Topological electromagnetic responses of bosonic quantum Hall, topological insulator, and chiral semimetal phases in all dimensions

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We calculate the topological part of the electromagnetic response of bosonic integer quantum Hall (BIQH) phases in odd (space-time) dimensions, and bosonic topological insulator (BTI) and bosonic chiral semimetal (BCSM) phases in even dimensions. To do this, we use the nonlinear sigma model (NLSM) description of bosonic symmetry-protected topological (SPT) phases, and the method of gauged Wess-Zumino (WZ) actions. We find the surprising result that for BIQH states in dimension $2m-1$ ($m = 1,2,\dots$), the bulk response to an electromagnetic field A_μ is characterized by a Chern-Simons term for A_μ with a level quantized in integer multiples of $m!$ (factorial). We also show that BTI states (which have an extra $\mathbb{Z}_2$ symmetry) can exhibit a $\mathbb{Z}_2$-breaking quantum Hall effect on their boundaries, with this boundary quantum Hall effect described by a Chern-Simons term at level $\frac{m!}{2}$. We show that the factor of $m!$ can be understood by requiring gauge invariance of the exponential of the Chern-Simons term on a general Euclidean manifold, and we also use this argument to characterize the electromagnetic *and* gravitational responses of fermionic SPT phases with U(1) symmetry in all odd dimensions. We then use our gauged boundary actions for the BIQH and BTI states to (i) construct a bosonic analog of a chiral semimetal (BCSM) in even dimensions, (ii) show that the boundary of the BTI state exhibits a bosonic analog of the parity anomaly of Dirac fermions in odd dimensions, and (iii) study anomaly inflow at domain walls on the boundary of BTI states. In a series of Appendixes we derive important formulas and additional results. In particular, in Appendix A we use the connection between equivariant cohomology and gauged WZ actions to give a mathematical interpretation of the actions for the BIQH and BTI boundaries constructed in this paper.

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I. INTRODUCTION

In the years since the theoretical prediction and experimental discovery of the electron topological insulators [1,2], the study of symmetry-protected topological (SPT) phases of matter [3–7] has emerged as an extremely rich subfield of condensed matter physics, with interesting and surprising connections to high-energy physics and mathematics. Although there has been tremendous progress in the understanding of these states of matter, some basic issues about these phases are still the subject of intense investigation. As illustrative examples we point to the question of which theories can describe a surface termination of the time-reversal invariant electron topological insulator in three spatial dimensions [8–14], as well as the analogous question for the surface of the bosonic topological insulator in three spatial dimensions [15,16].

A very useful definition of an SPT phase is as follows [17]. Consider a quantum many-body system with Hamiltonian H, where H has the symmetries of a group G and a gapped spectrum. Then, the ground state $|\Psi\rangle$ of H represents an SPT phase if it satisfies several properties. First, $|\Psi\rangle$ should be unique independently of the topology of the (closed) spatial manifold that H is defined on. This ensures that the ground state of H does not represent a phase with topological order (no excitations with fractional charge or statistics, etc.). Second, $|\Psi\rangle$ should be invariant under the action of G, i.e., $U(g)|\Psi\rangle = |\Psi\rangle$ for any $g \in G$, where $U(g)$ is a representation of G on the Hilbert space of the system. This means that the ground state of H does not spontaneously break the symmetry of the group G. Finally, $|\Psi\rangle$ cannot be continuously tuned to a *trivial* product state (e.g., by adding terms to the Hamiltonian) without (i) breaking the symmetry of G, or (ii) closing the gap in the spectrum of H. Despite the lack of anyon excitations in the bulk, interesting degrees of freedom will in general be present at the boundary of an SPT phase.

In this paper, we focus our primary attention on bosonic SPT phases and, in particular, on those bosonic SPT phases which are analogs of more familiar topological phases of fermions. We are especially interested in the bosonic integer quantum Hall (BIQH) effect [18–28], a bosonic analog of the ordinary $\nu = 1$ integer quantum Hall effect of fermions in three (space-time) dimensions, and in the time-reversal invariant bosonic topological insulator (BTI) [29–32], a bosonic analog of the time-reversal invariant electron topological insulator in four dimensions. In fact, the main goal of our paper is to consider generalizations of the BIQH and BTI states to *all* odd and even space-time dimensions, respectively, and then to study the physical properties of these higher-dimensional states. The reader should note that in the remainder of this paper the word "dimension" always refers to the *space-time* dimension. We always write "spatial dimension" when we want to discuss the dimension of space only.

BIQH phases are protected only by U(1) charge-conservation symmetry, while the BTI phase is protected by the symmetry group $\mathrm{U}(1) \rtimes \mathbb{Z}_2$, where, as we discuss later, the $\mathbb{Z}_2$ symmetry is unitary charge-conjugation symmetry $\mathbb{Z}_2^C$ in dimensions 2,6,10, etc., and antiunitary time-reversal symmetry $\mathbb{Z}_2^T$ in dimensions 4,8,12, etc. The symbol "$\rtimes$" means that the U(1) and $\mathbb{Z}_2$ symmetry operations do not commute with each other. Since both of these phases have U(1) charge-conservation symmetry, they can both be coupled to an external electromagnetic field A_μ. One can then study the electromagnetic response of these states.

One of our main results in this paper is an explicit derivation of the (topological part of) the electromagnetic response of BIQH phases in all odd dimensions and BTI phases in all even dimensions. From a physical standpoint, the magnitude of the electromagnetic response is extremely interesting, as it is known already in three dimensions that the requirement that a BIQH state has no topological order places a constraint on the allowed values of the Hall conductance of any putative BIQH state [18]. In particular, the Hall conductance must be a multiple of 2 (in units of $\frac{e^2}{h}$), i.e., a BIQH state has twice the Hall conductance that a free-fermion integer quantum Hall state can have. In higher dimensions, we also find that the electromagnetic response of the BIQH state is some integer multiple of the minimum value which can be realized by free fermions, and we find analogous results in even dimensions for BTI states.

To calculate the electromagnetic response of these states, we need a concrete model to work with. For reasons to be discussed in the next section, we choose to use the nonlinear sigma model (NLSM) description of bosonic SPT phases [33–38]. This allows us to use the theory of gauged Wess-Zumino (WZ) actions [39–44] to study the boundary of these states, and from our study of the boundary we are able to deduce the bulk response. As a by-product, our explicit construction of gauged WZ actions for the boundaries of these states allows us to study several physical properties of these states in more detail. We show that the boundary theory for the BTI displays a bosonic analog of the parity anomaly for Dirac fermions in odd dimensions [45–49], and we also use the boundary theory of the BIQH state to construct effective theories for bosonic analogs of Weyl (or chiral) semimetals in all even dimensions.

For the case of the BIQH state, we also provide an alternative derivation of the response by requiring the gauge invariance of (the exponential of) the Chern-Simons functional describing the electromagnetic response of the state. We also use this gauge-invariance argument to derive and discuss the electromagnetic and gravitational responses of fermionic integer quantum Hall (FIQH) phases in different dimensions. This gauge-invariance argument provides us with a general understanding of the difference in the quantization of response coefficients of BIQH and FIQH phases.

Before moving on, we take this opportunity to provide some justification for our study of bosonic SPT phases in dimensions higher than the physically relevant dimensions of two, three, and four. Studying a state of matter in generic dimensions can often reveal underlying organizational principles or mathematical structures which cannot be seen by studying low-dimensional examples on their own. An obvious example of this is the periodic table of topological insulators and superconductors [50,51], which exhibits an eightfold periodicity in the dimension of space (i.e., the pattern does not completely develop if one considers only low dimensions). In the case of bosonic SPT phases, low-dimensional examples suggest that the response of the bosonic analog of a given fermionic state (integer quantum Hall or electron topological insulator) is twice that of its fermionic counterpart. However, our results in this paper clearly show that this is *not* the case in higher dimensions. Finally, it is also worth mentioning that many new insights on four-dimensional physics can be gained by imagining that our four-dimensional space-time is the boundary of a five-dimensional SPT phase [52–54].

This paper is organized as follows. First, in Sec. II we outline our basic approach and summarize our main results. In Sec. III we review the relevant background information on BIQH and BTI phases, the NLSM description of SPT phases, and the method of gauged WZ actions. In Sec. IV we construct the gauged WZ action for the boundary of the BIQH phase, and we use the anomaly of the gauged boundary action to deduce the bulk response of the BIQH phase. We also give an alternative derivation of the BIQH response which relies on only the bulk physics of the NLSM. In Sec. V we use a general gauge-invariance argument to understand the electromagnetic response of BIQH states, and also the electromagnetic and gravitational responses of FIQH states in odd dimensions. In particular, we illuminate the important differences between the quantization of response coefficients in BIQH and FIQH phases. In Sec. VI we construct the gauged WZ action for the boundary of the BTI phase, and we use the gauged boundary action to study the symmetry-breaking BIQH response of the BTI boundary. In Sec. VII we use the results from Secs. IV and VI to (i) construct effective theories for bosonic analogs of Weyl, or chiral, semimetals in all even dimensions, (ii) show that the boundary of a BTI state displays an analog of the parity anomaly for Dirac fermions in odd dimensions, and (iii) study the physics of symmetry-breaking domain walls on the boundary of BTI states. Section VIII presents our conclusions. Finally, in a series of appendixes we examine the results of the paper from a more mathematical point of view, and also derive several important formulas which are used throughout the paper.

II. BASIC APPROACH AND SUMMARY OF RESULTS

In this section, we outline our basic approach to calculating the electromagnetic response of higher-dimensional bosonic SPT phases, and then we present our results. In this paper, we work in units where $\hbar = e = 1$, where e is the charge of the basic particles (bosons or fermions) which make up the state we are interested in. To restore e in any formula one can simply replace A_μ (the external electromagnetic field) with eA_μ.

Let us first discuss the general form that the topological part of the electromagnetic response is expected to take for BIQH and BTI states. In odd dimensions, the response of a higher-dimensional analog of a quantum Hall state to an external field $A = A_\mu dx^\mu$ (we use differential form notation) is characterized by a Chern-Simons (CS) term $S_{\text{CS}}[A]$ in the effective action for the external field. In $2m-1$ dimensions, this term takes the form

$$S_{\text{CS}}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A\wedge F^{m-1}, \tag{2.1}$$

where N_{2m-1} is called the *level* of the CS term, $F = dA$, F^{m-1} is shorthand for the wedge product of F with itself $m-1$

times, and $\mathcal{M}$ represents the space-time manifold. Let us also note here that all actions in the paper are written in Minkowski signature (real time) except in Sec. V and Appendix B, where we consider CS and other terms in Euclidean space-times. On the other hand, the response of an analog of a topological insulator in $2m$ dimensions is characterized by a "Chern character" term (we avoid using "theta term" here since that name is also used for a type of topological term in the NLSM action)

$$S_{\mathrm{CC}}[A] = \frac{\Theta_{2m}}{(2\pi)^m m!}\int_{\mathcal{M}} F^m. \quad (2.2)$$

Here, the coefficient Θ_{2m} should be interpreted as an angular variable, although its period is not necessarily 2π. We call this term a "Chern character" term as the quantity $\frac{1}{m!}(\frac{F}{2\pi})^m$ appears as the mth term in the expansion of the total Chern character $\mathrm{ch}[F] = e^{\frac{F}{2\pi}}$ of a U(1) principal bundle with curvature F [55]. Since locally we can write $F^m = d(A \wedge F^{m-1})$, we see that for a BTI state with a boundary, the term $S_{\mathrm{CC}}[A]$ can be interpreted as a CS term at level $\frac{\Theta_{2m}}{2\pi}$ on the $(2m-1)$-dimensional boundary of the BTI state (more precisely, this is only true when the bulk field configuration F has vanishing topological contributions).

