right\rangle_4' = \sum_{\nu_1,\nu_2} c_{\nu_1}^{\lambda^*} c_{\nu_2}^{\lambda^*} \left\langle s_{\nu_1'}, s_{\nu_2'}\right\rangle_4' \\ &= \sum_{\nu_1,\nu_2} c_{\nu_1}^{\lambda^*} c_{\nu_2}^{\lambda^*} \left\langle s_{\nu_1}, s_{\nu_2}\right\rangle_4' = \left\langle s_{\lambda^*}^2, s_{\lambda^*}^2 \right\rangle_4'. \end{aligned} \tag{91}$$

■

We have shown that conjugate partitions have asymptotically (assuming $\mathcal{N} \approx 1$) equal IPRs. This is not the case for finite M, when $\mathcal{N} \neq 1$. We thank the anonymous referee for suggesting the following interpretation.

The ground state is represented as an infinite sequence of particles at positions $-1,-2,\ldots$. In positions $0,1,\ldots$ are holes. A partition $\lambda = [\lambda_1,\lambda_2,\ldots]$ represents the configuration with respect to the ground state in which the ith particle from the top is raised by λ_i positions (see Fig. 1).

The map taking the particle configuration of partition λ to the configuration of λ^* corresponds to particle-hole duality: First particles and holes are exchanged, and then the configuration is flipped. We prove this in the following.

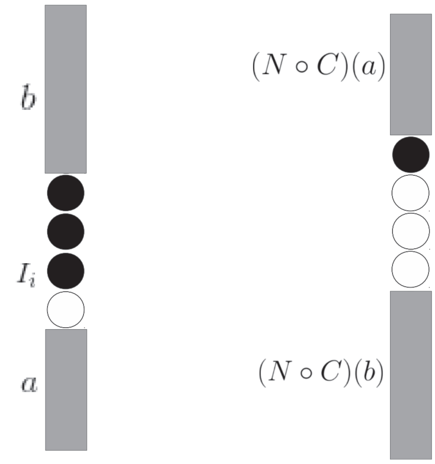

FIG. 2. Proof of Proposition 2.

Proposition 2. Suppose that M and L are infinite. Partitions λ correspond to infinite sequences of integers $I_1 > I_2 > \cdots$ via

$$I_j = \lambda_j - j. \tag{92}$$

We denote this map by ϕ. Let C be the map taking a set S to its complement in $\mathbb{Z}$ and let N be the map $x \mapsto -x-1$. The following diagram commutes.

$$\begin{array}{ccc} \text{Partitions} & \xrightarrow{\phi} & \text{Infinite integer sequences} \\ *\downarrow & & \downarrow N\circ C \\ \text{Partitions} & \xrightarrow{\phi} & \text{Infinite integer sequences} \end{array}$$

Proof. We prove the result by induction on $|\lambda|$. For the base case, corresponding to $\lambda = []$, $\phi(\lambda) = \{-1,-2,\ldots\}$. Then $N \circ C(\{-1,-2,\ldots\}) = N(\{0,1,2,\ldots\}) = \{-1,-2,\ldots\}$. On the other hand, $\phi([]^*) = \phi([]) = \{-1,-2,\ldots\}$, establishing the base case.

Now assume the result for all partitions with $|\lambda| \leqslant k-1, k \geqslant 1$ and consider a partition with $|\lambda| = k$. Since the partition has finite $|\lambda|$, there is some least index i for which $\lambda_i > \lambda_{i+1}$. Let μ be the partition defined by

$$\mu_j = \begin{cases} \lambda_j & j \geqslant i \\ \lambda_j - 1 & j < i, \end{cases}$$

a partition with $|\mu| < |\lambda|$. Label the regions of $\phi(\lambda)$ as in Fig. 2. The diagram of $\phi(\lambda)$ consists of a region a, a hole, a sequence of consecutive particles, and then a region b of holes. The diagram of $\phi(\mu)$ is similar: Region a is followed by the sequence of consecutive particles, then a hole, then region b. Applying the transformation $N \circ C$ to the diagram of $\phi(\lambda)$ yields $N \circ C(b)$, followed by a sequence of consecutive holes, then a particle, then $N \circ C(a)$. The region $N \circ C(b)$ consists solely of particles. The transformation $N \circ C$ takes the diagram of $\phi(\mu)$ to $N \circ C(b)$, followed by a particle, followed by a sequence of consecutive holes, followed by $N \circ C(a)$. By induction, this is the diagram of $\phi(\mu^*)$. Since $N \circ C(b)$ consists solely of particles, the diagram of $N \circ C(\phi(\lambda))$ corresponds to raising the next unraised particle of $\phi(\mu^*)$ by i positions. In particular, it is the diagram of the dual partition $\phi(\lambda^*)$. This completes the induction.

