Dual view on sliding phases in $U(1)$ symmetric systems

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The proposal of sliding phases (SP) is revisited from the perspective of duality. A generic argument is formulated as essentially a no-go theorem for SP in translationally invariant nonfrustrated systems with short-range interactions—classical or quantum. Its validity is demonstrated on an asymmetric bilayer and its multilayer variation models where the duality allows obtaining asymptotically exact analytical solution. This solution is in drastic contrast with the perturbative renormalization group prediction and is strongly supported by Monte Carlo simulations. An alternative path toward finding SP is suggested. Its key ingredient is a long-range gauge-type interaction suppressing the interlayer Josephson coupling.

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I. INTRODUCTION

The idea of the sliding phases SP has been emerging in several different contexts—liquid crystals, superconductors, 1D quantum systems, correlated disorder, and spin liquids—within a general theme of a possible dimensional decoupling (reduction) when a D-dimensional system breaks into a stack of systems of essentially lower dimensionality. It can be traced back to the suggestion of vanishing shear modulus in layered liquid crystals where each layer is a quasisolid positionally decoupled from its neighbors [1]. This mechanism has been further explored in Refs. [2,3]. In layered superconductors magnetic field parallel to the layers has been proposed to suppress the interlayer Josephson coupling [4]. However, the frustration due to magnetic field turned out to be insufficient to fully suppress the coupling as shown in Ref. [5].

In the context of quantum 1D chains the possibility of the decoupling between chains has been explored as a pathway toward non-Fermi liquid in high T_c materials [6,7]. The main argument for such a decoupling is based on using the scaling dimensions of the Josephson coupling determined with respect to the Luttinger liquid parameter in each chain: if it is larger than 2, the coupling should become irrelevant [7]. These proposals have been criticized in Refs. [8,9] where it was shown that the interchain tunneling is always relevant. In Refs. [10] it has been shown that the dimensional reduction is not possible due to pair tunneling in quantum wires. This analysis is based on RG developed for bosonized models with nonzero conformal spin (see Ref. [11]).

The results [1–3] refer to noncompact variables—translation of one layer against its neighbors. That SP can occur in the case of compact XY variables has been proposed in Ref. [12], where the interlayer gradient couplings between classical XY variables in each layer have been considered as a "knob" controlling scaling dimensions of the Josephson coupling and of the vortex fugacity in each layer. The SP would occur if the first one is irrelevant above some temperature T_d, while vortices in each layer are still bound into neutral pairs. This approach was also developed for the case of quantum 1D Luttinger liquids coupled by both the Josephson and the gradient terms [13–16] (which are the analog of the Andreev-Bashkin drag effect [17] in neutral superfluids or Biot-Savart interactions in superconductors[18]). More recently, the dimensional reduction was considered in the context of layered disorder [19,20] and non-Fermi liquids in the spin-liquid regime [21].

It is important to note that the proposal of SP is based on applying the renormalization group (RG) logic to compact variables characterized by global $U(1)$ symmetry. While these early suggestions were more of a purely academic interest, expanding capabilities of ultra-cold-atoms techniques in recent years emphasize the importance of these suggestions especially in the context of possible new phases in composite lattices [22] and in the presence of disorder [19,20]. In more general terms, the question is if it is possible to realize a phase transition, rather than a crossover, from a low- to higher-dimensional behavior.

Here we propose an alternative approach to the problem of SP. It is based on the dual formulation of a field model of compact variables in terms of positive defined statistics of random closed loops of integer currents obeying Kirchhoff's conservation law [23]. In this language, spontaneous symmetry breaking is equivalent to the formation of a "soup" of fully disordered macroscopic loops. Accordingly, the SP implies that, while proliferating along the layers, such loops do not proliferate perpendicular to them. This immediately leads to the generic requirement for the SP to exist: *The energy cost $E_\perp$ for a loop element to invade a neighboring layer must be macroscopically large with respect to the layer size L because otherwise the entropy for such an invasion will dominate and will cause simultaneous proliferation of the loops along and perpendicular to the layers.*

In order to illustrate the above general statement, we will consider a classical XY layered system characterized by gradient interlayer interactions and the Josephson coupling u. The gradient terms are chosen in such a way that the SP is supposed to exist in some range in accordance with the RG prediction for zero conformal spin. We will present results of the large-scale Monte Carlo simulations of this system in its dual representation—in terms of the closed loops. The main finding is that, in accordance with the generic argument, no SP state exists in such a system. As a comparison, the standard asymmetric XY layered model where no SP are expected to occur will be analyzed too. As will be seen, both models demonstrate essentially the same behavior. Furthermore, using dual representation, it becomes possible to find exact analytical solution for the renormalized Josephson coupling u_r in the asymptotic limit $u \to 0$. The validity of this solution will be demonstrated numerically for both models. Thus, the main conclusion is that, rather than following the RG prediction, the model of SP demonstrates 3D behavior.

Our paper is organized as follows. In Sec. II we introduce the bilayer model and provide the RG solution for SP. Then, we construct the dual representation in Sec. II B. The asymptotic analytical solution for the renormalized Josephson coupling u_r as well as the numerical results will be discussed in Sec. II C. Then, in Sec. III the stack of bilayers will be discussed. Finally, in Sec. IV we discuss the implications of our analytical and numerical results and also provide an alternative model for the SP.

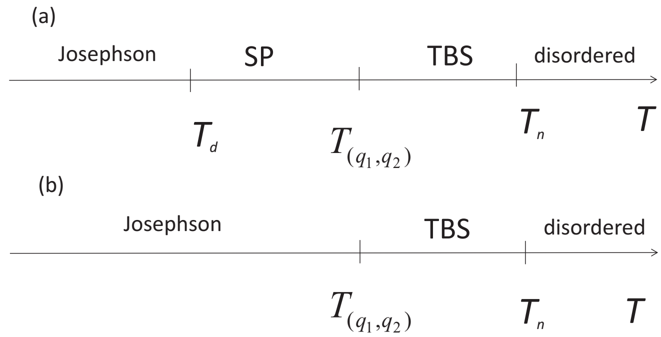

FIG. 1. Two options for phases in the bilayer model: (a) with the SP according to RG; (b) without SP.

II. BILAYER MODEL OF SP

The purpose of the following analysis is to introduce a simplest model that admits the RG solution predicting sliding phases. This result will then be tested numerically and analytically in the asymptotic limit of vanishingly small Josephson coupling.

Consider two classical asymmetric parallel layers, each being a square lattice of linear size L (in terms of the intersite shortest distance). These layers host two $U(1)$ fields $\psi_1 = \exp(i\phi_1)$ and $\psi_2 = \exp(i\phi_2)$ on the layers $z = 1,2$, respectively. The continuous (low-energy) action,

$$H_\phi = \int d^2x \left[\frac{1}{2} K_{zz'} \vec{\nabla}\phi_z \vec{\nabla}\phi_{z'} - u\cos(\phi_2 - \phi_1)\right], \quad (1)$$

features the gradient interaction represented by the (Luttinger parameter) matrix $K_{z,z'}$ as well as by the interlayer Josephson term $\sim u$. Here $\vec{\nabla}\phi_z$ refers to the x,y derivatives along the planar directions. The summation over the repeated indexes is used here and hereafter. Stability of the system is guaranteed if $\det(K_{zz'}) > 0$, that is,

$$K_{11}K_{22} - K_{12}^2 > 0. \quad (2)$$

In the partition function,

$$Z = \int D\phi_1 D\phi_2 \exp(-H_\phi), \quad (3)$$

the measure of the functional integration must account for the compactness of the phases ϕ_z. This can be achieved by using the discrete lattice formulation described in Appendix A and further discussed in Sec. II B. Let's first, however, discuss the RG approach to the system.

A. Scaling dimensional analysis for the bilayer

Here we will present the analysis to the system Eqs. (1) and (3) based on RG in line with the approach suggested in Refs. [12]. It is important that in this analysis the compact nature of the "angles" $\phi_{1,2}$ is ignored.

Derivation of the RG equations and their solutions for the bilayer are presented in Appendix B along the line as described in Refs. [24,25]. Despite the asymmetric nature of the system, the resulting RG flow Eqs. (B11) and (B12) for the Josephson coupling and Eqs. (B17) and (B18) for the vortex fugacity turn out to be identical to the standard RG equations for the Sine-Gordon model (see Refs. [24,25]). Thus, in order to identify the critical points (see Fig. 1) T_d of the SP and $T_{(q_1,q_2)}$ of the Berezinskii-Kosterlitz-Thouless (BKT) transition, it is enough to evaluate the scaling dimensions (see Ref. [26]) of the inter-layer Josephson and vortex fugacity, and find the range $T_d < T < T_{(q_1,q_2)}$ of parameters where both are irrelevant. We begin with the first critical point $T = T_d$ of the transition from the phase where the interlayer Josephson coupling is relevant (called "Josephson" in Fig. 1) to the SP.