For the analogs of integer quantum Hall states of fermions (FIQH states) in odd dimensions, the level N_{2m-1} of the CS term can be any integer [56–59], while for free-fermion topological insulators, and their generalizations to higher dimensions, the angle Θ_{2m} is 2π periodic and the value which represents a nontrivial topological insulator state is $\Theta_{2m} = \pi$ [60] (the result for fermionic topological insulators in any even dimension is easily established using the axial anomaly for a Dirac fermion in $2m$ dimensions). For bosonic SPT phases in low dimensions we know that $N_3 = 2k$, $k \in \mathbb{Z}$, for BIQH states in three dimensions [18,19], that Θ_4 has 4π periodicity, and $\Theta_4 = 2\pi$ for the nontrivial BTI state in four dimensions [29,30,61,62].

One of the main purposes of this paper is to calculate the values of the response coefficients N_{2m-1} and Θ_{2m} for BIQH and BTI states in all dimensions. There are (at least) two ways that one could go about doing this. The first way would be to formulate a general physical argument based, for example, on the consistency of the value of N_{2m-1} or Θ_{2m} and the fact that a bosonic SPT state should have no fractionalized excitations, and in this way determine a constraint on the possible values of N_{2m-1} or Θ_{2m}. In fact, such an argument has already been given for the BIQH state in the case $m = 2$ (three space-time dimensions). In Ref. [18] the authors showed that if the response coefficient N_3 (which is just the Hall conductance in units of $\frac{e^2}{h}$) is odd, then the underlying theory must contain an excitation of charge one (in units of the charge e of the underlying bosons) with fermionic exchange statistics. An excitation with fermionic statistics is not allowed in a state of bosons which has no fractionalized excitations, and so the authors of Ref. [18] concluded that N_3 must be an even integer for BIQH states in three dimensions. Generalizing this argument to higher dimensions clearly represents a significant conceptual difficulty, as in higher dimensions one is probably forced to consider generalized braiding processes for extended objects such as string or membrane excitations [63–65]. For this reason, we do not pursue this approach in this work, and instead use a second method.

The second method for answering this question, and the method that we choose to use, is to (i) start with a concrete field-theoretic model which is believed to accurately describe the low-energy physics of a BIQH or BTI state in the relevant dimension, (ii) couple this model to the external field A, and (iii) directly calculate the electromagnetic response for this particular model. In the literature, there are two main kinds of field-theoretic models that can describe SPT phases: topological quantum field theory (TQFT) in terms of gauge field variables (e.g., Chern-Simons theory in three dimensions [19,25,66,67] and twisted gauge theory [68–70] in four dimensions [32,71]) and the nonlinear sigma model (NLSM) description in terms of constrained bosonic fields [33–38]. In both approaches, the bulk topological order is trivial but global symmetry is imposed nontrivially on the field variables. In this paper, we choose to use the NLSM description since this description can be easily generalized to any space-time dimension.

In the NLSM description, a bulk bosonic SPT phase in $d+1$ space-time dimensions is described by an O$(d+2)$ NLSM with topological theta term having coefficient $\theta = 2\pi k$ where $k \in \mathbb{Z}$. In this description, the boundary of the SPT phase is then described by an O$(d+2)$ NLSM with Wess-Zumino (WZ) term, where the coefficient of the WZ term, known as the *level* of the WZ term, is equal to k. Conventionally, the construction of the WZ term for the boundary theory requires defining an extension of the NLSM field into an auxiliary direction of space-time. In a series of works [33–38], the NLSM description has been shown to accurately describe the structure of the ground-state wave function of SPT phases [17], the point and loop braiding statistics of excitations in gauged SPT phases [17,63,64,72,73], the decorated domain-wall construction of SPT phases [74], as well as several other properties of these phases. In addition, a mathematical classification of bosonic SPT phases based on the NLSM description has been shown to be completely identical to the group cohomology classification [6] in situations where both classification schemes can be applied. In fact, there is even a concrete procedure for calculating the cocycle which classifies an SPT phase in the group cohomology approach by starting with the NLSM description of that SPT phase [75]. Additional applications of NLSMs to the study of SPT phases with translation symmetry and to exotic quantum phase transitions in Weyl semimetals were considered recently in Refs. [76,77]. However, despite the many successes of the NLSM description, deriving the electromagnetic response of a bosonic SPT phase *directly* from its NLSM description remains a difficult problem. In the few instances in which the response of an SPT phase has been determined from its NLSM description, it has been by an indirect method such as an appeal to gauge invariance of the final effective action [78], a dual vortex description of the theory [29], or a description of the NLSM involving auxiliary fermions which also carry charge of the external field A [16,79]. The descriptions in terms of auxiliary fermions are in turn based on a set of formulas due to Abanov and Wiegmann [80] which allow one to generate an O$(d+2)$ NLSM with theta term by coupling the NLSM field to a set of auxiliary fermions and then integrating out those fermions.

In this paper, we overcome this difficulty and give a direct computation of the response of higher-dimensional generalizations of BIQH and BTI states in all dimensions from their NLSM description. To do this, we use a two-pronged approach. First, instead of focusing on the bulk of the SPT phase, we study the boundary and, in particular, the behavior of the gauged boundary theory. In the case of the BIQH state we find that the boundary has a perturbative U(1) anomaly, which we explicitly calculate. Since the CS action changes by a boundary term under a gauge transformation, requiring the entire system (bulk plus boundary) to be gauge invariant allows us to determine the bulk response coefficient N_{2m-1} from the boundary anomaly. In the BTI case we show that the boundary exhibits a quantum Hall response when the associated discrete symmetry (e.g., time reversal in four dimensions) of the BTI state is broken. Again, from this boundary response we can directly read off the coefficient Θ_{2m} using the fact that for a system with boundary, the action $S_{\text{CC}}[A]$ is equivalent to a CS action with level $\frac{\Theta_{2m}}{2\pi}$ on the boundary of the BTI.

To study the boundary theory coupled to the external field electromagnetic A we use the method of gauged WZ actions [39,40,42–44] (see also Refs. [81,82] for some recent applications of gauged WZ actions in condensed matter physics). This machinery can be applied to this problem since, in the NLSM description, the boundary of an SPT phase in $d+1$ dimensions is described by an O$(d+2)$ NLSM with WZ term. Therefore, we require knowledge of the proper way to gauge a WZ action in order to gauge the boundary theory of the SPT phase. For readers who are familiar with gauged WZ actions, it is also worth remarking that all terms in the gauged actions we write (with the sole exception of the original un-gauged WZ term) are expressed as integrals of fields only over the physical boundary space-time. That is, we do not assume an extension of the external field A into the auxiliary direction of space-time which is used to write the WZ term. This is to be contrasted with the general approach of Ref. [44], in which all terms in the gauged action are written as integrals over the extended space-time, and an analog of the method used to obtain the Chern-Simons form from the Chern character must then be used to reduce the terms in the action to integrals only over the physical space-time. This difficulty usually prevents one from writing an explicit local (i.e., not involving integrals over the extended space-time) form for the gauged WZ action in any dimension. We emphasize that here we do not encounter this difficulty. For the BIQH and BTI systems that we study, we give explicit local expressions for the gauged boundary action in all dimensions.

In Sec. IV we use this method to derive the unusual result that for BIQH states in $2m-1$ dimensions the level of the CS term in the effective action for A is quantized as

$$N_{2m-1} = (m!)k, \quad k \in \mathbb{Z} \tag{2.3}$$

where $m!$ denotes the factorial of m. This general formula agrees with existing results for the cases of three [18,19,25] and five [79] dimensions ($m=2$ and 3, respectively), and gives a prediction for all higher odd dimensions. In this case we also provide an alternative derivation of the value of N_{2m-1} using only the NLSM description of the *bulk* of the BIQH state, which confirms our result derived using the anomaly of the boundary theory.

Next, in Sec. V we show that the BIQH response computed in Sec. IV can be understood by requiring the exponential of the CS response action for the BIQH state to be invariant under large U(1) gauge transformations when the response theory is formulated on general closed, compact Euclidean manifolds. Furthermore, we apply these gauge-invariance arguments to study the electromagnetic *and* gravitational responses of fermionic SPT phases with U(1) symmetry in odd dimensions, and point out the distinctive features between the bosonic and fermionic cases.

Moving on to the BTI case, we show in Sec. VI, using the NLSM description of the BTI phase, that the nontrivial BTI state in $2m$ dimensions is characterized by a coefficient

$$\Theta_{2m} = 2\pi\left(\frac{m!}{2}\right). \tag{2.4}$$

Again, this general formula agrees with the known answer in four dimensions [29,30,61,62] ($m=2$) and gives a prediction for all higher even dimensions. It also suggests that the period of the parameter Θ_{2m} is $2\pi(m!)$ for BTI states in $2m$ dimensions.

In Sec. VII we use the gauged boundary actions derived in Secs. IV and VI to derive several other interesting results. First, we construct an effective theory for a bosonic analog of a two-node Weyl (or chiral) semimetal in all even dimensions d using two copies of the boundary action for the BIQH state. We refer to this state as a bosonic chiral semimetal (BCSM). The theory that we construct has an electromagnetic response of the form ($\mathbb{R}^{d-1,1}$ is d-dimensional Minkowski space-time)

$$S^{(b)}_{\text{eff}}[A,B] = -2\left(\frac{d}{2}+1\right)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1,1}} B\wedge A\wedge(dA)^{\frac{d}{2}-1}, \tag{2.5}$$

where $B = B_\mu dx^\mu$ is a one-form whose components B_μ represent the separation in energy and momentum of the two copies of the BIQH boundary theory (in the fermionic case the components of B_μ specify the separation in energy and momentum of the two Weyl cones). This response is larger than the response of the fermionic chiral semimetal in the same dimension by a factor of $(\frac{d}{2}+1)!$. This factor turns out to be identical to the factor of $m!$ discussed earlier for the BIQH state since our semimetal theory in d dimensions is constructed from two copies of the boundary theory for the BIQH state in $d+1 = 2m-1$ dimensions. Next, we show that the boundary theory of the BTI exhibits a bosonic analog of the parity anomaly of a single Dirac fermion in odd dimensions [45–49]. This parity anomaly is essentially the statement that although the boundary theory of the BTI is gauge invariant and possesses the $\mathbb{Z}_2$ symmetry of the BTI state, the $\mathbb{Z}_2$ symmetry can be *spontaneously* broken at the boundary of the BTI, resulting in a half-quantized BIQH response on the boundary. This anomaly then provides strong evidence that the boundary theory of the BTI (with the symmetries of the BTI phase) cannot be realized intrinsically in $2m-1$ dimensions. Finally, we analyze the physics of symmetry-breaking domain walls on the boundary of the BTI state, and we show that the physics of such domain walls provides a nice example of the phenomenon of *anomaly inflow* [83] in bosonic SPT phases.

The Appendixes of the paper contain several additional results, most of a more mathematical nature. In Appendix A we use the well-known connection between gauged WZ actions and equivariant cohomology to understand the mathematical structure of the gauged WZ actions that we construct for the boundaries of BIQH and BTI states. In particular, we show that the construction of these actions is related to the mathematical problem of constructing an *equivariant extension* of the volume form for the sphere S^{2m-1} (in the BIQH case) or S^{2m} in the BTI case, and we study this mathematical problem in detail. In Appendix B we show an example of the computation of the Chern character for the field strength F on the complex projective space $C\mathbb{P}^m$. This example serves to illustrate the necessity of the peculiar quantization of the CS level required for gauge invariance of the CS term on general manifolds as derived in Sec. V. In Appendix C we discuss a dimensional reduction procedure which allows one to obtain the response action for the BTI phase from the response action for the BIQH phase in one higher dimension. In Appendix D we derive a general dimensional reduction formula for topological theta terms in NLSMs. Finally, in Appendix E we compute the electromagnetic response of the O(2) NLSM in one dimension.

