

Landau-Zener transition driven by slow noise

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The effect of a slow noise in nondiagonal matrix element $J(t)$ that describes the diabatic level coupling on the probability of the Landau-Zener transition is studied. For slow noise, the correlation time τ_c of $J(t)$ is much longer than the characteristic time of the transition. Existing theory for this case suggests that the average transition probability is the result of averaging of the conventional Landau-Zener probability, calculated for a given constant J , over the distribution of J . We calculate a finite- τ_c correction for this classical result. Our main finding is that this correction is dominated by *sparse realizations* of noise for which $J(t)$ passes through zero within a narrow time interval near the level crossing. Two models of noise, random telegraph noise and Gaussian noise, are considered. Naturally, in both models the average probability of transition decreases upon decreasing τ_c . For Gaussian noise we identify two domains of this falloff with specific dependencies of average transition probability on τ_c .

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I. INTRODUCTION

A standard expression [1–4] for the probability of the Landau-Zener transition between the two diabatic levels $\pm vt/2$ reads

$$P_{\text{LZ}} = 1 - Q_{\text{LZ}}, \quad Q_{\text{LZ}} = \exp\left(-\frac{2\pi J^2}{v}\right), \quad (1)$$

where J is the off-diagonal matrix element. Correspondingly, Q_{LZ} is the probability to stay on the initial diabatic level.

The dynamics of the transition is governed by the following system:

$$\begin{aligned} i\dot{a}_1 &= \frac{vt}{2}a_1 + Ja_2, \\ i\dot{a}_2 &= -\frac{vt}{2}a_2 + Ja_1, \end{aligned} \quad (2)$$

where a_1, a_2 denote the amplitudes for the particle to reside on diabatic levels 1 and 2, respectively.

The question about the value of P_{LZ} when the transition is driven by noise so that J is a random function of time was first put forward and answered in Refs. [5,6]. The answer depends on whether the noise is slow or fast. More precisely, on whether the noise correlation time τ_c is longer or shorter than the time of the Landau-Zener transition. For large matrix element $J \gg v^{1/2}$ when P_{LZ} is close to 1, this transition time is given by

$$\tau_{\text{LZ}} = \frac{J}{v}. \quad (3)$$

Fast noise corresponds to $\tau_{\text{LZ}} \gg \tau_c$, i.e., the matrix element oscillates many times in the course of the transition. Then it is apparent that both outcomes of the transition are almost equally probable. In other words, Q_{LZ} differs from 1/2 only slightly. It was demonstrated in Ref. [6] that, for the fast noise,

$$\langle Q_{\text{LZ}} \rangle = \frac{1}{2} \left[1 + \exp\left(-\frac{4\pi J_c^2}{v}\right) \right], \quad (4)$$

where $\langle \dots \rangle$ denotes the averaging over the realizations of $J(t)$ and J_c^2 is defined via the noise correlator,

$$\langle J(t)J(t') \rangle = J_c^2 K(t-t'), \quad (5)$$

at coinciding times, where $K(0) = 1$.

In the opposite limit of slow noise $\tau_c \gg \tau_{\text{LZ}}$, the result obtained in Ref. [6] is also physically transparent. Namely, one can neglect the change in $J(t)$ during the transition. This suggests that the expression Eq. (1) should be simply averaged over the distribution $\mathcal{P}(J)$ of the matrix element,

$$\langle Q_{\text{LZ}} \rangle = \int_{-\infty}^{\infty} dJ \mathcal{P}(J) \exp\left\{-\frac{2\pi J^2}{v}\right\}. \quad (6)$$

Further developments in the theory of the noise-driven Landau-Zener transition in Refs. [7–23] included: (i) consideration of specific microscopic models of the environment leading to random $J(t)$, (ii) extension of Eq. (4) to the case when $J(t)$ has both constant and fluctuating components (Pokrovsky-Sinitsyn formula), and (iii) generalization of the theory of the fast noise to the case of three and more crossing levels.

With regard to the papers [7–23], we note that the results for the average transition probabilities were obtained only in two “extreme” cases: $\tau_c \rightarrow 0$ for the fast noise and $\tau_c \rightarrow \infty$ for the slow noise. The corrections in small parameters τ_c/τ_{LZ} for the fast noise and τ_{LZ}/τ_c for the slow noise were not found. There is actually a fundamental reason for this, which can be traced back to the techniques employed in Ref. [6] and subsequent works. Namely, in Ref. [6], Q_{LZ} was presented as expansion in powers of J^2 and then averaged over random realizations of $J(t)$ term by term. As we will see below, this technique does not allow to describe a crossover between the limits Eqs. (4) and (6) and even to capture finite- τ_c corrections.

Calculation of a finite- τ_c correction for the case of a slow noise is the main subject of the present paper. We will demonstrate that this correction comes from *nonperturbative* effects, or in other words, not from typical but rather from sparse noise realizations. To illustrate the message, consider the particular model of telegraph noise as in Ref. [17] when $J(t)$ switches randomly between the two values $\pm J_0$. We will

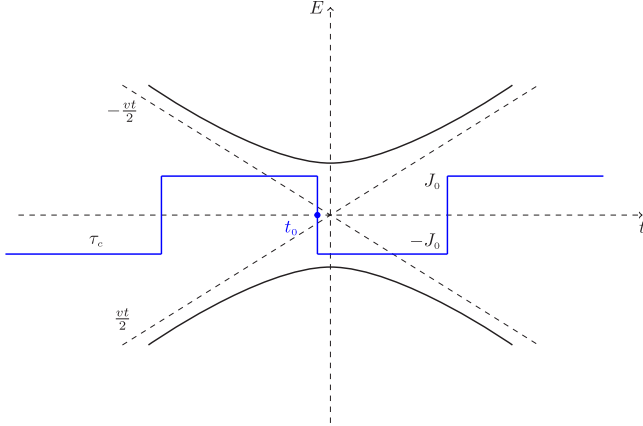


FIG. 1. Schematic of the Landau-Zener transition driven by slow telegraph noise. For a typical noise realization the matrix element switches from J_0 to $-J_0$ at time $t_0 \sim \tau_c \gg \tau_{LZ}$. For such realizations the probability Q_{LZ} to stay on the same diabatic level is exponentially small. However, for sparse realizations with $t_0 \lesssim \tau_{LZ}$, the value Q_{LZ} is close to 1. Thus, it is these sparse realizations that are responsible for the finite- τ_c correction.

demonstrate that it is particularly the switchings which take place near the level crossing $t = 0$ that have a dramatic effect on Q_{LZ} . This situation is illustrated in Fig. 1.