If the vortex fugacity is irrelevant, the compact nature of the phases is usually ignored. Then, introducing the variables $\phi_+ = \phi_1 + \phi_2$ and $\phi_- = \phi_2 - \phi_1$ in Eqs. (1) and (3) and, then, integrating out ϕ_+, the resulting partition function becomes

$$Z_- = \int D\phi_- e^{-H_-},$$
$$H_- = \int d^2x \left[\frac{K}{2}(\vec{\nabla}\phi_-)^2 - u\cos\phi_-\right], \quad (4)$$

where the notation

$$K = \frac{K_{11}K_{22} - K_{12}^2}{K_{11} + K_{22} + 2K_{12}} \quad (5)$$

is introduced. Equations (4) and (5) represent the standard Sine-Gordon model in 2D. The scaling dimension of the operator $\sim u$ is $\Delta_u = 1/(4\pi K)$. Thus, the Josephson term becomes irrelevant if $\Delta_u > 2$, that is, at $K < K_d = 1/(8\pi)$, so that the renormalized u should flow to zero as $u_r \sim uL^b \to 0$, $b = 2(1 - K_d/K) < 0$. Such a behavior is supposed to occur together with the persistence of the algebraic order along the planes. Without loss of generality let's assume $K_{11} < K_{22}$ and introduce the notations: $T = 1/K_{11}$ as a measure of temperature, and $Y = K_{22}/K_{11} > 1$, $X = K_{12}/K_{11}$. Then, the condition $K < 1/(8\pi)$ for SP becomes

$$T > T_d = \frac{8\pi(Y - X^2)}{1 + Y + 2X}. \quad (6)$$

In order to guarantee the algebraic order in each layer no BKT transition should occur in the layers. In other words, all backscattering harmonics V_{q_1,q_2} in the action Eq. (B1) must be irrelevant below some temperature $T_{(q_1,q_2)}$ exceeding T_d in Eq. (6). In order to determine possible types of vortices responsible for the transition, we examine the form Eq. (1) in the limit $u = 0$ using the Kosterlitz-Thouless argument for the BKT transition. Specifically, a composite vortex (q_1,q_2)

induced by the drag term $\sim K_{12}$ [12,27,28] with circulations q_1,q_2 in the layers 1 and 2, respectively, is introduced at the same position x,y along the layers. The free energy of such a composite vortex is

$$F_v = \{\pi[(q_1 + Xq_2)^2 + (Y - X^2)q_2^2] - 2T\}\ln L. \quad (7)$$

Then, the stability against the BKT transition is guaranteed by the positivity of F_v or

$$T < T_{(q_1,q_2)} = \frac{\pi}{2}[(q_1 + Xq_2)^2 + (Y - X^2)q_2^2], \quad (8)$$

where the minimization with respect to q_1,q_2 must be performed. This condition corresponds to the requirement that the scaling dimension $\Delta_{q_1,q_2} = \pi\sum_{a,b} K_{ab}q_aq_b$ of the most dangerous backscattering amplitude V_{q_1,q_2} in Eq. (B13) is above 2.

Proliferation of simple vortices $q_1 = \pm 1$, $q_2 = 0$ or $q_1 = 0$, $q_2 = \pm 1$ corresponds to $T_{(1,0)} = \pi/2$ and $T_{(0,1)} = \pi Y/2 > T_{(1,0)}$, respectively. The minimal solution with composite vortex can exist only as long as $X \neq 0$, that is, when $K_{12} \neq 0$.

Solutions for Eqs. (6) and (8) exist for integer values of $X \geqslant 3$. Introducing $\delta = Y - X^2 > 0$ [due to the stability requirement Eq. (2)], one should distinguish cases $\delta > 1$ and $\delta < 1$. In the first case, the dominant vortex is (1,0) and the solution for $T_d < T_{(1,0)}$ exists if $1 < \delta < (1 + X)^2/15$. If $0 < \delta < 1$, the dominant vortex is composite $(-X,1)$ and the condition Eqs. (6) and (8) become

$$\frac{8\pi\delta}{\delta + (1 + X)^2} < T < \frac{\pi}{2}\delta. \quad (9)$$

For $X \gg 1$, $T_d \to 0$ while $T_{(q_1,q_2)} \to (\pi/2)\min(1,\delta)$ as long as δ is kept constant. Such a limit corresponds to the largest range of T, where SP are to be anticipated for the two-layer model. However, for practical purposes of simulations using too large X leads to slower convergence. Thus, we choose $X = 5$, $Y = 25.5$ corresponding to a reasonably wide range where SP is anticipated to exist. Then, Eq. (9) becomes $8\pi/73 < T < \pi/4$ or $0.344 < T < 0.785$. [The simulations discussed below have been conducted at T in the middle of the interval Eq. (9), that is, $T \approx 0.565$. More specifically, $K_{11} = 1/T = 1.77$, $K_{22} = 25.5K_{11}$, $K_{12} = 5K_{11}$.]

Proliferation of the composite vortex pairs with vorticities (q_1,q_2) corresponds to disordering of the original fields $\exp(i\phi_{1,2})$. At the same time the composite field $\Psi = \exp[i(q_1\phi_1 + q_2\phi_2)]$ remains (algebraically) ordered. This mechanism constitutes the formation of thermally induced bound phases (or using the language of superfluidity—*thermally paired superfluid* [29]). For the values chosen above this composite field is $\Psi = \exp[i(\phi_1 + X\phi_2)]$. Since $X > 1$ we call such a composite phase as thermally bound superfluid (TBS). This effect does not require that X is necessarily integer. If X is noninteger, its closest integer part will determine the power of ψ_2. In Fig. 1 the TBS exists in the range $T_{(q_1,q_2)} < T < T_n$. Full symmetry is restored above T_n—that is, no algebraic order exists in any composite or original fields.

Concluding this section, the presented analysis based on the RG finds the range of temperatures where the sequence of phases is as presented in Fig. 1(a): at $T < T_d$ the Josephson coupling is relevant. At $T_d < T < T_{(q_1,q_2)}$ there is the SP where the symmetry $U(1)$ is promoted to $U(1)\times U(1)$. In the range $T_{(q_1,q_2)} < T < T_n$, the TBS phase is characterized by the composite field Ψ. Thus, the broken symmetry is partially restored through the subgroup Z_N, where $N = 1 + q_2$. At higher temperatures, $T > T_n$, the composite field Ψ becomes disordered too. In what follows we will show that the actual sequence of phases is correctly depicted in Fig. 1(b) rather than in Fig. 1(a).

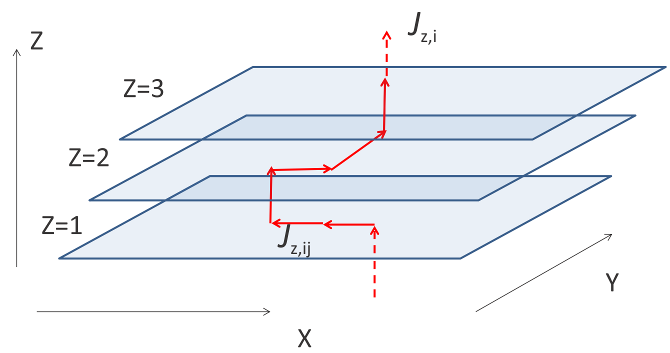

FIG. 2. A J-current configuration characterized by $W_z = 1$, $W_x = 0$, $W_y = 0$. Horizontal oriented arrows show J currents along planes, with $|J_{z,ij}| = 1$. The vertical ones indicate J currents between the planes, with $J_{z,i} = 1$, with the dashed lines showing currents that are completing periodic boundary conditions.

B. Dual representation

As described in detail in Appendix A, the bilayer model Eqs. (1) and (3) can be reformulated in terms of the dual variables that account for the compact nature of the variables $\phi_{1,2}$. (The logic behind this transformation is along the line of the J-current model, Ref. [23].) The partition function Eq. (3) is now represented as

$$Z = \sum_{\{J_{z,ij}\},\{J_{z,i}\}} e^{-H_J}, \quad (10)$$

with the action

$$H_J = \sum_{\langle ij\rangle} \frac{1}{2}(K^{-1})_{zz'} J_{z,ij}J_{z',ij} + \sum_i \frac{1}{2u_V} J_{z,i}^2, \quad (11)$$

where $(K^{-1})_{zz'}$ is the matrix inverse to $K_{zz'}$ introduced in Eq. (1) and u_V is the Villain value of the Josephson coupling u. Since we are interested in the limit $u \ll 1$, it is $u_V \approx 1/[2\ln(2/u)]$ [30,31] (for more details see Appendix A). The summation runs over the integer bond currents $J_{z,ij} = -J_{z,ji}$, $z = 1,2$ defined between neighboring sites i and j and oriented from site i to site j within each corresponding layer as well as over the integer currents $J_{z,i}$ oriented along the bond connecting the site i in the layer 1 to the site i in the layer 2. All the configurations are restricted by the Kirchhoff's current conservation rule—the total of all J currents incoming to any site must be equal to the total of all outcoming currents from the same site.

The resulting configurational space consists of closed loops of the bond currents as schematically depicted in Fig. 2. Further simulations can be effectively performed by the Worm Algorithm [32]. As will be shown, in addition to being very effective in numerics, the language of loops also allows obtaining analytic expression for the renormalized Josephson coupling u_r, which is exact in the asymptotic limit $u \to 0$, while strong algebraic order persists along the layers.

If $u = 0$, there are two sorts of loops—one in each layer. Thus, each configuration is characterized by definite values of the windings $W_{z,\alpha}$ in the zth layer along the $\alpha = \hat{x},\hat{y}$ directions of the planes. These quantities are defined as a total of all J currents crossing any line perpendicular to the direction α. (The Kirchhoff's rule guarantees that windings are independent of the choice of the line.) It is straightforward to show that statistics of these windings determine the renormalized values $\tilde{K}_{zz'}$ of the matrix $K_{zz'}$ along the line of the approach [33]. More specifically,

$$\tilde{K}_{zz'} = \frac{1}{2}\sum_{\alpha=\hat{x},\hat{y}} \langle W_{z,\alpha} W_{z',\alpha}\rangle. \tag{12}$$

This expression is valid for periodic boundary conditions (PBC). It is important to note that $\tilde{K}_{zz'}$ represents an exact linear response with respect to the Thouless phase twists. In other words, if there are externally imposed infinitesimal constant gradients $\nabla_\alpha \phi_{1,2} \to 0$ (violating the PBC) of the phases $\phi_{1,2}$, the free energy acquires the contribution $\delta F = \frac{1}{2}L^2 \sum_{z,z',\alpha} \tilde{K}_{zz'} \nabla_\alpha \phi_z \nabla_\alpha \phi_{z'}$. On the other hand, in the presence of the gradient the integrand of the partition function gets the factor $\exp(iL\sum_{z,\alpha} W_{z,\alpha}\nabla_\alpha \phi_z)$. Comparing both expressions leads to the relation Eq. (12).

