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Scaling laws of dipolar magnetic systems at finite temperature

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We report theoretical and numerical results on scaling laws for systems with multiple time scales in the context of kinetic dynamics of domain growth. The theoretical method has been tested by Monte Carlo simulations of a planar magnetic Heisenberg model with long-range interactions. We show that the characteristic length of the domain growth behaves according to a Lisfshitz-Allen-Cahn law with logarithmic corrections, such that the exponent of the correction depends on the temperature.

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I. INTRODUCTION

Scaling laws provide general properties of physical phenomena in a phenomenological manner. In fact, they can be viewed as proportionality relations between parameters characterizing a system and the spatiotemporal scales involved. The scaling laws appear in several branches, for instance in liquid crystals, polymers, fluids, and even in finance [1–6]. In complex systems, they can provide guides to some underlying general aspects of the system [6–9]. In particular, in magnetism they provide a useful tool to characterize experiments and theory based on Ising or (generalized) Heisenberg models [2,10].

One important application of scaling laws is the possibility to explain universal features of the domain growth in a wide range of physical systems [11–33]. In particular, the kinetic dynamics helps us to understand the pattern formation during domain growth, and typically coarsening or spinodal decomposition is found [22,23]. To describe a given system one can define an order parameter, which obeys a dynamical law. The asymptotical behavior of these dynamical laws can be algebraic laws [11–16,24,28], logarithmic ones [17,20,21,27], and in more intricate situations a combination of both can be found [25,29,33]. In particular, in the case of a magnetic Heisenberg system we will show that a power law with logarithmic corrections is obtained.

To characterize a system with two time scales we develop in this work a method that is based on a generalization of the differential equation for the characteristic length scale for the domain growth in a two-dimensional lattice with fixed distance d. We find a scaling law for the length scale, which is a combination of an algebraic power law with a logarithmic function. In order to gain more insight into the general model, we finally analyze the scaling laws for the planar Heisenberg model with dipole-dipole interaction through Monte Carlo simulations. We find that it obeys the same kind of scaling law with the coefficients depending on temperature, but being almost independent of the lattice size. In addition, if the interaction strength is varied by changing the lattice spacing d, generally the same type of scaling law is found with the scaling parameters depending on d. However, for larger d (smaller interaction strength) the logarithmic corrections tend to unity and the pure algebraic power is law is obtained. The manuscript is organized as follows: In Sec. II, the theoretical model for domain growth in a system with two time scales is presented. In Sec. III numerical simulations for a planar Heisenberg model are performed and the scaling laws are computed. Finally, conclusions are presented in Sec. IV.

II. THEORETICAL MODEL

Since pattern formation laws in nonequilibrium systems have been well established and characterized by dynamical scaling exponents, here we employ a method based on scaling arguments from the dynamical systems point of view. We assume for simplicity two time scales, although our arguments can be generalized to more time scales. For two-dimensional lattices one can define a characteristic length scale for the domain growth, which it is denoted by $L(t)$. This order parameter evolves as [17]

$$\frac{dL}{dt} = \frac{a(L,t,\lambda)}{L}, \tag{1}$$

where a is a function that depends on the physical system under study, such that λ represents the set of parameters involved, such as temperature, diffusion coefficient, or viscosity. This method reduces the information regarding spatiotemporal dynamics into a temporal dynamics, and it has been applied in many different systems. For example, when the order parameter in the system is conserved, L obeys $L \sim t^{1/3}$, called the Lifshitz-Slyozov (LS) scaling law [11]. This situation was derived in solid dynamics context, such that the rate of change of domain size is associated with the chemical potential gradient, $dL/dt \sim |\nabla\mu|$, μ being the chemical potential. Lifshitz and Slyozov found that the chemical potential is inversely proportional to the domain length $\mu \sim 1/L$, obtaining $dL/dt \sim 1/L^2$ [11]. When the order parameter of the system is not conserved, such as in chemical reactions [24], the characteristic length follows the Lifshitz-Allen-Cahn (LAC) law $L \sim t^{1/2}$ [12]. It is obtained by studying the dynamics of a droplet of radius L using a nonlinear diffusion equation [23]. Following the arguments of the aforementioned reference one can start with a spherical symmetric equation for the concentration field ϕ,

$$\frac{\partial\phi}{\partial t} = \nabla^2\phi - \frac{dV(\phi)}{d\phi} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{dV(\phi)}{d\phi}, \tag{2}$$