III. BACKGROUND

In this section we introduce the relevant background material necessary for understanding the later sections of the paper. We start with a brief review of the physics of the BIQH and BTI states, and also present definitions of higher-dimensional generalizations of these states. We then review the NLSM description of the bulk and boundary of bosonic SPT phases, and discuss the specifics of the NLSM descriptions of the BIQH and BTI states that we study in this paper. Finally, we give a general discussion of the tool of gauged WZ actions, and we describe in concrete terms the procedure that we use in this paper to construct gauged WZ actions for the boundaries of BIQH and BTI states.

A. BIQH and BTI phases

In its original formulation [18,19], the BIQH phase was conceived of as a gapped quantum phase of bosons in three space-time dimensions which exhibits a nonzero Hall conductance, but does not have any bulk topological order. As an SPT phase it is protected by only charge-conservation symmetry, i.e., we have $G = \mathrm{U}(1)$ where G is the symmetry group of the SPT phase. Physically, the BIQH state is characterized by a CS term in the effective action for the external field A,

$$S_{\text{eff}}[A] = \frac{N_3}{4\pi}\int_{\mathcal{M}} A \wedge dA, \tag{3.1}$$

in which the coefficient N_3 (which is just the Hall conductance in units of $\frac{e^2}{h}$) is quantized in integer multiples of 2. The authors of Ref. [18] gave a very appealing physical argument for why the value of $N_3 = 1$ is not allowed if the BIQH state is required to have no fractionalized excitations, and we now briefly review their argument. Consider a hypothetical BIQH state on flat space, and a configuration of A in which a thin tube of 2π flux pierces the spatial surface. According to the action $S_{\text{eff}}[A]$, the point in space where the flux tube pierces the plane will bind a charge equal to N_3. Now, one invokes a standard argument[1] that 2π flux is gauge equivalent to zero flux, and so the pointlike excitation created by threading the flux is in fact an excitation of the BIQH fluid and not an external defect. One can therefore ask about the phase obtained by the wave function of the system after a process in which two such excitations are exchanged. By the Aharanov-Bohm effect, taking one excitation completely around another results in a statistical phase of $2\pi N_3$. The phase for an exchange process is therefore half of that, $\vartheta_{ex} = \pi N_3$. From this result the authors of Ref. [18] concluded that the state described by the effective action of Eq. (3.1) contains a fermionic excitation if N_3 is odd, and so N_3 must be an even integer in order for the action of Eq. (3.1) to represent the electromagnetic response of a BIQH phase.

In this paper, we consider generalizations of the BIQH state to all odd space-time dimensions. One definition of a BIQH state in $2m-1$ dimensions which is sufficient for our purposes is that a BIQH state is an SPT phase of bosons which is protected by the symmetry group $G = \mathrm{U}(1)$, where U(1) is charge-conservation symmetry, and which exhibits a CS response to an applied electromagnetic field A of the form of Eq. (2.1). We should also mention here that in odd dimensions there is a countable infinity of different BIQH states, i.e., these states have a $\mathbb{Z}$ classification [6,34]. This means that the coefficient N_{2m-1} takes on a countable infinity of values which all have the form of some particular number times an integer.

On the other hand, the BTI phase [6,29,32] is a bosonic analog of the time-reversal invariant electron topological insulator in four space-time dimensions. As an SPT phase it is protected by the symmetry group $G = \mathrm{U}(1) \rtimes \mathbb{Z}_2^T$, where U(1) represents charge conservation and $\mathbb{Z}_2^T$ is time-reversal symmetry. If we write $\mathbb{Z}_2^T = (1,\mathcal{T})$ where $\mathcal{T}$ is the time-reversal operator, then we have $\mathcal{T}^2 = 1$ for the BTI. This should be contrasted with the relation $\mathcal{T}^2 = (-1)^F$ which holds for the electron topological insulator, where F is the fermion number. The semidirect product "$\rtimes$" indicates that the U(1) and $\mathbb{Z}_2^T$ symmetries do not commute with each other. In the next subsection we will see an explicit representation of the action of the group $\mathrm{U}(1) \rtimes \mathbb{Z}_2^T$ on the fields in the NLSM description of the BTI.

The bulk of the BTI phase is characterized by an effective action for A of the Chern character form

$$S_{\text{eff}}[A] = \frac{\Theta_4}{8\pi^2}\int_{\mathcal{M}} F \wedge F, \tag{3.2}$$

where $F = dA$, and $\Theta_4 = 2\pi$ for the BTI (compare with $\Theta_4 = \pi$ for the electron topological insulator [60]). The parameter Θ_4 has 2π periodicity in the case of the electron topological insulator [60] but 4π periodicity in the BTI case [29,30]. One way to understand this effective action is to consider what happens when the space-time $\mathcal{M}$ has a boundary $\partial\mathcal{M}$. In this

[1] In fact, this statement is only true on a lattice when we can couple to a compact U(1) gauge field, or in the continuum when the level of the CS term is an integer. To see what can go wrong, consider $N_3 = \frac{1}{q}$ for $q \in \mathbb{Z}$. Then, the object created by threading a thin 2π flux tube has charge $\frac{1}{q}$ and so only q such fluxes are a physical excitation of the system, in the sense that all states in the physical Hilbert space of the quantum mechanical system should have integer charge.

case, if the bulk field configuration F is topologically trivial, then we can write $F \wedge F = d(A \wedge dA)$ to find

$$S_{\mathrm{eff}}[A] = \frac{\Theta_4}{2\pi}\frac{1}{4\pi}\int_{\partial\mathcal{M}} A \wedge dA, \tag{3.3}$$

which is equivalent to a quantum Hall state with Hall conductance $\sigma_H = \frac{\Theta_4}{2\pi}$ on the boundary of $\mathcal{M}$. In particular, for the BTI we have $\Theta_4 = 2\pi$ so that the surface of the BTI exhibits a *half-quantized* BIQH effect (i.e., $\sigma_H = 1$ on the surface). Such a surface quantum Hall response breaks the time-reversal symmetry of the BTI.

Now, we turn to the question of how to generalize the BTI state to all even dimensions. The main issue with generalizing the BTI state to all even dimensions is that the discrete part of the symmetry group G, which was antiunitary time-reversal symmetry $\mathbb{Z}_2^T$ in four dimensions, should be chosen differently when the space-time dimension is equal to zero or two modulo four. Whenever the space-time dimension is equal to zero modulo four, we choose the discrete part of G to be antiunitary time-reversal symmetry $\mathbb{Z}_2^T$. On the other hand, whenever the space-time dimension is equal to two modulo four, we choose the discrete part of G to be *unitary* charge-conjugation (or particle-hole) symmetry $\mathbb{Z}_2^C$. This choice is consistent with the results of the group cohomology [6] and NLSM [34] classifications of SPT phases in these dimensions, and with the symmetries which protect the fermion topological insulators in two and four space-time dimensions, respectively [60].

We therefore choose to use the following definition of a BTI phase in all even dimensions. A BTI phase in space-time dimension $2m$ is an SPT phase of bosons with symmetry group

$$G = \begin{cases} \mathrm{U}(1) \rtimes \mathbb{Z}_2^T, & m = \text{even} \\ \mathrm{U}(1) \rtimes \mathbb{Z}_2^C, & m = \text{odd} \end{cases} \tag{3.4}$$

and which exhibits a bulk response to an external field A of the form of Eq. (2.2). As we noted earlier, when the space-time $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, and when the field configuration F is topologically trivial, this bulk response is equivalent to a boundary quantum Hall response of the form of Eq. (2.1) with coefficient $N_{2m-1} = \frac{\Theta_{2m}}{2\pi}$. In addition, this boundary quantum Hall response breaks the $\mathbb{Z}_2^T$ symmetry (for m even) or $\mathbb{Z}_2^C$ symmetry (for m odd) of the BTI phase. When we discuss the BTI phase in a general dimension $2m$, and when we do not have a particular m in mind, we just write $\mathbb{Z}_2$ for the discrete part of G. However, the reader should always keep in mind that the $\mathbb{Z}_2$ symmetry is different for the cases of m even and m odd as discussed in this section.

Finally, we also mention that based on the group cohomology [6] and NLSM [34] classification schemes, only the smallest value of Θ_{2m} is expected to represent a nontrivial BTI phase in $2m$ dimensions. This can be understood as follows. For SPT phases with $\mathrm{U}(1) \rtimes \mathbb{Z}_2^T$ symmetry in four dimensions, the group cohomology and NLSM classifications predict a $(\mathbb{Z}_2)^2$ classification. One of these $\mathbb{Z}_2$ factors corresponds to the BTI state, while the other corresponds to a state in which the U(1) symmetry plays no role [30] (so this second state cannot be interpreted as an insulator). This means that there is only a single nontrivial BTI state in four dimensions. In addition, in two dimensions the classification for SPTs with $\mathrm{U}(1) \rtimes \mathbb{Z}_2^C$ symmetry is $\mathbb{Z}_2$, and the U(1) symmetry does play a role in the nontrivial phase, so we identify that phase with the BTI phase in two dimensions. Based on this evidence we expect the existence of a single nontrivial BTI phase to generalize to all even dimensions. In the context of the NLSM classification, this can be understood as coming from the fact that in $2m$ dimensions the $\mathrm{O}(2m+1)$ NLSM theory with $\theta = 2\pi k$ can be smoothly connected to the theory with $\theta = 2\pi(k \pm 2)$ (see, e.g., the discussion in Ref. [34]).

B. NLSM description of the bulk and boundary of SPT phases

We now give a brief review of the NLSM description of SPT states, which was presented in its fully developed form in Ref. [34]. Let us consider bosonic SPT phases in $d+1$ space-time dimensions. The space-time coordinates are x^μ, $\mu = 0,\ldots,d$ ($x^0 = t$ is the time coordinate), and for now we focus on the case of flat Minkowski space-time $\mathbb{R}^{d,1}$ with the mostly minus metric $\eta = \mathrm{diag}(1,-1,\ldots,-1)$. Following the prescription of Ref. [34], a bosonic SPT phase in this dimension is described by an $\mathrm{O}(d+2)$ NLSM with topological theta term where the coefficient of the theta term is given by $\theta = 2\pi k$ with $k \in \mathbb{Z}$. The $\mathrm{O}(d+2)$ NLSM is a theory of a $(d+2)$-component unit vector field $\mathbf{n}$ (i.e., $\mathbf{n}\cdot\mathbf{n} = 1$) with components n_a, $a = 1,\ldots,d+2$. Because of the constraint, the configuration space (or target space) of the NLSM field is the $(d+1)$-dimensional sphere S^{d+1}. Latin indices $a,b,c,\ldots$, which label components of n_a, can be raised and lowered with the Euclidean metrics δ^{ab}, δ_{ab}, and so n^a and n_a are numerically equal to each other. In what follows, we use the summation convention for any indices (Latin or Greek) which appear once in an upper position and once in a lower position in any expression.