As a test of consistency, we have checked numerically that in the regime where the SP state is supposed to exist (that is, $X = 5$, $Y = 25.5$, $T \approx 0.565$), the deviations of $\tilde{K}_{zz'}$ from the bare values $K_{zz'}$ are within the statistical error less than 1% for all tested sizes of the layers $10 \leqslant L \leqslant 1000$. Significant deviations are observed only as the system approaches fully disordered state—that is, $T \to T_n$, Fig. 1, where the fields $\psi_{1,2}$ as well as the composite one Ψ become disordered. In this case, $\tilde{K}_{zz'}$ flow to zero as L increases. The deviations remain small (about 2–3%) even in the regime where Ψ is the only ordered field. The emergence of the TBS is detected by observing that windings $W_{z,\alpha}$ in the layers 1 and 2 are changing exactly by the increment $\Delta W_1 = 1$, $\Delta W_2 = X$ (plus or minus), respectively.

At finite values of u the loops belong to both layers so that no separate windings can be introduced. However, the sums $W_\alpha = W_{1,\alpha} + W_{2,\alpha}$ remain well defined and can be used to evaluate the rigidity ρ_α of the fields along the layers. In a general case of N_z layers $\rho = \rho_x = \rho_y$:

$$\rho = \frac{1}{2N_z}\sum_\alpha \langle W_\alpha^2\rangle, \tag{13}$$

$$W_\alpha = \frac{1}{L}\sum_{\langle ij\rangle, a=1,2,\ldots N_z} J_{a,ij}, \tag{14}$$

where for a given $\alpha = \hat{x},\hat{y}$ in Eq. (14) the bond $\langle ij\rangle$ (as well as $J_{a,ij}$) is oriented along the direction α.

Our focus here on the renormalized value u_r of the Josephson coupling u in the SP regime. In the case of N_z layers, if the periodic boundary conditions are also imposed perpendicular to the layers (along z direction), the interlayer response u_r is given by windings W_z along z direction:

$$u_r = \frac{N_z}{L^2}\langle W_z^2\rangle, \quad W_z = \frac{1}{N_z}\sum_i J_{z,i}, \tag{15}$$

where the summation $\sum_i$ of the currents $J_{z,i}$ (oriented along z direction) is performed over all sites of all layers. Similarly to the cases Eqs. (12) and (13), Eq. (15) represents the full linear response at zero momentum—that is, the renormalized value u_r of the Josephson coupling u.

At this point, we should comment on how to interpret the PBC for two layers, $N_z = 2$. While in the cases $N_z \geqslant 3$ it is a natural procedure to link the $z = N_z$th layer to the first one, $z = 1$, by the Josephson term, the case $N_z = 2$ needs an auxiliary construction because the layers 1 and 2 are coupled already directly. The formal procedure, then, consists of adding a third layer, $z = 3$, with no rigidity along x,y directions and coupled by the Josephson term to both layers, $z = 1,2$. If the coupling u_{13} between the layers 1 and 3 and the coupling u_{23} between the layers 2 and 3 add up as $1/u_{13} + 1/u_{23} = 1/u_V$, in the dual action Eq. (11) the sum in the last term can be extended to the layers $z = 1,2,3$ in the periodic manner. The key to this procedure is the Kirchhoff's rule: the J current from a site (x,y) along z direction from the layer 2 to the layer 3 must be exactly the same as the current from the site (x,y) in the layer 3 to the layer 1. Then, in the form Eq. (11) the same value u_V can be used for the currents from the layer 1 to the layer 2 directly or through the layer 3 as shown in Fig. 2.

C. Asymptotic expression for u_r

As mentioned above, the dual representation allows obtaining analytically asymptotic solution for u_r. Let's begin with the simplest case of zero stiffnesses $K_{zz'}$ and arbitrary number of layers, $N_z = 2, 3, 4$. The action in this case in the field representation becomes $\sim \sum_z \int d^2x[-u\cos(\phi_{z+1} - \phi_z)]$, or in the dual form,

$$H_A = \frac{N_z}{2u_V}\sum_i J_{z,i}^2, \quad J_{z,i} = 0, \pm 1, \pm 2, \ldots, \tag{16}$$

where the summation runs over all sites i of only *one* layer, say, $z = 1$. In this expression the Kirchhoff rule dictates that the current $J_{z,i}$ at a given site along z direction must be the same for all values of z. Thus, such a current with $J_{z,i}$ constitutes one closed loop characterized by the winding $W = J_{z,i}$. This allows constructing the partition function exactly as

$$Z_A = \left[\sum_{W=0,\pm1,\pm2,\ldots} \exp\left(-\frac{N_z}{2u_V}W^2\right)\right]^{L^2}, \tag{17}$$

where L^2 is the number of sites in one layer. The stiffness Eq. (15) can be found by taking into account that the total winding along z direction is $W_z = \sum_i J_{z,i}$, where the summation runs over L^2 sites of only one layer. Then, using statistical independence of different sites we find

$$u_r = \frac{2N_z \sum_{W=1,2,\ldots} W^2 \exp(-N_z W^2/2u_V)}{1 + 2\sum_{W=1,2,\ldots} \exp(-N_z W^2/2u_V)}. \tag{18}$$

This expression shows that, as long as N_z is finite, the Josephson coupling remains relevant even if there is no in-plane order. In the limit $u_V \ll 1$ only the term $W = 1$ is important, so that Eq. (18) becomes

$$u_r = \frac{2N_z}{2 + \exp(N_z/2u_V)} \sim 2N_z \exp(-N_z/2u_V). \quad (19)$$

The exponential decay versus N_z in Eq. (19) is a direct consequence of the absence of the stiffness along the layers, that is, $\rho = 0$ in Eq. (14), so that the shortest loop is "vertical" with the number $M = N_z$ of the vertical currents $J_{z=1,i} = J_{z=2,i} = \ldots = J_{z=M,i}$.

Thus, it is natural to anticipate that the dimensional decoupling in a strong sense, when u_r scales to zero as some negative power of L as prescribed by RG, should not occur even in the absence of the algebraic order along the planes, when $\rho \to 0$. The stiffness along z direction remains finite in the limit $L \to \infty$ as long as N_z is finite.

This example indicates that short-range interplane correlations rather than long-range intraplane coherences are responsible for finite interplane Josephson coupling. In terms of the original variables ϕ_z, the result Eq. (19) implies that $u_r \sim u^{N_z}$ (because $u_V \approx 1/[2\ln(2/u)]$ in the limit $u \to 0$). This can be viewed as the perturbative result of N_zth order with respect to u.

If there is a finite strong stiffness $\rho \gg 1$, Eq. (19) can also be used, with N_z substituted by some effective value $M = 1,2,3,\ldots$, that is

$$u_r = \frac{2M}{2 + \exp(M/2u_V)} \to 2M \exp(-M/2u_V). \quad (20)$$

The value of M is determined by the length of a "cheapest" string of J currents along z directions. The loop proliferation can be viewed from the perspective of the Worm Algorithm [32] where one open end of a string of J currents walks randomly until it meets another open end so that a closed loop is formed. Then, most of the path is residing in a layer with only occasional jumps between neighboring layers (in the limit $u_V \to 0$). Such an elementary jump has the probability $\sim \exp(-M/2u_V)$, so that all higher values M are exponentially suppressed. In other words, the situation is reminiscent of the "ideal gas" of rare fluctuations of the J currents of length M in z direction.

Thus, generically, it is expected that $u_r \propto u^M$ in the limit $u \to 0$ because then $u_V \approx 1/[2\ln(2/u)]$ [30,31]. Below we will show that for the model we consider $M = 2$ and, thus, $u_r \propto u^2$.

In the standard XY model (with no drag effect and no asymmetry between the layers) in its J-current representation [23], characterized by finite in-plane stiffness ρ and small interlayer coupling u_V, the "cheapest" string in z direction has $M = 1$ in Eq. (20). The standard XY model and its comparison with the multilayer extension of the bilayer model will be discussed in more detail in the Sec. III D. Below we will show that $M = 2$ in Eq. (20) for the bilayer in the SP regime and will present the numerical support for this. In other words, contrary to the RG prediction, the Josephson interlayer coupling u_r remains finite in the limit $L \to \infty$.

D. Numerical results for $N_z = 2$

Here we discuss the results of Monte Carlo simulations of the bilayer in the regime of SP. The action Eq. (11) can be represented in the notations T, X, Y, δ [introduced below Eq. (5)] as

$$H_J = \sum_{\langle ij\rangle}\left[\frac{T}{2}J_{1,ij}^2 + \frac{T}{2\delta}(J_{2,ij} - XJ_{1,ij})^2\right] + \sum_i \frac{J_{z,i}^2}{2u_V}, \quad (21)$$

where the values of the parameters have been discussed at the end of Sec. II A: $X = 5$, $\delta = 1/2$, $T = (T_d + T_{(X,-1)})/2 \approx 0.565$.

