Entanglement dynamics in quantum many-body systems

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The dynamics of entanglement has recently been realized as a useful probe in studying ergodicity and its breakdown in quantum many-body systems. In this paper, we study theoretically the growth of entanglement in quantum many-body systems and propose a method to measure it experimentally. We show that entanglement growth is related to the spreading of local operators in real space. We present a simple toy model for ergodic systems in which linear spreading of operators results in a universal, linear-in-time growth of entanglement for initial product states, in contrast with the logarithmic growth of entanglement in many-body localized (MBL) systems. Furthermore, we show that entanglement growth is directly related to the decay of the Loschmidt echo in a composite system comprised of several copies of the original system, in which connections are controlled by a quantum switch (two-level system). By measuring only the switch's dynamics, the growth of the Rényi entropies can be extracted. Our work provides a way of understanding entanglement dynamics in many-body systems and to directly measure its growth in time via a single local measurement.

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I. INTRODUCTION AND RESULTS

In the past decade, quantum entanglement has emerged as an indispensable tool for characterizing and classifying the ground states of many-body systems; for example, in the field of topological order [1,2].

Recently, it was realized that entanglement *dynamics* also exhibits universality, providing a useful probe of ergodicity and its breakdown in many-body systems. In particular, the growth of entanglement following a quantum quench can be used to distinguish between localized and ergodic phases. For many-body localized (MBL) systems [3,4], emergent local integrability [5–8] implies that entanglement grows logarithmically in time, $S(t) \sim \log t$, following a quantum quench from initial product states [9,10]. For generic quantum ergodic systems, entanglement appears to grow universally linearly [11]: $S(t) \sim t$.

Previous works predicted linear growth $S(t) \sim t$ in integrable systems, including $(1+1)$-dimensional CFTs [12,13], while other works studied entanglement growth in nonintegrable, higher-dimensional CFTs via holographic calculations [14,15]. However, an understanding of the entanglement dynamics in generic quantum chaotic or ergodic systems is lacking. One puzzling fact pointed out in Ref. [11] is that energy transport in a quantum chaotic or ergodic many-body system under a time-independent Hamiltonian is diffusive ($\sim\sqrt{t}$), yet entanglement growth is linear ($\sim t$)—clearly, the mechanisms of particle and quantum information transport are different, and so it is important to understand the dynamics of the latter.

The purpose of this paper is to provide (1) a theoretical description, and hence, a physical picture of the growth of entanglement entropy (EE) in a many-body system, and (2) a proposal to experimentally measure it. To be precise, we consider the dynamics of the nth Rényi EE $S_n(t)$ of pure states $|\psi\rangle$ that are initially random product states, evolving unitarily under a local, potentially time-dependent nonintegrable many-body Hamiltonian:

$$H = \sum_X H_X. \tag{1}$$

The Hamiltonian is defined on a lattice Λ of sites (labeled by i) in d spatial dimensions so that the local Hilbert space $\mathcal{H}_i$ is bounded: $\dim(\mathcal{H}_i) = k < \infty$, and X is a local region in Λ. The nth Rényi EE of A, for a bipartition of the system into two subregions A and B, is given by

$$S_n(t) = \frac{1}{1-n} \log \mathrm{Tr}\left[\rho_A^n(t)\right], \tag{2}$$

with $\rho_A(t) \equiv \mathrm{Tr}_B(U_t|\psi\rangle\langle\psi|U_t^\dagger)$ being the reduced density matrix of subsystem A, and U_t being the unitary time evolution operator generated by H according to the Schrödinger equation. The state, being initially a random product state, has $S_n(0) = 0$, but $S_n(t) > 0$ for $t > 0$ and grows in general. Also, $S_n \geqslant S_m$ for any $m > n$; in particular, S_2 is a lower bound for S, the von Neumann entropy.

A summary of our results is as follows: We show that the growth of entanglement as measured by $S_2(t)$ is directly related to the measurement of basis operators of subregion A. Under time evolution by a local Hamiltonian, a basis operator physically spreads in real space and, as it grows, the value of its measurement typically decreases, thus leading to entanglement growth. Furthermore, we introduce a simple toy model for ergodic many-body systems in which (1) operators spread at some maximal velocity v, (2) delocalize completely within this light cone $r \sim vt$ and show that $S_2(t)$ grows linearly in time with a velocity related to v. We believe such a model captures the salient features of the universal linear-in-time growth of entanglement seen by Ref. [11].