The NLSM action describing the SPT phase is

$$S_{\mathrm{bulk}}[\mathbf{n}] = \int d^{d+1}x\, \frac{1}{2g}(\partial^\mu n^a)(\partial_\mu n_a) + S_\theta[\mathbf{n}], \tag{3.5}$$

where $g > 0$ is the coupling constant of the NLSM [with units of $(\text{length})^{d-1}$], and $S_\theta[\mathbf{n}]$ is the theta term. To write the theta term in a compact way we first introduce some notation. Let ω_{d+1} be the volume form on S^{d+1}. Explicitly, we have

$$\omega_{d+1} = \sum_{a=1}^{d+2}(-1)^{a-1} n_a dn_1 \wedge \cdots \wedge \overline{dn_a} \wedge \cdots \wedge dn_{d+2}, \tag{3.6}$$

where the overline means to omit that term from the wedge product. We also use the notation $\mathcal{A}_{d+1} \equiv \mathrm{Area}[S^{d+1}] = \frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})}$ for the area of the sphere S^{d+1}. In terms of these quantities, the theta term can be written compactly in differential form notation as

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{d+1}}\int_{\mathbb{R}^{d,1}} \mathbf{n}^*\omega_{d+1}, \tag{3.7}$$

where $\mathbf{n}^*\omega_{d+1}$ denotes the pullback to space-time of the form ω_{d+1} via the map $\mathbf{n} : \mathbb{R}^{d,1} \to S^{d+1}$. In coordinates this becomes

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{d+1}}\int d^{d+1}x\, \epsilon^{a_1\ldots a_{d+2}} n_{a_1}\partial_{x^0}n_{a_2}\partial_{x^1}n_{a_3}\ldots\partial_{x^d}n_{a_{d+2}}. \tag{3.8}$$

For the description of SPT phases we have $\theta = 2\pi k$ for integer k. The reason for choosing $\theta = 2\pi k$ is that at these values of θ, the NLSM is expected to flow to a disordered ($g \to \infty$) fixed point under the renormalization group [34]. In addition, we note that the full action of Eq. (3.5) (including theta term) has an $\mathrm{SO}(d+2)$ global symmetry, where the action of the group on the NLSM field is given by $n_a \to R_a^b n_b$ for any matrix $R \in \mathrm{SO}(d+2)$. When the coefficient θ is set to zero, this symmetry is promoted to an $\mathrm{O}(d+2)$ global symmetry [under a general transformation $R \in \mathrm{O}(d+2)$ the theta term transforms only by acquiring the sign $\det[R] = \pm 1$]. The fixed-point theory (with $g \to \infty$ at $\theta = 2\pi k$) is gapped and has a unique ground state which does not break the $\mathrm{SO}(d+2)$ symmetry of the NLSM with theta term [33]. This property of the disordered ground state of the NLSM at $\theta = 2\pi k$ is one of the main reasons why these field theories are useful for describing SPT phases.

SPT phases are classified according to their symmetry group G. In the NLSM description of Ref. [34] this symmetry is encoded in a homomorphism $\sigma : G \to \mathrm{O}(d+2)$, which maps $g \in G$ to some $(d+2) \times (d+2)$ matrix $\sigma(g) \in \mathrm{O}(d+2)$. We refer to such a σ as a *symmetry assignment*. According to the NLSM classification of SPT phases, if $g \in G$ represents an *internal* unitary symmetry operation (i.e., g does not have any action on the space-time coordinates) then σ should be chosen so that $\det[\sigma(g)] = 1$. In this case it is then clear that the action of g leaves the theta term invariant. On the other hand, if $g \in G$ represents the time-reversal operation, then σ should be chosen so that $\det[\sigma(g)] = -1$. Since the time-reversal operation also sends $t \to -t$ (in addition to its action on the components of the NLSM field), the minus sign in the theta term from $\det[\sigma(g)]$ will be canceled by the minus sign from sending $\partial_t \to -\partial_t$. Thus, choosing $\det[\sigma(g)] = -1$ in this case ensures that the theta term is invariant under the time-reversal transformation.

Not all NLSMs with a symmetry assignment will describe a nontrivial SPT phase. For example, an NLSM with a symmetry assignment σ will describe a trivial phase if there exists a vector $\mathbf{v}$ such that $\sigma(g)\mathbf{v} = \mathbf{v}\ \forall\ g \in G$. This is because in this case we are allowed to add a term $\mathbf{n} \cdot \mathbf{v}$ to the NLSM action without breaking the symmetry of the group G. Such a term will then drive the system into a trivial phase in which $\mathbf{n}$ is parallel or antiparallel to $\mathbf{v}$ at all points in space. If a vector $\mathbf{v}$ with this property does not exist, then the NLSM with symmetry assignment σ can describe a nontrivial SPT phase.

When an SPT phase has a bulk description in terms of an $\mathrm{O}(d+2)$ NLSM with theta term and theta angle $\theta = 2\pi k$, its d-dimensional boundary is described by an $\mathrm{O}(d+2)$ NLSM with Wess-Zumino (WZ) term at level k. Let us for simplicity study the boundary perpendicular to the x^d direction, so on the boundary we have coordinates x^μ, $\mu = 0,\dots,d-1$, and the boundary space-time is $\mathbb{R}^{d-1,1}$. To construct the WZ term we need to extend the field configuration n_a into a fictitious extra dimension of the boundary space-time. We take $s \in [0,1]$ to be the coordinate for this extra direction, and define $\mathcal{B} = [0,1] \times \mathbb{R}^{d-1,1}$ to be the extended boundary space-time. Let $\tilde{n}_a(x^\mu,s)$ be an extension of the field n_a into the s direction. It is typical to choose boundary conditions in the extra direction so that $\tilde{n}_a(x^\mu,1) = \delta_{a,1}$ (i.e., a trivial configuration) and $\tilde{n}_a(x^\mu,0) = n_a(x^\mu)$ so that the physical boundary space-time is located at $s = 0$. Then, the action for the boundary theory takes the form

$$S_{\mathrm{bdy}}[\mathbf{n}] = \int d^d x\ \frac{1}{2g_{\mathrm{bdy}}}(\partial^\mu n^a)(\partial_\mu n_a) + S_{\mathrm{WZ}}[\mathbf{n}], \tag{3.9}$$

where the WZ term is

$$S_{\mathrm{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{d+1}} \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_{d+1}. \tag{3.10}$$

Here, g_{bdy} is the coupling constant for the boundary theory, and the WZ term now involves the pullback of ω_{d+1} to $\mathcal{B}$ (the extended boundary space-time) via the map $\tilde{\mathbf{n}} : \mathcal{B} \to S^{d+1}$. Again, in coordinates this takes the form

$$S_{\mathrm{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{d+1}} \int_0^1 ds \int d^d x\, \epsilon^{a_1 \dots a_{d+2}} \tilde{n}_{a_1} \times \partial_s \tilde{n}_{a_2} \partial_{x^0} \tilde{n}_{a_3} \dots \partial_{x^{d-1}} \tilde{n}_{a_{d+2}}. \tag{3.11}$$

We now discuss the specific symmetry assignments $\sigma : G \to \mathrm{O}(d+2)$ which will be used to construct NLSM descriptions of BIQH states in odd space-time dimensions and BTI states in even space-time dimensions. We start with the case of BIQH states in $2m-1$ space-time dimensions. In this case, the integer m is related to d by the relation $2m = d+2$, and the BIQH state is described by an $\mathrm{O}(2m)$ NLSM with theta term. In the BIQH case, the symmetry group is just $G = \mathrm{U}(1)$ and the particular $\mathrm{U}(1)$ symmetry that we are interested in is embedded in the full $\mathrm{O}(2m)$ group as follows. We first combine pairs of the $2m$ components n_a of the NLSM field to create the m boson fields

$$b_\ell = n_{2\ell-1} + i n_{2\ell}, \quad \ell = 1,\dots,m. \tag{3.12}$$

Then, the $\mathrm{U}(1)$ symmetry we consider acts on the NLSM field as

$$\mathrm{U}(1) : b_\ell \to e^{i\xi} b_\ell,\ \forall\ \ell, \tag{3.13}$$

where ξ is a constant parameter. We can consider the fields b_ℓ to be m complex scalar fields of charge 1, but subject to the constraint $\sum_{\ell=1}^m |b_\ell|^2 = 1$, which is equivalent to the constraint $\mathbf{n} \cdot \mathbf{n} = 1$ for the NLSM field n_a. This choice of $\mathrm{U}(1)$ transformation, and the corresponding pairing of the components of $\mathbf{n}$ into the bosons b_ℓ, is convenient, but it is not unique. Since the NLSM action with theta term (or WZ term) is still invariant under the group $\mathrm{SO}(2m)$, we can do any change of basis $n^a \to M_a^b n_b$ with $M \in \mathrm{SO}(2m)$ to obtain a theory with a different action of the $\mathrm{U}(1)$ symmetry, but with the same physical properties. As discussed above, the most important property of the symmetry assignment is that there should not be any vector $\mathbf{v}$ that remains fixed under the $\mathrm{U}(1)$ action. Indeed, if such a $\mathbf{v}$ exists, then the NLSM with this symmetry assignment describes a trivial phase. The choice above satisfies this requirement.

For the case of BTI states in even dimensions $2m$, the integer m is instead related to d by the formula $2m+1 = d+2$, so that these states are described by $\mathrm{O}(2m+1)$ NLSMs with theta term. As we discussed in the previous subsection the symmetry group in this case is $G = \mathrm{U}(1) \rtimes \mathbb{Z}_2^T$ for m even and $G = \mathrm{U}(1) \rtimes \mathbb{Z}_2^C$ for m odd. To define the symmetry assignment σ in this case we again take pairs of the first $2m$ components of the NLSM field and combine them into bosons b_ℓ, $\ell = 1,\dots,m$, as done for the BIQH case. The $\mathrm{U}(1)$

symmetry we consider again acts as in Eq. (3.13) on these bosons, but leaves the final component n_{2m+1} of the NLSM field fixed. Finally, in the BTI case the additional discrete $\mathbb{Z}_2$ symmetry (which is either $\mathbb{Z}_2^T$ or $\mathbb{Z}_2^C$ depending on the parity of m) is taken to act on the NLSM field as

$$\mathbb{Z}_2 : n_a \to n_a, \quad a = 1,3,\dots,2m-1 \tag{3.14a}$$

$$n_a \to -n_a, \quad a = 2,4,\dots,2m,2m+1. \tag{3.14b}$$

In the case where the $\mathbb{Z}_2$ symmetry is time reversal $\mathbb{Z}_2^T$, we also need to send $t \to -t$ in the argument of n_a and in the action. Under the transformation in Eq. (3.14), the theta term of the NLSM picks up the sign $(-1)^{m+1}$. So, we see that for m odd the theta term in the NLSM automatically has this symmetry, while in the case of m even it must be supplemented with the replacement $t \to -t$, which gives an extra minus sign in the theta term. So, the NLSM has the internal, unitary $\mathbb{Z}_2^C$ particle-hole symmetry in the case of m odd, while in the case of m even it has the antiunitary time-reversal symmetry $\mathbb{Z}_2^T$.

Now that we know how the fields in the NLSM description transform under the U(1) symmetry of the BIQH and BTI phases, we can considering coupling the NLSM theory, and in particular the boundary theory which involves a WZ term, to the external electromagnetic field $A = A_\mu dx^\mu$. In order to do this, we are going to need the tool of gauged WZ actions.

C. Gauged Wess-Zumino actions

We now give a discussion of the theory of gauged WZ actions, mostly focusing on the general philosophy behind the construction of a gauged WZ action. The details of this construction will be worked out explicitly for the boundary theories of the BIQH and BTI phases in all dimensions in later sections of this paper. In addition, in Appendix A we review the relation between gauged WZ actions and equivariant cohomology, and we reexamine the gauged WZ actions constructed in this paper from this more mathematical point of view.