Without switching near $t = 0$, the probability Q_{LZ} is exponentially small. On the other hand, *with switching*, as we will demonstrate below, Q_{LZ} is close to 1. Since the probability that the switching takes place at $t_0 \lesssim \tau_{LZ}$ is $\sim \tau_{LZ}/\tau_c$, the correction to Q_{LZ} can be estimated as $\sim \tau_{LZ}/\tau_c$. This correction becomes important even at large enough τ_c since in the limit $\tau_c \rightarrow \infty$ the value of Q_{LZ} is exponentially small. In the next section we justify the above picture by a rigorous calculation. In Sec. III we study the case of a Gaussian noise when $J(t)$ is a smooth function of time. We show that in this case the finite- τ_c correction to Q_{LZ} , again, originates from the sparse realizations of noise when $J(t)$ passes through zero near the moment of the level crossing. In addition to the finite- τ_c correction, the realizations with $J(t_0) = 0$ for $t_0 \ll \tau_c$ determine the behavior of Q_{LZ} upon decreasing in the correlation time. We identify two domains of τ_c with distinctively different $Q_{LZ}(\tau_c)$ dependence. In Sec. IV, which concludes the paper, we speculate on the form of the finite- τ_c correction to Q_{LZ} in the limit of fast noise $\tau_c \ll \tau_{LZ}$.

II. TELEGRAPH NOISE: SWITCHING NEAR $t = 0$

Consider the system Eq. (2) and assume that the coupling constant is equal to J_0 for $t < t_0$, whereas for $t > t_0$ it switches

to $-J_0$, see Fig. 1. Solutions of the system for $t < t_0$ are expressed via the parabolic cylinder functions [24] as follows:

$$\begin{aligned} a_1 &= D_\nu(z), \\ a_2 &= -i\sqrt{\nu}D_{\nu-1}(z), \end{aligned} \quad (7)$$

where the argument z is defined as $z = \sqrt{\nu}e^{i\pi/4}t$ and the index ν is given by

$$\nu = -\frac{J^2}{v}. \quad (8)$$

The solution Eq. (7) ensures that $a_2(-\infty) = 0$, i.e., that the particle is in state a_1 away from the crossing.

For $t > t_0$ the solution of the system represents a linear combination of two parabolic-cylinder functions,

$$\begin{aligned} a_1 &= AD_\nu(z) + BD_\nu(-z), \\ a_2 &= i\sqrt{\nu}AD_{\nu-1}(z) - i\sqrt{\nu}BD_{\nu-1}(-z). \end{aligned} \quad (9)$$

The fact that the coefficient in front of $D_{\nu-1}(z)$ in a_2 is equal to $-A$, unlike Eq. (7), is the consequence of the switching. The coefficients A and B are found from continuity of a_1 and a_2 at $t = t_0$. The corresponding system reads

$$\begin{aligned} D_\nu(z_0) &= AD_\nu(z_0) + BD_\nu(-z_0), \\ D_{\nu-1}(z_0) &= -AD_{\nu-1}(z_0) + BD_{\nu-1}(-z_0), \end{aligned} \quad (10)$$

where $z_0 = \sqrt{\nu}e^{i\pi/4}t_0$. From this system we readily derive the following expressions:

$$A = \frac{\frac{D_{\nu-1}(-z_0)}{D_{\nu-1}(z_0)} - \frac{D_\nu(-z_0)}{D_\nu(z_0)}}{\frac{D_{\nu-1}(-z_0)}{D_{\nu-1}(z_0)} + \frac{D_\nu(-z_0)}{D_\nu(z_0)}}, \quad B = \frac{2}{\frac{D_{\nu-1}(-z_0)}{D_{\nu-1}(z_0)} + \frac{D_\nu(-z_0)}{D_\nu(z_0)}}. \quad (11)$$

Without switching, we would have $A = 1$, $B = 0$. Suppose now that the switching took place at $t_0 = 0$. Then from Eq. (11) we have $A = 0$ and $B = 1$. This illustrates the dramatic effect of switching on the probability Q_{LZ} . Indeed, in the limit $z_0 \rightarrow \infty$, the ratio $|D_\nu(z_0)/D_\nu(-z_0)|^2$ approaches a small value of $\exp(-2\pi J_0^2/v)$, which is the value of Q_{LZ} without switching. By contrast, with $B = 1$ as a result of switching, Q_{LZ} is equal to 1.