IPRs and diagonals of permutahedra

An alternate expression for $\sum_{\mathbf{x}} |c(\mathbf{x})|^4$ is obtained by multiplying out the terms of the determinant:

$$\sum_{\mathbf{x}} |c(\mathbf{x})|^4 = \frac{1}{M!} \sum_{P,Q,R,S\in S_M} \mathrm{sgn}(PQRS) \prod_{j=1}^{M} \sum_{x_1,\ldots,x_M=1}^{L} e^{ix_j(k_{P(j)}+k_{Q(j)}-k_{R(j)}-k_{S(j)})} \tag{93}$$

$$= \frac{L^M}{M!} \sum_{P,Q,R,S\in S_M} \mathrm{sgn}(PQRS)\, \delta(k_{P(j)} + k_{Q(j)} - k_{R(j)} - k_{S(j)} \equiv 0 \mod 2\pi, \forall j). \tag{94}$$

Using $k_j = \frac{2\pi I_j}{L}$, this sum is equal to

$$\frac{L^M}{M!} \sum_{P,Q,R,S\in S_M} \mathrm{sgn}(PQRS)\, \delta(I_{P(j)} + I_{Q(j)} - I_{R(j)} - I_{S(j)} \equiv 0 \mod L, \forall j) \tag{95}$$

$$= \frac{L^M}{M!} \sum_{P,Q,R,S\in S_M} \mathrm{sgn}(PQRS)\, \delta(I_P + I_Q - I_R - I_S \equiv 0 \mod L). \tag{96}$$

Consequently,

$$\mathbf{t}_\lambda = \frac{1}{L^M M!} \sum_{P,Q,R,S\in S_M} \mathrm{sgn}(PQRS)\, \delta(I_P + I_Q - I_R - I_S \equiv 0 \mod L). \tag{97}$$

We may interpret the sum as follows. A permutahedron is the polytope obtained by taking the convex hull of all permutations of a fixed vector $(I_1, I_2, \ldots, I_M)$ with distinct entries. We can interpret $I_P - I_R$ as a diagonal vector of this permutahedron. Then a condition of the form

$$I_P - I_R = I_S - I_Q \tag{98}$$

would be a condition on two diagonals to be mutual translates, and a congruence of the form

$$I_P - I_R \equiv I_S - I_Q \pmod{L} \tag{99}$$

is interpreted similarly.

V. FUTURE DIRECTIONS

Several interesting questions remain to be studied.

(1) Numerical experiments suggest that

$$\lim_{M\to\infty} \frac{\langle J_\lambda^{(1/2)}, J_\lambda^{(1/2)} \rangle_{2M;4}}{\langle J_\lambda^{(1/2)}, J_\lambda^{(1/2)} \rangle_4} = 1. \tag{100}$$

In other words, for a finite number of variables, the discrete Hall inner product is asymptotically equal to the Hall inner product when $L = 2M$. The reason this is not a simple consequence of the convergence of a Riemann sum to the corresponding integral is that even though the number of sample points L on each torus increases, the number of torii M does as well. Nonetheless, numerical experiment suggests that the error decreases (see Fig. 3). If true, this will allow us to extend the methods to compute the asymptotics of IPRs when $L = 2M$.

(2) There exists a dynamical interpretation of the IPRs (described in Ref. [3]). In the case when the spectrum is nondegenerate, it is given by summing the IPRs:

$$T := \sum_k t_k = \sum_{i,k=1}^{D} |\langle a_i|\psi_k\rangle|^4. \tag{101}$$

This quantity measures how much the eigenstates are localized in the preferential basis. It can range from $T_{\min} = 1$ (the eigenstates are spread maximally over the whole basis) to $T_{\max} = D$ (each eigenstate matches a basis vector). The ratio T/D measures the stationary return probability to an initial basis state, averaged over all the basis states. The minimum value is reached if the dynamics connects any initial basis state to all the other basis states. On the other hand, if it takes on the maximum value, then the system does not evolve if it is initialized from a basis state. What can be said about the quantity T for XX?