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where V is a potential function of ϕ. In the case of a double-well potential, this equation has two simple homogeneous solutions $\phi = \pm 1$, hence one can consider a single spherical domain of $\phi = -1$ immersed in a sea of $\phi = 1$. Therefore, the concentration has a heteroclinic connection between both homogeneous states, such that its trajectory can be written as $\phi(r,t) = f(r - L(t))$ where L is the droplet radius. Inserting this ansatz into Eq. (2), one finds that the droplet obeys an equation of the form [12] $dL/dt \sim 1/L$. A generalization of the LAC law for the drop curvature in reaction-diffusion systems was proposed by Binder and Stauffer [13]. The authors essentially assumed that the average distance between neighboring droplets is proportional to the average droplet radius and the dynamics obeys $dL/dt \sim D/L^2$ such that the diffusion coefficient depends on L as $D \propto L^{-x}$, producing a general algebraic law [13,22] $L \sim t^{1/(x+2)}$. In addition, specific power laws such as $L \sim t$ and $L \sim t^{2/3}$ can found in fluids in the viscous and inertial regimes, respectively [26]. On the other hand, for slow dynamics, the typical behavior is logarithmic, $L \sim \ln(t)$ [20]. It was also found from Monte Carlo simulations of Ising systems with first- and second-nearest neighbor interactions [17]. Finally, we remark that some classifications as a function of the temperature have been done in Ref. [21], in which a generalization the logarithmic law, $L \sim [\ln(t)]^p$, was performed.

In all of the aforementioned cases, one needs only one a function. In fact, to derive the LAC, LS and the logarithmic laws, the function a can be written as $a(L,t) = h(L)n(t) = L^\eta n(t)$. For this case, Eq. (1) is reduced to $L^{1-\eta}dL = n(t)dt$. The LAC and SL cases are found for $\eta = 0$ and $\eta = -1$, respectively, with $n(t) = a_0$ in both cases. The logarithmic law is obtained for $\eta = 3$ and $n(t) = a_0 t^{-1}$. Such an inversely proportional time dependence has been found in coarsening models of crystal surfaces [34]. Another interesting way to derive asymptotically the logarithmic law was made by Evans [27].

In situations with several spatiotemporal scales, we assume that $a(L,t)$ is a superposition of several functions. For two time scales as a minimum, this is a combination of two functions,

$$a(L,t) = h_1(L)n_1(t) + h_2(L)n_2(t), \tag{3}$$

where the two time-dependent components of the order parameter are (n_1,n_2) and the coefficients (h_1,h_2) are scaling functions of L. We assume that one of the time scales is slow and the other one is significantly faster. We know that, for slow dynamics, the function must contain a term of the form $n_j(t) = t^{-1}\tilde{n}_j(t)$, where $\tilde{n}_j$ is a function of t. The additional time scale is assumed to have a power law behavior, such that $\tilde{n}_1(t) \propto p$ and $\tilde{n}_2(t) \propto t^p$. The slow component also requires nonlinear terms of L in the h functions. Hence, without loss of generality one can write the a function as $a(L,t) = a_0 t^{-1} L^2(p + \varepsilon\, t^{p/\varepsilon} L^{-\varepsilon})$. In this case, Eq. (1) is reduced to

$$\frac{dL}{dt} - \frac{pL}{t} = \varepsilon\, t^{-1+p/\varepsilon} L^{1-\varepsilon}, \tag{4}$$

which is a particular Bernoulli differential equation [35]. Note that Eq. (4) can be solved analytically. To obtain the solution, one introduces the new variable $\psi = L^{1/\varepsilon}$ leading to $d\psi/dt - (p\psi)/(t\varepsilon) = t^{-1+p/\varepsilon}$. For $\mu = t^{p/\varepsilon}$, multiplying both sides by μ, one obtains after some algebra $d(t^{-p/\varepsilon}\psi)/dt = 1/t$. This equation can be directly integrated $\psi = t^{p/\varepsilon}\ln(\beta t)$, where β is an arbitrary integration constant. In terms of the original variable one has