The situation $t_0 = 0$ is most favorable for enhancement of Q_{LZ} . In order to find the average enhancement, we should analyze the expressions for A and B , given by Eq. (11), as a function of z_0 . The fact which allows for carrying out this analysis analytically is that the typical value of Q_{LZ} is small. This is also equivalent to the condition $J_0^2 \gg v$ or $|\nu| \gg 1$. Conventionally, the behavior of the parabolic cylinder functions at large $|z_0| \gg 1$ is obtained from the semiclassics. The condition $|\nu| \gg 1$ justifies the semiclassical analysis even for small z_0 . This analysis is carried out in the Appendix. Substituting Eqs. (A8) and (A10) into Eq. (11), we obtain the following expression for B :

$$B = 2 \left[\frac{\left(\frac{\sqrt{4\nu-z_0^2-i z_0}}{\sqrt{4\nu-z_0^2+i z_0}} \right)^{\nu+1/2} + e^{i\nu\pi+(i z_0/2)\sqrt{4\nu-z_0^2}}}{e^{(i z_0/2)\sqrt{4\nu-z_0^2}} + e^{i\nu\pi} \left(\frac{\sqrt{4\nu-z_0^2-i z_0}}{\sqrt{4\nu-z_0^2+i z_0}} \right)^{\nu+1/2}} + \frac{\left(\frac{\sqrt{4\nu-z_0^2-i z_0}}{\sqrt{4\nu-z_0^2+i z_0}} \right)^{\nu-1/2} - e^{i\nu\pi+(i z_0/2)\sqrt{4\nu-z_0^2}}}{e^{(i z_0/2)\sqrt{4\nu-z_0^2}} - e^{i\nu\pi} \left(\frac{\sqrt{4\nu-z_0^2-i z_0}}{\sqrt{4\nu-z_0^2+i z_0}} \right)^{\nu-1/2}} \right]^{-1}. \quad (12)$$

The above expression for B allows a dramatic simplification. It is achieved upon adding the two fractions and subsequently rewriting the result in the form of a single fraction. The result reads

$$B = \frac{iz_0 + i\sqrt{4v}\sin(\phi + v\pi)}{\sqrt{4v - z_0^2}\sinh(|v|\pi)}, \quad (13)$$

where the phase ϕ is defined as

$$\phi = -\frac{z_0}{2}\sqrt{4v - z_0^2} - iv \ln \frac{(\sqrt{4v - z_0^2} - iz_0)^2}{4v}. \quad (14)$$

In a similar way, from Eq. (11) we get a simplified expression for A ,

$$A = -\frac{i\sqrt{4v}\sin\phi + iz_0\cosh(|v|\pi)}{\sqrt{4v - z_0^2}\sinh(|v|\pi)}. \quad (15)$$

The exact expression for the probability to stay on the same diabatic level in the presence of the switching is $Q_{LZ} = |a_1(\infty)/a_1(-\infty)|^2$. From Eq. (9) we can express this probability in terms of A and B as follows:

$$Q_{LZ} = |A|^2 e^{-2\pi|v|} + (A^*B + B^*A)e^{-\pi|v|} + |B|^2. \quad (16)$$

Let us first check that for large $|z_0| \gg |v|^{1/2}$ Eq. (16) reproduces $Q_{LZ} = \exp(-2\pi|v|)$ as is expected on physical grounds. Indeed, in this limit, we can replace A by -1 and B by $2\exp(-\pi|v|)$. Then the second and the third terms in Eq. (16) cancel out.

To study the behavior of Q_{LZ} in the domain of $|z_0| \sim |v|^{1/2}$, we notice that only the third term does not contain any exponentially small factor. Hence, for this term, we can use the asymptotic expression in which corrections on the order of $\exp(-\pi|v|)$ are neglected. Namely, $B \approx (z_0^2/4v - 1)^{-1/2}e^{-i\phi}$. This immediately leads to the following expression for Q_{LZ} in which we return to the original notations t_0 and τ_{LZ} :

$$Q_{LZ} = \frac{4}{4 + \left(\frac{t_0}{\tau_{LZ}}\right)^2}. \quad (17)$$

In Fig. 2 the dependence of Q_{LZ} on the moment of switching is shown schematically. It is described by the Lorentzian Eq. (17) for $t_0 \sim \tau_{LZ}$ and approaches $\exp(-2\pi|v|)$ for $t_0 \gg \tau_{LZ}$.

As we established above, the switching at $t_0 = 0$ leads to the probability $Q_{LZ} = 1$. Then Eq. (17) suggests that the

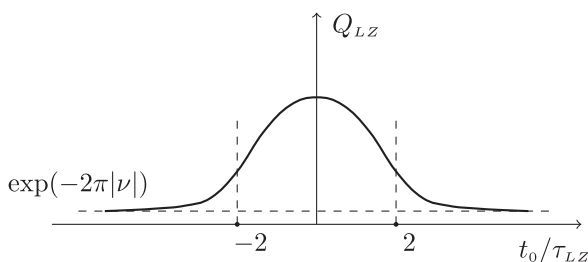


FIG. 2. The dependence of the probability to stay on the initial diabatic level at $t \rightarrow \infty$ is shown schematically versus the moment of switching t_0 . The probability $Q_{LZ}(t_0)$ is described by the Lorentzian Eq. (17) for $t_0 \sim \tau_{LZ}$ and approaches asymptotically to $\exp(-2\pi|v|)$ for t_0 much bigger than the time τ_{LZ} of the Landau-Zener transition.

