Anomalous Hall effects beyond Berry magnetic fields in a Weyl metal phase

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Applying time-varying magnetic fields to Weyl metals, a pair of Weyl points becomes oscillating. This oscillating monopole and antimonopole pair gives rise to ac Berry magnetic fields, responsible for the emergence of Berry electric fields, which have not been discussed before at least in the context of Weyl metals. Introducing this information into Boltzmann transport theory, we find anomalous Hall effects beyond Berry magnetic fields as a fingerprint of Berry electric fields.

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I. INTRODUCTION

Recently, Weyl metals [1–3] are of interest not only in condensed-matter physics but also in particle physics, involved with their topologically identified nontrivial properties [4]. Their anomalous metallic properties originate from a pair of Weyl bands, separated in momentum space, where each Weyl band describes emergent relativistic Weyl electrons. The band structure itself may be regarded to be a three-dimensional version of a graphene. Interestingly, the pair of Weyl points can be identified with a magnetic monopole and antimonopole pair in momentum space. Accordingly, the Berry curvature, or more accurately the Berry magnetic field, is assigned by this monopole pair, which turns out to play an essential role in anomalous transport phenomena of Weyl metals [5–27].

The present paper starts from an idea that the relative position of the monopole pair can be controlled by external magnetic fields [4]. Applying time-varying magnetic fields to Weyl metals, we investigate the role of an oscillating monopole pair in the transport. This oscillating monopole pair is expected to cause ac Berry magnetic fields [28]. If such Berry magnetic fields are governed by Berry-Maxwell equations, the Maxwell equation in momentum space, Berry electric fields would be generated. In this paper we investigate the role of the emergent Berry electric field in the transport of Weyl metals beyond the Berry magnetic field.

Based on Boltzmann transport theory with a topologically modified Drude model, which takes into account the novel information of the Berry electric field, we find that the oscillating monopole pair gives rise to anomalous Hall currents. These anomalous Hall currents should be distinguished from "conventional" anomalous Hall currents described by Berry magnetic fields. We classify these Hall currents in all possible situations. We reveal that these anomalous Hall effects are involved with an extended chiral anomaly given by a field theory, where a time-varying chiral gauge field appears to describe an "oscillating" relative-distance vector of the monopole pair in the anomaly equation.

II. CHIRAL GAUGE FIELD AS THE GRADIENT OF AN AXION ANGLE OF THE TOPOLOGICAL-IN-ORIGIN $\boldsymbol{E}\cdot\boldsymbol{B}$ TERM

The chiral anomaly [29] is an essential ingredient in anomalous transport phenomena of Weyl metals [9]. It is encoded into an inhomogeneous topological-in-origin θ term [30], given by

$$Z = \int D\psi_{\alpha a} \exp\bigg[-\int_0^\beta d\tau \int d^3\boldsymbol{r} \times \bigg\{\psi_{\alpha a}^\dagger\big((\partial_\tau - \mu)\boldsymbol{I}_{\alpha\beta}\otimes\boldsymbol{I}_{ab} - i v_D(\boldsymbol{\partial_r} - i\boldsymbol{A})\cdot\boldsymbol{\sigma}_{\alpha\beta}\otimes\boldsymbol{\tau}^z_{ab} + m\boldsymbol{I}_{\alpha\beta}\otimes\boldsymbol{\tau}^x_{ab}\big)\psi_{\beta b} + \frac{\theta(\boldsymbol{r})}{2\pi}\frac{\alpha}{2\pi}\boldsymbol{E}\cdot\boldsymbol{B}\bigg\}\bigg]. \tag{1}$$

Here, $\psi_{\alpha a}$ is a four-component Dirac spinor with spin α and orbital a. $\boldsymbol{\sigma}_{\alpha\beta}$ and $\boldsymbol{\tau}_{ab}$ are two-by-two Pauli matrices, acting on spin and orbital spaces, respectively. v_D is a velocity, m is a mass parameter, and μ is a chemical potential. $\boldsymbol{A}$ is an electromagnetic vector potential, regarded to be externally applied. $\boldsymbol{E} = -\frac{1}{c}\partial_\tau\boldsymbol{A}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ are externally applied electric field and magnetic field, respectively. $\theta(\boldsymbol{r})$ is an axion angle, and α is a fine structure constant.

One can represent this effective theory in terms of four-by-four Dirac gamma matrices, given by

$$\gamma^0 = \boldsymbol{I}_{\alpha\beta}\otimes\boldsymbol{\tau}^x_{ab}, \quad \gamma^k = -i\boldsymbol{\sigma}^k_{\alpha\beta}\otimes\boldsymbol{\tau}^y_{ab}. \tag{2}$$

Then, the partition function reads

$$Z = \int D\psi \exp\bigg[-\int_0^\beta d\tau \int d^3\boldsymbol{r}\bigg\{\bar\psi(i\gamma^0(\partial_\tau - \mu) - i v_D\boldsymbol{\gamma}\cdot(\boldsymbol{\partial_r} - i\boldsymbol{A}) + m)\psi + \frac{\theta(\boldsymbol{r})}{2\pi}\frac{\alpha}{2\pi}\boldsymbol{E}\cdot\boldsymbol{B}\bigg\}\bigg] \tag{3}$$

with $\bar\psi = \psi^\dagger\gamma^0$.

In order to determine the angle parameter, we recall the chiral anomaly equation [29]

$$\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = \frac{\alpha}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}. \tag{4}$$

This equation states that the classically conserved chiral current is not preserved any more in the quantum level, described by applied electromagnetic fields. Replacing the topological-in-origin $\boldsymbol{E}\cdot\boldsymbol{B}$ term with the chiral current based on this anomaly equation and performing the integration by parts, we rewrite the effective action as follows:

$$\mathcal{S}_{\text{eff}} = \int_0^\beta d\tau \int d^3\boldsymbol{r}\{\bar\psi(i\gamma^0(\partial_\tau - \mu) - i v_D\boldsymbol{\gamma}\cdot(\boldsymbol{\partial_r} - i\boldsymbol{A} - i\gamma^5\boldsymbol{c}) + m)\psi\}. \tag{5}$$

Here, $\boldsymbol{c}$ is a chiral gauge field, given by

$$\boldsymbol{c} = \nabla_{\boldsymbol{r}}\theta(\boldsymbol{r}), \tag{6}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \boldsymbol{I}_{\alpha\beta} \otimes \boldsymbol{\tau}^z_{ab}$ is the chiral matrix. It turns out that the band structure of this effective action describes that of a Weyl metal phase, where the distance between a pair of Weyl points is given by $2\boldsymbol{c}$ in the case of $m = 0$ [4].

III. MAXWELL EQUATIONS IN MOMENTUM SPACE

In order to describe the dynamics of electromagnetic fields in Weyl metals, we start from the following effective action for electromagnetic fields [30]:

$$\mathcal{S}_{\rm EM} = \int dt d^3\boldsymbol{r}\bigg\{\frac{1}{8\pi}(\boldsymbol{E}^2 - \boldsymbol{B}^2) + \frac{\theta}{2\pi}\frac{\alpha}{2\pi}\boldsymbol{E}\cdot\boldsymbol{B} - \frac{1}{c}\boldsymbol{j}\cdot\boldsymbol{A} - \rho\Phi\bigg\}. \tag{7}$$

Here, both electric and magnetic fields are given by

$$\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\partial_t\boldsymbol{A}, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{8}$$

respectively, where $\boldsymbol{A}$ and Φ are electromagnetic vector and scalar potentials. The topological-in-origin $\boldsymbol{E}\cdot\boldsymbol{B}$ term breaks time-reversal symmetry generally speaking, encoding chiral anomaly. α is a fine structure constant, as introduced before. $\boldsymbol{j}$ and ρ represent electrical current and charge density, respectively. Dynamics of these matter fields are described by Eq. (5).

Applying the variational principle to this effective action with respect to electromagnetic vector and scalar potentials, we obtain modified Maxwell equations to describe axion electrodynamics [30]:

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E} = 4\pi\rho + \frac{\alpha}{\pi}(\nabla\theta\cdot\boldsymbol{B}),\\
&\nabla\times\boldsymbol{B} - \frac{1}{c}\partial_t\boldsymbol{E} = \frac{4\pi}{c}\boldsymbol{j} - \frac{\alpha}{\pi}\bigg((\nabla\theta\times\boldsymbol{E}) + \frac{1}{c}(\partial_t\theta)\boldsymbol{B}\bigg),\\
&\nabla\times\boldsymbol{E} + \frac{1}{c}\partial_t\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{B} = 0.
\end{aligned} \tag{9}$$

If we redefine both electric and magnetic fields as $\mathbb{E} = \boldsymbol{E} - \frac{\alpha}{\pi}\theta\boldsymbol{B}$ and $\mathbb{B} = \boldsymbol{B} + \frac{\alpha}{\pi}\theta\boldsymbol{E}$, respectively, these equations are reduced to conventional Maxwell equations to describe Maxwell electrodynamics. In other words, the topological-in-origin $\boldsymbol{E}\cdot\boldsymbol{B}$ term gives rise to mixing between electric and magnetic fields.

In order to understand the axion electrodynamics, we should find how both the electrical current and charge density are represented in terms of electric and magnetic fields, referred to as constituent equations. An essential point is that the conventional Ohm's law does not work in Weyl metals [9–27]. Novel constituent equations should be uncovered. Actually, they can be found, based on Boltzmann transport theory for Weyl metals:

$$\partial_t f_\chi + \dot{\boldsymbol{r}}_\chi\cdot\nabla_{\boldsymbol{r}} f_\chi + \dot{\boldsymbol{p}}_\chi\cdot\nabla_{\boldsymbol{p}} f_\chi = -\frac{f_\chi - f_{\rm eq}}{\tau_{\rm eff}}. \tag{10}$$

Here, $f_\chi = f_\chi(\boldsymbol{p};\boldsymbol{r},t)$ is a distribution function of chiral fermions near a chiral Fermi surface, given by the chirality $\chi = \pm$, where $\boldsymbol{p}$ is a momentum, the Fourier transformed coordinate of a relative distance between a particle-hole pair, and $\boldsymbol{r}$ and t are center-of-mass coordinates of the particle-hole pair [31]. $f_{\rm eq} = f_{\rm eq}(\boldsymbol{p})$ is an equilibrium distribution function. $\tau_{\rm eff}$ is an effective relaxation time in terms of disorder scattering between intra-Fermi surfaces of the same chirality and that between inter-Fermi surfaces of the opposite chirality.