(3) The scalar product $\langle \cdot,\cdot \rangle_{L;\beta}$ is a discretization of the Hall inner product. Using the Gram-Schmidt orthonormalization procedure on the symmetric monomial functions, we may define an orthonormal basis which is triangular with respect to the monomial symmetric basis. Such "discrete Jack polynomials" may have interesting properties.

(4) The data table suggests that the IPR of the ground state is largest. The next few excited states, labeling by partitions, have IPRs which are approximately $1/2, 1/3, 1/4$, and $1/5$ of the ground state IPR, for $|\lambda| = 1,2,3,4$. It is unlikely that this pattern will continue. It would be interesting to determine the distribution of the values of the IPRs.

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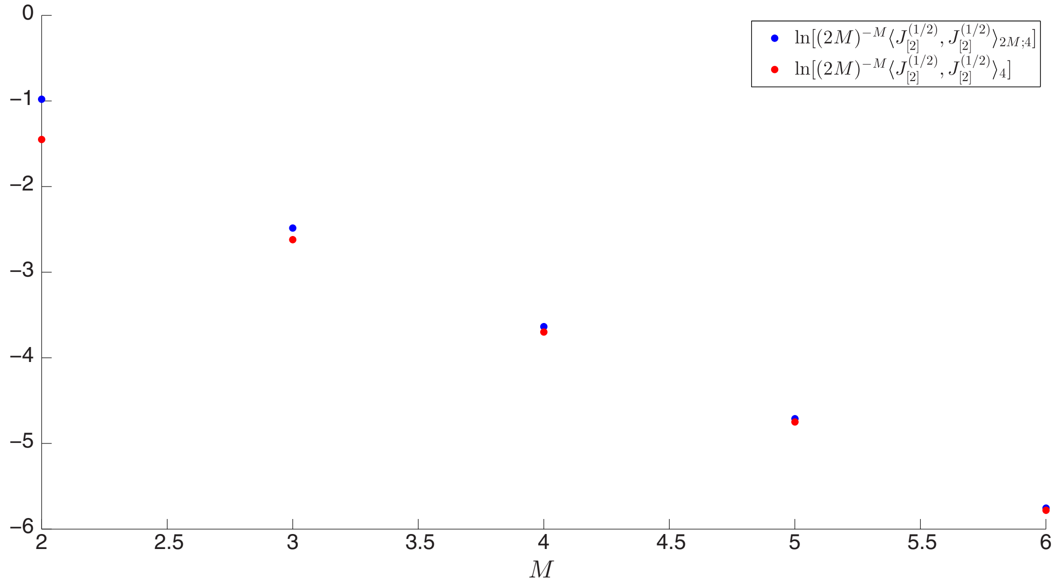


FIG. 3. Plotted are the logs of $\frac{\langle J_{[2]}^{(1/2)}, J_{[2]}^{(1/2)}\rangle_{L;4}}{L^M}$ and $\frac{\langle J_{[2]}^{(1/2)}, J_{[2]}^{(1/2)}\rangle_{4}}{L^M}$ for $L = 2M$.

APPENDIX A: CONNECTION BETWEEN THE IPRS OF XX, THE CIRCULAR SYMPLECTIC ENSEMBLE, AND THE DYSON-GAUDIN COULOMB GAS

We show in this section how to interpret the IPRs of XX in terms of expectation values of quantities for two-dimensional Coulomb gas on a one-dimensional lattice, or equivalently, as expectation values over a discretization of the circular symplectic ensemble.

In studying his famous random matrix ensembles, Dyson [8] developed a physical model for the eigenvalues of the matrices of the circular ensemble consisting of charges distributed on a unit circle in two dimensions and experiencing Coulomb forces. In Ref. [9], Gaudin studied a discrete formulation of the problem, in which the gas particles lie on lattice sites of the circle. This formulation corresponds to the finite L situation of the IPRs above, whereas Dyson's to the limit as $L \to \infty$. We will now explain the details.