Before we start, let us note that the kinetic term for the NLSM is easily gauged using ordinary minimal coupling (also known as a "Peierls substitution" in a condensed matter context). In fact, the gauged kinetic term is most simply written in terms of the b_ℓ as

$$S_{\text{kin,gauged}}[\mathbf{n},A] = \int d^dx \, \frac{1}{2g_{\text{bdy}}} \sum_{\ell=1}^{m} (D^\mu b_\ell)^*(D_\mu b_\ell) \tag{3.15}$$

for the boundary of the BIQH state ($d+2=2m$), or

$$S_{\text{kin,gauged}}[\mathbf{n},A] = \int d^dx \, \frac{1}{2g_{\text{bdy}}} \Bigg[\sum_{\ell=1}^{m} (D^\mu b_\ell)^*(D_\mu b_\ell) + (\partial^\mu n_{2m+1})(\partial_\mu n_{2m+1}) \Bigg] \tag{3.16}$$

for the boundary of a BTI state ($d+2=2m+1$), where $D_\mu = \partial_\mu - iA_\mu$ is the usual covariant derivative. Note here that since we are only interested in enforcing a U(1) subroup of the full SO$(d+2)$ symmetry group of the NLSM, we could allow a different boundary coupling constant $g_{bdy,\ell}$ for each species b_ℓ of boson. This type of anisotropy in the coupling constant will not affect the results in the rest of the paper since those results only depend on the form of the WZ term.

Gauging the WZ term is more subtle. The main problem we face in attempting to gauge this term is the fact that the WZ term is written as an integral of an expression involving the field $\tilde{n}_a$ over the $(d+1)$-dimensional extended space-time $\mathcal{B}$. One method [44] for gauging a WZ term involves defining an extension $\tilde{A}$ of the gauge field A into the extra s direction, and then applying the usual minimal coupling procedure (but using the extended field $\tilde{A}$) inside the WZ term. This has the effect of replacing the integrand $\tilde{\mathbf{n}}^*\omega_{d+1}$ of the WZ term in Eq. (3.10) with $\tilde{\mathbf{n}}^*\omega_{d+1}^{\tilde{A}}$, where $\omega_{d+1}^{\tilde{A}}$ represents the volume form on S^{d+1} but with the ordinary exterior derivative d replaced with a gauge-covariant exterior derivative D (the precise form of D is not important for the general discussion here). However, minimal coupling alone is not sufficient, as varying the minimally coupled WZ action does not lead to d-dimensional equations of motion, i.e., the resulting equations of motion depend on the extensions $\tilde{n}_a$ and $\tilde{A}$. To remedy this, the authors of Ref. [44] used the following prescription. They suggested that one should add a second term $U(\tilde{n}_a,\tilde{A})$ to the integrand of the WZ term such that the combination $\omega_{d+1}^{\tilde{A}} + U$ is a closed form on the extended space-time. Since a closed form is locally exact (i.e., a closed form ω can be written as $\omega = d\gamma_i$ for some γ_i on each coordinate patch $\mathcal{U}_i$ of the manifold), variation of this new WZ term leads to d-dimensional equations of motion on each coordinate patch of the original space-time manifold. There is, however, one conceptual issue with this method, which the authors of Ref. [44] point out [see their discussion in the paragraph after Eq. (4.7)]. The problem is that in the usual setup of the WZ term, the form $\omega_{d+1}^{\tilde{A}}$ (and also ω_{d+1}) is a $(d+1)$ form on the $(d+1)$-dimensional extended space-time $\mathcal{B}$, and so it is trivially closed. Therefore, in order to apply the method of Ref. [44] one has to imagine that the extended space-time $\mathcal{B}$ is embedded in a space-time $\mathcal{X}$ of even higher dimension so that $d\omega_{d+1}^{\tilde{A}}$ is not trivially equal to zero.

From this discussion it is clear that gauging a WZ is in general a difficult procedure. However, for the problems encountered in this paper, in which we only deal with a U(1) subgroup of the full O$(d+2)$ symmetry of the NLSM theories, we do not need the complicated machinery developed in Ref. [44]. Instead, we use the following concrete procedure (which is similar in spirit to the methods used in Refs. [42,43]) to gauge the U(1) symmetry of our theories. First, we consider how the WZ term changes under the transformation $b_\ell \to e^{i\xi}b_\ell$ (with a space-time dependent ξ). We will see that it changes by a term which is a total derivative, which means that the change of the WZ term can be written as an integral only over the physical boundary space-time $\mathbb{R}^{d-1,1}$ instead of over the extended space-time $\mathcal{B}$. Next, we attempt to cancel this change in the action by adding an integral over space-time of the NLSM field coupled to A. We will see that this procedure usually needs to be iterated several times because the counterterms that we add to the action may not transform nicely under a gauge transformation, where "nicely" is defined below by Eq. (3.17). We use the following criterion, inspired by the discussion in Ref. [43], for determining when the action has been properly gauged.

Gauging principle. The correctly gauged action $S_{\text{gauged}}[\mathbf{n},A]$, if it is not completely gauge invariant, must transform under a gauge transformation $b_\ell \to e^{i\xi}b_\ell$, $A \to A + d\xi$, as

$$S_{\text{gauged}}[\mathbf{n},A] \to S_{\text{gauged}}[\mathbf{n},A] + \delta_\xi S_{\text{gauged}}[A,\xi], \tag{3.17}$$

where we have used the notation $\delta_\xi S_{\text{gauged}}$ to indicate the change in S_{gauged} under a gauge transformation. The key point here is that the change in the action under a gauge transformation depends only on A and ξ, but not on the matter field $\mathbf{n}$.

Let us also note here that in this paper we use the word "anomaly" to refer to the change in the action (or action plus path-integral measure) under a U(1) gauge transformation. There is no anomaly if the action (plus path-integral measure) is gauge invariant. The gauging principle stated above then simply asserts that the anomaly $\delta_\xi S_{\text{gauged}}[A,\xi]$ of the gauged action $S_{\text{gauged}}[\mathbf{n},A]$ should only depend on A and ξ.

We will see in the following sections that we may need to add several counterterms to the WZ action to get Eq. (3.17) to hold. In the BIQH case, the correctly gauged action still transforms under a gauge transformation, and so the U(1) symmetry of the boundary theory of the BIQH phase is anomalous. This fact is what allows us to deduce the bulk CS response of the BIQH state. On the other hand, for the surface of the BTI it is possible to construct a completely gauge-invariant action. However, from the form of the gauge-invariant action we will be able to see that if the NLSM field condenses in a way that preserves the U(1) symmetry, but breaks the $\mathbb{Z}_2$ symmetry of the BTI phase, then the surface of the BTI will exhibit a $\mathbb{Z}_2$ symmetry-breaking quantum Hall response.

IV. ELECTROMAGNETIC RESPONSE OF BIQH STATES IN ALL ODD DIMENSIONS

In this section, we construct the gauged WZ action for the boundary of BIQH states in all odd dimensions. The action we construct satisfies the gauging principle of Eq. (3.17), but is still not completely gauge invariant, as evidenced in the U(1) anomaly of the boundary theory of the BIQH state. We then use the U(1) anomaly of the gauged boundary action to calculate the bulk CS response of the BIQH state in all odd dimensions. As we discussed in the Introduction, we find that for the BIQH state in $2m-1$ dimensions the level N_{2m-1} of the CS term appearing in the effective action is quantized in units of $m!$. We then give a more intuitive derivation of the BIQH response using only the dimensional reduction properties of CS terms and of theta terms in NLSMs. This second derivation relies on results which we derive in Appendixes D and E. This intuitive picture confirms our more technical derivation using gauged WZ actions.

The result in this section is related to the results of several other sections of this paper. In Sec. V, we show that the factor of $m!$ for the CS response of the BIQH state computed in this section can be understood by requiring that partition functions containing the CS response action be invariant under large U(1) gauge transformations on general Euclidean manifolds. Later, in Appendix A, we reexamine the gauged WZ action constructed in this section in light of the well-known connection between gauged WZ actions and equivariant cohomology of the target space of the NLSM. The construction of a gauged WZ action for the boundary of the BIQH state is equivalent to the problem of constructing an *equivariant extension* [with respect to the U(1) symmetry] of the volume form ω_{2m-1} for S^{2m-1}. In Appendix A we attempt to construct such an extension, and then show that the construction fails at the last step. The fact that such an extension does not exist is mathematically equivalent to our finding that the gauged action for the boundary of the BIQH state still has a U(1) anomaly. In Appendix A, we also show that the differential forms $\Omega^{(r)}$, which appear later in this section in the counterterms of Eq. (4.29), are the same forms which appear in the construction of the equivariant extension of ω_{2m-1} (although the extension fails at the last step in this case as mentioned above).

Let us make a few remarks on the notation used in this section and in later sections of the paper. In what follows, we omit the pullback symbol $\mathbf{n}^*$ so as not to clutter the notation, but one should always remember that the integrand of any integral should be pulled back to space-time (or the extended space-time, in which case one would write $\tilde{\mathbf{n}}^*$). In addition, we will express many quantities in terms of the integer m instead of d. Recall that these are related by $2m = d + 2$ in the BIQH case. So, for example, we write the WZ term as

$$S_{\text{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_{\mathcal{B}} \omega_{2m-1}. \tag{4.1}$$

For later use we also define several differential forms which are constructed from the components of the NLSM field. We define the one-form $\mathcal{J}_\ell$ and two-form $\mathcal{K}_\ell$ by

$$\mathcal{J}_\ell = n_{2\ell-1}dn_{2\ell} - n_{2\ell}dn_{2\ell-1}, \tag{4.2a}$$

$$\mathcal{K}_\ell = dn_{2\ell-1} \wedge dn_{2\ell}. \tag{4.2b}$$

Under a gauge transformation $b_\ell \to e^{i\xi}b_\ell$ these forms transform as

$$\mathcal{J}_\ell \to \mathcal{J}_\ell + \left(n_{2\ell-1}^2 + n_{2\ell}^2\right)d\xi, \tag{4.3a}$$

$$\mathcal{K}_\ell \to \mathcal{K}_\ell + (n_{2\ell-1}dn_{2\ell-1} + n_{2\ell}dn_{2\ell}) \wedge d\xi. \tag{4.3b}$$

We also note here that

$$\mathcal{K}_\ell = \tfrac{1}{2} d\mathcal{J}_\ell \tag{4.4}$$

and so

$$d\mathcal{K}_\ell = 0, \tag{4.5}$$

i.e., $\mathcal{K}_\ell$ is an *exact* differential form.

A. O(4) NLSM with WZ term in two space-time dimensions

Before presenting the gauged action for any integer m, we warm up with an explicit calculation for the simplest possible case, which is the O(4) NLSM with WZ term which appears at the two-dimensional boundary of the BIQH state in three dimensions. We also mention here that an O(4) NLSM with WZ term in two dimensions is equivalent to a model of an SU(2) matrix field $U = n_4\mathbb{I} + \sum_{a=1}^{3} n_a\sigma^a$ (where σ^a are the three Pauli matrices) with WZ term for U, so the analysis in this section is actually a special case of the analysis done in

Refs. [42,43]. Although we focus on the case of a continuous symmetry [namely, the U(1) charge-conservation symmetry], we also note here that anomalies in the two-dimensional boundary theories of SPT phases protected by the symmetry of a *finite* Abelian group were considered previously in Ref. [84].

In the O(4) case the volume form can be written as

$$\omega_3 = \mathcal{J}_1 \wedge \mathcal{K}_2 + \mathcal{J}_2 \wedge \mathcal{K}_1. \tag{4.6}$$

Under the transformation $b_\ell \to e^{i\xi} b_\ell$ we have

$$\begin{aligned}\delta_\xi \omega_3 &= \mathcal{K}_1 \wedge d\xi + \mathcal{K}_2 \wedge d\xi \\ &= \tfrac{1}{2} d\mathcal{J}_1 \wedge d\xi + \tfrac{1}{2} d\mathcal{J}_2 \wedge d\xi \\ &= \tfrac{1}{2} d[\mathcal{J}_1 \wedge d\xi + \mathcal{J}_2 \wedge d\xi],\end{aligned} \tag{4.7}$$

which is a total derivative. So, we find (neglecting any terms coming from the boundary of the physical space-time $\mathbb{R}^{1,1}$)

$$\delta_\xi S_{\mathrm{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_3} \frac{1}{2} \int_{\mathbb{R}^{1,1}} (\mathcal{J}_1 + \mathcal{J}_2) \wedge d\xi. \tag{4.8}$$

We attempt to cancel this variation by adding the counterterm

$$S^{(1)}_{ct}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_3} \frac{1}{2} \int_{\mathbb{R}^{1,1}} (\mathcal{J}_1 + \mathcal{J}_2) \wedge A. \tag{4.9}$$

It is clear that when we send $A \to A + d\xi$ in $S^{(1)}_{ct}$ it will cancel the gauge variation of the WZ term.