The structure of the loops is determined by the energy of creating a J-current element along a given direction. A typical energy to create a J-current element along a bond in the plane 2 can be estimated as $\delta E_2 \approx T/(2\delta) \approx 0.5$. Thus, large loops with a typical values $|\vec{J}_2| = 1$ can exist in the plane 2. In contrast, the energy to create an isolated element in the plane 1 (with no J_2 currents along the same bond in the layer 2) requires much more energy: $\delta E_1 \approx T(1 + X^2/\delta)/2 \approx 15$. Accordingly, the probability to create such an element is exponentially suppressed as $\sim\exp(-15)$, and no entropy contribution (due to four optional directions along the plane) can compensate for such a low value. This implies that no large isolated loops can exist in the layer 1. The only option to create a large loop in the layer 1 is if each element $J_{1,ij}$ is mirrored by the current $J_{2,ij} = XJ_{1,ij}$ along the same bonds in the layer 2. A typical energy of this combined element is $\delta E_{12} \approx T/2 \approx 0.25$. This strong asymmetry between the layers has immediate implication for the windings along z direction—the minimal length M of the element $J_{z,i}$ must be $M = 2$ in Eq. (20). Thus, the stiffness u_r in the limit $u \ll 1$ becomes

$$u_r = \frac{4}{2 + \exp(1/u_V)} \approx 4e^{-1/u_V} = u^2, \quad (22)$$

where the asymptotic expression $u_V = \frac{1}{2\ln(2/u)}$ [30,31] has been used. Accordingly, for the simple XY model (with no drag interaction) the corresponding dependence is $u_r \propto u^1$. This will be discussed below.

As discussed above, the power u^2 stems from the value $M = 2$ in Eq. (20). Formally speaking, Eq. (22) appears to be as though the weak layer ($z = 1$) is incoherent and, thus, is eliminated in second order of perturbation with respect to u—very much like the situation discussed in Sec. II C. It is, however, important to note that the weak layer is coherent and the application of the perturbative approach in terms of the original variables—the phases ϕ_z—is not that apparent. In contrast, the dual representation leading to the picture of the "ideal gas" of the vertical currents gives the result $u_r \sim u^2$ quite naturally.

The results of the simulations is shown in Fig. 3. The first striking feature to notice is that u_r, while changing over 7 orders of magnitude, does not depend on the layers size L. Second, u_r versus u_V follows the analytical result Eq. (22) with high accuracy—even for values $u_V \sim 1$. Both features are in the striking conflict with the RG prediction stating that u_r should scale as $\propto L^{2(1-T/T_d)} \approx L^{-1.28} \to 0$ in the SP regime. It should be also noted that the stiffness along the layers Eq. (13)

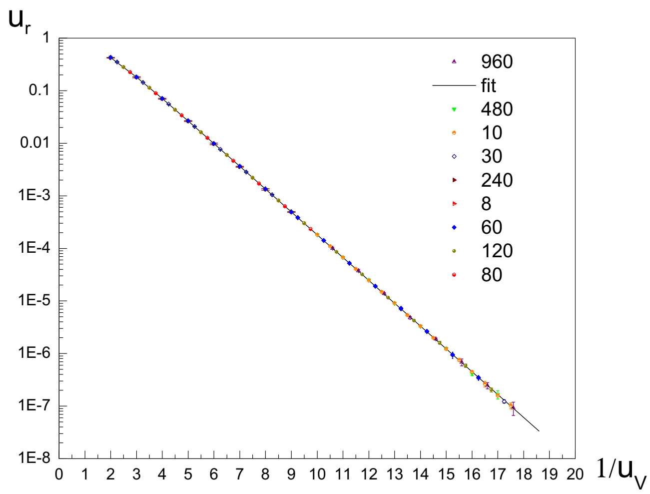

FIG. 3. Monte Carlo results for the interlayer stiffness u_r vs. its bare value u_V for the bilayer for various layer sizes (shown in the legend). Error bars are shown, and for the majority of the data points these are smaller than symbols. The fit line is the solution Eq. (22).

remains finite and much larger than u_r, that is, $\rho = 32.3 \pm 0.1$ for all simulated sizes from $L = 8$ to $L = 960$. This justifies the validity of Eq. (22) even in the case $u_V \sim 1$.

III. STACK OF BILAYERS

As it became evident from the previous analysis, no SP can occur in the double layer. Referring to the sketch of the possibilities, Fig. 1(b) is realized instead of Fig. 1(a). Here we will address a possibility of SP in a N_z-layers setup. In other words, we will be looking for a behavior where the renormalized Josephson stiffness u_r decays as a function of N_z in the limit $L \to \infty$, while the stiffness along planes remains finite. This would be a "weaker" version of the SP in a clean system (cf. the SP in the layered disorder case [20]).

We consider the PBC setup: the odd $z = 1,3,5,7,\ldots$ and the even $z = 2,4,6,..$ layers are characterized by the inplane stiffnesses K_{11} and $K_{22} > K_{11}$, respectively, with the nearest layers coupled by the current-current term $\propto K_{12}$ (the same for all pairs of layers) as well as by the Josephson coupling $-u\sum_{x,y,z}\cos(\phi_{z+1} - \phi_z)$, where $\psi_z(x,y) = \exp[i\phi_z(x,y)]$ is the XY variable defined on a site (x,y) belonging to the layer z.

In the linearized with respect to the gradients of ϕ_z approximation analogous to Eq. (1) the model becomes

$$H_N = \sum_{z=1,3,5,\ldots}\left\{H_z - \int d^2xu[\cos(\phi_{z+1} - \phi_z) + \cos(\phi_{z-1} - \phi_z)]\right\}, \quad (23)$$

where the summation runs over odd values of z and the notation

$$H_z = \int d^2x\left[\frac{K_{11}}{2}(\vec{\nabla}\phi_z)^2 + \frac{K_{22}}{2}(\vec{\nabla}\phi_{z+1})^2 + K_{12}\vec{\nabla}\phi_z(\vec{\nabla}\phi_{z+1} + \vec{\nabla}\phi_{z-1})\right] \quad (24)$$

is used. The Gaussian part of the action can be diagonalized by using Fourier representation along z direction with doubled unit cell. Then, the matrix $K_{zz'}$ becomes dependent on the wave vector $q_z = 4\pi n_z/N_z$, $n_z = 0,1,2,\ldots,Nz/2 - 1$ along z axis. The corresponding partition function becomes

$$Z = \int D\phi_z \exp(-H_N), \quad (25)$$

where the measure of functional integration $D\phi_z$ must explicitly account for the definition of the phases $\phi_z(x,y)$ modulo 2π.

A. RG solution

The corresponding RG equation for u_r Eq. (B19) is analogous to that discussed in Appendix B for the bilayer. Then the critical temperature of the dimensional decoupling becomes

$$T_d^{-1} = \frac{1}{4\pi N_z}\sum_{m=0}^{(Nz/2)-1}\frac{1 + Y + 4X\cos^2 q_m}{Y - 4X^2\cos^2 q_m}, \quad (26)$$

where the wave vectors along z take values dictated by the periodic boundary conditions $q_m = 4\pi m/N_z$, $m = 0,1,2,\ldots,(N_z/2) - 1$. Here we use the same notations $T = 1/K_{11}$, $X = K_{12}/K_{11}$, $Y = K_{22}/K_{11}$ introduced in Sec. II A. Thus, RG predicts irrelevance of u_r at $T > T_d$.

The upper limit on T is determined by the loss of algebraic order along the layers. (At $u_r = 0$ there should be no 3D vortices.) Clearly, if T is as high as $> \pi Y/2 \gg 1$, all layers will become disordered. Less drastic situation occurs when only weak layers (odd) are disordered $\pi/2 < T < \pi Y/2$. In this case the Josephson coupling between even layers will be supported by the proximity effect. We, however, will be considering the situation $T < \pi/2$, which implies algebraic order in all layers.

We considered also a possibility of proliferation of the composite vortices. One option is a composite vortex characterized by phase windings $q_1 = 1$ and $q_2 = X > 1$ in odd and even layers, respectively, forming a string of length N_z perpendicular to the layers. In this case the vortex energy will have a factor $\sim N_z$, which makes such vortices too energetically costly to play any role in the limit $N_z \gg 1$, provided the system is not too close to the instability (when the matrix of the gradient interactions acquires zero eigenvalue, that is, $Y - 4X^2 = 0$). In our simulations we have been avoiding this region. Thus, such "infinite" vertical vortices are excluded. Another option is when composite vortices occur as finite length vertical strings—say, of length 2 (along z axis) with $q_1 = \pm 1$ in an odd layer and $q_2 = -[2X]q_1$ in the neighboring (even) layer. However, a simple analysis shows that energy of such (and longer) composite vortices turns out to be higher than that of the simple vortex with $q_1 \pm 1$, $q_2 = 0$ destroying order in the odd layers. Thus, we impose the requirement $T_d < \pi/2$ in order to have a finite range $T_d < T < \pi/2$ for the SP to exist within the RG approximation. This implies

$$\frac{1}{N_z}\sum_{m=0}^{Nz/2-1}\frac{1 + Y + 4X\cos^2 q_m}{Y - 4X^2\cos^2 q_m} > 8. \quad (27)$$

It can surely be achieved for large enough X in the limit $N_z \gg 1$. Replacing the summation by integration in this limit and considering $\delta \ll 1$, Eq. (27) gives $\delta < X^2/64$. For the simulations we have chosen $\delta = 0.3$ and $X = 6$, which gives $T_d \approx 0.983$ with $T = 1.28$ chosen in the middle of the interval between $T_{(1,0)} = \pi/2 \approx 1.57$ and T_d. The chosen value of T_d corresponds to the limit $N_z \to \infty$, and for any finite N_z, the actual T_d from Eq. (26) is below this value.