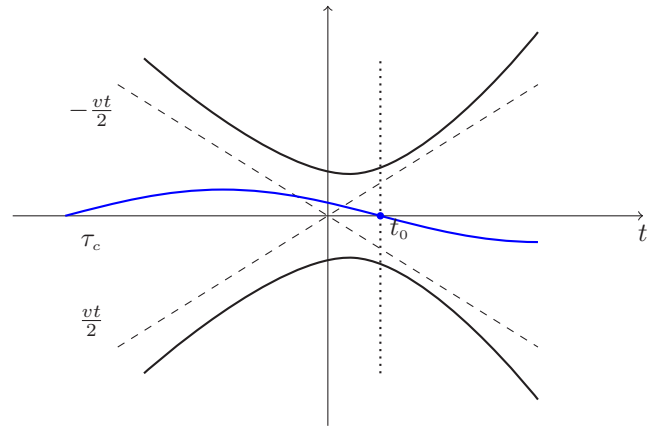


FIG. 3. If the matrix element $J(t)$ turns to zero at some $t = t_0$ close to the crossing of the diabatic levels (blue curve), the calculation of the transition probability reduces to the Landau-Zener problem with renormalized velocity and coupling both depending on t_0 . The moment t_0 also defines the shift of minima of adiabatic levels from $t = 0$. This shift is smaller than t_0 , see Eq. (22).

enhancement of Q_{LZ} takes place in the domain of $t_0 \sim \pm 2\tau_{LZ}$. Since t_0 is random, we average Eq. (17) over t_0 to get the net enhancement of Q_{LZ} and present the final result in the form

$$\begin{aligned} 1 - P_{LZ} &= \exp\left\{-\frac{2\pi J^2}{v}\right\} + \frac{1}{\tau_c} \int_{-\infty}^{\infty} dt_0 \frac{4}{4 + \left(\frac{t_0}{\tau_{LZ}}\right)^2} \\ &= \exp\left\{-\frac{2\pi J^2}{v}\right\} + 2\pi \frac{\tau_{LZ}}{\tau_c}. \end{aligned} \quad (18)$$

It is important to note that the condition of applicability of Eq. (17) is $\tau_{LZ} \ll \tau_c$. Under this condition, the second term of Eq. (18) can greatly exceed the first term, which is the result corresponding to no switching.

III. GAUSSIAN NOISE

Suppose that $J(t)$ changes continuously as in Fig. 3 and that the typical J is much bigger than $v^{1/2}$ so that the typical P_{LZ} is close to 1. As J slowly changes with time, the maximum contribution to Q_{LZ} will come from the time domain when it takes small values of $J \lesssim v^{1/2}$. Since the portion of these domains is $\sim v^{1/2}/J$, the ratio $v^{1/2}/J$ determines the average Q_{LZ} . This is how the analytical result of Ref. [6] can be interpreted. This result applies for very long correlation times $\tau_c \rightarrow \infty$.

As we are interested in a finite- τ_c correction, we will study the domains of small J in more detail. A nontrivial consequence of small J values is that, when the noise takes these values, the Landau-Zener transition time is much shorter than the typical τ_{LZ} . This, in turn, suggests that even when the condition of slow noise is violated for typical J , it can still be met for anomalously small J , which determines Q_{LZ} . Thus, one can expect a nontrivial behavior of Q_{LZ} upon decreasing the correlation time.

Naturally, small J values are realized in the vicinity of zeros of $J(t)$. Similar to the previous section, the most relevant are the realizations of noise when a zero occurs in the vicinity of $t = 0$ when the levels cross. For these realizations we can

linearize $J(t)$ as follows:

$$J(t) = (t - t_0)J', \quad (19)$$

where J' is the slope and t_0 is much smaller than the correlation time.

Our prime observation is that the system Eq. (2) can be solved exactly with $J(t)$ in the form Eq. (19). This is achieved by introducing new variables,

$$\begin{aligned} b_1 &= a_1 \cos \varphi + a_2 \sin \varphi, \\ b_2 &= a_1 \sin \varphi - a_2 \cos \varphi, \end{aligned} \quad (20)$$

where angle φ is defined as

$$\tan(2\varphi) = \frac{2J'}{v}. \quad (21)$$

It is straightforward to check that the system of equations for b_1, b_2 has the form

$$\begin{aligned} i\dot{b}_1 &= \frac{v}{2} \frac{[t - t_0 \sin^2(2\varphi)]}{\cos(2\varphi)} b_1 + \frac{v}{2} t_0 \sin(2\varphi) b_2, \\ i\dot{b}_2 &= -\frac{v}{2} \frac{[t - t_0 \sin^2(2\varphi)]}{\cos(2\varphi)} b_2 + \frac{v}{2} t_0 \sin(2\varphi) b_1. \end{aligned} \quad (22)$$

We see that the system has reduced to the original system Eq. (2) with renormalized velocity $\tilde{v} = v/\cos(2\varphi)$ and the coupling $\tilde{J} = vt_0 \sin(2\varphi)/2$. The established mapping allows us to write the answer for Q_{LZ} straightaway [25],

$$Q_{LZ}(t_0) = \exp \left\{ -\frac{2\pi J'^2 t_0^2}{v} |\cos^3(2\varphi)| \right\}. \quad (23)$$

Using Eq. (21), the above expression can be recast into the form

$$Q_{LZ} = \exp \left\{ -\frac{2\pi J'^2 t_0^2}{v} \left(\frac{v^2}{v^2 + 4J'^2} \right)^{3/2} \right\}. \quad (24)$$

The crucial assumption made in the course of deriving Eq. (24) is that the linearization Eq. (19) is valid during the entire renormalized transition time $\tilde{\tau}_{LZ}$. We are now in position to check this assumption. Indeed,

$$\tilde{\tau}_{LZ} = \frac{\tilde{J}}{\tilde{v}} = \frac{J'v}{v^2 + 4J'^2} t_0. \quad (25)$$

Since the first factor does not exceed 1, we conclude that the condition $t_0 \ll \tau_c$ is the only condition necessary for Landau-Zener transition to be dominated by a local zero of $J(t)$.