The effective velocity of $\dot{\boldsymbol{r}}_\chi$ and the effective force of $\dot{\boldsymbol{p}}_\chi$ are described by

$$\dot{\boldsymbol{r}}_\chi = \boldsymbol{v}_\chi + \chi\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi + \dot{\boldsymbol{p}}_\chi\times\boldsymbol{\mathcal{B}}_\chi \tag{11}$$

and

$$\dot{\boldsymbol{p}}_\chi = e\boldsymbol{E} + \frac{e}{c}\dot{\boldsymbol{r}}_\chi\times\boldsymbol{B}, \tag{12}$$

respectively. If the second and third terms are neglected in Eq. (11), these two equations are referred to as the Drude model. Here, $\boldsymbol{v}_\chi$ is the group velocity. The equation for $\dot{\boldsymbol{p}}_\chi$ describes the Lorentz force. On the other hand, the third term in Eq. (11) gives rise to the contribution of anomalous velocity, where $\boldsymbol{\mathcal{B}}_\chi$ is Berry magnetic field [32,33]. The second term is our main discovery, describing the Berry electric field. This contribution will be derived in the next section.

Resorting to this topologically modified Boltzmann transport theory, one can find constituent equations for charge density and electric current as follows:

$$\begin{aligned}
\rho &= \sum_\chi \rho_\chi \equiv e\sum_\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}G_\chi f_\chi,\\
\boldsymbol{j} &= \sum_\chi \boldsymbol{j}_\chi \equiv e\sum_\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}G_\chi\dot{\boldsymbol{r}}_\chi f_\chi,
\end{aligned} \tag{13}$$

where the phase-space measure is modified as

$$G_\chi = 1 + \frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_\chi. \tag{14}$$

We recall $\sum_\chi \chi = 0$.

It is not surprising to observe that both Berry magnetic and Berry electric fields in the topologically modified Drude model satisfy Maxwell equations in momentum space, described by

$$\begin{aligned}
&\nabla_{\boldsymbol{p}}\cdot\boldsymbol{\mathcal{B}} = 2\pi\sum_\chi\chi\delta(\boldsymbol{p}+\chi\boldsymbol{c}),\\
&\nabla_{\boldsymbol{p}}\times\boldsymbol{\mathcal{E}} - \frac{1}{\mathcal{C}}\partial_t\boldsymbol{\mathcal{B}} = -\frac{2\pi}{\mathcal{C}}\sum_\chi\dot{\boldsymbol{c}}\delta(\boldsymbol{p}+\chi\boldsymbol{c}),\\
&\nabla_{\boldsymbol{p}}\times\boldsymbol{\mathcal{B}} + \frac{1}{\mathcal{C}}\partial_t\boldsymbol{\mathcal{E}} = 0, \quad \nabla_{\boldsymbol{p}}\cdot\boldsymbol{\mathcal{E}} = 0.
\end{aligned} \tag{15}$$

Here, both Berry magnetic and electric fields are given by the sum of all chiral charges:

$$\boldsymbol{\mathcal{B}} = \sum_\chi\boldsymbol{\mathcal{B}}_\chi, \quad \boldsymbol{\mathcal{E}} = \sum_\chi\boldsymbol{\mathcal{E}}_\chi. \tag{16}$$

The vector field $\boldsymbol{c}$ in momentum space corresponds to the distance between a pair of Weyl points, given by Eq. (6). The chirality is identified with a magnetic monopole in momentum space. The right-hand side in the second equation of the Berry-Maxwell equation Eq. (15) describes a monopole current in momentum space. The third and fourth equations may be regarded to be vector-field identities, referred to as the Bianchi identity [34]. Here, we do not find quantities that correspond to the electrical current and charge density of Maxwell equations. An interesting quantity $\mathcal{C}$ is proposed to play the role of the speed of light in momentum space, i.e., the propagating speed of Berry electromagnetic waves in momentum space. Derivation of the speed of the Berry electromagnetic wave is in progress [35].

In order to figure out Berry-Maxwell equations, we start from a Lorentz invariant solution for the Berry magnetic field, given by [36]

$$\mathfrak{B}_\chi = \frac{\chi}{2}\frac{\boldsymbol{p}+\chi\boldsymbol{c}}{|\boldsymbol{p}+\chi\boldsymbol{c}|^3\gamma^2[1-\beta^2\sin^2\Psi_\chi]^{3/2}}. \tag{17}$$

where

$$\cos\Psi_\chi = -\chi\frac{\dot{\boldsymbol{c}}}{|\dot{\boldsymbol{c}}|}\cdot\frac{\boldsymbol{p}+\chi\boldsymbol{c}}{|\boldsymbol{p}+\chi\boldsymbol{c}|}, \tag{18}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}},\quad \beta = \frac{|\dot{\boldsymbol{c}}|}{\mathcal{C}}. \tag{19}$$

This expression is reduced to

$$\mathfrak{B}_\chi \approx \boldsymbol{\mathcal{B}}_\chi + \delta\boldsymbol{\mathcal{B}}_\chi, \tag{20}$$

taking into account $\beta \ll 1$. We obtain $\boldsymbol{\mathcal{B}}_\chi = \frac{\chi}{2}\frac{\boldsymbol{p}+\chi\boldsymbol{c}}{|\boldsymbol{p}+\chi\boldsymbol{c}|^3}$ in the $\mathcal{O}(\beta^0)$ order and $\delta\boldsymbol{\mathcal{B}}_\chi = \beta^2(\frac{3}{2}\sin^2\Psi_\chi - 1)\boldsymbol{\mathcal{B}}_\chi$ in the $\mathcal{O}(\beta^2)$ order. There do not appear to be any Berry magnetic fields in the $\mathcal{O}(\beta^1)$ order.

We introduce the $\mathcal{O}(\beta^1)$ order as follows:

$$\boldsymbol{\mathcal{E}}_\chi \approx -\frac{1}{\mathcal{C}}\chi\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi, \tag{21}$$

identified with Berry electric field. Then, the curl of the Berry electric field Eq. (21) in the second equation of Eq. (15) is given by

$$\nabla_{\boldsymbol{p}}\times\boldsymbol{\mathcal{E}}_\chi = -\chi\frac{\dot{\boldsymbol{c}}}{\mathcal{C}}\nabla_{\boldsymbol{p}}\cdot\boldsymbol{\mathcal{B}}_\chi + \chi\frac{\dot{\boldsymbol{c}}}{\mathcal{C}}\cdot\nabla_{\boldsymbol{p}}\boldsymbol{\mathcal{B}}_\chi \tag{22}$$

in the $\mathcal{O}(\beta^1)$ order. The time derivative of the Berry magnetic field Eq. (17) is

$$\frac{1}{\mathcal{C}}\partial_t\mathfrak{B}_\chi = \chi\frac{\dot{\boldsymbol{c}}}{\mathcal{C}}\cdot\nabla_{\boldsymbol{p}}\boldsymbol{\mathcal{B}}_\chi + \mathcal{O}(\beta^3). \tag{23}$$

As a result, we confirm that the second equation of Eq. (15) holds up to the $\mathcal{O}(\beta^1)$ order.

In order to check out the third equation of Eq. (15), we consider $\ddot{\boldsymbol{c}} = 0$ for simplicity and $\dot{\boldsymbol{c}} = |\dot{\boldsymbol{c}}|\hat{z}$ without loss of generality. The curl of the Berry magnetic field Eq. (17) is given by

$$\begin{aligned}\nabla_{\boldsymbol{p}}\times\mathfrak{B}_\chi &= \nabla_{\boldsymbol{p}}\times\delta\boldsymbol{\mathcal{B}}_\chi\\ &= -\frac{3}{2}\beta^2\chi\frac{p_z+\chi c_z}{|\boldsymbol{p}+\chi\boldsymbol{c}|^5}[(p_x+\chi c_x)\hat{y}\\ &\quad -(p_y+\chi c_y)\hat{x}].\end{aligned} \tag{24}$$

The time derivative of the Berry electric field Eq. (21) is

$$\frac{1}{\mathcal{C}}\partial_t\mathcal{E}_\chi = \frac{3}{2}\beta^2\chi\frac{p_z+\chi c_z}{|\boldsymbol{p}+\chi\boldsymbol{c}|^5}[(p_x+\chi c_x)\hat{y}-(p_y+\chi c_y)\hat{x}]. \tag{25}$$

As a result, we find that the third equation of the Berry-Maxwell equation Eq. (15) holds up to the $\mathcal{O}(\beta^2)$ order.

IV. DERIVATION OF THE TOPOLOGICALLY MODIFIED DRUDE MODEL

In order to prove the topologically modified Drude model, in particular, the emergence of the Berry electric field, we start from an effective Hamiltonian for a Weyl metal phase, given by

$$H_\chi = \chi\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}+\chi\boldsymbol{c}+\frac{e}{c}\boldsymbol{A}\right)+e\Phi. \tag{26}$$

$\chi = \pm$ is a chiral charge. $\boldsymbol{\sigma}$ is a two-by-two Pauli matrix. $\boldsymbol{p}$ is a momentum. $\boldsymbol{c}$ is a chiral gauge field, given by Eq. (6). $\boldsymbol{A}$ and Φ are electromagnetic vector and scalar potentials with an electric charge e and the speed of light c. This effective Hamiltonian gives rise to the transition amplitude:

$$\begin{aligned}\langle f|e^{-iH_\chi(t_f-t_i)}|i\rangle &= \int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} D\boldsymbol{r}\int D\boldsymbol{p}\,\exp\Bigg[i\int_{t_i}^{t_f}dt\Bigg\{\boldsymbol{p}\cdot\dot{\boldsymbol{r}}\\ &\quad -\chi\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}+\chi\boldsymbol{c}+\frac{e}{c}\boldsymbol{A}\right)-e\Phi\Bigg\}\Bigg],\end{aligned} \tag{27}$$

where $\hbar = 1$.