1. Circular symplectic ensemble

The circular symplectic ensemble (CSE) is the space of self dual unitary quaternion matrices with a probability measure defined as follows. Each element is a unitary matrix, so it has eigenvalues on the unit circle $e^{i\theta_1},\dots,e^{i\theta_M}$. The probability density function for the phases in the circular symplectic ensemble is given by

$$p(\theta_1,\dots,\theta_M) = \frac{1}{Z_{M,4}} \prod_{1\leqslant i<j\leqslant M} |e^{i\theta_i} - e^{i\theta_j}|^4. \tag{A1}$$

The normalization constant is given by

$$Z_{M,4} = (2\pi)^M \frac{(2M)!}{2^M}. \tag{A2}$$

2. Dyson-Gaudin Coulomb gas

The positions which a unit charge can occupy on the circumference of a unit circle are restricted to L equidistant points $\exp(i\theta_j)$, $\theta_j = 2\pi j/L$, $1 \leqslant j \leqslant L$. One considers three distinguished values of the inverse temperature $\beta = 1,2,4$ [10]. The joint probability density for M unit charges to occupy positions $j_1,\dots,j_M$ is given by

$$P_\beta(j_1,\dots,j_M) = C_{LM\beta}^{-1} L^{-M} \exp(-\beta W). \tag{A3}$$

Here W is the potential energy, calculated as follows. If we place point unit charges at angles $\theta_1,\dots,\theta_M$ on a circle in two dimensions of radius 1, then the potential energy is equal to

$$W = -\sum_{1\leqslant j<k\leqslant M} \log|e^{i\theta_k} - e^{i\theta_j}|, \quad \theta_l = 2\pi j_l/L. \tag{A4}$$

Note in particular that

$$\exp(-\beta W) = \prod_{1\leqslant j<k\leqslant M} |e^{i\theta_k} - e^{j\theta_j}|^\beta. \tag{A5}$$

The expected value of a quantity $f(e^{i\theta_1},\dots,e^{i\theta_M})$ is given by

$$\begin{aligned}\mathbb{E}_\beta(f) &= \sum_{\boldsymbol{\theta}} f(e^{i\theta_1},\dots,e^{i\theta_M}) P_\beta(j_1,\dots,j_M)\\ &= C_{LM\beta}^{-1} L^{-M} \sum_{\boldsymbol{\theta}} f(e^{i\theta_1},\dots,e^{i\theta_M})\\ &\quad\times \prod_{1\leqslant j<k\leqslant M} |e^{i\theta_k} - e^{i\theta_j}|^\beta.\end{aligned} \tag{A6}$$

In Ref. [9], Gaudin calculates the partition function of the discrete Coulomb gas. In particular, he determines the normalization constant for $\beta = 4$:

$$\sum_{\boldsymbol{\theta}} \prod_{1\leqslant j<k\leqslant M} |e^{i\theta_k} - e^{i\theta_j}|^4 = \frac{(2M)!L^M}{2^M M!}. \tag{A7}$$

Hence

$$\mathbb{E}_{\beta=4}(f) = \frac{2^M M!}{(2M)!L^M} \sum_{\boldsymbol{\theta}} f(e^{i\theta_1},\ldots,e^{i\theta_M}) \times \prod_{1\leqslant j<k\leqslant M} |e^{i\theta_k} - e^{i\theta_j}|^4. \quad \text{(A8)}$$

We recognize that

$$\mathbf{t}_\lambda = \frac{(2M)!}{2^M M! L^M} \mathbb{E}_{\beta=4}(|s_\lambda|^4). \quad \text{(A9)}$$

Thus the IPR of the ground state is equal to

$$\mathbf{t}_0 = \frac{(2M)!}{2^M M! L^M} \quad \text{(A10)}$$

as determined previously in Refs. [1,3].

APPENDIX B: JACK AND SCHUR POLYNOMIALS

In this section we gather the results from symmetric function theory that we will use to analyze the IPRs. The primary sources are Refs. [11] and [7].