At this point, our candidate for the gauged WZ term is then

$$S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = S_{\mathrm{WZ}}[\mathbf{n}] + S^{(1)}_{ct}[\mathbf{n}, A]. \tag{4.10}$$

However, this action is not completely gauge invariant, and under a gauge transformation we find

$$\begin{aligned}\delta_\xi S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] &= -\frac{2\pi k}{\mathcal{A}_3} \frac{1}{2} \int_{\mathbb{R}^{1,1}} (\delta_\xi \mathcal{J}_1 + \delta_\xi \mathcal{J}_2) \wedge A \\ &= -\frac{2\pi k}{\mathcal{A}_3} \frac{1}{2} \int_{\mathbb{R}^{1,1}} d\xi \wedge A \\ &= -\frac{k}{2\pi} \int_{\mathbb{R}^{1,1}} d\xi \wedge A \\ &= k \int_{\mathbb{R}^{1,1}} \xi \left(\frac{F}{2\pi}\right),\end{aligned} \tag{4.11}$$

where we used the formula for $\delta_\xi \mathcal{J}_\ell$ from Eq. (4.3), the fact that $\mathbf{n}$ is a unit vector field, $\mathcal{A}_3 = 2\pi^2$, and also performed an integration by parts in the last line ($F = dA$). We conclude that the U(1) symmetry here is anomalous and, since the kinetic term has been made completely gauge invariant, the total anomaly of the boundary theory is given by Eq. (4.11). We also note that the anomaly in Eq. (4.11) is exactly what is needed to cancel the gauge variation of the bulk CS action of Eq. (3.1) with $N_3 = -2k$.

B. The O(2m) NLSM with WZ term in 2m − 2 space-time dimensions

Now, we move on to the general case of an O($2m$) NLSM with WZ term on the ($2m - 2$)-dimensional boundary of a BIQH state in $2m - 1$ dimensions (recall that m is related to the integer d in the BIQH case by $d = 2m - 2$, so that d is also the dimension of the boundary space-time). In this case, we find that a total of $m - 1$ counterterms are needed in order for the gauged WZ action to transform as in Eq. (3.17) under a gauge transformation. To start, we note that the volume form ω_{2m-1} can be rewritten using the forms $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$ as

$$\omega_{2m-1} = \frac{1}{(m-1)!} \sum_{\ell_1,\dots,\ell_m=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_m}. \tag{4.12}$$

To see it, simply note that if any of $\ell_2, \dots, \ell_m$ are equal to each other or to ℓ_1, then the wedge product vanishes. So, each index ℓ_s can be summed over the full range of 1 to m. However, this means that we are actually overcounting in the sum over all ℓ_s. This is not a problem though as $\mathcal{K}_{\ell_s}$ can be commuted past each other in the wedge products (they are all two-forms), so all we need to do to remedy this is to divide by the factor of $(m - 1)!$, where $m - 1$ is the number of factors of $\mathcal{K}_\ell$ appearing in the expression.

Now, for any integer r in the range $0, \dots, m - 1$, we introduce the form

$$\Omega^{(r)} = \sum_{\ell_1,\dots,\ell_{m-r}=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}}. \tag{4.13}$$

In particular, we have $\omega_{2m-1} = \frac{1}{(m-1)!}\Omega^{(0)}$ and $\Omega^{(m-1)} = \sum_{\ell_1=1}^{m} \mathcal{J}_{\ell_1}$. In Appendix A, we give a mathematical interpretation of these forms in terms of U(1)-equivariant cohomology of S^{2m-1}. The following formula for the change in $\Omega^{(r)}$ under a gauge transformation is the essential ingredient in our construction of the full gauged WZ action.

Claim. Under a gauge transformation $b_\ell \to e^{i\xi} b_\ell$ we have $\Omega^{(r)} \to \Omega^{(r)} + \delta_\xi \Omega^{(r)}$ with

$$\delta_\xi \Omega^{(r)} = \tfrac{1}{2} d\Omega^{(r+1)} \wedge d\xi. \tag{4.14}$$

Proof. Using Eqs. (4.3), we can show

$$\begin{aligned}\delta_\xi \Omega^{(r)} = &\sum_{\ell_1,\dots,\ell_{m-r}=1}^{m} \left(n^2_{2\ell_1-1} + n^2_{2\ell_1}\right) \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \wedge d\xi \\ &+ \sum_{s=2}^{m-r} \sum_{\ell_1,\dots,\ell_{m-r}=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \overline{\mathcal{K}_{\ell_s}} \wedge \cdots \wedge \\ &\times \mathcal{K}_{\ell_{m-r}} \wedge \left(n_{2\ell_s-1} dn_{2\ell_s-1} + n_{2\ell_s} dn_{2\ell_s}\right) \wedge d\xi,\end{aligned} \tag{4.15}$$

where the overline again means to omit that term from the wedge product. Next, we use the two properties

$$\sum_{\ell=1}^{m} \left(n^2_{2\ell-1} + n^2_{2\ell}\right) = 1, \tag{4.16a}$$

$$\sum_{\ell=1}^{m} (n_{2\ell-1} dn_{2\ell-1} + n_{2\ell} dn_{2\ell}) = 0, \tag{4.16b}$$

which follow from the fact that $\mathbf{n}$ is a unit vector field with $2m$ components, to find that

$$\delta_\xi \Omega^{(r)} = \sum_{\ell_2,\dots,\ell_{m-r}=1}^{m} \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-r}} \wedge d\xi \tag{4.17}$$

or, after reindexing,

$$\delta_\xi \Omega^{(r)} = \sum_{\ell_1,\ldots,\ell_{m-(r+1)}=1}^{m} \mathcal{K}_{\ell_1} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-(r+1)}} \wedge d\xi. \tag{4.18}$$

So, in fact, only the term in the first line of Eq. (4.15) has contributed. Next, we write $\mathcal{K}_{\ell_1} = \frac{1}{2} d\mathcal{J}_{\ell_1}$ and use the fact that $\mathcal{K}_\ell$ is closed to find

$$\begin{aligned}\delta_\xi \Omega^{(r)} &= \frac{1}{2} \sum_{\ell_1,\ldots,\ell_{m-(r+1)}=1}^{m} d\mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_{m-(r+1)}} \wedge d\xi \\ &= \frac{1}{2} d\Omega^{(r+1)} \wedge d\xi\end{aligned} \tag{4.19}$$

which completes the proof. ■

With Eq. (4.14) in hand, we can now construct the properly gauged action step by step. We go through the first few steps explicitly, and then write the final answer. To start, the change of the WZ term under a gauge transformation is

$$\begin{aligned}\delta_\xi S_{\mathrm{WZ}}[\mathbf{n}] &= \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \int_{\mathcal{B}} \delta_\xi \Omega^{(0)} \\ &= \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathcal{B}} d\Omega^{(1)} \wedge d\xi \\ &= \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(1)} \wedge d\xi.\end{aligned} \tag{4.20}$$

So, the first counterterm we should add is

$$S_{ct}^{(1)}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(1)} \wedge A. \tag{4.21}$$

The part of the action containing the WZ term is now

$$S'_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = S_{\mathrm{WZ}}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n}, A], \tag{4.22}$$

and under a gauge transformation we find

$$\begin{aligned}&\delta_\xi S'_{\mathrm{WZ,gauged}}[\mathbf{n}, A] \\ &\quad = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2} \int_{\mathbb{R}^{d-1,1}} \delta_\xi \Omega^{(1)} \wedge A,\end{aligned} \tag{4.23}$$

which becomes

$$\begin{aligned}&\delta_\xi S'_{\mathrm{WZ,gauged}}[\mathbf{n}, A] \\ &\quad = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^2} \int_{\mathbb{R}^{d-1,1}} d\Omega^{(2)} \wedge d\xi \wedge A.\end{aligned} \tag{4.24}$$

Now, we note that

$$d(\Omega^{(2)} \wedge d\xi \wedge A) = d\Omega^{(2)} \wedge d\xi \wedge A + \Omega^{(2)} \wedge d\xi \wedge F, \tag{4.25}$$

and we use this to do an integration by parts. Neglecting boundary terms (in general, we neglect all terms coming from the boundaries of the physical boundary space-time), we now have

$$\begin{aligned}&\delta_\xi S'_{\mathrm{WZ,gauged}}[\mathbf{n}, A] \\ &\quad = \frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(2)} \wedge d\xi \wedge F.\end{aligned} \tag{4.26}$$

Therefore, we should choose the second counterterm to be

$$S_{ct}^{(2)}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^2} \int_{\mathbb{R}^{d-1,1}} \Omega^{(2)} \wedge A \wedge F, \tag{4.27}$$

and the total gauged action is now

$$S''_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = S_{\mathrm{WZ}}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n}, A] + S_{ct}^{(2)}[\mathbf{n}, A]. \tag{4.28}$$

At this point, the pattern is clear. After iterating this procedure we find that a total of $m-1$ counterterms are needed to construct a gauged WZ action which satisfies Eq. (3.17). The rth counterterm (for $r = 1, \ldots, m-1$) is given by

$$S_{ct}^{(r)}[\mathbf{n}, A] = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^r} \int_{\mathbb{R}^{d-1,1}} \Omega^{(r)} \wedge A \wedge F^{r-1}, \tag{4.29}$$

where F^{r-1} is shorthand for the wedge product of F with itself $r-1$ times. The total gauged action is then

$$S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = S_{\mathrm{WZ}}[\mathbf{n}] + \sum_{r=1}^{m-1} S_{ct}^{(r)}[\mathbf{n}, A]. \tag{4.30}$$

In Appendix A we discuss this gauged WZ action from the point of view of U(1)-equivariant cohomology over the sphere S^{2m-1}.