In order to describe low-energy dynamics of electrons near a Fermi surface, we do not need to know the information of high-energy electrons deep inside the Fermi surface, generally speaking. However, we are not allowed to neglect high-energy dynamics of electrons in a Weyl metal phase when we deal with a pair of chiral Fermi surfaces. In particular, the topological information involved with the pair of Weyl points should be taken into account, integrating over such high-energy electrons. In order to understand the low-energy dynamics near a pair of chiral Fermi surfaces, we should integrate over high-energy electron fields near the pair of Weyl points.

The integration of high-energy electrons can be performed, rewriting the effective Weyl Hamiltonian in terms of a diagonalized basis:

$$U_{\tilde{\boldsymbol{p}}}^\dagger\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{p}}U_{\tilde{\boldsymbol{p}}} = |\tilde{\boldsymbol{p}}|\sigma^3, \tag{28}$$

where $\tilde{\boldsymbol{p}} \equiv \boldsymbol{p}+\chi\boldsymbol{c}$. $U_{\tilde{\boldsymbol{p}}}$ is a two-by-two unitary matrix, expressed by $U_{\tilde{\boldsymbol{p}}} = (u_{\tilde{\boldsymbol{p}}}\ v_{\tilde{\boldsymbol{p}}})$, where two-component column vectors are determined by

$$(\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{p}})u_{\tilde{\boldsymbol{p}}} = |\tilde{\boldsymbol{p}}|u_{\tilde{\boldsymbol{p}}},\quad (\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{p}})v_{\tilde{\boldsymbol{p}}} = -|\tilde{\boldsymbol{p}}|v_{\tilde{\boldsymbol{p}}}, \tag{29}$$

respectively.

In order to describe the low-energy dynamics of electrons near a pair of chiral Fermi surfaces, we neglect off-diagonal terms and take the 11 component for $\chi = +$ and 22 component for $\chi = -$. In other words, we consider [13,15]

$$\left\{U_{\tilde{p}'}^{\dagger} \exp\left[-i\boldsymbol{\sigma}\cdot\left(\tilde{\boldsymbol{p}}+\frac{e}{c}\boldsymbol{A}\right)\Delta t\right]U_{\tilde{p}}\right\}_{11}$$
$$\approx u_{\tilde{p}'}^{\dagger}u_{\tilde{p}} \exp\left[-i\frac{|\tilde{\boldsymbol{p}}|+|\tilde{\boldsymbol{p}}'|}{2}\Delta t - i\frac{e}{c}\frac{\hat{\tilde{\boldsymbol{p}}}+\hat{\tilde{\boldsymbol{p}}}'}{2}\cdot\boldsymbol{A}\Delta t - \frac{e}{c}\frac{\Delta\tilde{\boldsymbol{p}}\times\hat{\tilde{\boldsymbol{p}}}}{2|\tilde{\boldsymbol{p}}|}\cdot\boldsymbol{A}\Delta t\right]$$
$$\approx u_{\tilde{p}'}^{\dagger}u_{\tilde{p}} \exp\left[-\frac{i}{2}\left(\left|\tilde{\boldsymbol{p}}+\frac{e}{c}\boldsymbol{A}\right|+\left|\tilde{\boldsymbol{p}}'+\frac{e}{c}\boldsymbol{A}\right|\right)\Delta t - i\frac{e}{c}\frac{\boldsymbol{B}\cdot\hat{\tilde{\boldsymbol{p}}}}{2|\tilde{\boldsymbol{p}}|}\Delta t\right]$$
$$\approx u_{\tilde{p}'}^{\dagger}u_{\tilde{p}} \exp\left[-i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}{}^{+}\right)|\tilde{\boldsymbol{p}}|\Delta t\right] \tag{30}$$

in the path-integral representation, where $\hat{\tilde{\boldsymbol{p}}} \equiv \tilde{\boldsymbol{p}}/|\tilde{\boldsymbol{p}}|$ and $\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}{}^{+} = \frac{\hat{\tilde{\boldsymbol{p}}}}{2|\tilde{\boldsymbol{p}}|}$. Here, we used the Gordon identity, given by

$$u_{\boldsymbol{p}'_\chi}^{\dagger}\sigma^i u_{\boldsymbol{p}_\chi} = u_{\boldsymbol{p}'_\chi}^{\dagger}\left[\frac{p_\chi^i + p'^{\,i}_\chi}{|\boldsymbol{p}_\chi|+|\boldsymbol{p}'_\chi|} - \frac{i\epsilon_{ijk}\Delta p_\chi^j}{|\boldsymbol{p}_\chi|+|\boldsymbol{p}'_\chi|}\sigma^k\right]u_{\boldsymbol{p}_\chi}, \tag{31}$$

up to the linear order in Δp_χ. We also assume the semiclassical regime that the magnetic field is small enough for us to neglect the Landau-level splitting. The Berry gauge field appears from

$$u_{\tilde{p}+\Delta\tilde{p}}^{\dagger}u_{\tilde{p}} \approx \exp(-i\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{+}\cdot\Delta\tilde{\boldsymbol{p}}) = \exp[-i\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{+}\cdot(\Delta\boldsymbol{p}+\Delta\boldsymbol{c})], \tag{32}$$

represented by

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{+} = iu_{\tilde{p}}^{\dagger}\nabla_{\boldsymbol{p}}u_{\tilde{p}}. \tag{33}$$

Similarly, we find

$$\left\{U_{\tilde{p}'}^{\dagger} \exp\left[i\boldsymbol{\sigma}\cdot\left(\tilde{\boldsymbol{p}}+\frac{e}{c}\boldsymbol{A}\right)\Delta t\right]U_{\tilde{p}}\right\}_{22} \approx v_{\tilde{p}'}^{\dagger}v_{\tilde{p}} \exp\left[-i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}{}^{-}\right)|\tilde{\boldsymbol{p}}|\Delta t\right] \tag{34}$$

with $\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}{}^{-} = -\frac{\hat{\tilde{\boldsymbol{p}}}}{2|\tilde{\boldsymbol{p}}|}$. The Berry gauge field at $\chi = -$ results from

$$v_{\tilde{p}+\Delta\tilde{p}}^{\dagger}v_{\tilde{p}} = \exp[-i\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{-}\cdot(\Delta\boldsymbol{p}-\Delta\boldsymbol{c})], \tag{35}$$

given by

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{-} = iv_{\tilde{p}}^{\dagger}\nabla_{\boldsymbol{p}}v_{\tilde{p}} = iu_{-\tilde{p}}^{\dagger}\nabla_{\boldsymbol{p}}u_{-\tilde{p}}. \tag{36}$$

An effective action for the low-energy dynamics of chiral fermions in a Weyl metal phase reads

$$\mathcal{S}_\chi^{\text{eff}} = \int_{t_i}^{t_f} dt\left\{\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)\cdot\dot{\boldsymbol{r}} - e\Phi - \boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{\chi}\cdot(\dot{\boldsymbol{p}}+\chi\dot{\boldsymbol{c}}) - \left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}^{\chi}\right)|\boldsymbol{p}+\chi\boldsymbol{c}|\right\}, \tag{37}$$

where

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{\chi} = iu_{\chi\tilde{p}}^{\dagger}\nabla_{\boldsymbol{p}}u_{\chi\tilde{p}} \tag{38}$$

is the Berry gauge field, originating from the high-energy dynamics near the Weyl point. It is trivial to check out $\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}^{\chi} = \nabla_{\boldsymbol{p}}\times\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{\chi}$. Here, a novel ingredient beyond the previous study is a coupling term $-\chi\boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{\chi}\cdot\dot{\boldsymbol{c}}$.

It is straightforward to read the low-energy effective Hamiltonian for a pair of chiral Fermi surfaces from the effective action Eq. (37) as follows:

$$\mathcal{H}_\chi = -\frac{e}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + e\Phi + \boldsymbol{\mathcal{A}}_{\boldsymbol{p}}^{\chi}\cdot(\dot{\boldsymbol{p}}+\chi\dot{\boldsymbol{c}}) + \left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}^{\chi}\right)|\boldsymbol{p}+\chi\boldsymbol{c}|. \tag{39}$$

Hamiltonian equations of motion $\dot{\boldsymbol{r}}_\chi = \frac{\partial\mathcal{H}_\chi}{\partial\boldsymbol{p}_\chi}$ and $\dot{\boldsymbol{p}}_\chi = -\frac{\partial\mathcal{H}_\chi}{\partial\boldsymbol{r}_\chi}$ give rise to the topologically modified Drude model Eqs. (11) and (12) with an emergent Berry electric field, where

$$\boldsymbol{r}\to\boldsymbol{r}_\chi,\quad \boldsymbol{p}\to\boldsymbol{p}_\chi \tag{40}$$

have been considered. The group velocity in Eq. (11) is given by

$$\boldsymbol{v}_\chi = \nabla_{\boldsymbol{p}}\left\{\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_{\boldsymbol{p}}^{\chi}\right)|\boldsymbol{p}+\chi\boldsymbol{c}|\right\}. \tag{41}$$

This completes the derivation of the topologically modified Drude model.