Contribution of the realizations with $t_0 \ll \tau_{LZ}$ to the probability Q_{LZ} is given by the average,

$$\overline{Q}_{LZ} = \int_{-\infty}^{\infty} \frac{dt_0}{\tau_c} Q_{LZ}(t_0) = \left(\frac{v}{2} \right)^{1/2} \frac{1}{|J'|\tau_c} \left(\frac{v^2 + 4J'^2}{v^2} \right)^{3/4}. \quad (26)$$

We are interested in the τ_c dependence of the probability Q_{LZ} . The correlation time enters into Eq. (26) in two ways: First, it is present directly in the denominator, and second, via J' since its typical value is $J' \sim J_c/\tau_c$. The remaining task is to average Eq. (26) over the random slopes J' . We will perform this averaging in the domains of small and large J' 's separately:

(i) Small J' . In this limit, we can replace the brackets in Eq. (26) by 1. The average over J' does not diverge since

J' cannot be smaller than the minimal value determined by the condition $t_0 \ll \tau_c$. Since the typical t_0 is $\sim v^{1/2}/J'$, the estimate for the minimal J' is $J' < v^{1/2}/\tau_c$. This minimal J' fixes the lower limit in the integral,

$$\langle \overline{Q}_{LZ} \rangle = \left(\frac{v}{\pi} \right)^{1/2} \frac{1}{J'_c \tau_c} \int_{v^{1/2}/\tau_c}^{\infty} \frac{dJ'}{J'} \exp \left(-\frac{J'^2}{2J_c^2} \right), \quad (27)$$

where J'_c is the width of the Gaussian distribution of J' ,

$$J_c'^2 = J_c^2 \left. \frac{\partial^2 K}{\partial t_1^2} \right|_{t_1=t_2}, \quad (28)$$

and the correlator K is defined in Eq. (5). Within a number under the logarithm, the final result for the double average reads

$$\langle \overline{Q}_{LZ} \rangle = \left(\frac{v}{\pi} \right)^{1/2} \frac{1}{J'_c \tau_c} \ln \left(\frac{J'_c \tau_c}{v^{1/2}} \right). \quad (29)$$

Let us compare this result with a standard expression derived in Ref. [6]. In our limit $J_c \gg v^{1/2}$, the result of Gaussian averaging in Eq. (6) reads

$$\langle Q_{LZ} \rangle = \left(\frac{v}{4\pi} \right)^{1/2} \frac{1}{J_c}. \quad (30)$$

Since $J'_c \sim J_c/\tau_c$, the prefactor in Eq. (29) is on the same order as in Eq. (30). In the particular case of a Gaussian correlator, we have $J'_c \tau_c = 2^{1/2} J_c$ so that the prefactor in Eq. (29) is two times bigger than in Eq. (30). However, our result contains an additional big factor $\ln(J_c/v^{1/2})$. This factor originates from realizations of noise for which $J(t)$ passes through zero near $t = 0$. The importance of these realizations leading to the enhancement of Q_{LZ} is mentioned here.

The result Eq. (29) yields the average Q_{LZ} in the limit $\tau_c \rightarrow \infty$. Calculation of the finite- τ_c correction is straightforward. Expanding the brackets in Eq. (26) to the first power in the small parameter J'^2/v^2 and averaging over J' , we get

$$\begin{aligned} \langle \overline{Q}_{LZ}(\tau_c) \rangle - \langle \overline{Q}_{LZ}(\infty) \rangle &= \frac{3}{(\pi v^3)^{1/2}} \int_0^{\infty} dJ' \frac{|J'|}{\tau_c J'_c} \\ &\times \exp \left(-\frac{J'^2}{2J_c^2} \right). \end{aligned} \quad (31)$$

It follows from Eq. (31) that upon decreasing the correlation time the correction to \overline{Q}_{LZ} grows as $1/\tau_c^2$. It is instructive to rewrite this correction as

$$\langle \overline{Q}_{LZ}(\tau_c) \rangle - \langle \overline{Q}_{LZ}(\infty) \rangle = 3 \left(\frac{2}{\pi} \right)^{1/2} \left(\frac{J_c}{v^{1/2}} \right) \left(\frac{1}{v^{1/2} \tau_c} \right)^2. \quad (32)$$

Naturally, the applicability of Eq. (32) is terminated as τ_c becomes smaller than J_c/v .

(ii) Large J' . In this limit the brackets in Eq. (26) should be replaced by $(2|J'|/v)^{3/2}$. Then the averaging over J' yields

$$\langle \overline{Q}_{LZ} \rangle \approx \frac{2}{\pi} \Gamma \left(\frac{3}{4} \right) \left(\frac{J_c}{v^{1/2}} \right)^{1/2} \left(\frac{1}{v^{1/2} \tau_c} \right)^{3/2}, \quad (33)$$

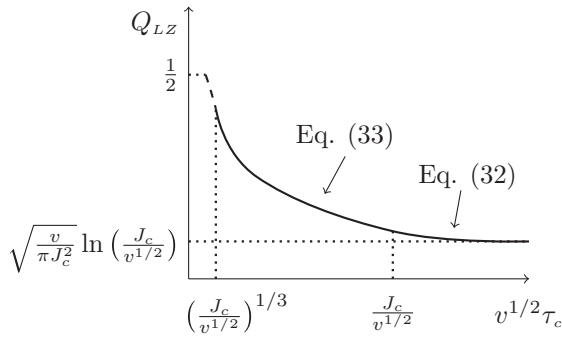


FIG. 4. Probability to stay on the same diabatic level after the transition is plotted schematically versus the dimensionless correlation time $v^{1/2}\tau_c$. In the domain of a “true” slow noise $v^{1/2}\tau_c \gg J_c/v^{1/2}$, the probability Q_{LZ} deviates from the asymptotic value of $(\frac{v}{\pi J_c^2})^{1/2} \ln(\frac{J_c}{v^{1/2}})$ only slightly. In the intermediate domain $(J_c/v^{1/2})^{1/3} \ll v^{1/2}\tau_c \ll J_c/v^{1/2}$ the crossover from the slow to the fast noise takes place.

$$\langle \bar{Q}_{LZ} \rangle = \frac{1}{(\pi|v|)^{1/2}} \begin{cases} \ln|v|^{1/2} + 3 \times 2^{1/2} \frac{|v|}{\tilde{\tau}_c^2}, & \tilde{\tau}_c \gg |v|^{1/2}, \\ \frac{2\Gamma(\frac{3}{4})}{\pi^{1/2}} \left(\frac{|v|}{\tilde{\tau}_c^2}\right)^{3/4}, & |v|^{1/2} \gg \tilde{\tau}_c \gg |v|^{1/6}. \end{cases} \quad (35)$$

The second line describes the crossover between the slow-noise and the fast-noise regimes upon decreasing the correlation time.