It is worth reminding that, according to RG, u_r should exhibit suppression as some power of $L \to \infty$ in the range $T_d < T < T_{(1,0)}$. However, as shown below analytically and then numerically, there is no such suppression in the asymptotic limit $u_V \ll 1$.

B. Dual formulation

The dual formulation of the model Eqs. (23), (24), and (25) in terms of the closed loops of integer J currents (along bonds in and between the layers) can be achieved similarly to the case $N_z = 2$. Using Villain approach (see Appendix A) to the discrete gradients : $\vec{\nabla}\phi_z \to (\nabla_{ij}\phi_z + 2\pi m_{z,ij})$ along the planes and $-u\cos(\phi_{z+1} - \phi_z) \to (u_V/2)(\phi_{z+1} - \phi_z + 2\pi m_{i,z})^2$ for the Josephson terms, where $m_{z,ij}$ refers to integer defined on the bond ij belonging to the plane z and $m_{z,i}$ stands for an integer on a bond connecting site i in the plane z to the same site in the plane $z+1$, the partition function Eq. (25) follows as a result of explicit integration over all $\phi_z(i)$ and summations over all bond integers.

The J currents enter through the Poisson identity $\sum_{m=0,\pm1,\pm2,..} f(m) \equiv \sum_{J=0,\pm1,\pm2,..} \int dx \exp(2\pi \mathrm{i} Jx) f(x)$ applied to each bond integer. This allows explicit integration over all phases ϕ_z as well as over the bond integers $m_{z,ij}$,$m_{i,z}$. There are two types of J currents: inplane $J^{(a)}_{z,ij}$, $a = 1,2$ within each "elementary cell" (along z) and between the planes $J_{i,z}$. The label $a = 1,2$ refers to J current defined on the bond ij belonging to a plane with odd and even z, respectively. $J_{z,i}$ denotes the current from the site i from the plane z to the plane $z+1$. The integration over phases ϕ generates the Kirchhoff constraint—similarly to the bilayer case.

Finally, the J-current ensemble can be represented as

$$Z = \sum_{\{\vec{J}\}} \exp(-H_J),$$

$$H_J = \frac{1}{2}\sum_{ij;z,z'} V_{ab}(z-z') J^{(a)}_{z,ij} J^{(b)}_{z',ij'} + \frac{1}{2u_V}\sum_{i,z} J^2_{z,i}, \qquad (28)$$

where the matrix $V_{ab}(z-z')$ is defined in terms of the matrix $K_{zz'}$. It reflects the asymmetry between odd and even layers. Explicitly, $V_{11}(z) = Y V_{22}(z)$, for $z = z - z'$ being even, describes the interaction between odd layers, and $V_{22}(z)$ is defined between even layers; $V_{12}(z) = -X[V_{22}(z+1) + V_{22}(z-1)]$ refers to the interaction between odd and even layers (that is, z is odd), and

$$V_{22}(z) = \frac{2T}{N_z}\sum_{q_m} \frac{\cos(q_m z)}{Y - 4X^2\cos^2(q_m)}, \qquad (29)$$

with $z = 0, \pm 2, \pm 4, \ldots$ and the summation running over $q_m = 4\pi m/N_z, m = 0,1,\ldots,N/2-1$.

C. Asymptotic solution

Analogously to the case of the bilayer, the dual representation allows constructing the asymptotic solution for u_r for arbitrary N_z. We begin by finding the renormalized Josephson coupling in the asymptotic limit $u \to 0, L \to \infty$ with N_z kept fixed. The dual formulation Eq. (28) for N_z layers allows obtaining the asymptotic expression for u_r within the same logic used for deriving Eq. (22). We will repeat it here. The loop formation can be viewed as a process of random walks of two ends of a broken loop—exactly along the line of the Worm Algorithm [32]. Such a walk of each end is controlled by energetics of creating one bond element $|J| = 1$ in a randomly chosen direction—either along a given plane or perpendicular to it. Similarly to the case of the two layers, the energy to create such an element alone along an odd layer costs energy $\gg T \sim 1$, while the same element along an even layer costs energy $\sim$1. The only option for creating a loop in an odd layer is if its energy is compensated by parallel elements in the even plane. This feature is caused by the strong current-current interaction $\sim X$. Thus, if the walk occurs along z direction from some even layer z toward the neighboring odd layer $z+1$, the subsequent move along the odd layer will be too energetically costly so that the walker would either move further toward $z+2$ layer or will go back to the original layer z. Thus, the interlayer elements are characterized by either $J_{i,z} = J_{i,z+1} = \pm 1$ or $J_{i,z} = J_{i,z+1} = 0$. The weight of such a process is either $\exp(-1/u_V)$ or 1, respectively. Even if the walker makes a step or two along the layer $z+1$ (which is a highly improbable event) and then chooses to go toward the layer $z+2$, the contribution to the partition function will be further reduced exponentially by the energy of the element J along the odd plane. Thus, such processes can be ignored, and we arrive at the conclusion that u_r given by Eq. (22) must be valid for arbitrary N_z in the asymptotic limit. The validity of this solution will be verified numerically as explained below.

It is instructive to discuss the dependence on N_z in the situation when $L \gg 1$ is fixed and $u_V \to 0$. In this situation the renormalized Josephson stiffness u_r does exhibit the SP-like behavior $u_r \sim \exp(-\ldots N_z)$ (which, however, transforms into the solution Eq. (22) as $L \to \infty$). The reason for this, however, is of a purely geometrical nature (which has nothing to do with the drag interactions). Indeed, for any finite $N_z \gg 1$, the system becomes essentially of (quasi-) 1D nature as long as $u_V \to 0$. In this case, there is such a value u^* of u_V below which there is essentially only one macroscopic vertical loop with $W_z = \pm 1$ for a given area L^2, with higher ones exponentially suppressed. This situation corresponds to the contribution of zero modes to the stiffness along z direction. These modes are characterized by $\nabla_{x,y}\phi_z(x,y) = 0$, which leads to the effective Hamiltonian $H_0 = \sum_z u_r L^2 \cos(\phi_{z+1} - \phi_z)$, with $u_r \approx 2\exp(-1/u_V)$ being the renormalized mesoscopic stiffness. Zero modes become dominant excitations as long as $u_r L^2 \ll K_{11}$.

The dual form of the zero mode action takes a form

$$\tilde{H}_0 = \frac{N_z}{2L^2 u_r} W_z^2, \qquad (30)$$

where the duality procedure has been implemented as explained earlier. Calculation of the Josephson stiffness

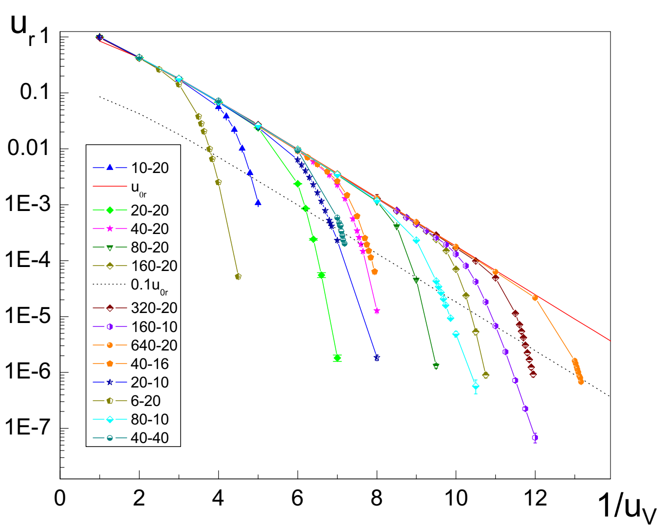

FIG. 4. Monte Carlo results for the interlayer stiffness u_r of the model Eqs. (28) and (29) in the SP regime. Dashed orange line is the analytical solution Eq. (22). Dotted black line represents the offset $u_r = 0.1$ of the analytical solution Eq. (22). The first and the second numbers in the legend indicate values of L and N_z, respectively

according to Eq. (15) gives the resulting stiffness

$$u_r^{(0)} \sim \exp\left(-\frac{N_z \exp(1/u_V)}{8L^2}\right) \quad (31)$$

in the main exponential approximation in the limit $u_r^{(0)} \ll u_r \approx 4e^{-1/u_V}$. Thus, if u_V is taken to zero, there is such a value $u_V = u^*$ below which this inequality will be satisfied for fixed $L, N_z \gg 1$. The corresponding value can be obtained from $\frac{N_z \exp(1/u^*)}{8L^2} \geqslant 1$, which gives

$$u^* \approx \frac{1}{\ln(8L^2/N_z)} \quad (32)$$

in the main logarithmic approximation. Thus, for fixed L, N_z the solution Eq. (22) is valid as long as $u_V > u^*$ and it must cross over to Eq. (31) as long as $u_V \ll u^*$. However, as $L \to \infty$, the crossover value of u_V, Eq. (32), goes to zero, which means the recovery of the asymptotic solution Eq. (22) for any finite u_V. This effect will be seen in the simulations as discussed below. It is important to mention, though, that such a suppression has nothing to do with SP because the RG solution (discussed above) implies the suppression of $u_r \to 0$ in the limit $L \to \infty$ for fixed u_V, while the asymptotic solution gives finite u_r, Eq. (22), in the same limit.