V. CURRENT CONSERVATION LAW AND CHIRAL ANOMALY

Solutions of the topological Drude model are given by

$$G_\chi\dot{\boldsymbol{p}}_\chi = e\boldsymbol{E} + \frac{e}{c}(\boldsymbol{v}_\chi + \chi\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi)\times\boldsymbol{B} + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\mathcal{B}}_\chi,$$
$$G_\chi\dot{\boldsymbol{r}}_\chi = \boldsymbol{v}_\chi + \chi\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi + e\boldsymbol{E}\times\boldsymbol{\mathcal{B}}_\chi + \frac{e}{c}(\boldsymbol{v}_\chi\cdot\boldsymbol{\mathcal{B}}_\chi)\boldsymbol{B}. \tag{42}$$

It is interesting to observe the symmetric structure between these two solutions, where the correspondence is

$$e\boldsymbol{E}\longleftrightarrow\boldsymbol{v}_\chi+\chi\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi,\quad \frac{e}{c}\boldsymbol{B}\longleftrightarrow\boldsymbol{\mathcal{B}}_\chi. \tag{43}$$

Inserting these solutions into the Boltzmann equation Eq. (10), it is straightforward to find the current conservation equation

$$\partial_t\rho_\chi + \nabla_{\boldsymbol{r}}\cdot\boldsymbol{j}_\chi = e\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}(\partial_t G_\chi)f_\chi + \frac{\chi}{4\pi^2}\frac{e^3}{c}(\boldsymbol{E}\cdot\boldsymbol{B}) + \frac{e^2}{c}\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}[\nabla_{\boldsymbol{p}}\cdot\{(\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi)\times\boldsymbol{B}\}f_\chi], \tag{44}$$

where both electric charge density and current density are defined in Eq. (13).

The first term in the right-hand side is

$$e\int\frac{d^3p}{(2\pi)^3}(\partial_t G_\chi)f_\chi = \frac{e^2}{c}\chi\int\frac{d^3p}{(2\pi)^3}f_\chi\dot{\boldsymbol{c}}\cdot\nabla_{\boldsymbol{p}}(\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_\chi). \tag{45}$$

The third term in the right-hand side is

$$\frac{e^2}{c}\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}[\nabla_{\boldsymbol{p}}\cdot\{(\dot{\boldsymbol{c}}\times\boldsymbol{\mathcal{B}}_\chi)\times\boldsymbol{B}\}f_\chi] = \frac{1}{4\pi^2}\frac{e^2}{c}\boldsymbol{B}\cdot\dot{\boldsymbol{c}} - \frac{e^2}{c}\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}f_\chi\dot{\boldsymbol{c}}\cdot\nabla_{\boldsymbol{p}}(\boldsymbol{B}\cdot\boldsymbol{\mathcal{B}}_\chi). \quad (46)$$

As a result, we obtain a modified current conservation law, given by

$$\partial_t\rho_\chi + \nabla_{\boldsymbol{r}}\cdot\boldsymbol{j}_\chi = \frac{\chi}{4\pi^2}\frac{e^3}{c}\boldsymbol{E}\cdot\boldsymbol{B} + \frac{1}{4\pi^2}\frac{e^2}{c}\boldsymbol{B}\cdot\dot{\boldsymbol{c}}. \quad (47)$$

Actually, this current conservation law has been known for a long time in the context of quantum field theory. An effective action for a Weyl metal phase is given by

$$\mathcal{S}_{\mathrm{WM}} = \int d^4x\bar{\psi}i\gamma_\mu[\partial_\mu - iA_\mu - i\gamma_5 c_\mu(t)]\psi. \quad (48)$$

Here, ψ is a four-component Dirac spinor. γ^μ is a four-by-four Dirac gamma matrix to satisfy the Clifford algebra with $\mu = 0,1,2,3$ and $x^\mu = (t,x,y,z)$. γ_5 is a chiral matrix.

Based on Fuzikawa's path-integral method [29], one can find modified anomaly equations in the presence of the chiral gauge field:

$$\partial_\mu\langle\bar{\psi}\gamma^\mu\gamma_5\psi\rangle = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}(F_{\mu\nu}F_{\alpha\beta} + f_{\mu\nu}f_{\alpha\beta}) = \frac{1}{2\pi^2}\boldsymbol{E}\cdot\boldsymbol{B} \quad (49)$$

and

$$\partial_\mu\langle\bar{\psi}\gamma^\mu\psi\rangle = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}(F_{\mu\nu}f_{\alpha\beta} + f_{\mu\nu}F_{\alpha\beta}) = \frac{1}{2\pi^2}\dot{\boldsymbol{c}}\cdot\boldsymbol{B}, \quad (50)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu$ are field strength tensors for electromagnetic and chiral gauge fields, respectively [37,38]. Here, $\langle\bar{\psi}\gamma^\mu\gamma_5\psi\rangle = \boldsymbol{j}_+ - \boldsymbol{j}_-$ is the chiral current and $\langle\bar{\psi}\gamma^\mu\psi\rangle = \boldsymbol{j}_+ + \boldsymbol{j}_-$ is the electromagnetic U(1) current, where $\boldsymbol{j}_\chi$ is given by Eq. (13). We recall that the chiral gauge field is $(0,\boldsymbol{c}(t))$, resulting in the last equation Eq. (50). Since the electromagnetic U(1) current is not conserved, one may suspect the validity of this result. However, this originates from the definition of the electromagnetic current. An essential point is that the right-hand side in this anomaly equation is given by a total derivative term. Redefining the electromagnetic U(1) current from $\langle\bar{\psi}\gamma^\mu\psi\rangle$ to $J_\mu = \langle\bar{\psi}\gamma^\mu\psi\rangle - \frac{1}{2\pi^2}\boldsymbol{c}\cdot\boldsymbol{B}\delta_{\mu\tau}$, we obtain the current conservation law $\partial_\mu J_\mu = 0$. This conserved current satisfies the Maxwell equation, real to be observed in experiments [39].

VI. ROLE OF THE EMERGENT BERRY ELECTRIC FIELD IN TRANSPORT

A. Anomalous Hall currents driven by the Berry electric field

An essential question is on the role of the Berry electric field in anomalous transport of a Weyl metal phase. Resorting to the framework of Boltzmann transport theory, we resolve this issue. Solving the Boltzmann equation, we obtain the distribution function up to the linear order in the electric field as follows:

$$f_\chi(t,\boldsymbol{p}) \approx f_\chi^{\mathrm{eq}}(\boldsymbol{p}) - \frac{\partial f_\chi^{\mathrm{eq}}(\varepsilon)}{\partial\varepsilon} \times\left(\int_{-\infty}^{\infty}d\omega e^{i\omega t}\frac{\tau_{\mathrm{eff}}}{1-i\omega\tau_{\mathrm{eff}}}\dot{\boldsymbol{p}}_\chi(\omega)\right)\cdot\boldsymbol{v}_\chi(t) \quad (51)$$

where

$$f_\chi^{\mathrm{eq}}(\boldsymbol{p}) = \frac{1}{e^{\beta(G_\chi(\boldsymbol{p})|\boldsymbol{p}+\chi\boldsymbol{c}|-\mu)}+1}, \quad (52)$$

$$\frac{\partial f_\chi^{\mathrm{eq}}(\boldsymbol{p})}{\partial\epsilon} \equiv \frac{\partial}{\partial\epsilon}\left(\frac{1}{e^{\beta(\epsilon-\mu)}+1}\right)\Bigg|_{\epsilon=G_\chi(\boldsymbol{p})|\boldsymbol{p}+\chi\boldsymbol{c}|}. \quad (53)$$

Here, $f_\chi^{\mathrm{eq}}(\boldsymbol{p})$ is an equilibrium distribution function, and its derivative with respect to energy is $\frac{\partial f_\chi^{\mathrm{eq}}(\boldsymbol{p})}{\partial\epsilon}$. Then, the electric current reads

$$\boldsymbol{j}_\chi = e\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}G_\chi\dot{\boldsymbol{r}}_\chi f_\chi^{\mathrm{eq}}(\boldsymbol{p}) + e\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\mathrm{eq}}(\varepsilon)}{\partial\varepsilon}\right) \times G_\chi\dot{\boldsymbol{r}}_\chi\left(\int_{-\infty}^{\infty}d\omega e^{i\omega t}\frac{\tau_{\mathrm{eff}}}{1-i\omega\tau_{\mathrm{eff}}}\dot{\boldsymbol{p}}_\chi(\omega)\right)\cdot\boldsymbol{v}_\chi(t). \quad (54)$$

In this study we focus on the adiabatic regime, defined by

$$\omega\tau_{\mathrm{eff}} \ll 1. \quad (55)$$

Then, the distribution function can be simplified as

$$f_\chi(t,\boldsymbol{p}) \approx f_\chi^{\mathrm{eq}}(\boldsymbol{p}) + \tau_{\mathrm{eff}}[\dot{\boldsymbol{p}}_\chi(t)\cdot\boldsymbol{v}_\chi(t)]\left(-\frac{\partial f_\chi^{\mathrm{eq}}(\varepsilon)}{\partial\varepsilon}\right). \quad (56)$$

Accordingly, the electric current is given by

$$\boldsymbol{j}_\chi = e\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}G_\chi\dot{\boldsymbol{r}}_\chi f_\chi^{\mathrm{eq}}(\boldsymbol{p}) + e\tau_{\mathrm{eff}}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f_{\chi\mathrm{eq}}(\boldsymbol{p})}{\partial\varepsilon}\right)G_\chi\dot{\boldsymbol{r}}_\chi\dot{\boldsymbol{p}}_\chi(t)\cdot\boldsymbol{v}_\chi(t). \quad (57)$$

As a result, we obtain an electrical current in a Weyl metal phase:

$$\boldsymbol{j} = \boldsymbol{j}_{\mathrm{AHE}} + \boldsymbol{j}_{\mathrm{CME}} + \boldsymbol{j}_{\mathrm{LMC}} + \Delta\boldsymbol{j}_{BE}. \quad (58)$$

Here,

$$\boldsymbol{j}_{\mathrm{AHE}} = e^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{\chi=\pm}(\boldsymbol{E}\times\mathfrak{B}_\chi)f_\chi^{\mathrm{eq}}(\boldsymbol{p}) \quad (59)$$

describes an anomalous Hall effect, resulting from the Berry magnetic field [23–25,32,33]:

$$\boldsymbol{j}_{\mathrm{CME}} = \frac{e^2}{c}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{\chi=\pm}(\boldsymbol{v}_\chi\cdot\mathfrak{B}_\chi)\boldsymbol{B}f_\chi^{\mathrm{eq}}(\boldsymbol{p}) \quad (60)$$

gives rise to the chiral magnetic effect [10–16], which can occur when the chiral chemical potential exists. These two types of currents are dissipationless in nature, where even high-energy electrons deep inside a pair of chiral Fermi surfaces are involved:

$$\boldsymbol{j}_{\rm LMC} = e\tau_{\rm eff}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)G_\chi^{-1}\boldsymbol{v}_\chi \cdot\left\{e\boldsymbol{E}+\frac{e}{c}\boldsymbol{v}_\chi\times\boldsymbol{B}+\frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\mathfrak{B}_\chi\right\}\left\{\boldsymbol{v}_\chi + e\boldsymbol{E}\times\mathfrak{B}_\chi+\frac{e}{c}(\boldsymbol{v}_\chi\cdot\mathfrak{B}_\chi)\boldsymbol{B}\right\} \quad (61)$$

results in the longitudinal negative magnetoresistivity, which occurs when the electric current is driven along the direction of the pair of Weyl points. Although this electric current is a Fermi-surface contribution, the chiral anomaly plays a central role in this transport coefficient, regarded to be a fingerprint of a Weyl metal phase [5–9,17–22].