We also propose a way to experimentally measure $S_n(t)$ via a local measurement. We introduce a quantum switch (a two-level system) that allows tunneling between different parts of a replicated system consisting of n disjoint copies of the original system, depending on the state of the switch [16]. By preparing the replicated system and the quantum switch appropriately, and by subsequently measuring $\sigma^x(t)$ of the quantum switch only, the entropy growth $S_n(t)$ corresponding to the entanglement entropy of the original system can be measured; see Fig. 1.

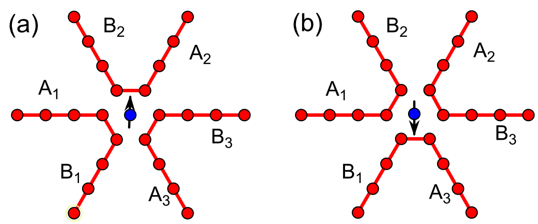

FIG. 1. Setup to measure the growth of the nth Rényi entropy $S_n(t)$. Here $n = 3$ and we have shown it for a $d = 1$ chain. There is a quantum switch in the middle of the setup which governs tunneling between different subsystems. (a) When the quantum switch is in the state $|\uparrow\rangle$, tunneling between A_i and B_i is allowed while tunneling between A_i and B_{i+1} is prohibited. (b) When the quantum switch is in the state $|\downarrow\rangle$, the allowed and prohibited tunnelings are swapped. A composite state in an n-copy product state of the chains and in a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ of the quantum switch will have two parts evolving in time differently according to the quantum switch; measuring $\sigma^x(t)$ of the quantum switch gives the Loschmidt echo and hence the nth Rényi entropy.

We refine and expound upon these ideas below. For clarity of argument, we focus on the case of a spin-$\frac{1}{2}$ system, so $\dim(\mathcal{H}_i) = 2$, but our results are general and hold for systems with bounded local Hilbert spaces of other dimensions [17]. We also choose subregion A to be a ball of radius r_A with volume $\gamma_d r_A^d = N_A$ in d dimensions [$\gamma_d = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of a unit d ball], but the analysis can be readily adapted to other geometries.

II. LOCAL MEASUREMENTS OF BASIS OPERATORS GIVES ENTANGLEMENT

We first show that the second Rényi entropy $S_2(t)$ is directly related to measuring all time-evolved basis operators $\mathcal{O}_A(t) = U_t^\dagger \mathcal{O}_A U_t$ in the initial state $|\psi\rangle$, such that the operator $\mathcal{O}_A$ at $t = 0$ has support strictly in A (i.e., where it acts nontrivially on the lattice). That is, for a spin-$\frac{1}{2}$ system of N_A sites, $S_2(t) = -\log \mathrm{Tr}_A[\rho_A^2(t)]$, with

$$\mathrm{Tr}_A\left[\rho_A^2(t)\right] = \frac{1}{2^{N_A}} \sum_{\mathcal{O}_A} \langle\psi|\mathcal{O}_A(t)|\psi\rangle^2, \tag{3}$$

where the sum goes over all operators $\mathcal{O}_A$ with support strictly in A, assumed to be Hermitian and independent as defined by the Hilbert–Schmidt inner product: $\frac{1}{2^{N_A}}\mathrm{Tr}_A(\mathcal{O}_A^\dagger \mathcal{O}'_A) = \delta_{\mathcal{O}_A,\mathcal{O}'_A}$, and which form a basis for all Hermitian operators on A. A potential basis set is $\otimes_{i=1}^{N_A}\{\mathbb{I}_i, \sigma_i^x, \sigma_i^y, \sigma_i^z\}$, where $\mathbb{I}_i, \sigma_i^\alpha$ are the identity matrix and Pauli matrices acting on site i, respectively, and we will use this basis for our subsequent analysis. The proof of this statement is straightforward—one notes that the reduced density matrix $\rho_A(t) = \mathrm{Tr}_B\,|\psi(t)\rangle\langle\psi(t)|$ is in particular a Hermitian operator, and so must be a linear combination of $\mathcal{O}_A$ operators with real coefficients given by $\frac{1}{2^{N_A}}\langle\psi(t)|\mathcal{O}_A|\psi(t)\rangle = \frac{1}{2^{N_A}}\langle\psi|\mathcal{O}_A(t)|\psi\rangle$. Then, squaring $\rho_A(t)$ and using the orthonormality of $\mathcal{O}_A$ under the inner product, one obtains the claimed result.