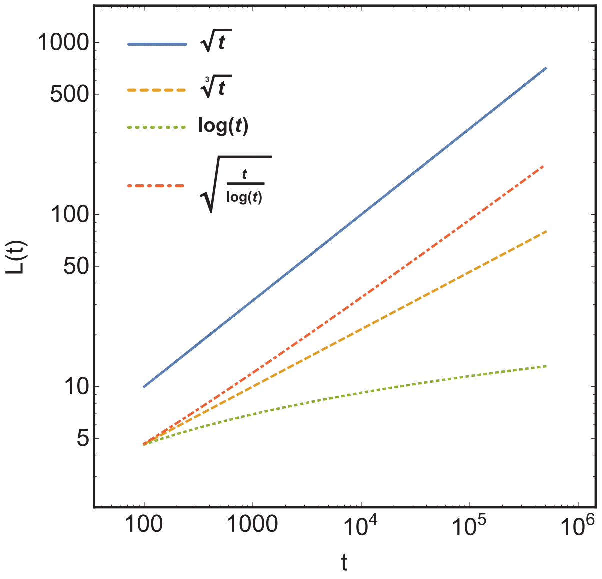

FIG. 1. Different behaviors of L as a function of time.

$$L(t) = L_0 t^p [\ln(\beta t)]^\varepsilon. \tag{5}$$

We remark that this formula can reproduce all the known cases; for example with $p \to 0$ and $\varepsilon \to 1$ one obtains the slow time scale behavior $\ln(\beta t)$, whereas for $\varepsilon \to 0$ the standard power law t^p of scaling results. We remark that there are several situations where particular cases of Eq. (5) have been found [23,25,29,30,32,33]. For instance, in binary mixtures subject to a uniform shear the domain growth follows $L \sim [t^5/\ln(t)]^{1/4}$ and $L \sim [t/\ln(t)]^{1/4}$ [25]. In addition, in two-dimensional liquid crystals with long-range interactions the domain growth shows a Lisfshitz-Allen-Cahn law with a logarithmic correction [23,32,33] $L \sim [t/\ln(t)]^{1/2}$. Figure 1 shows the typical scaling laws of L as a function of time. We can clearly observe that the amplitude of the $t^{1/2}$ law is larger than the logarithmic corrections $\ln(t)$ or $[\ln(t)]^{-1/2}$ as well others power laws like $t^{1/3}$. The pure logarithmic laws slowly increase in time with respect to all the rest.

III. SIMULATIONS

In order to study a particular system that contains multiple time scales, we consider a magnetic system with long-range interactions at a finite temperature, T. This system has at least two time scales. The first one is given by the collective behavior due to the interaction among the particles. It is lower than the second one, given by motions produced by thermal fluctuations. Let us consider a two-dimensional square array of $N \times N$ magnetic moments $\mathbf{m}_i$. The Hamiltonian of the system is given by [36]

$$\mathcal{H} = \sum_{j>i}^{N,N} \frac{1}{r_{ij}^3}[\mathbf{m}_i \cdot \mathbf{m}_j - 3(\mathbf{m}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{m}_j \cdot \hat{\mathbf{n}}_{ij})], \tag{6}$$

where $\mathbf{r}_i$ denotes the vector position of the particle i and $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = r_{ij}\hat{\mathbf{n}}_{ij}$ is the relative position of i with respect to j. We only consider planar spin rotations, hence the magnetic moments can be written as $\mathbf{m}_i = \{\cos\theta_i, \sin\theta_i\}$. Planar Heisenberg models with different static magnetic properties, taking into account dipole-dipole interaction, have been studied in Refs [36–46]. Note that the proper limit of this system is an antiferromagnetically ordered lattice [38,40]. Kinetic properties of planar models with exchange interactions have been studied by Monte Carlo simulations [15,16] as well as by molecular dynamics [18]. The authors found that the characteristic length behaves as $L \sim t^n$ with the exponent, n, depending on the system's parameters. In addition, planar systems with long-range interactions in in the context of liquid crystals show a characteristic length behavior as $L \sim [t/\ln(t)]^{1/2}$ [33].