Other current contributions turn out to result from the Berry electric field. They are classified in the following way:

$$\Delta\boldsymbol{j}_{BE} = \Delta\boldsymbol{j}^{(1)}_{BE}+\Delta\boldsymbol{j}^{(2)}_{BE}+\Delta\boldsymbol{j}^{(3)}_{BE}+\Delta\boldsymbol{j}^{(4)}_{BE}, \quad (62)$$

where

$$\Delta\boldsymbol{j}^{(1)}_{BE} = e\sum_\chi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\chi\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi f^{\rm eq}_\chi(\boldsymbol{p}) \quad (63)$$

is a dissipationless current, given by the total electrons inside both chiral Fermi surfaces, and

$$\Delta\boldsymbol{j}^{(2)}_{BE} = e\tau_{\rm eff}\sum_\chi\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)G_\chi^{-1}\boldsymbol{v}_\chi \cdot\left\{e\boldsymbol{E}+\frac{e}{c}\boldsymbol{v}_\chi\times\boldsymbol{B}+\frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\mathfrak{B}_\chi\right\}\chi\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi, \quad (64)$$

$$\Delta\boldsymbol{j}^{(3)}_{BE} = e\tau_{\rm eff}\sum_\chi\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)\times G_\chi^{-1}\left\{\boldsymbol{v}_\chi+e\boldsymbol{E}\times\mathfrak{B}_\chi+\frac{e}{c}(\boldsymbol{v}_\chi\cdot\mathfrak{B}_\chi)\boldsymbol{B}\right\}\times\frac{e}{c}\chi\left((\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi)\times\boldsymbol{B}\right)\cdot\boldsymbol{v}_\chi \quad (65)$$

and

$$\Delta\boldsymbol{j}^{(4)}_{BE} = e\tau_{\rm eff}\sum_\chi\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)\times G_\chi^{-1}\chi\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi\frac{e}{c}(\chi(\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi)\times\boldsymbol{B})\cdot\boldsymbol{v}_\chi \quad (66)$$

are dissipative currents with the effective scattering time, given by chiral fermions near the pair of chiral Fermi surfaces. We emphasize that all these currents are proportional to $\dot{\boldsymbol{c}}$.

We observe symmetry properties under the variable change of $\boldsymbol{p}\to-\boldsymbol{p}$ as follows:

$$f^{\rm eq}_\chi(-\boldsymbol{p}) = f^{\rm eq}_{-\chi}(\boldsymbol{p}), \quad (67)$$

$$\frac{\partial f^{\rm eq}_\chi(-\boldsymbol{p})}{\partial\epsilon} = \frac{\partial f^{\rm eq}_{-\chi}(\boldsymbol{p})}{\partial\epsilon}, \quad (68)$$

$$\mathfrak{B}_\chi(-\boldsymbol{p}) = \mathfrak{B}_{-\chi}(\boldsymbol{p}), \quad (69)$$

$$G_\chi(-\boldsymbol{p}) = G_{-\chi}(\boldsymbol{p}), \quad (70)$$

$$\boldsymbol{v}_\chi(-\boldsymbol{p}) = -\boldsymbol{v}_{-\chi}(\boldsymbol{p}). \quad (71)$$

Applying these symmetry properties to electric currents driven by the Berry electric field, we find that many terms vanish identically. First, we obtain $\Delta\boldsymbol{j}^{(1)}_{BE}=0$ and $\Delta\boldsymbol{j}^{(4)}_{BE}=0$. Both the second and third contributions are also simplified as follows:

$$\Delta\boldsymbol{j}^{(2)}_{BE} = e\tau_{\rm eff}\sum_\chi\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)G_\chi^{-1}\boldsymbol{v}_\chi \cdot\left\{e\boldsymbol{E}+\frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\mathfrak{B}_\chi\right\}\chi\dot{\boldsymbol{c}}\times\mathfrak{B}_\chi \quad (72)$$

and

$$\Delta\boldsymbol{j}^{(3)}_{BE} = e\tau_{\rm eff}\sum_\chi\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(-\frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\varepsilon}\right)\times G_\chi^{-1}\boldsymbol{v}_\chi\cdot\left\{(\boldsymbol{B}\cdot\dot{\boldsymbol{c}})\mathfrak{B}_\chi-(\boldsymbol{B}\cdot\mathfrak{B}_\chi)\dot{\boldsymbol{c}}\right\}\frac{e^2}{c}\chi\boldsymbol{E}\times\mathfrak{B}_\chi, \quad (73)$$

respectively. In the following discussion we will consider these transport coefficients up to the first order in β, described by $\boldsymbol{\mathcal{B}}_\chi$ and $\boldsymbol{\mathcal{E}}_\chi$ introduced before.

In order to figure out the direction of electrical currents in terms of applied electric fields, magnetic fields, and the direction of $\dot{\boldsymbol{c}}$, we separate each vector quantity such as $\boldsymbol{E}$, $\boldsymbol{\mathcal{B}}_\chi$, $\dot{\boldsymbol{c}}$, etc., into two components as follows: Parallel and perpendicular to $\boldsymbol{B}$, respectively,

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_{||}, \quad \boldsymbol{c}=\boldsymbol{c}_{||}(\because \boldsymbol{c}||\boldsymbol{B}),\\ \boldsymbol{E} &= \boldsymbol{E}_{||}+\boldsymbol{E}_\perp, \quad \dot{\boldsymbol{c}}=\dot{\boldsymbol{c}}_{||}+\dot{\boldsymbol{c}}_\perp,\\ \boldsymbol{\mathcal{B}}_\chi &= \frac{\chi}{2}\frac{\boldsymbol{p}_{||}+\chi\boldsymbol{c}_{||}}{R_\chi(\theta)^3}+\frac{\chi}{2}\frac{\boldsymbol{p}_\perp}{R_\chi(\theta)^3}\\ &\equiv \boldsymbol{\mathcal{B}}_{\chi||}(p,\theta)+\boldsymbol{\mathcal{B}}_{\chi\perp}(p,\theta,\phi), \end{aligned} \quad (74)$$

where

$$R_\chi(\theta)\equiv|\boldsymbol{p}+\chi\boldsymbol{c}| = \sqrt{(p\cos\theta+\chi\boldsymbol{c}_{||})^2+p^2\sin^2\theta} \quad (75)$$

with $\boldsymbol{p}=(p\sin\theta\cos\phi,p\sin\theta\sin\phi,p\cos\theta)$. θ and ϕ are inclination and azimuth, respectively, in the polar coordinate, where $\hat{\boldsymbol{B}}_{||}$ is identified with $\hat{z}$. Accordingly, we have

$$\begin{aligned} f^{\rm eq}_\chi(\boldsymbol{p}) &= f^{\rm eq}_\chi(p,\theta),\\ \frac{\partial f^{\rm eq}_\chi(\boldsymbol{p})}{\partial\epsilon} &= \frac{\partial f^{\rm eq}_\chi(p,\theta)}{\partial\epsilon},\\ G_\chi(\boldsymbol{p}) &= G_\chi(p,\theta) = 1+\frac{e}{c}B_{||}\mathcal{B}_{\chi||}(p,\theta). \end{aligned} \quad (76)$$

We introduce a modified group velocity as follows:

$$\begin{aligned} \tilde{\boldsymbol{v}}_\chi(\boldsymbol{p}) &\equiv G_\chi^{-1}(\boldsymbol{p})\boldsymbol{v}_\chi(\boldsymbol{p})\\ &= G_\chi^{-1}|\boldsymbol{p}+\chi\boldsymbol{c}|\frac{e}{c}(\boldsymbol{B}\cdot\nabla_{\boldsymbol{p}})\boldsymbol{\mathcal{B}}_\chi+\frac{\boldsymbol{p}+\chi\boldsymbol{c}}{|\boldsymbol{p}+\chi\boldsymbol{c}|}. \end{aligned} \quad (77)$$