(ii) In obtaining the results Eqs. (29), (32), and (33) we assumed that it is only the single zero in $J(t)$ closest to the level crossing, that is responsible for the probability Q_{LZ} . This Q_{LZ} , which is the probability for the particle to stay on the diabatic level, can also be viewed as the probability to change the adiabatic level. In fact, in the domain $J_c^{1/3}/v^{2/3} \ll \tau_c \ll J_c/v$, many ($\sim J_c/v\tau_c$) zeros of $J(t)$ will cause local minima (maxima) in the upper (lower) adiabatic energy level as illustrated in Fig. 5. In other words, many local Landau-Zener transitions precede the transition near $t = 0$. For our results to apply it is necessary that, in the course of each of this transition, the particle does not change the adiabatic level. Calculation of probability of changing the levels is performed in full analogy to that described above for $J(t_0) = 0$ with $t_0 \ll \tau_c$. One has to linearize $J(t)$ near actual zero and perform the rotation from (a_1, a_2) to (b_1, b_2) . It is important that the time of the local transition turns out to be $\sim v\tau_c^2/J_c$, which is much smaller than τ_c , thus confirming that the transition is indeed local. For the probability to stay on the same adiabatic level, the above procedure yields the result,

$$1 - \exp \left[-2\pi^3 \left(\frac{v^2 \tau_c^3}{J_c} \right) \left(1 - \frac{v^2 \tau_c^2}{J_c^2} \right) \right] \quad (36)$$

(the actual numerical factor in the exponent depends on the location of the transition point within τ_c). For the lower boundary $\tau_c \sim J_c^{1/3}/v^{2/3}$, the number in the exponent is close to 1 indicating that at this boundary the regime of the fast noise takes over. At the upper boundary $\tau_c \sim J_c/v$ the number in the exponent is large ($\sim |v|$). Thus the particle does not

i.e., $\langle \bar{Q}_{LZ} \rangle$ grows as $\tau_c^{-3/2}$ upon the decreasing in the correlation time. The lower boundary of the large J' domain is determined by the condition $\langle \bar{Q}_{LZ} \rangle \sim 1$, which yields $\tau_c \sim J_c^{1/3}/v^{2/3}$. Different behaviors of $\langle \bar{Q}_{LZ} \rangle$ are illustrated in Fig. 4.

IV. CONCLUDING REMARKS

(i) The main results of the present paper are the expressions Eqs. (29), (32), and (33) for the probability Q_{LZ} to stay on the same diabatic level after the transition. These results pertain to the domain where this probability is small, i.e., for $J_c^2 \gg v$ or equivalently for $|v| \gg 1$ where the parameter v is defined by Eq. (8). Below we combine these results into a single expression for $Q_{LZ}(\tilde{\tau}_c)$, where

$$\tilde{\tau}_c = v^{1/2} \tau_c \quad (34)$$

is the dimensionless correlation time. Then one obtains

change the adiabatic level at the moments when $J(t)$ passes through zero at times $\gtrsim \tau_c$, Fig. 5.

(iii) It is important to relate our results to the pioneering calculation in Ref. [6]. The expression for Q_{LZ} in the presence of noise in this paper was based on the expansion of Q_{LZ} in powers of J_c^2 ,

$$Q_{LZ} = \sum_{n=0}^{\infty} (-1)^n J_c^{2n} L^{(n)}, \quad (37)$$

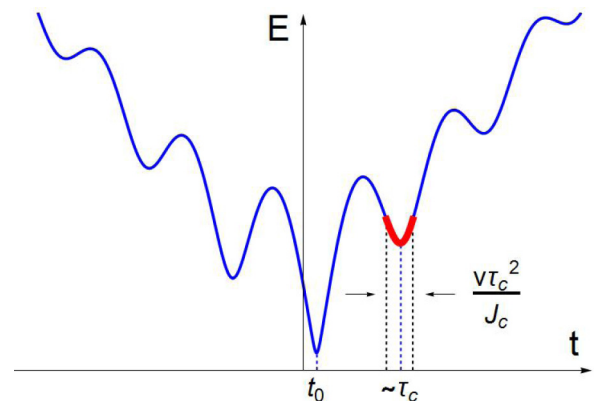


FIG. 5. In the domain of correlation times $J_c^{1/3}/v^{2/3} \ll \tau_c \ll J_c/v$ a crossover between the slow-noise and the fast-noise regimes takes place. In this domain, time-dependent adiabatic levels acquire local minima due to the randomness of $J(t)$. The duration of the Landau-Zener transition (shown with red) in the vicinity of each minimum is much shorter than τ_c . The probability to remain on the same adiabatic level after the transition, given by Eq. (36), is close to 1 so that only the transition in the vicinity of $t = t_0$ is responsible for Q_{LZ} .