The Monte Carlo (MC) simulations are performed with a standard Metropolis algorithm, such that in a MC move a randomly chosen moment $\mathbf{m}_i$ is rotated by an angle $\Delta\theta_i \in [-\pi/100,\pi/100]$. This move is accepted or rejected according to the probability $p = 1$, if $\Delta\mathcal{H} > 0$, or $p = \exp(-\Delta\mathcal{H}/k_B T)$, if $\Delta\mathcal{H} < 0$, where $\Delta\mathcal{H}$ is the change in energy and k_B Boltzmann's constant. Here we have used $N = 90$ and run up to 2×10^4 MC steps. All statistical quantities presented in the manuscript have been computed as averages over 15 independent runs. In dimensionless variables we explore a set of temperatures in the range $k_B T \in (0.08,0.28)$ with with a fixed lattice constant $d = 1$. For a Py (permalloy) array with $M_s = 840\,\mathrm{kA/m}$ and a lattice constant of $d_0 = 250$ nm the temperature range becomes approximately (115 K, 403 K). From Fig. 2 one can observe that the system exhibits a complex pattern formation with some clustering.

The magnetic domains growth with spatiotemporal scaling laws that are characterized by the length $L(t)$. A consequence the correlation function, $C(r,t)$, scales as [33]

$$C(r,t) = \alpha\langle\cos^2\theta_0\cos^2\theta_r\rangle - \beta = \mathcal{G}(r/L(t)), \tag{7}$$

where $\mathcal{G}$ is a time-independent function. The factors (α,β) are calculated to ensure that $C(0,t) = 1$ and $C(r,0) \to 0$ for $r \to \infty$. The length scale $L(t)$ is defined as the distance over which the correlation function falls to half its maximum value.

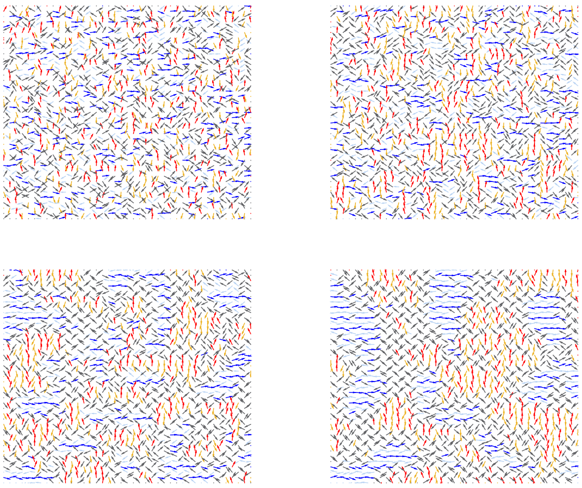

FIG. 2. Snapshots of the two-dimensional magnetic array at different subsequent times at $T = 173$ K.

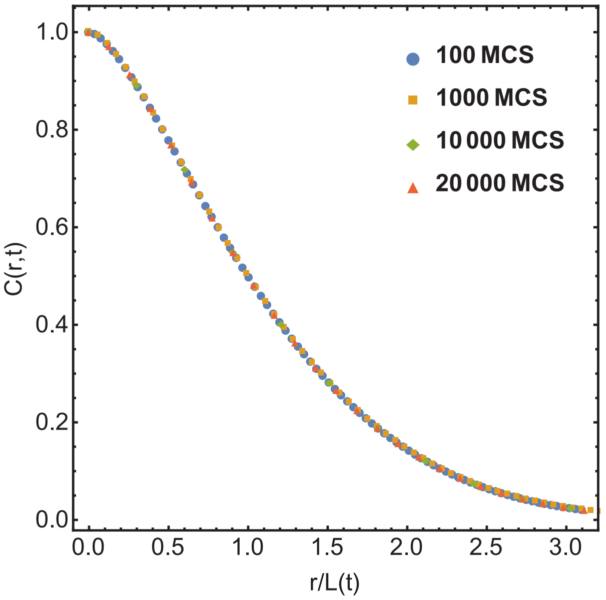

FIG. 3. Scaling plot of the correlation function as a function r/L for different MC steps at $T = 173$ K.