When we look at the change of the full action $S_{\mathrm{WZ,gauged}}[\mathbf{n}, A]$ under a gauge transformation we find that it is not completely gauge invariant. In other words, the U(1) symmetry of the boundary theory of the BIQH state is anomalous, as we expect on physical grounds. The anomaly is controlled only by the final counterterm $S_{ct}^{(m-1)}[\mathbf{n}, A]$ since all other contributions cancel by construction. Under a gauge transformation we have

$$\begin{aligned}&\delta_\xi S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] \\ &\quad = -\frac{2\pi k}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^{m-1}} \int_{\mathbb{R}^{d-1,1}} \delta_\xi \Omega^{(m-1)} \wedge A \wedge F^{m-2}.\end{aligned} \tag{4.31}$$

Now, we use $\delta_\xi \Omega^{(m-1)} = d\xi$, the formula $\mathcal{A}_{2m-1} = \frac{2\pi^m}{(m-1)!}$, and integrate by parts to arrive at the final formula

$$\delta_\xi S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = k \int_{\mathbb{R}^{d-1,1}} \xi \left(\frac{F}{2\pi}\right)^{m-1}, \tag{4.32}$$

or in terms of the boundary space-time dimension d,

$$\delta_\xi S_{\mathrm{WZ,gauged}}[\mathbf{n}, A] = k \int_{\mathbb{R}^{d-1,1}} \xi \left(\frac{F}{2\pi}\right)^{\frac{d}{2}}. \tag{4.33}$$

C. Chern-Simons effective action for bulk electromagnetic response

We now use the result of the previous subsection to understand the bulk electromagnetic response of BIQH states in all odd space-time dimensions. As we discussed in the Introduction, a quantum Hall state in $2m-1$ dimensions is characterized by the presence of a CS term in the effective action $S_{\text{eff}}[A]$ for the electromagnetic field A. Recall that on $(2m-1)$-dimensional space-time the CS term takes the form

$$S_{\text{CS}}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A\wedge(dA)^{m-1}. \tag{4.34}$$

Now, it is well known that under a gauge transformation $A\to A+d\xi$ the CS action changes by a boundary term

$$\delta_\xi S_{\text{CS}}[A] = \frac{N_{2m-1}}{m!}\int_{\partial\mathcal{M}} \xi\left(\frac{F}{2\pi}\right)^{m-1}. \tag{4.35}$$

We can then deduce the coefficient N_{2m-1} for the bulk response of BIQH states by matching the variation of the bulk CS effective action for A with the anomaly of the boundary theory of the BIQH state [the O$(2m)$ NLSM with WZ term] which we calculated in the previous subsection. The gauge transformation of the bulk CS term must cancel the anomaly of the boundary theory in order for the entire system (bulk plus boundary) to be gauge invariant. This is exactly the concept of anomaly inflow [83] which we mentioned in the Introduction. Comparing Eq. (4.35) to (4.32) for the U(1) anomaly of the O$(2m)$ theory with WZ term, we deduce that the coefficient N_{2m-1} must be given by

$$N_{2m-1} = -(m!)k, \quad k\in\mathbb{Z} \tag{4.36}$$

in order to cancel the anomaly of the boundary theory. Therefore, we find that the level N_{2m-1} of the CS effective action for BIQH states in $2m-1$ space-time dimensions is quantized in units of $m!$. This answer agrees with the known cases for three and five space-time dimensions and gives a prediction for all odd dimensions beyond those. In Sec. V we discuss this peculiar quantization of the CS level from a mathematical point of view by studying the transformation of the CS term under large U(1) gauge transformations on general Euclidean manifolds (including manifolds which do not admit a spin structure).

We also remark here that based on the form of the CS response for the BIQH state in $2m-1$ dimensions, we can conclude that the chiral anomaly of the boundary theory of the BIQH state is $m!$ times larger than the chiral anomaly of the boundary theory for a fermionic SPT phase in $2m-1$ dimensions with a bulk CS response at level one. So, for example, the anomaly of the boundary theory is twice as large when the bulk is three dimensional ($m=2$ case) and six times as large when the bulk is five dimensional ($m=3$).

D. A derivation of the response from the bulk physics

To close this section, we present an alternative derivation of the response of the BIQH state. This derivation uses only bulk properties of the BIQH state, which should be contrasted with our derivation using gauged WZ actions which was based on the anomaly of the boundary theory. Recall again that the bulk of the BIQH state is described by an O$(2m)$ NLSM with theta term and theta angle $\theta=2\pi k$ (so we have a theta term and not a WZ term in the bulk description). The main reason for including this alternative derivation is that it provides a clear physical reason for the appearance of the $m!$ factor in the response. The derivation in this section uses only the dimensional reduction properties of the CS response action for the external field, and the theta term of the NLSM, which we now review.

We start by considering the CS response action at level N in $2m-1$ dimensions,

$$S_{\text{CS}}[A] = \frac{N}{(2\pi)^{m-1}m!}\int_{\mathbb{R}^{D,1}} A\wedge(dA)^{m-1}, \tag{4.37}$$

where D is the spatial dimension so that $D+1=2m-1$. Let $\mathbf{x}=(x^1,\ldots,x^D)$ be the spatial coordinates. Now, suppose we thread a delta function of 2π flux at a point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane (i.e., $x_0^j=0$, $j=1,\ldots,D-2$). Concretely, we set

$$F_{x^{D-1}x^D} = 2\pi\delta\big(x^{D-1}-x_0^{D-1}\big)\delta\big(x^D-x_0^D\big), \tag{4.38}$$

and we assume that $F_{x^jx^{D-1}}=F_{x^jx^D}=0\forall\ j=1,\ldots,D-2$, and that $F_{x^jx^k}$ is independent of (x^{D-1},x^D) for $j,k=1,\ldots,D-2$. Then, for this configuration, the CS response action reduces to

$$S_{\text{CS}}[A]\to\frac{N}{(2\pi)^{m-2}(m-1)!}\int_{\mathbb{R}^{D-2,1}}\tilde{A}\wedge(d\tilde{A})^{m-2}. \tag{4.39}$$

The key point is that it reduces to a CS term *at the same level* N on the $(D-2)$-dimensional space located at the point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane.

Now that we know what happens in the CS response action when we thread a 2π delta function flux of F in a particular plane, let us also see what happens in the NLSM description of the BIQH phase when this flux is inserted. In the NLSM description, the m bosons b_ℓ are all charged under the U(1) symmetry. Therefore, threading a 2π delta function flux at the point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane will cause all of the bosons b_ℓ to have a vortex configuration in that plane around the point $\mathbf{x}_0$. By a vortex configuration we just mean that the phases of the complex numbers b_ℓ all wind by 2π as one encircles the point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane. So, we conclude that threading a 2π delta function flux of F will create m vortex excitations in the O$(2m)$ NLSM which describes the bulk of the BIQH.

On the other hand, we are going to show that if a *single boson* b_ℓ for some ℓ has a vortex configuration at a point $\mathbf{x}_0$ in the (x^{D-1},x^D) plane, then the O$(2m)$ NLSM action with $\theta=2\pi k$ reduces to an O$(2m-2)$ NLSM with $\theta=2\pi k$ living on the $(D-2)$-dimensional space at $\mathbf{x}_0$. So, if we have a vortex in one boson only, then the NLSM theory for the BIQH state in $2m-1$ dimensions reduces to the NLSM theory for the BIQH state in $2m-3$ dimensions (inside the vortex core) and *with the same theta angle*.

We now prove the assertion in the previous paragraph that a vortex in one boson b_ℓ in the O$(2m)$ NLSM traps an O$(2m-2)$ NLSM with the same theta angle inside the vortex core. To do this, we consider an explicit vortex ansatz for the NLSM field in which the last boson $b_m=n_{2m-1}+in_{2m}$ takes on a vortex configuration. To set up the notation let (r,ϕ) be polar coordinates for the (x^{D-1},x^D) plane, and let $\mathbf{y}=(x^1,\ldots,x^D)$ be the coordinates for the remaining directions of space. Then, our vortex ansatz has the form

$$\mathbf{n}(t,\mathbf{x}) = \{\sin[f(r)]\mathbf{N}(t,\mathbf{y}),\cos[f(r)]\mathbf{m}(\phi)\}, \tag{4.40}$$

where $\mathbf{N}(t,\mathbf{y})$ is a $(2m-2)$-component unit vector field depending only on t and $\mathbf{y}$, and $\mathbf{m}(\phi)=[\cos(\phi),\sin(\phi)]$ represents the vortex configuration of the last two components of $\mathbf{n}$. The function $f(r)$ is assumed to satisfy the boundary conditions

$$f(0) = \frac{\pi}{2}, \tag{4.41}$$

$$\lim_{r\to\infty} f(r) = 0, \tag{4.42}$$

which means that the field $\mathbf{N}(t,\mathbf{y})$ lives in the core of the vortex. This vortex ansatz is equivalent to the $q = 1$, $n_q = 1$ case of the more general defect configurations for NLSMs considered in Appendix D. Using the dimensional reduction formula from Eq. (D10) of Appendix D, we immediately derive that on this configuration the theta term of the O($2m$) NLSM reduces to

$$\begin{aligned} S_\theta[\mathbf{n}] &= \frac{\theta}{\mathcal{A}_{2m}} \int_{\mathbb{R}^{D,1}} \mathbf{n}^*\omega_{2m} \\ &\to \frac{\theta}{\mathcal{A}_{2m-2}} \int_{\mathbb{R}^{D-2,1}} \mathbf{N}^*\omega_{2m-2}. \end{aligned} \tag{4.43}$$

This is the theta term for the O($2m-2$) NLSM with field $\mathbf{N}$ living in the vortex core, and we see that the theta angle is the same as for the original O($2m$) NLSM. This proves our claim from the previous paragraph.

From the discussion above we see that threading a 2π flux of F in the O($2m$) NLSM theory will produce m copies of the O($2m-2$) theory since the 2π flux creates a vortex in all m species of bosons, and a vortex in just one species produces one copy of the O($2m-2$) NLSM with theta term. We should mention a technical point that the m vortices cannot all be localized at a point and should spread or separate slightly in space after we thread the 2π flux. This is because the amplitude $|b_\ell|$ should vanish at the core of a vortex in the phase of b_ℓ, but the NLSM constraint $\sum_\ell |b_\ell|^2$ does not allow the amplitudes $|b_\ell|$ for all ℓ to simultaneously vanish at a particular point. However, this subtlety does not effect the basic physical point which is that threading the 2π flux of F produces m vortices (at nearly the same point), each of which carries a copy of the lower-dimensional BIQH state.

Let us denote the CS level for the response of the O($2m$) NLSM with $\theta = 2\pi k$ in $2m-1$ dimensions by N_{2m-1}. From what we have just learned, and from Eq. (4.39) for the reduction of the CS term after threading 2π flux, we find that the CS levels for the response of the NLSMs in dimensions $2m-1$ and $2m-3 = 2(m-1)-1$ must obey the recursion relation

$$N_{2m-1} = m N_{2m-3}. \tag{4.44}$$

We can now iterate this equation to generate

$$N_{2m-1} = (m!)N_1. \tag{4.45}$$

This equation gives the electromagnetic response of the O($2m$) NLSM with $\theta = 2\pi k$ in terms of the response of the O(2) NLSM in one dimension with $\theta = 2\pi k$. In Appendix E, we directly calculate N_1 for the O(2) NLSM (in the limit of large coupling g) and show that $N_1 = -k$ in that case. This then implies that

$$N_{2m-1} = -(m!)k, \tag{4.46}$$

and this agrees (in magnitude and in sign) with our boundary calculation using gauged WZ actions. Thus, the dimensional reduction approach employed in this section gives a clear physical picture for the $m!$ factor in the response, and crucially depends on the fact that all the bosons b_ℓ carry a U(1) charge.

V. GENERAL GAUGE-INVARIANCE ARGUMENT FOR THE BIQH RESPONSE AND COMPARISON WITH THE FERMIONIC CASE

In this section we show that the factor of $m!$ in the BIQH response derived in Sec. IV can be understood by studying large U(1) gauge transformations of the CS action on general (closed, compact) Euclidean manifolds which do not necessarily admit a spin structure. Physically, we require the *exponential* of the CS term to be gauge invariant since this object is part of the partition function of a short-range entangled (gapped) phase coupled to the external field A. In such phases, since the ground state is always unique, one can always safely integrate out the matter field and obtain a gauge-invariant action. In contrast, if we do the same thing for a topologically ordered state, for example a Laughlin state, we will indeed get a non-gauge-invariant response theory. This is because the calculation to arrive at a response theory is only perturbatively defined around a single ground state.

The level N_{2m-1} of the CS term must be quantized for the exponential of the CS term to be gauge invariant, but we find that the required quantization of N_{2m-1} is different depending on whether or not the Euclidean manifold admits a spin structure. Bosonic theories may be formulated on any generic manifold, but the Dirac equation cannot be formulated properly on a manifold which does not admit a spin structure, and so we cannot place fermions on these manifolds. In particular, we find that the CS action will be gauge invariant on a generic manifold if the level N_{2m-1} is quantized in integer multiples of $m!$, which agrees with our direct calculation for the NLSM theory from Sec. IV. For the fermionic case we use the Atiyah-Singer index theorem for the twisted Dirac complex [55] to show that the CS response action will not, in general, be U(1) gauge invariant unless suitable gravitational terms are also included in the response action. We also discuss an explicit example of how these gravitational terms can contribute to the response of a fermionic SPT phase with U(1) symmetry. Furthermore, using these examples, we compare the quantization of FIQH and BIQH states, as well as another type of bosonic SPT state with nontrivial topological electromagnetic- gravitational response.