D. Numerical results for $N_z > 2$

The model Eq. (28) has been simulated by the Worm Algorithm [32]. The renormalized interlayer stiffness u_r was found for a range of layer sizes $6 \leqslant L \leqslant 640$ and layer numbers $10 \leqslant N_z \leqslant 40$. The resulting data is presented in Figs. 4 and 5. As can be seen in Fig. 4, the solution Eq. (22) plays the role of the envelop for the family of the curves u_r versus $1/u_V$ for various L and N_z. We note that the stiffness ρ along the layers [as determined by Eq. (13)] remains independent of the sizes and much larger ($\rho = 22.6 \pm 0.5$) than u_r. This justifies the applicability of the asymptotic limit for Eq. (22).

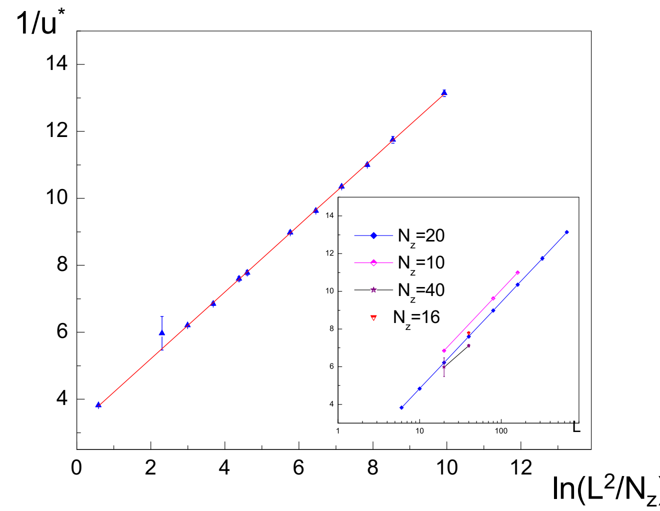

FIG. 5. The values of u^* determined numerically from the data shown in Fig. 4 by finding the crossings of the curves u_r with the offset (dotted) line in Fig. 4. The linear fit of this line gives the slope $\gamma = 1.00 \pm 0.02$.

We have also controlled that the system is far enough from any possible composite phases [28] state by measuring the lowest order correlator $\langle \exp(i\phi_z(x,y)) \exp(-i\phi_{z'}(x',y'))\rangle$ and observing that it exhibits long-range order. (In the composite phase state such a correlator is short ranged.) Thus, the system is well in the putative SP state. Its behavior, however, is drastically different from the RG prediction.

At this point we should discuss the deviations of the numerical curves from the analytical result seen in Fig. 4. As discussed above, this behavior is a consequence of zero modes. The value of $u_V = u^*$ below which the suppression begins *decreases* as

$$(u^*)^{-1} = \gamma \ln(L^2/N_z), \ \gamma = 1.00 \pm 0.02, \quad (33)$$

for $L^2/N_z \gg 1$ in the main logarithmic approximation. This behavior is demonstrated in Fig. 5, where the value u^* corresponds the offset for u_r taken at 1/10 of the value given by the analytical Eq. (22). The result Eq. (33) is consistent with the analytical formula Eq. (32).

Clearly, such a quasi-1D suppression (zero modes effect) is also present in the standard XY model (where no SP are anticipated to exist). In order to demonstrate this explicitly we have also simulated a simple XY model given by the system

$$Z_{XY} = \int D\phi_z \exp(-H_{XY}),$$
$$H_{XY} = -\sum_{\langle ij\rangle,z}[\tilde{K}\cos(\nabla_{ij}\phi_z) + u\cos(\nabla_z\phi_z)], \quad (34)$$

with some $\tilde{K} \gg 1$ (guaranteeing that no BKT transition occurs in each layer for $u = 0$), and $0 < u \ll \tilde{K}$. In the dual representation this system is described by

$$H_{XY} \to \tilde{H}_{XY} = \sum_{\langle ij\rangle,z} \frac{1}{2\tilde{K}} J_{z,ij}^2 + \sum_{i,z} \frac{1}{2u_V} J_{i,z}^2, \quad (35)$$

where $J_{ij,z}$ and $J_{i,z}$ are the same J currents introduced above for the model Eq. (28). The results of the simulations of

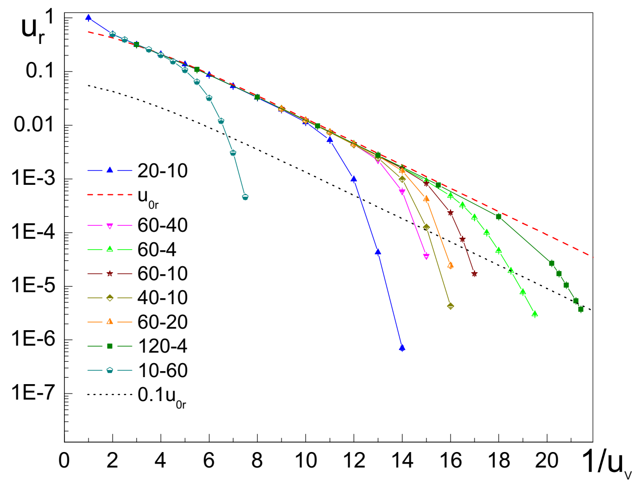

FIG. 6. Monte Carlo results for the interlayer stiffness u_r of the strongly asymmetric XY model. Dashed orange line is the analytical solution Eq. (20) with $M = 1$. Dotted black line represents the offset $u_r = 0.1$ of the analytical value. The first and second numbers in the legend indicate values of L and N, respectively.

this model are presented in Figs. 6 and 7. In the asymptotic limit the interlayer stiffness is described by Eq. (20) with $M = 1$. Then, according to the above discussion, the value u^* determining where the deviations from the analytical formula begin is given by $(u^*)^{-1} = 2\ln(L^2/N_z)$, that is, with the slope $\gamma = 2$, which should be compared with the numerical value $\gamma = 1.95 \pm 0.05$ in Fig. 7. Thus, both models demonstrate essentially the same 3D behavior, with the only difference being the slope of the renormalized Josephson coupling $\ln u_r$ versus its bare value u_V.

IV. DISCUSSION

The RG approach to 2D systems proves to be very effective in many cases, including 2D XY model when it can be mapped on the Sine-Gordon (SG) one [34]. A successful implementation of the RG analysis to the Josephson coupling was demonstrated in Ref. [35] where a single weak link can make one channel Luttinger liquid insulating.

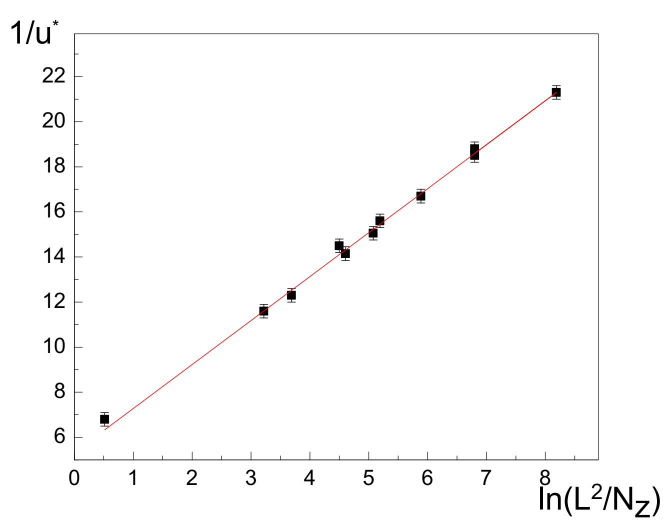

FIG. 7. The values of u^* determined numerically from the data shown in Fig. 6 by finding the crossings of the curves u_r with the offset (dotted) line in Fig. 6. The linear fit of this line gives the slope 1.95 ± 0.05.

The merit of RG, however, should be taken with caution when applied to the dimensional reduction situations in layered systems hosting compact $U(1)$ variables. In this case there is no exact mapping between XY and SG representations at finite interlayer Josephson coupling, and the approximation ignoring the compact nature of the variables becomes uncontrolled. As our analysis of one particular layered system shows, no SP exists in such a system despite the RG prediction: the system shows essentially the 3D behavior of the asymmetric XY model. Clearly, the simplest example presented here reveals the flaw in extending RG to the dimensional decoupling situations [12–16] when the effective model corresponds to zero conformal spin [11]. (At nonzero spin, tunneling of pairs can take over [10].) As shown in Sec. II C, the interlayer Josephson coupling exists even when there is no intralayer order—which is consistent with the proximity effect.

The dual formulation in terms of the closed loops gives a very important insight. Specifically, the SP means that as layer size $L \to \infty$, a suppression of the Josephson coupling between layers would require that the number of times elements of closed loops fluctuate between layers must scale slower than L^2 so that the density of such events is zero in the limit $L = \infty$. The loops statistics, however, is controlled by local energies of creating finite elements and the entropy due to six directions in 3D versus four along layers. Thus, as long as there is a finite energy to cross between neighboring layers, the entropy will lead to a finite density of crossings for large enough L. Similar argument can be applied to quantum wires in terms of the quantum to classical mapping where imaginary time is treated as an extra dimension.

The dual approach and the argumentation along the line of the numerical algorithm [32], treating closed loops formation as a process of the worm head wandering around and eventually finding its tail, allowed us to expose the actual stages of the renormalization of the Josephson coupling: (i) At small scales Josephson coupling is controlled by exponentially suppressed random and independent (in the asymptotic limit) events of crossings between layers. It can be viewed as an ideal gas of J currents between the layers. This stage leads to the renormalized coupling, in general, represented by Eq. (20) with $M = 1,2,3, \ldots$. (ii) If the number of layers N_z increases, with L being fixed, quasi-1D fluctuations further suppress the coupling exponentially as demonstrated in Eq. (31).