Figure 3 shows the correlation function, Eq. (7), as a function of r/L for different MC steps. We can observe that all the curves collapse to a single one, exposing its universal property. In addition, one can fit this curve by $C_{GAF}(r,t) = 1 + (1-\gamma^2)\ln(1-\gamma^2)/\gamma^2$ with $\gamma = \exp[-r^2/(6L^2)]$. We remark that a similar fitting function has been found in nematic liquid crystals with long-range interactions in the auxiliary Gaussian field approximation [33]. Figure 4 shows the characteristic length L as a function of MC steps for different values of the temperature. We can observe that L increases as the time increases and all curves follow the law $L(t) \sim \sqrt{t}/\ln(t)^\epsilon$, where ϵ depends on the temperature as shown in the inset of Fig. 4. This law is of the LAC type, but

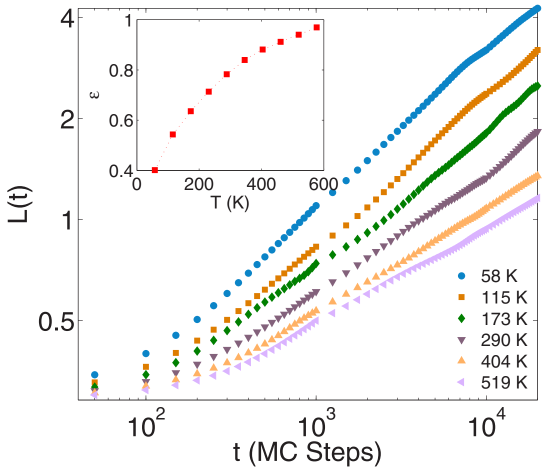

FIG. 4. L as a function of time for different values of the temperature. The inset shows the exponent ϵ in $L(t) \sim \sqrt{t}/\ln(t)^\epsilon$ as a function of temperature.

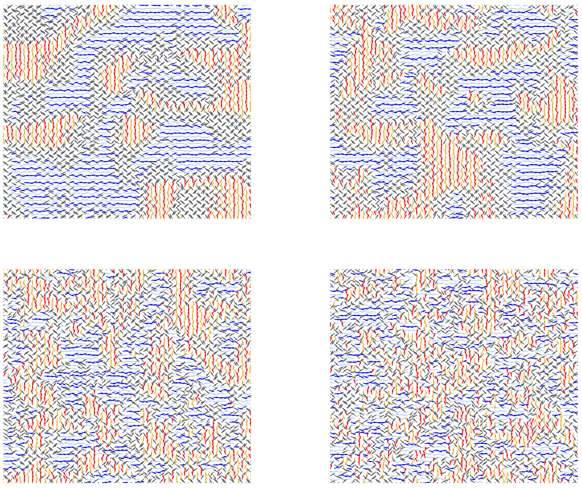

FIG. 5. Snapshots of the two-dimensional magnetic array for different temperatures $T = 58,115,290,519$ K or 2×10^4 MC steps and $d = 1$.

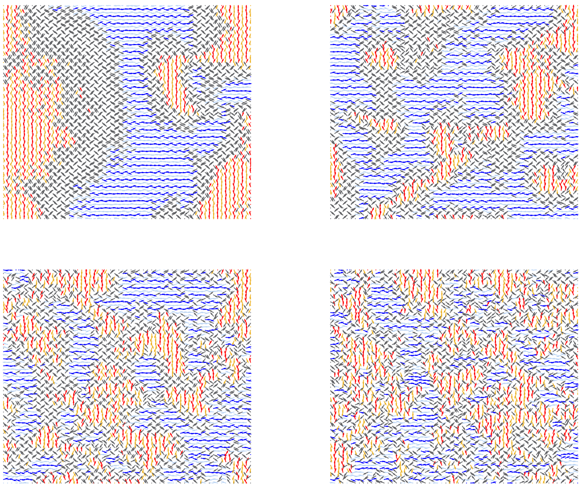

FIG. 7. Snapshots of the two-dimensional magnetic array for different distances $d = (0.5,0.75,1.0,1.25)$ at 2×10^4 MC steps and $T = 173K$.

with a logarithmic correction. At the same number of the MC steps, the increment of the temperature produces a decrease of L, and consequently of the correlation. Figure 5 shows that when the temperature increases the size of the dipolar domains decreases, producing a loss of magnetic order. This issue is consistent with the behavior of L.

In addition, one can define the Fourier transform of the correlation function, $S(\mathbf{k},t) = \int d^2r \exp{(i\mathbf{k}\cdot\mathbf{r})}C(r,t)$, as the *structure factor*. It is commonly related to experimental measurements and is also connected to the characteristic length with the scaling $S = [L(t)]^q \mathcal{F}[k\,L(t)]$, where q is the dimensionality ($q = 2$ in our case). Figure 6 shows $S(k,t)/L^2$ as a function of kL. We can observe that all the curves collapse in a single one, similar to the case of the correlation function. In addition, the structure factor decays asymptotically with a power law in kL.