III. PHYSICAL SPREADING OF BASIS OPERATORS LEADS TO ENTANGLEMENT GROWTH

We now use the above result to understand why entanglement typically grows in many-body systems. We consider the ensemble average of $S_2(t)$ over initial pure product states $|\psi\rangle = |\vec{\sigma}_1\vec{\sigma}_2\cdots\vec{\sigma}_N\rangle$, where $\vec{\sigma}_i$ is the random direction that the ith spin is pointing to on its Bloch sphere.

It is convenient to switch from an average over an ensemble of initial states which are random product states to an average over an ensemble of locally rotated Hamiltonians. A given initial product state can be written as $|\vec{\sigma}_1\vec{\sigma}_2\cdots\vec{\sigma}_N\rangle = \prod_i V_i|\uparrow_1\uparrow_2\cdots\uparrow_N\rangle$ where V_i is the local unitary that rotates $|\uparrow_i\rangle$ into $|\sigma_i\rangle$. We can rewrite Eq. (3) as

$$\begin{aligned}\mathrm{Tr}_A\left[\rho_A^2(t)\right] &= \frac{1}{2^{N_A}}\sum_{\mathcal{O}_A}\langle\uparrow\uparrow\uparrow|\tilde{U}_t^\dagger \mathcal{V}^\dagger \mathcal{O}_A \mathcal{V}\tilde{U}_t|\uparrow\uparrow\uparrow\rangle^2 \\ &= \frac{1}{2^{N_A}}\sum_{\mathcal{O}_A}\langle\uparrow\uparrow\uparrow|\tilde{U}_t^\dagger \mathcal{O}_A \tilde{U}_t|\uparrow\uparrow\uparrow\rangle^2,\end{aligned} \tag{4}$$

where $\mathcal{V} \equiv \prod_i V_i$ and $\tilde{U}_t$ is the time evolution operator generated by the locally rotated Hamiltonian $\tilde{H} = \mathcal{V}^\dagger H \mathcal{V}$. This local rotation generates a new Hamiltonian $\tilde{H}$ that has the same locality properties as the original Hamiltonian H, i.e., $\tilde{H} = \sum_X \tilde{H}_X$, which implies that both H and $\tilde{H}$ have the same Lieb–Robinson (LR) velocities v_{LR} [18–20], bounds which govern the speed of information and operator spreading quantum many-body systems. The second equality arises from a straightforward statement of "basis invariance" of measurements [17]. Thus, taking an ensemble average of $S_2(t)$ over initial product states in Eq. (3) is equivalent to taking an ensemble average over locally rotated Hamiltonians $\tilde{H}$ in Eq. (4) with the measurement done in the particular state $|\uparrow\uparrow\uparrow\rangle$.

We are now in a position to gain a physical understanding of how entanglement grows in a many-body system: under time evolution, a basis operator $\mathcal{O}_A$ with support in A spreads on the lattice to become $\mathcal{O}_A(t) \equiv \tilde{U}_t^\dagger \mathcal{O}_A \tilde{U}_t$ (note the evolution under $\tilde{H}$), which has not only larger support but is also a complicated sum of other basis operators, although with conserved "weight"; for example,

$$\begin{aligned}\sigma_i^x \xrightarrow{t} \tilde{U}_t^\dagger \sigma_i^x \tilde{U}_t &= c_i^x \sigma_i^x + c_{i,i+1}^{x,y}\sigma_i^x\sigma_{i+1}^y + c_{i-1}^z\sigma_i^z \\ &\quad + c_{i-1,i,i+1}^{z,x,z}\sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z + \cdots \equiv \sum_{X,\mu} c_X^\mu \sigma_X^\mu,\end{aligned} \tag{5}$$