Here we have discussed a local model characterized by short range interactions between the interlayer J-current elements. This feature in combination with the low density of such elements justifies the "ideal gas," which in its turn leads to finite values of the renormalized interlayer Josephson coupling.

The question may be raised if a presence of long-range forces between the interplane J currents $J_{z,i}$ can change the situation and lead to the SP or its weaker version—where $u_r \to 0$ with the growing number of layers N_z in the limit $L = \infty$. In this respect we note that in order to realize this, fluctuations of the difference of the J currents with positive and negative orientations must be macroscopically suppressed. In this case, the fluctuation of the winding numbers in z direction $\langle W_z^2 \rangle$ will scale slower than L^2 so that $u_r \sim \langle W^2\rangle/L^2 \to 0$. This may be caused by interactions between the interlayer J currents decaying not faster than second power of their separation along planes. More specifically, the following additional repulsive term,

$$H_{\rm SP} = \frac{1}{2}\sum_{i,j;z} U(\vec{x}_i - \vec{x}_j) J_{z,i} J_{z,j}, \tag{36}$$

in the simple XY J-current model Eq. (35) with $U(\vec{x})$ having the long-range tail $\sim 1/|\vec{x}|^\sigma$ with $\sigma < 2$ will generate the energy contribution $\sim W_z^2 L^{-\sigma}$ in terms of the windings in z direction. Consequently, the renormalized Josephson coupling Eq. (15) would scale as $u_r \sim L^{\sigma-2} \to 0$.

As one particular example, long-range forces can be introduced in the standard XY model Eq. (34) by some effective gauge-type term $-u\cos(\nabla_z\phi - g_z A_z) + (\vec{\nabla} A_z)^2$, where $\vec{\nabla} A_z$ refers to the derivatives along the layers of some soft mode A_z, with g_z being a constant. The resulting interaction in the dual form Eq. (36) becomes $U \sim g_z^2 \ln(|\vec{x}|)$ and, thus, it suppresses the interlayer Josephson as $u_r = N_z\langle W_z^2\rangle/L^2 \sim 1/(L^2\ln L)$ in the limit $L \to \infty$ for fixed N_z. At the moment we do not comment on how realizable in practice such a mechanism is.

Here we have analyzed a clean system and found no SP. The situation is completely different in the presence of layered disorder [19,20] when the weakly sliding phases occur due to rare fluctuations of disorder resulting in a large stack of insulating layers simply blocking the flow perpendicular to the layers. The number of such layers scales logarithmically with the total number of layers, so that $u_r \sim N_z^{-c}$ with some nonuniversal exponent $c > 0$. This effect does not need any drag-type interactions and can occur in a simple XY model.

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APPENDIX A: LATTICE FORMULATION

In order to introduce the cross-gradient term $\sim K_{12}$ in Eq. (1) (cf. Refs. [3,13,14,16]) consistent with the compact nature of the phases, we use an effective gauge-type field A_{ij} defined on bonds of the lattice:

$$H_{A,\phi} = -\sum_{\langle ij\rangle}\left[t_1\cos(\nabla_{ij}\phi_1 - A_{ij}) + t_2\cos(\nabla_{ij}\phi_2 - g_2 A_{ij}) + \frac{1}{2g}A_{ij}^2\right] - \sum_i u\cos[\phi_2(i) - \phi_1(i)], \tag{A1}$$

where $t_1 > 0, t_2 > 0, g > 0$ and g_2 are parameters; $\langle ij\rangle$ denotes summation over nearest-neighbor sites within each layer; $\nabla_{ij}\phi_z \equiv \phi_z(i) - \phi_z(j)$; A_{ij} is a bond vector field (that is, $A_{ij} = -A_{ji}$) oriented along the bond $\langle ij\rangle$. Accordingly, the partition function Eq. (3) should be rewritten as

$$Z = \int DAD\phi_1 D\phi_2 \exp(-H_{A,\phi}), \tag{A2}$$

where the temperature is absorbed into the the parameters t_1, t_2, u, g. Our focus here is on verifying the applicability of the RG analysis to the renormalization of the Josephson coupling u. Hence, we will not discuss physical origins of the variables and the parameters.

If the fugacity of the inplane vortices is irrelevant, the terms $-\cos(\nabla_{ij}\phi_1 - A_{ij})$ and $-\cos(\nabla_{ij}\phi_2 - g_2 A_{ij})$ in Eq. (A1) can be replaced by $(\nabla_{ij}\phi_1 - A_{ij})^2/2$ and $(\nabla_{ij}\phi_2 - g_2 A_{ij})^2/2$, respectively. Then, the Gaussian integration over A_{ij} can be carried out explicitly in Eq. (A2), so that Eq. (A1) in terms of the phases becomes exactly Eq. (1). where the 2×2 matrix $K_{zz'}, z,z' = 1,2$ is related to the original parameters as

$$K_{11} = \frac{t_1\left(1 + g g_2^2 t_2\right)}{1 + g\left(t_1 + g_2^2 t_2\right)},\; K_{22} = \frac{t_2(1 + g t_1)}{1 + g\left(t_1 + g_2^2 t_2\right)},$$
$$K_{12} = -\frac{g g_2 t_1 t_2}{1 + g\left(t_1 + g_2^2 t_2\right)}. \tag{A3}$$

(As a matter of taste, we will keep $g_2 < 0$ in order to have $K_{12} > 0$.)

The stability requirement Eq. (2) is guaranteed by

$$K_{11}K_{22} - K_{12}^2 = \frac{t_1 t_2}{1 + g\left(t_1 + g_2^2 t_2\right)} > 0. \tag{A4}$$

The condition Eq. (9) for the existence of SP for the chosen values $Y = 25.5$, $X = 5$ in terms of the parameters t_1, t_2, g_2, g, implies that $g t_2|g_2|(1 - 5|g_2|) = 5$, $g t_1(5.1|g_2| - 1) = 1$, $t_1 \approx 0.177|g_2|/[(1 - 4|g_2|)(5.1|g_2| - 1)$, and $10/51 < |g_2| < 1/5$.

The partition function Z, Eq. (A2), with the full action Eq. (A1) can be evaluated by the high-temperature expansion method (see, e.g., Ref. [36]) in terms of t_1, t_2, u with further explicit integration over the variables. This approach allows obtaining Z in terms of the integer bond variables—powers of the corresponding Taylor series. We will be utilizing the Villain approximation [30] for the cosines to obtain the so-called J-current version [23] of Eqs. (A2) and (A1):

$$Z = \sum_{\{m_{a,ij},m_i\}}\int D\phi \int DA e^{-H_V}, \tag{A5}$$

$$H_V = \sum_{\langle ij\rangle}\left[\frac{\tilde{t}_1}{2}(\nabla_{ij}\phi_1 - A_{ij} + 2\pi m_{1,ij})^2 + \frac{\tilde{t}_2}{2}(\nabla_{ij}\phi_2 - g_2 A_{ij} + 2\pi m_{1,ij})^2 + \frac{1}{2g}A_{ij}^2\right] + \sum_i \frac{u_V}{2}(\phi_2(i) - \phi_1(i) + 2\pi m_i)^2, \tag{A6}$$

where $m_{a,ij} = -m_{a,ji} = 0, \pm 1, \pm 2, \ldots$ $(a = 1,2)$ are integer numbers defined along bonds between two nearest sites i and j along the planes, and $m_i = 0, \pm 1, \pm 2, \ldots$ is an integer assigned to a site i and oriented from the layer 1 to the layer 2.

The Villain approximation proves to be very accurate for establishing the transition points as well as in general if the effective constants $\tilde{t}_1, \tilde{t}_2, u_V$ are properly expressed in terms of the corresponding bare values t_1, t_2, u (see Ref. [31]). The "renormalization" can be essentially ignored for $t_1, t_2 \geqslant 1$, so that in what follows we will be using $\tilde{t}_1 = t_1, \tilde{t}_2 = t_2$. Similarly, for the Josephson coupling $u \sim 1$ one should take $u_V = u$ and, if $u \ll 1$, the corresponding relation is $u_V = 1/[2\ln(2/u)]$ [30,31]. After using the Poisson identity, $\sum_m f(m) = \int dm f(m)\exp(2\pi i m J)$ for arbitrary function f, for each integer and performing the integrations over ϕ_i and A, the resulting expression becomes the dual formulation Eqs. (10) and (11) of the original system Eqs. (1) and (3).

APPENDIX B: RG EQUATIONS FOR THE BILAYER

Here we will provide the derivation of the RG equations based on the quantum to classical mapping. In our case it should rather be viewed as classical to quantum mapping. Treating one of the layers direction (say y) as imaginary time τ and using Haldane's approach [37] in terms of the phases ϕ_i and the "angles" θ_i counting particles as mutually conjugated variables, the corresponding action in $D = 1+1$ becomes [20,22]

$$H_Q = \int_0^L dx \int_0^\beta d\tau \Big\{ \frac{i}{\pi}\partial_x\theta_z\partial_\tau\phi_z + \frac{1}{2}K_{zz'}\partial_x\phi_z\partial_x\phi_{z'} + \frac{1}{2\pi^2}(K^{-1})_{zz'}\partial_x\theta_z\partial_x\theta_{z'} - u\cos(\phi_1-\phi_2) - \sum_{q_1,q_2} V_{q_1,q_2}\cos[2(q_1\theta_1+q_2\theta_2)]\Big\}, \tag{B1}$$

where $(K^{-1})_{zz'}$ are the matrix elements of the matrix inverse of $K_{zz'}$; $\beta = L$ (that is, the "speed of sound" $V_s = L/\beta = 1$) and the summation over the repeated indexes ($z,z' = 1,2$) labeling layers is used here and below. The last summation terms account for the backscattering events with q_1,q_2 being arbitrary integers (from $-\infty$ to $+\infty$), which represent charges of the instantons (or composite vortices—in the "language" of the original classical layers).