We also decompose this modified group velocity into

$$\tilde{\boldsymbol{v}}_\chi \equiv \tilde{\boldsymbol{v}}_{\chi||}(p,\theta)+\tilde{\boldsymbol{v}}_{\chi\perp}(p,\theta,\phi), \quad (78)$$

where

$$\tilde{\boldsymbol{v}}_{\chi||}(p,\theta) = G_\chi^{-1}(p,\theta)\chi\frac{e}{c}\frac{|\boldsymbol{p}_\perp|^2 - 2|\boldsymbol{p}_{||} + \chi\boldsymbol{c}_{||}|^2}{2R_\chi^4(p,\theta)}\boldsymbol{B}_{||} + \frac{\boldsymbol{p}_{||} + \chi\boldsymbol{c}_{||}}{R_\chi(p,\theta)},$$
$$\tilde{\boldsymbol{v}}_{\chi\perp}(p,\theta) = \left(-\frac{3}{2}G_\chi^{-1}(p,\theta)\chi|\boldsymbol{B}_{||}|\frac{e}{c}\frac{(\boldsymbol{p}_{||} + \chi\boldsymbol{c}_{||})\cdot\boldsymbol{B}_{||}}{R_\chi^4(p,\theta)|\boldsymbol{B}_{||}|} + \frac{1}{R_\chi(p,\theta)}\right)\boldsymbol{p}_\perp. \tag{79}$$

As a result, both contributions for electrical currents are given by

$$\Delta\boldsymbol{j}_{BE}^{(2)} = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\frac{e^2}{c}\chi \times\Bigg\{(\mathcal{B}_{\chi||}E_{||})(\tilde{v}_{\chi||}\mathcal{B}_{\chi||} + \tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{B}_{||}) + \frac{c}{e}(\tilde{v}_{\chi||}E_{||})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{\mathcal{B}}_{\chi||}) + \frac{c}{e}(\tilde{v}_{\chi\perp}E_\perp)(\dot{\boldsymbol{c}}_{||}\times\boldsymbol{\mathcal{B}}_{\chi\perp}) + \frac{e}{c}(\tilde{v}_{\chi\perp}E_\perp)(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{\mathcal{B}}_{\chi\perp})\Bigg\} \tag{80}$$

and

$$\Delta\boldsymbol{j}_{BE}^{(3)} = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\frac{e^2}{c}\chi \times\{(\tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\mathcal{B}_{\chi||}\dot{c}_{||})(\boldsymbol{E}_\perp\times\boldsymbol{B}_{||}) - (\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_{||}\mathcal{B}_{\chi||})(\boldsymbol{E}_{||}\times\boldsymbol{\mathcal{B}}_{\chi\perp}) - (\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_{||}\mathcal{B}_{\chi||})(\boldsymbol{E}_\perp\times\boldsymbol{\mathcal{B}}_{\chi\perp})\}, \tag{81}$$

where we have utilized the following properties:

$$\begin{aligned}
\boldsymbol{\mathcal{B}}_{\chi\perp}(p,\theta,\phi+\pi) &= -\boldsymbol{\mathcal{B}}_{\chi\perp}(p,\theta,\phi),\\
\boldsymbol{\mathcal{B}}_{\chi||}(p,\pi-\theta) &= \boldsymbol{\mathcal{B}}_{-\chi||}(p,\theta),\\
\boldsymbol{\mathcal{B}}_{\chi\perp}(p,\pi-\theta,\phi) &= -\boldsymbol{\mathcal{B}}_{-\chi\perp}(p,\theta,\phi),\\
\tilde{\boldsymbol{v}}_{\chi\perp}(p,\theta,\phi+\pi) &= -\tilde{\boldsymbol{v}}_{\chi\perp}(p,\theta,\phi),\\
\tilde{\boldsymbol{v}}_{\chi||}(p,\pi-\theta,\phi) &= -\tilde{\boldsymbol{v}}_{-\chi||}(p,\theta,\phi),\\
\tilde{\boldsymbol{v}}_{\chi\perp}(p,\pi-\theta,\phi) &= \tilde{\boldsymbol{v}}_{-\chi\perp}(p,\theta,\phi),\\
f_\chi^{\text{eq}}(p,\pi-\theta) &= f_{-\chi}^{\text{eq}}(p,\theta),\\
\frac{\partial f_\chi^{\text{eq}}(p,\pi-\theta)}{\partial\epsilon} &= \frac{\partial f_{-\chi}^{\text{eq}}(p,\theta)}{\partial\epsilon},\\
R_\chi(p,\pi-\theta) &= R_{-\chi}(p,\theta),\\
G_\chi(\pi-\theta) &= G_{-\chi}(\theta).
\end{aligned} \tag{82}$$

It is more convenient to reexpress these currents in the following way:

$$\Delta\boldsymbol{j}_{BE}^{(2)} + \Delta\boldsymbol{j}_{BE}^{(3)} \equiv \boldsymbol{j}_{B,E} + \boldsymbol{j}_E, \tag{83}$$

considering the $\boldsymbol{B}_{||}$ dependence. Here, $\boldsymbol{j}_{B,E}$ and $\boldsymbol{j}_E$ are

$$\boldsymbol{j}_{B,E} = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\frac{e^2}{c}\chi \times\{(\mathcal{B}_{\chi||}E_{||})(\tilde{v}_{\chi||}\mathcal{B}_{\chi||} + \tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{B}_{||}) + (\tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\mathcal{B}_{\chi||}\dot{c}_{||})(\boldsymbol{E}_\perp\times\boldsymbol{B}_{||}) - (\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_{||}\mathcal{B}_{\chi||})(\boldsymbol{E}_{||}\times\boldsymbol{\mathcal{B}}_{\chi\perp}) - (\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_{||}\mathcal{B}_{\chi||})(\boldsymbol{E}_\perp\times\boldsymbol{\mathcal{B}}_{\chi\perp})\} \tag{84}$$

and

$$\boldsymbol{j}_E = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\chi e \times\Bigg\{(\tilde{v}_{\chi||}E_{||})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{\mathcal{B}}_{\chi||}) + (\tilde{v}_{\chi\perp}E_\perp)(\dot{\boldsymbol{c}}_{||}\times\boldsymbol{\mathcal{B}}_{\chi\perp}) + (\tilde{v}_{\chi\perp}E_\perp)(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{\mathcal{B}}_{\chi\perp})\Bigg\}, \tag{85}$$

respectively.

Based on these equations, we determine the direction of an anomalous current associated with the Berry electric field. For simplicity, we assume

$$\boldsymbol{B}(t) = \boldsymbol{B}_0 + \boldsymbol{\delta B}(t) \tag{86}$$

with $|\boldsymbol{B}_0| \gg |\boldsymbol{\delta B}(t)|$. Note that $\dot{\boldsymbol{c}} \propto \delta\dot{\boldsymbol{B}}(t)$ since $\boldsymbol{c} \propto \boldsymbol{B}$.

First, we consider the case of $\boldsymbol{B}_0 \parallel \boldsymbol{E} \parallel \dot{\boldsymbol{c}}$. Then, we obtain

$$\boldsymbol{j}_{B,E} = \boldsymbol{j}_E = 0. \tag{87}$$

Second, we consider the case of $\boldsymbol{B}_0 \parallel \boldsymbol{E} \perp \dot{\boldsymbol{c}}$. Then, we find

$$\boldsymbol{j}_{B,E} = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\frac{e^2}{c}\chi \times\{(\mathcal{B}_{\chi||}E_{||})(\tilde{v}_{\chi||}\mathcal{B}_{\chi||} + \tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{B}_0) - (\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_0\mathcal{B}_{\chi||})(\boldsymbol{E}_{||}\times\boldsymbol{\mathcal{B}}_{\chi\perp})\}, \tag{88}$$

parallel with $(\hat{\boldsymbol{c}}\times\hat{\boldsymbol{E}})$, and

$$\boldsymbol{j}_E = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_{\chi\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\chi e \times(\tilde{v}_{\chi||}E_{||})(\dot{\boldsymbol{c}}_\perp\times\boldsymbol{\mathcal{B}}_{\chi||}), \tag{89}$$

parallel with $(\hat{\boldsymbol{c}}\times\hat{\boldsymbol{E}})$. We emphasize that these anomalous Hall currents are in the order of $\mathcal{O}[\{\boldsymbol{\delta B}(t)\}^0]$, which should be distinguished from conventional anomalous Hall currents in the order of $\mathcal{O}[\{\boldsymbol{\delta B}(t)\}^1]$. These are novel anomalous Hall currents beyond the Berry magnetic field.

Third, we consider the case of $\boldsymbol{B}_0 \perp \boldsymbol{E}||\dot{\boldsymbol{c}}$. Then, we have

$$\boldsymbol{j}_{B,E} = \boldsymbol{j}_E = 0. \tag{90}$$

Of course, there exists a conventional anomalous Hall current, described by Eq. (59), since the applied electric field $\boldsymbol{E}$ is orthogonal to the applied magnetic field $\boldsymbol{B}_0$.

Fourth, we consider the case of $\boldsymbol{B}_0 \perp \boldsymbol{E}$ with $\boldsymbol{B}_0||\dot{\boldsymbol{c}}$. Then, we obtain

$$\boldsymbol{j}_{B,E} = e\tau_{\text{eff}}\sum_\chi\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}\right)\frac{e^2}{c}\chi \times(\tilde{v}_{\chi\perp}\mathcal{B}_{\chi\perp})(\mathcal{B}_{\chi||}\dot{c}_{||})(\boldsymbol{E}_\perp\times\boldsymbol{B}_0), \tag{91}$$

TABLE I. Anomalous Hall currents $\boldsymbol{j}_E$ [Eq. (85)] and $\boldsymbol{j}_{B,E}$ [Eq. (84)] driven by the Berry electric field.