such that the total weight $\sum_{X,\mu}(c_X^\mu)^2 = 1$ for all times [17]. Then, the value of the measurement $\langle\uparrow\uparrow\uparrow|\mathcal{O}_A(t)|\uparrow\uparrow\uparrow\rangle$ typically decreases, as the complicated sum of operators will have many "off-diagonal" operators, such as $\sigma_i^x\sigma_{i+1}^y$, etc., that do not contribute, when only the "diagonal" operators which are products of σ_i^z do, leading to an increase in entanglement entropy as calculated by Eq. (4). Thus we see that entanglement growth in a many-body system is intimately related to the physical spreading of basis operators in real space: in some sense, EE increases because quantum information is "lost" in the inability of the state $|\uparrow\uparrow\uparrow\rangle$ to measure the increasingly complicated operator $\mathcal{O}_A(t)$.

IV. A TOY MODEL FOR EXPLAINING UNIVERSAL LINEAR GROWTH IN ERGODIC SYSTEMS

Equations (3) and (5) are true regardless of the system in question, be it ergodic or otherwise. The differences in entanglement growth between different systems are completely captured in how an operator spreads and is decomposed in terms of other basis operators; cf. Eq. (5). Here, let us introduce a simple toy model where operators completely "scramble" in a linear light cone, which gives a linear growth of entanglement. We believe that such a model captures the salient features of universal linear entanglement growth in generic [21] ergodic many-body systems, as seen by Ref. [11].

For local Hamiltonians with a bounded local Hilbert space, the velocity at which basis operators spreads can be at most linear, with an upper bound given by v_{LR}. In other words, the operator can only spread within the LR light cone $r \sim v_{LR}t$. The precise distribution of the coefficients $\{c_X^\mu\}$ in Eq. (5) depends on the Hamiltonian in question. In our toy model, we make a statistical statement about the distribution of coefficients. Let us assume that, because of ergodicity, (1) a basis operator spreads linearly at some fixed velocity $v \leqslant v_{LR}$ (for all $\tilde{H}$) so that an initially local operator at i spreads to become a sum of operators contained within a ball of radius $(vt)^d$ centered at i, and (2) it is effectively scrambled in this ball. Precisely, we assume that an initially localized basis operator at i has a decomposition as in Eq. (5) under time evolution, such that $\{c_X^\mu\}$ is a unit random vector of $4^{\gamma_d (vt)^d}$ coefficients, where X, the support of each basis operator in the decomposition is contained entirely within a ball of radius $(vt)^d$ centered at i. Such a picture is indeed supported by Ref. [22], and we assume this behavior to be generically true for ergodic systems. We note that such a scrambling assumption, tied to ergodicity, will manifestly not hold for MBL systems: basis operators that have significant overlap with the local integrals of motions stay localized near site i, invalidating the assumption.

Next, we estimate the measurement $\langle\uparrow\uparrow\uparrow|\mathcal{O}_A(t)|\uparrow\uparrow\uparrow\rangle$ in Eq. (4). In principle, we have to sum over all operators $\mathcal{O}_A$, but we can replace all $\mathcal{O}_A \to \mathcal{O}_{A,\mathrm{typ.}}$, where $\mathcal{O}_{A,\mathrm{typ.}}$ is a typical operator in subregion A having support of size $\bar{L} = 3/4N_A$. (Recall that the support of an operator are the sites where $\mathcal{O}_A$ acts nontrivially on; for example, $\mathcal{O}_A = \sigma_1^z \otimes \mathbb{I}_2 \otimes \sigma_3^x \otimes \mathbb{I}_4 \otimes \cdots \otimes \mathbb{I}_{N_A}$ has support on sites $1 \cup 3$ and size 2.) The statement of the size being $\bar{L} = 3/4N_A$ follows because, in constructing some $\mathcal{O}_A$, there is a choice of three nontrivial operators $\{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$ compared with a choice of a single trivial operator $\mathbb{I}_i$ for every site, so there are $\binom{N_A}{L}3^L$ operators with support of size L, a skewed binomial distribution, which gives the average size $\bar{L} = 3/4N_A$. Typicality follows because the standard deviation of the size of an operator $\sim\sqrt{N_A} \ll N_A$, coming from the same skewed binomial distribution. Thus, $\mathcal{O}_A(t)$ can be replaced by $\mathcal{O}_{A,\mathrm{typ.}}(t)$ because the latter dominates the sum.