where the coefficients $L^{(n)}$ are the $2n$ -fold integrals of the type,

$$L^{(n)} = \sum_{m=1}^n \underbrace{\int_{-\infty}^{\infty} d\tau_1 \int_{\tau_1}^{\infty} d\tau_2 \cdots \int_{-\infty}^{\infty} d\tau_{2m} \cdots \int_{-\infty}^{\tau_{2n-1}} d\tau_{2n}}_{2m \quad 2n-2m} \times F^{(n)}(\tau_1, \dots, \tau_{2n}) \exp \left[i \frac{v}{2} \sum_{j=1}^{2n} (-1)^j \tau_j^2 \right], \quad (38)$$

where $F^{(n)}$ is the product of the correlators $\exp(|\tau_i - \tau_j|/\tau_c)$ “forbidding” the variables τ_i and τ_j to differ more than τ_c . It is seen from Eq. (38) that typical values of τ_i are $v^{-1/2}$. Thus the restrictions imposed by the correlator start to matter for the terms with $n \gtrsim \tau_c v^{1/2}$. For smaller n the restriction is not important, and $F^{(n)}$ manifests itself only in the form of combinatorial factor $= (2n - 1)!!$. With this factor in front of $L^{(n)}$, the sum Eq. (37), instead of $\exp(-2\pi J_c^2/v)$, reduces to

$$Q_{\text{LZ}} = \frac{1}{(1 + 4\pi J_c^2/v)^{1/2}}. \quad (39)$$

The above result does not depend on the form of correlator and is interpreted in Ref. [6] as the average of $\exp(-2\pi J^2/v)$ with Gaussian distribution $\mathcal{P}(J) = (2\pi)^{-1/2} J_c \exp(-J^2/2J_c^2)$. This already reveals an inconsistency since the particular form $\exp(-|t_i - t_j|/\tau_c)$ of the correlator chosen in Ref. [6] corresponds to the telegraph noise with $\mathcal{P}(J) = \frac{1}{2}[\delta(J - J_c) + \delta(J + J_c)]$. Correspondingly, averaging over the distribution of J yields $\exp(-2\pi J_c^2/v)$ instead of Eq. (39), correctly implying that for $\tau_c \rightarrow \infty$ switchings of J do not affect the transition probability.

For Gaussian noise, our result Eq. (29) does not coincide with Eq. (39). The reason is that the logarithmic factor in Eq. (29), originating from sparse realizations, is a *singular* function of J_c^2 and cannot be captured by expansion in powers of J_c^2 . Equally, for the telegraph noise, the finite- τ_c correction Eq. (31) $\sim \tau_{\text{LZ}}/\tau_c$ is proportional to $|J_c|$ and, thus, is a singular function of J_c^2 . In general, the effects due to sparse realizations cannot be captured within the perturbative expansion.

(iv) Throughout the paper we assumed that the noise is slow and searched for finite- τ_c^{-1} corrections to the probability $1 - P_{\text{LZ}}$ to stay on the same diabatic level. We argued that these corrections are dominated by sparse realizations of noise. For the fast noise the probability to stay exceeds the probability to make a transition by $\frac{1}{2} \exp(-4\pi J_c^2/v)$. The small parameter τ_c/τ_{LZ} ensures that the noise is fast. Below we argue qualitatively that sparse realizations of noise can dominate the finite- τ_c correction to the standard result.

Consider the case of the fast telegraph noise. During the time τ_{LZ} , the off-diagonal matrix element switches from J_0 to $-J_0$ approximately τ_{LZ}/τ_c times. However, for certain sparse realizations, the switching happens only once at the moment $t_0 \sim \tau_c$. The probability of this realization can be estimated as $t_0/\tau_c \exp(-\tau_{\text{LZ}}/\tau_c)$. However, as was demonstrated in Sec. II, for such realizations, the system will stay on the initial diabatic level. This suggests the following form of P_{LZ} in the limit of fast noise:

$$\frac{1}{2} - P_{\text{LZ}} = \frac{1}{2} \exp \left\{ -\frac{4\pi J_c^2}{v} \right\} + \exp \left\{ -\frac{J_c}{v\tau_c} \right\}, \quad (40)$$

where the second term accounts for the sparse realizations. Note that this term dominates the first term for $\tau_c \gg 1/4\pi J_c$. Remarkably, this τ_c belongs to the domain of fast noise. Indeed, for $\tau_c = 1/4\pi J_c$, the ratio τ_c/τ_{LZ} is equal to $v/4\pi J_c^2$, i.e., it is small. Consequently, as in the case of the slow telegraph noise, we again come to the conclusion that the finite- τ_c correction can dominate the result.

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APPENDIX: ASYMPTOTIC BEHAVIOR OF $D_v(z)/D_v(-z)$

Since we are interested in the domain $z \sim |v|$, standard $z \rightarrow \pm\infty$ asymptotes of the parabolic cylinder function [24] are insufficient. Therefore, we start from the following integral representation:

$$D_v(z) = \sqrt{\frac{2}{\pi}} e^{z^2/4} \int_0^{\infty} e^{-t^2/2} t^v \cos \left(zt - \frac{v\pi}{2} \right) dt. \quad (A1)$$

It is convenient to divide the integral Eq. (A1) into two parts,

$$D_v(z) = I_+(z) e^{-iv\pi/2} + I_-(z) e^{iv\pi/2}, \quad (A2)$$

where the functions $I_+(z)$ and $I_-(z)$ are defined as

$$I_{\pm}(z) = \sqrt{\frac{1}{2\pi}} e^{z^2/4} \int_0^{\infty} e^{-t^2/2} t^v e^{\pm izt} dt. \quad (A3)$$

The advantage of introducing I_+ and I_- is that they are suited for evaluation using the steepest descent method. To apply this method, we rewrite I_+ in the form

$$I_+(z) = \sqrt{\frac{1}{2\pi}} e^{z^2/4} \int_0^{\infty} e^{f(t)} dt, \quad f(t) = -\frac{t^2}{2} + izt + v \ln t. \quad (A4)$$

The extrema of $f(t)$ correspond to $t_{\pm} = \frac{iz}{2} \pm \frac{\sqrt{4v - z^2}}{2}$. The real part of t_+ is positive and thus lies within the domain of integration. By contrast, $\text{Re } t_-$ is negative, and t_- should therefore be excluded.