A basic aspect of the statistical mechanics of magnetic systems is the influence of the interaction strength on the magnetic order. Indeed, if the interaction strength among particles is weak, magnetic order appears only at very low temperatures. In our case the distance among magnetic moments, d, is the control parameter of the interaction strength. Figure 7 shows the magnetic domains for four different distances at the same number of MC steps and temperature. Obviously, when d increases the size of the magnetic domains strongly decreases.

Figure 8 shows the characteristic length L as a function of MC steps for different values of the distance, d. We can observe that L increases when the distance among the particles is reduced, for the same number of MC steps, consistent with what is observed in the Fig. 7. In addition, we find that all the curves follow the law $L(t) \sim t^b/\ln{(t)}^\epsilon$, where b and ϵ depend on the distance as shown in the inset of Fig. 8. Let us remark that in contrast to what has been observed regarding the temperature dependence, the effect of the interaction strength requires not only a logarithmic generalization of the LAC law,

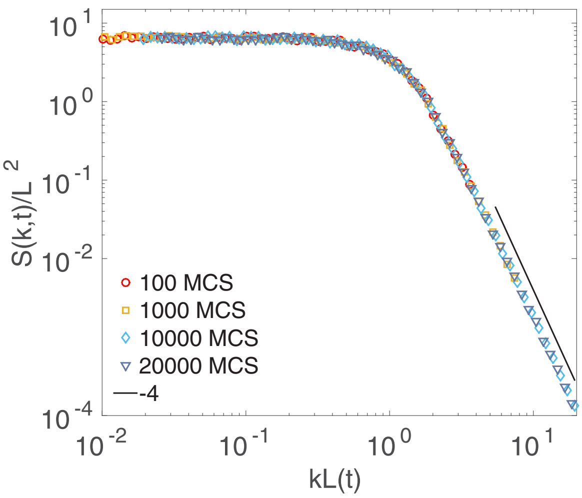

FIG. 6. Scaling plot of the structure factor as a function of kL for different MC steps at $T = 173$ K. The straight line with slope -4 denotes the generalized Porod law $S(k,t) \sim L^2(kL)^{-4}$.

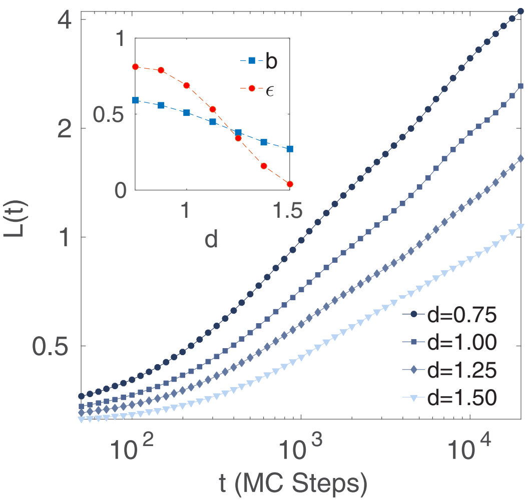

FIG. 8. L as a function of time for different values of the distance d at $T = 173$ K. The inset show the exponents b and ϵ in $L(t) \sim t^b/\ln(t)^\epsilon$ as functions of the distance.

but also an exponent of the power law that depends on d. Finally, one can observe that when d increases the coefficient ϵ tends to zero ($\epsilon \rightarrow 0$), and b tends to values lower than 0.5. Indeed, at $d = 1.5$ we obtain $\epsilon = 0.04$ and $b = 0.27$. Hence, large values of d implies that the two time scales disappears, the particles behaves like noninteracting ones, and, consequently, the characteristic length behaves with a power law, $L \sim t^b$.

IV. CONCLUSIONS

In summary, we show how multiple time scales can be incorporated into a generalized scaling law for domain growth. This law is fully consistent with diffusional growth in a droplet model. The approach is illustrated by an analysis of scaling lengths and correlation functions obtained from Monte Carlo simulations of a two-dimensional antiferromagnet with dipole-dipole interaction. We found that for a fixed distance the domain growth behaves according to a generalized LAC law with a logarithmic correction.

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