We now have to understand the time-evolution of $\mathcal{O}_{A,\mathrm{typ.}}$. Considered as an operator on the subregion A, $\mathcal{O}_{A,\mathrm{typ.}}$ has $N_A - L$ sites on which it acts trivially, i.e., with the identity operator. We call groups of contiguous $\mathbb{I}_i$ "clusters". A typical cluster has size $\sim O(1)$. Over time, each nontrivial single-site operator that makes up the typical operator spreads, emitting a spherical "wavefront", and so the clusters quickly get filled up after $vt \sim O(1)$. After such time, the resulting operator will therefore have grown to become a complicated operator within a ball of radius $(r_A + vt)^d$; see Fig. 2. Following the previous discussion, we assume that the vector of coefficients $\{c_X^\mu\}$ denoting the decomposition of a typical time-evolved operator into the basis operators of the ball is also a random vector but now with dimension $4^{\gamma_d (r_A+vt)^d}$.

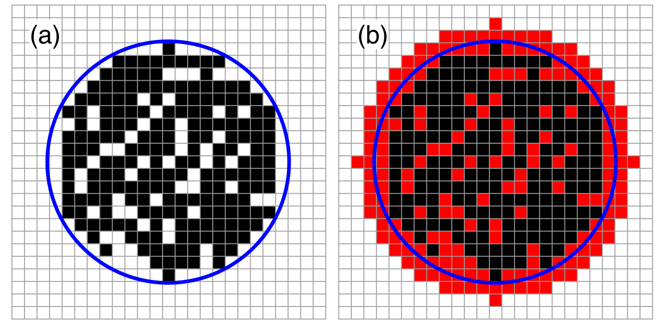

FIG. 2. (a) The support of a typical basis operator $\mathcal{O}_A$. Here Λ is a $d = 2$ square lattice, and the subregion A is a circle of radius r_A demarcated by the blue circle. Black sites represent the presence of a nontrivial operator, σ_i^x, σ_i^y, or σ_i^z, while white sites represent the presence of an identity operator $\mathbb{I}_i$. One can see clusters of white sites within the circle. A site within a cluster is $\sim O(1)$ site away from a black site. (b) Under time evolution, each black site spreads with velocity v, and so the clusters quickly get filled up (red sites) in time $vt \sim O(1)$, and simultaneously the operator also grows in physical size to become a ball of radius $r_A + vt$.

Next, we estimate the measurement of a typical operator. This is given by

$$\langle\uparrow\uparrow\uparrow|\mathcal{O}_{A,\mathrm{typ.}}(t)|\uparrow\uparrow\uparrow\rangle^2 = \frac{2^{\gamma_d (r_A+vt)^d}}{4^{\gamma_d (r_A+vt)^d}}, \tag{6}$$

which can be understood as follows: In the decomposition $\mathcal{O}_{A,\mathrm{typ.}}(t) = \tilde{U}_t^\dagger \mathcal{O}_{A,\mathrm{typ.}} \tilde{U}_t = \sum_{\mathcal{O}} c_{\mathcal{O}}(t)\mathcal{O}$, the only operators $\mathcal{O}$ which give a nonzero overlap with $|\uparrow\uparrow\uparrow\rangle$ are the diagonal operators $\mathbb{I}, \sigma_i^z, \sigma_i^z\sigma_j^z, \ldots$. There are $2^{\gamma_d (r_A+vt)^d}$ such operators contained in a ball of radius $(r_A + vt)^d$. The coefficient $c_{\mathcal{O}}$ that accompanies such an operator is a random variable (which follows from our scrambling assumption) and typical has magnitude $1/\sqrt{\mathrm{dim(Ball)}(t)} = [4^{\gamma_d (r_A+vt)^d}]^{-1/2}$. Furthermore, since the coefficients $c_{\mathcal{O}}(t)$ are independent random variables for different operators $\mathcal{O}$, the cross terms in the left-hand side (LHS) of Eq. (6), $[\sum_{\mathcal{O}} c_{\mathcal{O}}(t)]^2$, vanish, reducing the sum to $\sum_{\mathcal{O}} c_{\mathcal{O}}(t)^2 = 2^{\gamma_d (r_A+vt)^d}/4^{\gamma_d (r_A+vt)^d}$, the right-hand side (RHS).