		Direction of $\boldsymbol{j}_E$	Direction of $\boldsymbol{j}_{B,E}$
$\boldsymbol{B}_0 \| \boldsymbol{E}$	$\boldsymbol{B}_0 \| \dot{\boldsymbol{c}}$		
	$\boldsymbol{B}_0 \perp \dot{\boldsymbol{c}}$	$\dot{\boldsymbol{c}} \times \boldsymbol{E}$	$\dot{\boldsymbol{c}} \times \boldsymbol{E}$
$\boldsymbol{B}_0 \perp \boldsymbol{E}$	$\dot{\boldsymbol{c}} \| \boldsymbol{E}$		
	$\dot{\boldsymbol{c}} \| \boldsymbol{B}_0$	$\dot{\boldsymbol{c}} \times \boldsymbol{E}$	$\dot{\boldsymbol{c}} \times \boldsymbol{E}$
	$\dot{\boldsymbol{c}} \perp \boldsymbol{E}$ & $\dot{\boldsymbol{c}} \perp \boldsymbol{B}_0$	$\boldsymbol{B}_0$	$\boldsymbol{B}_0$

parallel with $(\hat{E} \times \hat{\boldsymbol{c}})$, and

$$\boldsymbol{j}_E = e\tau_{\text{eff}} \sum_\chi \int \frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial \epsilon}\right)\chi e \times (\tilde{v}_{\chi\perp} E_\perp)(\dot{\boldsymbol{c}}_{||} \times \boldsymbol{\mathcal{B}}_{\chi\perp}), \tag{92}$$

parallel with $-(\hat{E} \times \hat{\boldsymbol{c}})$. Here, we also point out the existence of the conventional anomalous Hall current, which does not depend on $\dot{\boldsymbol{c}}_{||}$. These anomalous Hall currents are driven by the emergent Berry electric field. Finally, we consider the case of $\boldsymbol{B}_0 \perp \boldsymbol{E}$, $\boldsymbol{E} \perp \dot{\boldsymbol{c}}$, and $\boldsymbol{B}_0 \perp \dot{\boldsymbol{c}}$. Then, we obtain

$$\boldsymbol{j}_{B,E} = e\tau_{\text{eff}} \sum_\chi \int \frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial \epsilon}\right)\frac{e^2}{c}\chi \times \{-(\tilde{v}_{\chi\perp}\dot{c}_\perp)(B_0\mathcal{B}_{\chi||})(\boldsymbol{E}_\perp \times \boldsymbol{\mathcal{B}}_{\chi\perp})\}, \tag{93}$$

parallel with $(-\hat{\boldsymbol{B}}_0)$, and

$$\boldsymbol{j}_E = e\tau_{\text{eff}} \sum_\chi \int \frac{d^3p}{(2\pi)^3}\left(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial \epsilon}\right)\chi e \times (\tilde{v}_{\chi\perp} E_\perp)(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{\mathcal{B}}_{\chi\perp})\Big\}, \tag{94}$$

parallel with $(-\hat{\boldsymbol{B}}_0)$. We recall that the conventional anomalous Hall current is given along the direction of $\boldsymbol{B}_0 \times \boldsymbol{E}$. On the other hand, these anomalous Hall currents are driven in parallel with the applied magnetic field $\boldsymbol{B}_0$, quite surprisingly.

All these situations are summarized in Table I.

Here, we considered the overall magnitude of these anomalous Hall currents in the "adiabatic regime" where the time it takes the distribution to relax is small compared to the period of the magnetic field driving the chiral Fermi surfaces. Amplitude of the current is proportional to the relaxation time and to the rate of change of the distance between the two chiral Fermi surfaces that goes as the frequency of the drive in the steady state. In this respect one may be concerned that the overall size of these effects is suppressed in the adiabatic regime by the ratio of the relaxation time to the period of the drive. However, there exists another aspect to enhance the present anomalous Hall effect. As shown explicitly in mathematical formulas, the Berry magnetic field contributes to this anomalous Hall effect. An essential point is that the Berry magnetic field becomes enhanced near the band touching point. Combining this Berry-flux enhancement with the contribution of finite density of states, there can exist parameter regimes to overcome the suppression effect.

B. Explicit expressions for $\boldsymbol{j}_E$ and $\boldsymbol{j}_{B,E}$ near zero temperature

We perform momentum integrals explicitly in Eq. (85) ($\boldsymbol{j}_E$) and Eq. (84) ($\boldsymbol{j}_{B,E}$). We have $(-\frac{\partial f_\chi^{\text{eq}}(\boldsymbol{p})}{\partial\epsilon}) \approx \delta[\epsilon_\chi(\boldsymbol{p}) - \mu]$ near zero temperature in these equations, implying that these anomalous Hall currents are Fermi-surface contributions. In order to perform momentum integrals in Eqs. (84) and (85), we introduce

$$G_\chi(\boldsymbol{p}) = 1 + \frac{e}{c}\boldsymbol{B}_0 \cdot \frac{\chi}{2}\frac{\boldsymbol{p}}{|\boldsymbol{p}|^3} = 1 + \chi A\frac{\cos\theta}{p^2}, \tag{95}$$

$$\tilde{v}_{\chi||} = \left(G_\chi^{-1}(\boldsymbol{p})\chi A\frac{\sin^2\theta - 2\cos^2\theta}{p^2} + \cos\theta\right)\hat{z}, \tag{96}$$

$$\tilde{v}_{\chi\perp} = \left(-3\chi G_\chi^{-1}(\boldsymbol{p})A\frac{\cos\theta}{p^2} + 1\right)\sin\theta\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right), \tag{97}$$

$$\boldsymbol{\mathcal{B}}_{\chi||} = \frac{\chi}{2}\frac{\cos\theta}{p^2}\hat{z}, \quad \boldsymbol{\mathcal{B}}_{\chi\perp} = \frac{\chi}{2}\sin\theta\frac{\cos\phi\hat{x} + \sin\phi\hat{y}}{p^2}, \tag{98}$$

$$\epsilon_\chi(\boldsymbol{p}) = G_\chi(\boldsymbol{p})p, \tag{99}$$

$$\delta(\epsilon_\chi(\boldsymbol{p}) - \mu) = \frac{p}{A}\delta\left(\cos\theta - \chi\frac{p(\mu - p)}{A}\right) \tag{100}$$

with $\boldsymbol{p} + \chi\boldsymbol{c} \to \boldsymbol{p}$, where $\boldsymbol{p} = p\sin\theta\cos\phi\hat{x} + p\sin\theta\sin\phi\hat{y} + p\cos\theta\hat{z}$ is taken into account. Here, we choose $\boldsymbol{B}_0 = B_0\hat{z}$. $A = \frac{eB_0}{2c}$ is an externally applied magnetic field.

Performing tedious but straightforward calculations, we obtain

$$\Delta\boldsymbol{j}_{\text{total}} = \boldsymbol{j}_E + \boldsymbol{j}_{B,E} = \frac{e^2\tau_{\text{eff}}}{(2\pi)^2}\left[(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_{||})F_1\left(\frac{\mu^2}{A}\right) + (\dot{\boldsymbol{c}}_{||} \times \boldsymbol{E}_\perp)F_2\left(\frac{\mu^2}{A}\right) + (\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_\perp)F_3\left(\frac{\mu^2}{A}\right)\right], \tag{101}$$

where

$$F_1(x) \equiv 1 - \frac{x}{8} + \frac{x^3}{1680} - \frac{1}{840}\sqrt{1 - \frac{4}{x}}(x-4)(x^2 + 6x - 180)\Theta(x-4) + \frac{1}{1680}\sqrt{1 + \frac{4}{x}}(x+4)(x^2 - 6x - 180), \tag{102}$$

$$F_2(x) \equiv -\frac{1}{2} + \frac{x}{4} - \frac{x^3}{210} + \frac{1}{210}\sqrt{1 - \frac{4}{x}}(x-4)(2x^2 + 12x - 45)\Theta(x-4) - \frac{1}{420}\sqrt{1 + \frac{4}{x}}(x+4)(2x^2 - 12x - 45), \tag{103}$$

$$F_3(x) \equiv \frac{1}{2} - \frac{x}{8} + \frac{x^3}{240} - \frac{1}{120}\sqrt{1 - \frac{4}{x}}x(x-4)(x+6)\Theta(x-4) + \frac{1}{240}\sqrt{1 + \frac{4}{x}}x(x+4)(x-6), \tag{104}$$

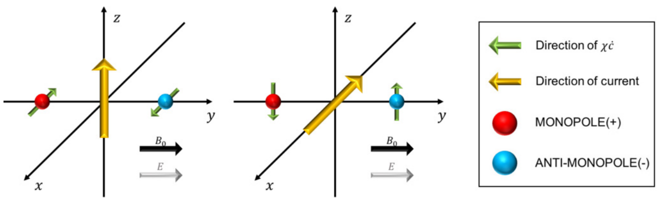

FIG. 1. Direction of the anomalous Hall currents driven by the Berry electric field for the case of $\boldsymbol{B}_0||\boldsymbol{E} \perp \dot{\boldsymbol{c}}$.

and $\Theta(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$. An interesting point is that the argument x in the special functions is given by the ratio between the external magnetic field A and the chemical potential μ. We remember to set $\hbar = 1$ and $v_{\rm Dirac} = 1$ in the above derivation. Recovering these constants in the magnetic field A, we find $\frac{\mu^2}{A} = \frac{2\mu^2}{eB_0 v_{\rm Dirac}^2 \hbar}$ in the SI unit. For example, if we consider $Bi_{1-x}Sb_x$ at $B_0 = 1$ T, $v_{\rm Dirac} = 10^5$ m/s, and $\mu = 0.02$ eV, we obtain $\frac{\mu^2}{A} = (\frac{0.02\text{eV}}{2.57\times10^{-2}\text{eV}})^2 \approx 0.6$.