Expanding $f(t)$ near t_+ , we get from Eq. (A4),

$$I_+(z) \approx \sqrt{\frac{1}{2\pi}} e^{z^2/4} e^{f(t_+)} \int_{-\infty}^{\infty} d(t - t_+) \times \exp \left[-\frac{1}{2} \left(1 + \frac{v}{t_+^2} \right) (t - t_+)^2 \right]. \quad (A5)$$

Upon performing the Gaussian integration, the result can be cast in the form

$$I_+(z) \approx \exp \left(\frac{iz\sqrt{4v - z^2}}{4} - \frac{v}{2} \right) (4v - z^2)^{-1/4} \times \left(\frac{iz + \sqrt{4v - z^2}}{2} \right)^{v+1/2}. \quad (A6)$$

The condition of applicability of the steepest descent method is that typical $(t - t_+)$ contributing to the integral Eq. (A5) is much smaller than t_+ . It is easy to see that for $z \sim |v|$ this condition is satisfied when $|v| \gg 1$, i.e., in the case we are interested in. The calculation of $I_-(z)$ is similar to

the above calculation. The saddle points are $-t_{\pm}$, and only $t = -t_-$ contributes to the integral. The result reads

$$I_-(z) \approx \exp\left(-\frac{iz\sqrt{4v-z^2}}{4} - \frac{\nu}{2}\right)(4v-z^2)^{-1/4} \times \left(\frac{-iz + \sqrt{4v-z^2}}{2}\right)^{\nu+1/2}. \quad (\text{A7})$$

Naturally, the result Eq. (A7) is consistent with the relation $I_+(z) = I_-(-z)$, which follows from Eq. (A3). If one takes a limit $z \gg |\nu|$ in Eqs. (A6) and (A7) and substitutes the result into Eq. (A2), one would recover the textbook asymptotes of $D_\nu(z)$. We are interested in the ratio $D_\nu(-z)/D_\nu(z)$. Using Eqs. (A5) and (A6), this ratio can be presented in a concise form

$$\frac{D_\nu(-z)}{D_\nu(z)} = \frac{I_-(z) + e^{i\nu\pi} I_+(z)}{I_+(z) + e^{i\nu\pi} I_-(z)} \approx \frac{\left(\frac{\sqrt{4v-z^2}-iz}{\sqrt{4v-z^2}+iz}\right)^{\nu+1/2} + e^{i\nu\pi+(iz/2)\sqrt{4v-z^2}}}{e^{(iz/2)\sqrt{4v-z^2}} + e^{i\nu\pi}\left(\frac{\sqrt{4v-z^2}-iz}{\sqrt{4v-z^2}+iz}\right)^{\nu+1/2}}. \quad (\text{A8})$$

To verify that Eq. (A8) has the right limit we recall that for $z \rightarrow \infty$ it should reproduce $Q_{LZ}^{-1/2}$. Indeed, in this limit, the denominator in the fraction turns to $\exp\left(\frac{iz}{2}\sqrt{4v-z^2}\right)$, whereas the numerator turns to $\exp\left(\frac{iz}{2}\sqrt{4v-z^2} + i\nu\pi\right)$. Consequently, Eq. (A8) yields $\exp(\pi|\nu|)$. Furthermore, for $z \rightarrow 0$, the ratio becomes 1 as expected.

Next we turn to the calculation of $D_{\nu-1}(-z)/D_{\nu-1}(z)$. Since the integral representation of $D_\nu(z)$ contains ν in the form of t^ν in the integrand, evaluation of the asymptote of $D_{\nu-1}(-z)$ with the steepest descent simply amounts to dividing the result for $I_+(z)$ by t_+ and the result for $I_-(z)$ by $-t_-$. This yields

$$D_{\nu-1}(z) \approx \frac{2e^{-i(\nu-1)\pi/2}}{iz + \sqrt{4v-z^2}} I_+(z) + \frac{2e^{i(\nu-1)\pi/2}}{-iz + \sqrt{4v-z^2}} I_-(z). \quad (\text{A9})$$

Then the ratio $D_{\nu-1}(-z)/D_{\nu-1}(z)$ can be cast in a form similar to Eq. (A8),

$$\frac{D_{\nu-1}(-z)}{D_{\nu-1}(z)} = \frac{I_-(z) + e^{i\nu\pi} I_+(z)}{I_+(z) + e^{i\nu\pi} I_-(z)} \approx \frac{\left(\frac{\sqrt{4v-z^2}-iz}{\sqrt{4v-z^2}+iz}\right)^{\nu-1/2} - e^{i\nu\pi+(iz/2)\sqrt{4v-z^2}}}{e^{(iz/2)\sqrt{4v-z^2}} - e^{i\nu\pi}\left(\frac{\sqrt{4v-z^2}-iz}{\sqrt{4v-z^2}+iz}\right)^{\nu-1/2}}. \quad (\text{A10})$$

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- [25] Strictly speaking, Eq. (23) relates the values of $|b_1|^2$ at $t \rightarrow \pm\infty$, provided that $b_2(-\infty) \rightarrow 0$. We are interested in the transition

probability from the particular initial-state $|a_1(-\infty)|^2 = 1$ with a_1 being a linear combination of b_1 and b_2 . The applicability of Eq. (23) is ensured by the fact that Eq. (21) determines two values of φ , which differ by $\pi/2$. Transition probabilities for these two values are the same. The proper initial state can thus be “arranged” by combining the solutions for b_1 , b_2 corresponding to the two φ values.