Finally, we arrive at the expression for $S_2(t)$ and obtain the universal linear growth. Plugging the above result into Eq. (4) and simplifying, we get

$$\mathrm{Tr}_A\big[\rho_A^2(t)\big] = 2^{-\gamma_d[(r_A+vt)^d - r_A^d]}. \tag{7}$$

This calculation is valid until $vt \lesssim (2^{1/d} - 1)r_A$, at which point $\mathrm{Tr}_A[\rho_A^2(t)] \sim 1/2^{N_A}$. Then, the error from neglecting the contribution of the global identity term $\mathbb{I}_{\mathcal{H}_A}$ becomes non-negligible, since the identity operator never spreads under time evolution and will always contribute a factor of $1/2^{N_A}$. Thus, the EE saturates to the maximum allowed for region A.

The second Rényi EE is thus

$$S_2(t) = \begin{cases} \gamma_d\left[(r_A + vt)^d - r_A^d\right]\log 2 & \text{for } \frac{\mathcal{O}(1)}{v} < t \lesssim \frac{f_d r_A}{v} \\ \gamma_d r_A^d \log 2 = N_A \log 2 & \text{for } t \gtrsim \frac{f_d r_A}{v}, \end{cases} \tag{8}$$

where $f_d = (2^{1/d} - 1)$. For times $t < 2r_A/v(d-1)$, $S_2(t)$ can be approximated by a linear function

$$S_2(t) \sim \left[\left(\gamma_d d r_A^{d-1}\right) vt\right] \log 2. \tag{9}$$

In fact, this linear approximation is always valid as long as the $S_2(t)$ calculation is valid, since $2r_A/v(d-1) > f_d r_A/v$ for all d. We see that the expression (9) for the dynamics of entanglement has three terms: (1) a geometric factor $\gamma_d d r_A^d$ which gives the area of an entanglement "wavefront", (2) the speed of the wavefront v, and (3) a universal linear time dependence t. Thus this has the interpretation of entanglement spreading in a "tsunami", similar to a view put forth in Ref. [14].

V. PROPOSAL TO EXPERIMENTALLY MEASURE $s_n(t)$ AND RELATION TO LOSCHMIDT ECHO

Here we propose a way to experimentally measure $S_n(t)$ via a local measurement. We consider an replicated system consisting of n disjoint copies of the original system, each with a similar bipartition A_i and B_i. We arrange them in a way similar to Ref. [16]: a star geometry such that the boundaries between subsystems A_i and B_i are placed near each other; see Fig. 1. We then introduce a quantum switch (a two-level system), which if in the $|\uparrow\rangle$ state allows tunneling only between subregions A_i and B_i such that the full Hamiltonian on the composite system is H, while if in the $|\downarrow\rangle$ state allows tunneling only between subregions A_i and B_{i+1} such that the full Hamiltonian is $H + V$. Here $H = \sum_{i=1}^n H_i = \sum_{i=1}^n H_{A_i} + H_{B_i} + H_{A_i B_i}$, where each H_i is a copy of the Hamiltonian (1) acting on the ith copy of the Hilbert space $\mathcal{H}_i = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$, and V is the local "reconnection" operator, $V = \sum_{i=1}^n H_{A_i B_{i+1}} - H_{A_i B_i}$, with the cyclic condition $n + 1 = 1$, so that $H + V = \sum_{i=1}^n H_{A_i} + H_{B_i} + H_{A_i B_{i+1}}$. If we prepare the state of the replicated system as $|\psi^n\rangle \equiv \otimes_{i=1}^n |\psi\rangle_i$ where $|\psi\rangle$ is the initial product state of the original system and also the quantum switch as a maximally entangled Bell state, so that the full state is $|\Psi\rangle = |\psi^n\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, then under unitary time evolution,

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-iHt}|\psi^n \uparrow\rangle + e^{-i(H+V)t}|\psi^n \downarrow\rangle). \tag{10}$$