These special functions are simplified as follows: In the $x \gg 1$ limit we obtain

$$F_1(x) \approx -1 + \frac{5}{8x} + \frac{6}{5x^2} + \frac{7}{12x^3}, \tag{105}$$

$$F_2(x) \approx 2 - \frac{1}{2x} - \frac{3}{5x^2} - \frac{1}{6x^3}, \tag{106}$$

$$F_3(x) \approx -\frac{1}{8x} - \frac{3}{5x^2} - \frac{5}{12x^3}, \tag{107}$$

and in the $x \ll 1$ limit we have

$$F_1(x) \approx -\frac{6}{7\sqrt{x}} + 1, \tag{108}$$

$$F_2(x) \approx \frac{6}{7\sqrt{x}} - \frac{1}{2}, \tag{109}$$

$$F_3(x) \approx \frac{1}{2} - \frac{\sqrt{x}}{5}. \tag{110}$$

Considering the case of $\boldsymbol{B}_0||\boldsymbol{E} \perp \dot{\boldsymbol{c}}$, we find

$$\Delta \boldsymbol{j}_{\rm total} \equiv \boldsymbol{j}_1 = \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_{||})F_1\left(\frac{\mu^2}{A}\right). \tag{111}$$

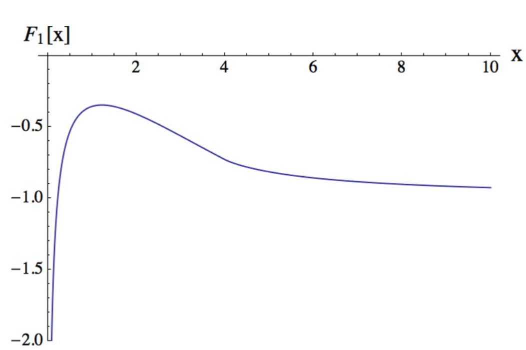

FIG. 2. x vs $F_1(x)$.

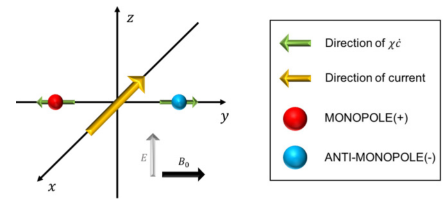

FIG. 3. Direction of the anomalous Hall current driven by the Berry electric field for the case of $\boldsymbol{B}_0 \perp \boldsymbol{E}$ and $\boldsymbol{B}_0||\dot{\boldsymbol{c}}$.

See Fig. 1 for the direction of these anomalous Hall currents. The behavior of $F_1(x)$ is shown in Fig. 2. In the $\mu^2 \gg A$ limit we obtain

$$\boldsymbol{j}_1 \approx \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_{||})\left[-1 + \frac{5}{8}\frac{A}{\mu^2} + \frac{6}{5}\left(\frac{A}{\mu^2}\right)^2\right], \tag{112}$$

and in the $\mu^2 \ll A$ limit we have

$$\boldsymbol{j}_1 \approx \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_{||})\left[-\frac{6}{7}\left(\frac{A}{\mu^2}\right)^{1/2} + 1\right]. \tag{113}$$

One may be concerned with the divergence of these anomalous Hall currents in the $\mu^2 \ll A$ limit (Fig. 2). However, this does not happen. We note that our Boltzmann transport theory does not apply to the $\mu \to 0$ limit.

Considering $\boldsymbol{B}_0 \perp \boldsymbol{E}$ and $\boldsymbol{B}_0||\dot{\boldsymbol{c}}$, we find

$$\Delta \boldsymbol{j}_{\rm total} \equiv \boldsymbol{j}_2 = \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_{||} \times \boldsymbol{E}_\perp)F_2\left(\frac{\mu^2}{A}\right). \tag{114}$$

See Fig. 3 for the direction of this anomalous Hall current. The behavior of $F_2(x)$ is shown in Fig. 4. In the $\mu^2 \gg A$ limit, we obtain

$$\boldsymbol{j}_2 \approx \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_{||} \times \boldsymbol{E}_\perp)\left[2 - \frac{1}{2}\frac{A}{\mu^2} - \frac{3}{5}\left(\frac{A}{\mu^2}\right)^2\right], \tag{115}$$

and in the $\mu^2 \ll A$ limit we have

$$\boldsymbol{j}_2 \approx \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_{||} \times \boldsymbol{E}_\perp)\left[\frac{6}{7}\left(\frac{A}{\mu^2}\right)^{1/2} - \frac{1}{2}\right]. \tag{116}$$

Considering $\boldsymbol{B}_0 \perp \boldsymbol{E}$, $\boldsymbol{E} \perp \dot{\boldsymbol{c}}$, and $\boldsymbol{B}_0 \perp \dot{\boldsymbol{c}}$, we find

$$\Delta \boldsymbol{j}_{\rm total} \equiv \boldsymbol{j}_3 = \frac{e^2\tau_{\rm eff}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_\perp)F_3\left(\frac{\mu^2}{A}\right). \tag{117}$$

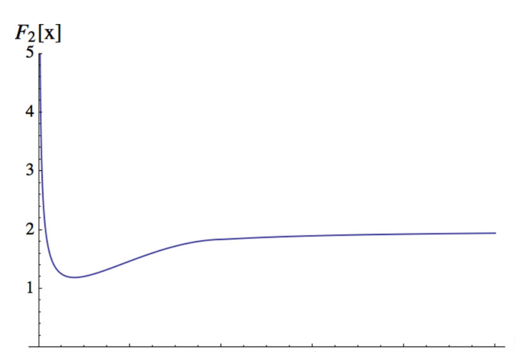

FIG. 4. x vs $F_2(x)$.

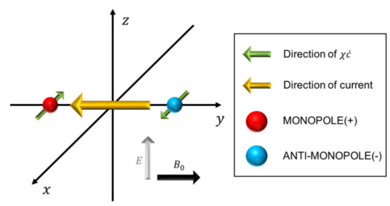

FIG. 5. Direction of the anomalous Hall current driven by the Berry electric field for the case of $\boldsymbol{B}_0 \perp \boldsymbol{E}$, $\boldsymbol{E} \perp \dot{\boldsymbol{c}}$, and $\boldsymbol{B}_0 \perp \dot{\boldsymbol{c}}$.

See Fig. 5 for the direction of this anomalous Hall current. The behavior of $F_3(x)$ is shown in Fig. 6. In the $\mu^2 \gg A$ limit, we obtain

$$\boldsymbol{j}_3 \approx \frac{e^2\tau_{\text{eff}}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_\perp)\left[-\frac{1}{8}\frac{A}{\mu^2} - \frac{3}{5}\left(\frac{A}{\mu^2}\right)^2\right], \tag{118}$$

and in the $\mu^2 \ll A$ limit we have

$$\boldsymbol{j}_3 \approx \frac{e^2\tau_{\text{eff}}}{(2\pi)^2}(\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_\perp)\left[\frac{1}{2} - \frac{1}{5}\left(\frac{\mu^2}{A}\right)^{1/2}\right]. \tag{119}$$

If we consider the case of $|\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_{||}| = |\dot{\boldsymbol{c}}_{||} \times \boldsymbol{E}_\perp| = |\dot{\boldsymbol{c}}_\perp \times \boldsymbol{E}_\perp|$, we see relative magnitudes of $F_1(\frac{\mu^2}{A})$, $F_2(\frac{\mu^2}{A})$, and $F_3(\frac{\mu^2}{A})$, shown in Fig. 7. As a result, we expect

$$|\boldsymbol{j}_2| > |\boldsymbol{j}_1| > |\boldsymbol{j}_3|. \tag{120}$$

VII. CONCLUSION

The Berry electric field is a novel ingredient in a Weyl metal phase. When the distance between a pair of Weyl points changes as a function of time, the Berry electric field arises. In this situation both the Berry magnetic field and Berry electric field are governed by the Berry-Maxwell equation [Eq. (15)]. This Berry electric field should be introduced into the topologically modified Drude model Eq. (11). As a result, we revealed the existence of anomalous Hall effects proportional to the Berry electric field, which should be distinguished from conventional anomalous Hall currents given by the Berry magnetic field. Current directions of such anomalous Hall effects are classified systematically in all possible cases.

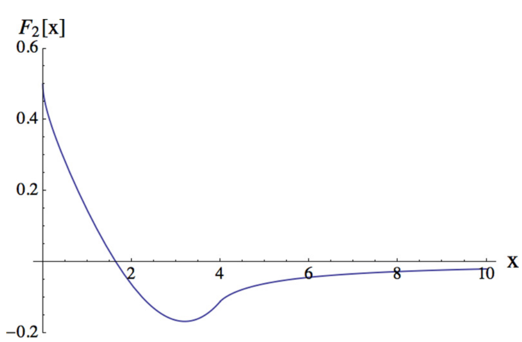

FIG. 6. x vs $F_3(x)$.

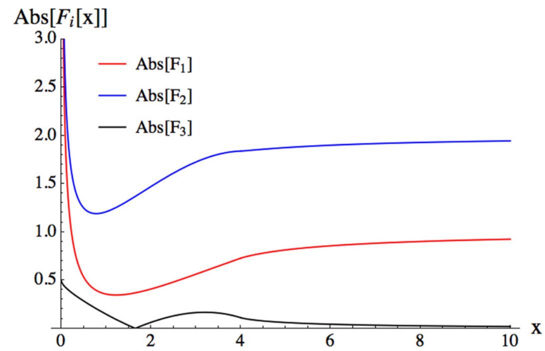

FIG. 7. x vs $|F_1(x)|$ (red), $|F_2(x)|$ (blue), and $|F_3(x)|$ (black).

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[35] Finishing this study, we were aware of an interesting work to consider an emergent Berry electric field [H. Ishizuka, T. Hayata, M. Ueda, and N. Nagaosa, Phys. Rev. Lett. **117**, 216601 (2016)]. The authors take into account a time-dependent band structure, essentially the same as the present situation. Following the standard procedure, they construct an effective time-dependent Berry gauge field. In this respect emergent Maxwell equations in momentum space are satisfied automatically. We point out that this study also considers the nonrelativistic limit. It is not a simple task to derive the Berry-Maxwell equation in a relativistically covariant way. We are trying to derive the Berry-Maxwell equation in the field theoretical framework, more convenient to keep the relativistic covariance. An idea is to derive an effective field theory for low-energy chiral fermions in the second quantization, where such low-energy chiral fermions see effects of Berry electromagnetic fields explicitly [Yong-Soo Jho, Jae-Ho Han, and Ki-Seok Kim, arXiv:1409.0414]. This can be achieved, integrating over high-energy electrons deep inside the chiral Fermi surface. Integrating over such low-energy chiral fermions, one can find an effective action for Berry electromagnetic fields as the kinetic energy of Maxwell electromagnetic fields arises, integrating over electrons.
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