1. Partitions

A partition $\lambda = [\lambda_1,\lambda_2,\ldots]$ is a weakly decreasing sequence of nonnegative integers. The length $l(\lambda)$ is the number of nonzero entries. Let $m_i(\lambda) = m_i$ be the number of times i appears in λ. Set

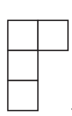
$$z_\lambda = \prod_{i\geqslant 1} i^{m_i} m_i!. \quad \text{(B1)}$$

The weight of a partition $|\lambda|$ is $\sum_i \lambda_i$. Given two partitions λ,μ of equal weight, the dominance partial order is defined by

$$\lambda \succeq \mu \iff \sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i \ \forall k. \quad \text{(B2)}$$

The Young diagram of a partition λ is a convenient graphical representation. It is obtained by putting λ_i boxes left aligned at row i. The conjugate partition λ^* of λ is obtained by reflecting the Young diagram of λ about the diagonal.

Example 5. Young diagrams and conjugate partitions. The Young diagram of $\lambda = [2,1,1]$ is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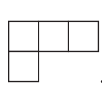
The conjugate partition λ^* is read off from the Young diagram after reflection in the diagonal: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Thus $\lambda^* = [3,1]$.

2. Jack polynomials

The Jack polynomials $J_\lambda^{(2/\beta)}$ are a notable family of symmetric polynomials parameterized by a real parameter β. For $\beta = 2$, they specialize to scaled Schur functions. For $\beta = 1$ one obtains the zonal polynomials and for $\beta = 4$ the quaternion zonal polynomials [12]. We will be using the "J" normalization. The Jack polynomials will be useful for us due to their property of orthogonalizing the circular ensembles [7, (10.36)]. We state this more precisely now.

Let λ and κ be a pair of partitions. Let λ^* denote the transpose of the partition λ and $l(\lambda)$ the number of nonzero parts in λ. Let $(i,j) \in \lambda$ refer to a cell of the Young diagram of λ [the indexing begins at (1,1)]. Set

$$\mathcal{N}_\lambda^{(\alpha)}(M) = \prod_{(i,j)\in\lambda} \frac{M + (j-1)\alpha - (i-1)}{M + j\alpha - i} \quad \text{(B3)}$$

and

$$C_\lambda(\alpha) = \prod_{(i,j)\in\lambda} (\alpha(\lambda_i - j) + \lambda'_j - i + 1) \times (\alpha(\lambda_i - j) + \lambda'_j - i + \alpha). \quad \text{(B4)}$$

The Jack polynomials satisfy [13]

$$\int_{[0,2\pi]^M} \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_M}{2\pi} J_\kappa^{(2/\beta)}(e^{i\theta_1},\ldots,e^{i\theta_n}) \times \overline{J_\lambda^{(2/\beta)}(e^{i\theta_1},\ldots,e^{i\theta_n})} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta = \delta_{\kappa,\lambda}\delta(l(\lambda)\leqslant M) C_\lambda(2/\beta) \mathcal{N}_\lambda^{(2/\beta)}(M) \frac{\Gamma(1+M\beta/2)}{\Gamma(1+\beta/2)^M}. \quad \text{(B5)}$$

The factor $\frac{\Gamma(1+M\beta/2)}{\Gamma(1+\beta/2)^M}$ is the one from Dyson's ex-conjecture [8],

$$\int_{[0,2\pi]^M} \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_M}{2\pi} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta = \frac{\Gamma(1+M\beta/2)}{\Gamma(1+\beta/2)^M}. \quad \text{(B6)}$$

The factor $\delta(l(\lambda)\leqslant M)$ is simply the statement that $J_\lambda^{(2/\beta)} = 0$ if the number of parts of λ is greater than M.

We define a corresponding inner product

$$\langle f,g\rangle_\beta = \frac{1}{M!} \int_{[0,2\pi]^M} \frac{d\theta_1\cdots d\theta_M}{(2\pi)^M} f\bar{g} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \quad \text{(B7)}$$

with respect to which

$$\langle J_\kappa^{(2/\beta)}, J_\lambda^{(2/\beta)}\rangle_\beta = \delta_{\kappa,\lambda} C_\lambda(2/\beta) \mathcal{N}_\lambda^{(2/\beta)}(M) \frac{\Gamma(1+M\beta/2)}{M!\Gamma(1+\beta/2)^M}. \quad \text{(B8)}$$