Measuring $\sigma^x(t)$ of the quantum switch, a local measurement gives

$$\mathrm{Tr}_A\left[\rho_A^n(t)\right] = \langle\psi^n|e^{i(H+V)t}e^{-iHt}|\psi^n\rangle \equiv \mathcal{F}(t), \tag{11}$$

from which the nth Rényi entropy can be obtained: $S_n(t) = \frac{1}{1-n}\log \mathcal{F}(t)$. Thus we see that the dynamics of entanglement is indeed an observable local quantity if we work in an extended system. See also Refs. [23,24] on other theoretical proposals to measure EE and its growth. Additionally, Ref. [25] has experimentally measured EE in a Bose–Hubbard system also in a replicated setup. However, the difference of our proposal with their experimental technique is that we only require a single *local* measurement whereas Ref. [25] requires an extensive number of measurements (namely, measuring the parity on all sites).

The above claim is based on an alternative reformulation of $S_n(t)$, relating it to a Loschmidt echo $\mathcal{F}(t)$ [26–31] on the replicated system. It is a well-known trick [17,23,32] that the trace of the nth power of ρ_A can be calculated by replicating n copies of the original Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and calculating the swap operator S that cyclically swaps subregions $A_i \to A_{i \bmod n+1}$: if $\Omega = \rho^{\otimes n}$ is the n-replicated density matrix, then $\mathrm{Tr}_A(\rho_A^n) = \mathrm{Tr}_{\mathcal{H}^{\otimes n}}(\Omega S)$.

Now let us specialize to the case considered before: when the initial state on one system is a random product state $|\psi\rangle = |\vec{\sigma}_1\vec{\sigma}_2\cdots\vec{\sigma}_N\rangle$ and $\rho = |\psi\rangle\langle\psi|$. If each ρ evolves independently under the Hamiltonian given by Eq. (1), then Ω evolves under the composite Hamiltonian $H = \sum_{i=1}^n H_i$. Then,

$$\mathrm{Tr}_A\left[\rho_A^n(t)\right] = \langle\psi^n|e^{iHt}Se^{-iHt}|\psi^n\rangle. \tag{12}$$

However, because the initial state is a product state, $\langle\psi^n|S^\dagger = \langle\psi^n|$, and since $S^\dagger e^{iHt}S = e^{i(H+V)t}$, we end up with the claimed result (11), which is a Loschmidt echo $\mathcal{F}(t)$ with "perturbation" V to H. Note that an analogous statement also holds for time-dependent Hamiltonians even though the unitary time evolution operator will not have the form e^{-iHt}.

VI. CONCLUSION

We have provided a theoretical description and a physical picture of the entanglement dynamics in a many-body system by relating entanglement growth to the physical spreading of basis operators on a lattice. We also introduced a simple toy model where an initial basis operator effectively scrambles in its light cone, thus producing a linear entanglement growth, which we believe captures the salient features of the universal linear-in-time growth of entanglement seen in ergodic many-body systems. Furthermore, the entanglement growth in this model has an interpretation of an "entanglement tsunami": there is a prefactor which depends on the geometry of the subregion A, which propagates outwards in a wavefront at speed v. We remark that, while entanglement growth in MBL systems has been successfully explained through a local integrals of motion picture, it is an interesting question to explain this dynamics in terms of our operator-spreading language.

We have also provided an alternative interpretation of the growth of the nth Rényi entropies as a Loschmidt echo in a composite system, subject to a perturbation that reconnects different subregions. By using this reformulation, we have proposed an experimental way of measuring the growth of the Rényi entropies: one can effect the reconnection by using a quantum switch and measure the state of the quantum switch to extract the Loschmidt echo and hence the EE. This proposal can be used, for example, to directly detect MBL phases by measuring the logarithmic growth of entanglement.

Note added. Recent works [33,34] also studied entanglement dynamics in chaotic many-body systems and argued that entanglement growth speed is bounded from above by the operator-spreading speed. In our operator counting language picture, this can be accounted for in our toy model by refining the scrambling assumption which gives rise to the measurement value (6): there will be a distribution of coefficients that presumably depends upon the size of the operators in the decomposition (5), which we leave for future work to explore.

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