We begin by looking for a solution where all the harmonics amplitudes V_{q_1,q_2} are irrelevant. In this case the Gaussian integration of the θ_i variables leads the effective low-energy action Eq. (1). In this regime the renormalization of u and $K_{zz'}$ can be obtained within the standard RG procedure (see, e.g., Ref. [25]). It consists of the repeated elimination of the high momenta harmonics from some cutoff Λ to $\Lambda/(1+s)$ with $s \to 0$ and further rescaling of the unit of length (and time) by the factor $(1+s)$. More specifically, the variables ϕ_z,

$$\phi_z = \phi_z^{(<)} + \phi_z^{(>)}, \tag{B2}$$

are separated into the low-energy $\phi_a^{(<)}$ and the high-energy $\phi_a^{(>)}$ parts, where the latter is to be integrated out from the partition function $Z = \int D\phi D\theta \exp(-H_Q)$. This (with the rescaling) will generate the effective action $H_Q^{(<)}$, which depends on the low-energy harmonics only and the renormalized values of $K_{zz'}$ and u. To the lowest order the resulting RG equation for u can be represented as

$$\frac{du}{dl} = \left(2 - \frac{1}{2s}\langle(\phi_1^{(>)} - \phi_2^{(>)})^2\rangle_s\right)u, \tag{B3}$$

where the averaging $\langle\ldots\rangle_s$ is performed over the harmonics in the narrow shell $\Lambda < |\vec{q}| < \Lambda/(1+s)$ in the gaussian part of the action Eq. (1).

The renormalization of $K_{zz'}$ is determined by the terms $\sim u^2$ in the lowest order. The resulting equations are

$$\frac{dK_{11}}{dl} = \frac{Cu^2}{s}(\langle(\phi_1^{(>)} - \phi_2^{(>)})^2\rangle_s), \tag{B4}$$

$$\frac{dK_{22}}{dl} = \frac{Cu^2}{s}(\langle(\phi_1^{(>)} - \phi_2^{(>)})^2\rangle_s), \tag{B5}$$

$$\frac{dK_{12}}{dl} = -\frac{Cu^2}{s}(\langle(\phi_1^{(>)} - \phi_2^{(>)})^2\rangle_s), \tag{B6}$$

where $C > 0$ stands for a constant which depends on the cutoff procedure. As usual, this constant can be absorbed into the definition of u, and we choose it as $C = 1$.

Using the notations $K_{22} = K_{11}Y$, $K_{12} = K_{11}X$ in the Gaussian integral $\langle(\phi_1^{(>)} - \phi_2^{(>)})^2\rangle_s/s$, the above equations become

$$\frac{du}{dl} = \left(2 - \frac{1}{4\pi K_{11}}\frac{1+Y+2X}{Y-X^2}\right)u, \tag{B7}$$

$$\frac{dK_{11}}{dl} = \frac{u^2}{2\pi K_{11}}\frac{1+Y+2X}{Y-X^2}, \tag{B8}$$

$$\frac{d(K_{11}Y)}{dl} = \frac{u^2}{2\pi K_{11}}\frac{1+Y+2X}{Y-X^2}, \tag{B9}$$

and

$$\frac{d(K_{11}X)}{dl} = -\frac{u^2}{2\pi K_{11}}\frac{1+Y+2X}{Y-X^2}. \tag{B10}$$

Equations (B8), (B9), and (B10) imply $K_{11} = C_1/(1+X)$, $Y = 1 + C_2(1+X)$, where $C_1 > 0, C_2 > 0$ are constants of integration. Finally, Eqs. (B7) and (B8) can be expressed in terms of two variables u and $K_\phi \equiv K_{11} - C_1/(2+C_2)$ as

$$\frac{du}{dl} = \left(2 - \frac{1}{4\pi K_\phi}\right)u, \tag{B11}$$

and

$$\frac{dK_\phi}{dl} = \frac{u^2}{2\pi K_\phi}. \tag{B12}$$

These are the standard RG equations, which are fully integrable. The SP phase corresponds to $K_\phi < 1/(8\pi)$, which is represented by Eq. (6) (with $T \equiv 1/K_{11}$). In this phase u flows to zero and the Luttinger matrix $K_{zz'}$ remains essentially scale independent.

The SP implies that Luttinger liquids in both wires remain gapless. Thus, the condition $K_\phi < 1/(8\pi)$ should be consistent with the requirement that all the harmonics V_{q_1,q_2} are irrelevant. In the regime $K_\phi < 1/(8\pi)$ (where u is irrelevant), Eq. (B1) can be expressed in terms of the "angles" θ_a as

$$H_\theta = \int_0^L dx \int_0^\beta d\tau \left\{ \frac{1}{2\pi^2}(K^{-1})_{ab}\vec{\nabla}\theta_a\vec{\nabla}\theta_b - \sum_{q_1,q_2} V_{q_1,q_2}\cos[2(q_1\theta_1 + q_2\theta_2)] \right\}. \tag{B13}$$

The RG equation for the most relevant harmonic can be obtained along the same lines as discussed above. It is

$$\frac{dV_{q_1,q_2}}{dl} = \left(2 - \frac{1}{2s}\langle(q_1\theta_1^{(>)} + q_2\theta_2^{(>)})^2\rangle_s\right)V_{q_1,q_2}. \tag{B14}$$

Evaluation of the correlator $\langle(q_1\theta_1^{(>)} + q_2\theta_2^{(>)})^2\rangle_s$ within the Gaussian part of the action Eq. (B13) gives

$$\frac{dV_{q_1,q_2}}{dl} = [2 - \pi K_{ab}q_a q_b]V_{q_1,q_2}, \;\rightarrow\; \frac{dV_{q_1,q_2}}{dl} = \{2 - \pi K_{11}[(q_1 + Xq_2)^2 + (Y - X^2)q_2^2]\}V_{q_1,q_2}. \tag{B15}$$

As can be immediately seen, this equation features the critical point of the transition into the composite phase described by Eq. (8) (where $T \equiv 1/K_{11}$).

The renormalization of the K matrix in the second order in the amplitude V_{q_1,q_2} is given by

$$\frac{d(K^{-1})_{ab}}{dl} = q_a q_b V_{q_1,q_2}^2 K_{rs} q_r q_s. \tag{B16}$$

Equations (B16) have two integrals. Using the notations $(K^{-1})_{22} = \tilde{Y}(K^{-1})_{11}$ and $(K^{-1})_{12} = \tilde{X}(K^{-1})_{11}$ (which are related to the previously introduced variables as $\tilde{Y} = 1/Y$ and $\tilde{X} = -X/Y$), we find $\tilde{Y} = q_2^2 q_1^{-2} - B_1 q_2^2/(K^{-1})_{11}$ and $\tilde{X} = q_2 q_1^{-1} - B_2 q_1 q_2/(K^{-1})_{11}$, where B_1, B_2 are constants of integration. Using these relations in Eqs. (B15) and (B16), we find

$$\frac{dV_{q_1,q_2}}{dl} = \left[2 - \frac{\pi q_1^2}{K_\theta}\right]V_{q_1,q_2}, \tag{B17}$$

$$\frac{dK_\theta}{dl} = \frac{q_1^4}{K_\theta}V_{q_1,q_2}^2, \tag{B18}$$

where the notation $K_\theta = (K^{-1})_{11} - q_1^2 B_2^2/(2B_2 - B_1)$ is introduced.

Equations (B17) and (B18) are also the standard RG equations. For $K_\theta < \pi q_1^2/2$ the most "dangerous" harmonic V_{q_1,q_2} is irrelevant, that is, the system remains in the superfluid regime with two gapless modes (provided the SP phase exists).

The above analysis implies that the SP phase exists if two conditions hold: $K_\theta < \pi q_1^2/2$ and $K_\phi < 1/(8\pi)$. These conditions are represented by Eqs. (8) and (6), respectively. As further analysis in the main text (Sec. II A) has shown, Eq. (9) is one of the solutions satisfying both inequalities.

RG for arbitrary N_z

The equation for u_r in the case of a stack of bilayers, as discussed in Sec. III, can be obtained along the same line as for the bilayer (see also Ref. [38] in the context of the bosonic composite phases in a layered system):

$$\frac{du_r}{d\ln l} = \left(2 - \frac{1}{2s}\langle(\phi_{z+1} - \phi_z)^2\rangle_s\right)u_r. \tag{B19}$$

We note that, due to the PBC along z direction, the mean $\langle(\phi_{z+1} - \phi_z)^2\rangle$ does not depend on z. Using discrete Fourier representation along z direction with doubled unit cell containing two layers (the odd and the even) with two sorts of phases $\phi_z = \phi^{(1)}(z)$ and $\phi_z = \phi^{(2)}(z)$ along odd and even layers, respectively, the part H_z in Eq. (23) can be diagonalized and the correlator in Eq. (B19) found. This gives Eq. (B19) rewritten as

$$\frac{du_r}{d\ln l} = 2\left(1 - \frac{T}{T_d}\right)u_r, \tag{B20}$$

where T_d is given by Eq. (26).

The flow equations for the matrix K can also be found along the same line as described in Ref. [38]. In this case the matrix $K_{zz'}$, which now depends on the wave vector q_z, remains essentially unrenormalized as long as $T > T_d$.

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