This inner product can be thought of as a specialization of the Hall inner product to finitely many variables. It may also be written as the extraction of a constant term (see, e.g., Ref. [7, VI, 10.35])

$$\langle f,g\rangle_\beta = \frac{1}{M!} \int_{T^M} f(z)\overline{g(z)} \Delta(z;\beta) = \frac{1}{M!} \mathrm{CT}[f\bar{g}\Delta(z;\beta)] \quad \text{(B9)}$$

$$\Delta(z;\beta) = \prod_{i\neq j} (1 - z_i z_j^{-1})^{\beta/2}, \qquad T = \{z : |z| = 1\}. \quad \text{(B10)}$$

A word about notation. Since our main interest will be with $\beta = 4$, we will reserve the shorter $\mathcal{N}_\lambda, \mathcal{C}_\lambda$ for that case.

The Jack polynomials also form an orthogonal basis of the symmetric polynomials with respect to the Hall inner product [13, (4.1),(4.3)],

$$\left\langle J_\lambda^{(2/\beta)}, J_\mu^{(2/\beta)} \right\rangle_\beta' = \delta_{\lambda\mu} C_\lambda(2/\beta) \tag{B11}$$

defined on power sum polynomials by

$$\langle p_\lambda, p_\mu \rangle_\beta' = \delta_{\lambda\mu} (2/\beta)^{l(\lambda)} z_\lambda. \tag{B12}$$

Conceptually, we will think of this inner product as being obtained from $\langle \cdot, \cdot \rangle_\beta$, upon approximating $\mathcal{N}_\lambda^{(\beta)}(M) \approx 1$, which occurs for large M. The Γ factors are a matter of normalization.

3. Schur polynomials

Let $s_\lambda(x_1, \ldots, x_M)$ denote the Schur polynomial associated to the partition $\lambda = (\lambda_1, \ldots, \lambda_M)$:

$$s_\lambda(x_1, \ldots, x_M) := \frac{\det\left[\left(x_i^{\lambda_j + M - j}\right)_{1 \leqslant i,j \leqslant M}\right]}{\det\left[\left(x_i^{M-j}\right)_{1 \leqslant i,j \leqslant M}\right]}. \tag{B13}$$

The Schur polynomials are (rescaled) Jack polynomials $J^{(1)}$ and form a linear basis for the symmetric polynomials. There is a combinatorial description for the product of two Schur functions. Namely, writing

$$s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu, \tag{B14}$$

the Littlewood-Richardson rule states that $c_{\lambda,\mu}^\nu$ is equal to the number of Littlewood-Richardson tableaux of skew shape ν/λ and of weight μ. The coefficients are known as the Littlewood-Richardson coefficients and appear in many other mathematical contexts.

4. Transition matrices

When discussing transition matrices between $\mathbb{Q}$ bases of symmetric polynomials [7, I §6], we index rows and columns by partitions of a positive integer n, arranged in reverse lexicographical order (so that $[n]$ is first and $[1^n]$ is last). A matrix $(M_{\lambda\mu})$ is *strictly upper triangular* if $M_{\lambda\mu} = 0$ unless $\mu \preceq \lambda$ in the dominance order on partitions. The strictly upper triangular matrices form a group. Given two $\mathbb{Q}$ bases $(u_\lambda), (v_\lambda)$, we denote by $M(u,v)$ the matrix $(M_{\lambda\mu})$ of coefficients in the equations

$$u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu. \tag{B15}$$

$M(u,v)$ is called the *transition matrix* from the basis (u_λ) to the basis (v_λ).

The following result in the special case when one of the Jack polynomials is specialized to be a Schur polynomial makes an appearance in experimental form in Ref. [14] and is derived using physical arguments in Ref. [15].

Proposition 3. The transition matrix $M(J^{\alpha_1}, J^{\alpha_2})$ is strictly upper triangular with respect to the dominance ordering of partitions.

Proof. Let (m_λ) denote the the monomial basis for the symmetric polynomials. The transition matrices $M(J^{\alpha_1}, m)$ and $M(J^{\alpha_2}, m)$ are strictly upper triangular. Consequently,

$$\begin{aligned} M(J^{\alpha_1}, J^{\alpha_2}) &= M(J^{\alpha_1}, m) M(m, J^{\alpha_2}) \\ &= M(J^{\alpha_1}, m) M(J^{\alpha_2}, m)^{-1} \end{aligned} \tag{B16}$$

is strictly upper triangular.

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