A. Gauge-invariance argument for bosonic and fermionic states

In Euclidean space-time the CS term takes the form

$$S_{\mathrm{CS}}[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A\wedge F^{m-1}. \tag{5.1}$$

Here, $\mathcal{M}$ is a ($2m-1$)-dimensional closed, compact manifold, and for the moment let us assume that N_{2m-1} is some number, not necessarily an integer. A more careful way to define the CS term is to consider an extension of the field configuration A into a $2m$-dimensional manifold $\mathcal{B}$ such that $\partial\mathcal{B} = \mathcal{M}$ (this type of analysis of CS terms dates back at least to Ref. [85]). Let $\tilde{A}$ denote this extension. Then, the CS term is more properly written as

$$S_{\mathrm{CS}}[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}} \tilde{F}^m, \tag{5.2}$$

where $\tilde{F} = d\tilde{A}$. In this formulation, a large U(1) gauge transformation of the action can be understood as a change of the extension of A into the larger space $\mathcal{B}$. Suppose $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ are two different extensions of A. In order for the CS term to be well defined, we require that the difference

$$-i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}}(\tilde{F}^{(1)})^m - \left(-i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_{\mathcal{B}}(\tilde{F}^{(2)})^m\right) \tag{5.3}$$

be an integer multiple of $2\pi i$ so that the exponential of the difference of the two Euclidean actions is equal to one. This is equivalent to the requirement that the exponential of the CS term be invariant under a large U(1) gauge transformation. This difference can in turn be written as the integral of the field strength F of a gauge field in $2m$ dimensions over the closed manifold $2m$-dimensional manifold X constructed by gluing $\mathcal{B}$ to another copy of $\mathcal{B}$ (with the opposite orientation) along their boundary (which is the original lower-dimensional manifold $\mathcal{M}$). So, the requirement for a well-defined CS term is to check that

$$I[A] = -i\frac{N_{2m-1}}{(2\pi)^{m-1}m!}\int_X F^m \tag{5.4}$$

is equal to $2\pi k$ for some integer k, where X is a $2m$-dimensional closed, compact manifold, and F is now the field strength of a gauge field A living in $2m$ dimensions.

We must also make one crucial assumption about the configuration of F on X, which is that F should be chosen to satisfy the Dirac quantization condition

$$\int_{\mathcal{C}}\frac{F}{2\pi} \in \mathbb{Z}\ , \tag{5.5}$$

where $\mathcal{C}$ is any nontrivial two-cycle on X [i.e., an element of the second homology group $H_2(X,\mathbb{R})$]. This requirement tells us how a general background field F on X can be expanded in terms of the elements of the second cohomology group $H^2(X,\mathbb{R})$ of X [more precisely, we expand F in terms of elements of the second de Rham cohomology group $H^2_{dR}(X)$, which is in turn isomorphic to $H^2(X,\mathbb{R})$ by de Rham's theorem].

If we enforce the Dirac quantization condition of Eq. (5.5), then on a generic closed, compact Euclidean manifold X we have

$$\int_X\left(\frac{F}{2\pi}\right)^m \in \mathbb{Z}. \tag{5.6}$$

Briefly, this comes from the fact that (assuming the Dirac quantization condition) $\frac{F}{2\pi}$ is the first Chern class c_1 of a complex line bundle over X. The integral over X of its mth power $(c_1)^m$ is then one of the Chern numbers of this complex line bundle, and is therefore an integer [86]. Note that here we also need to assume that X is orientable. From this result we deduce that the (exponential of the) CS term will be invariant under large U(1) gauge transformations on any Euclidean manifold provided that

$$N_{2m-1} = (m!)k, \quad k \in \mathbb{Z} \tag{5.7}$$

which agrees with our result from Sec. IV derived using the NLSM description of the BIQH state. In Appendix B we show that the minimum value with $\int_X(\frac{F}{2\pi})^m = 1$ can be achieved for $X = C\mathbb{P}^m$ if we thread 2π flux of F through the nontrivial two-cycle on $C\mathbb{P}^m$.

We can also compare this result with the result for FIQH phases with U(1) symmetry in the same dimension. In any odd dimension, we can consider the massive Dirac fermion as a model for a FIQH state with the global U(1) symmetry associated to charge conservation. The Lagrangian of a massive Dirac fermion on flat, $(2m-1)$-dimensional Minkowski space-time takes the form

$$\mathcal{L}_{\text{Dirac}}[\psi,A] = \overline{\psi}(i\not{\partial} - \not{A} - M)\psi, \tag{5.8}$$

where γ^μ, $\mu = 0,\dots,2m-2$, are the standard gamma matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta = \text{diag}(1,-1,\dots,-1)$, $\overline{\psi} = \psi^\dagger\gamma^0$, and $M > 0$ is the mass of the Dirac fermion. We also used the Feynman slash notation $\not{\partial} \equiv \gamma^\mu\partial_\mu$, etc. Here, we have also coupled the fermion ψ to the background U(1) gauge field (electromagnetic field) A_μ. After integrating out the massive Dirac fermion, we arrive at a topological response theory given by the CS theory at level one:

$$S_{\text{Dirac}}[A] = -i\frac{1}{(2\pi)^{m-1}m!}\int_{\mathcal{M}} A\wedge F^{m-1}, \tag{5.9}$$

where in this case the space-time manifold $\mathcal{M}$ is just $(2m-1)$-dimensional Minkowski space-time. In deriving this response theory we have employed a Pauli-Villars regularization procedure (see Ref. [45] or the more recent discussion in Ref. [48]) such that integrating out a Dirac fermion with a negative mass M does not produce any topological term (i.e., a CS term with level zero). Also, we have omitted all the nontopological terms, for example the Maxwell term, from the final response action. Since a single massive Dirac fermion gives rise to a CS term for A at level one, we have the result that

$$N_{2m-1} \in \mathbb{Z} \tag{5.10}$$

for general U(1) fermionic SPT phases in $2m-1$ dimensions.

However, as we know from the discussion of the CS term earlier in this section, on a generic manifold $\mathcal{M}$ the CS term will not be invariant under large U(1) gauge transformations unless the level N_{2m-1} is an integer multiple of $m!$. Thus, one might naively conclude that the response action for the FIQH state on a generic manifold $\mathcal{M}$ is not invariant under large U(1) gauge transformations. Of course, this is not the case. The resolution of this problem is to recall that on a curved manifold $\mathcal{M}$ a Dirac fermion also has nontrivial gravitational and (when coupled to the gauge field A) mixed gauge and gravitational responses. The gravitational part of the response comes from the coupling of the Dirac fermion to the metric $g_{\mu\nu}$ of the curved space-time $\mathcal{M}$. The response action for the FIQH state (as modeled by the massive Dirac fermion) will include these additional terms. The effective action for a massive Dirac fermion on a $(2m-1)$-dimensional closed, compact manifold $\mathcal{M}$ can be written in the form [47]

$$S_{\text{FIQH}}[A,g] = 2\pi i\int_{\mathcal{B}}\text{ch}(\tilde{F})\wedge\hat{A}(\mathcal{B}), \tag{5.11}$$

where $\partial\mathcal{B} = \mathcal{M}$, $\text{ch}(\tilde{F}) = e^{\frac{\tilde{F}}{2\pi}}$ is the Chern character of the extended field strength $\tilde{F}$, and $\hat{A}(\mathcal{B})$ is the *A-roof genus* (or Dirac genus) on $\mathcal{B}$. Since we are focusing on fermionic phases here, we should only consider spin manifolds $\mathcal{M}$ and $\mathcal{B}$. The A-roof genus $\hat{A}(\mathcal{B})$ can be expressed in terms of the Pontryagin classes $p_i(\mathcal{B})$ of $\mathcal{B}$ as [87]

$$\hat{A}(\mathcal{B}) = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2) + \cdots, \tag{5.12}$$

with

$$p_1 = -\frac{1}{8\pi^2}\mathrm{Tr}\tilde{\mathcal{R}}^2, \tag{5.13}$$

$$p_2 = -\frac{1}{64\pi^4}\mathrm{Tr}\tilde{\mathcal{R}}^4 + \frac{1}{128\pi^4}(\mathrm{Tr}\tilde{\mathcal{R}}^2)^2. \tag{5.14}$$

Here, $\tilde{R}$ is the $2m \times 2m$ matrix of two-forms (curvature two-form) on $\mathcal{B}$:

$$\tilde{R}^{\nu}_{\mu} = \tfrac{1}{2}\tilde{R}_{\alpha\beta\mu}{}^{\nu}dx^{\alpha}\wedge dx^{\beta} \tag{5.15}$$

which depends on the Riemann curvature tensor $\tilde{R}^{\nu}_{\alpha\beta\mu}$ in the extended space $\mathcal{B}$. In Eq. (5.11) it is understood that the integral is only over the terms of (differential form) degree $2m$ in the product $\mathrm{ch}(\tilde{F})\wedge\hat{A}(\mathcal{B})$ on $\mathcal{B}$. It is easy to see that when we only consider the electromagnetic response in $S_{\mathrm{FIQH}}[A,g]$ (e.g., by setting all p_i to 0 on $\mathcal{B}$), it recovers the response theory (5.9) of the massive Dirac fermion in $2m-1$ dimensions. More importantly, the response theory $S_{\mathrm{FIQH}}[A,g]$ is fully gauge invariant. This is because on any closed, compact $2m$-dimensional spin manifold X, the Atiyah-Singer index theorem for the twisted Dirac complex (see, for example, Ref. [55]) states that

$$\int_X \mathrm{ch}(\tilde{F})\wedge\hat{A}(X) = \mathrm{index}(\not{D}) \in \mathbb{Z}, \tag{5.16}$$

where $\mathrm{index}(\not{D})$ is the index (the difference between the number of positive and negative chirality zero modes) of the Dirac operator on X, and is necessarily an integer. Although we originally derived Eq. (5.11) by using the theory of a massive Dirac fermion on the curved manifold $\mathcal{M}$ as a model for the FIQH state, we argue that due to the requirement of large U(1) gauge invariance, Eq. (5.11) is the minimal (or "level 1") nontrivial gauge and gravitational response theory of any putative FIQH phase with U(1) symmetry in $(2m-1)$ dimensions.

There is one more subtlety here. When m is even (i.e., when the space-time dimension is $4k-1$ with $k\in\mathbb{Z}$), the object $\mathrm{ch}(\tilde{F})\wedge\hat{A}(\mathcal{B})$ contains a purely gravitational term that comes from $\hat{A}(\mathcal{B})$ alone. Such a term itself can be well defined (the index theorem for the untwisted Dirac complex guarantees that it integrates to an integer on a closed, compact spin manifold) and can capture the nontrivial gravitational response of certain short-range entangled states even without the inclusion of a global U(1) symmetry. For example, for $m=2$ the purely gravitational term is given by $-\frac{1}{24}p_1$ on $\mathcal{B}$, which is equivalent to the three-dimensional gravitational Chern-Simons term on $\mathcal{M}$. This term is tied to the chiral central charge. Hence, we can separately consider the purely gravitational term $\hat{A}(\mathcal{B})$ and the rest of the terms $[\mathrm{ch}(\tilde{F})\wedge\hat{A}(\mathcal{B}) - \hat{A}(\mathcal{B})]$ in Eq. (5.11).

In general, we can consider the FIQH phase at level $N_{2m-1}\in\mathbb{Z}$, whose topological response theory (minus the purely gravitational term) is given by

$$S'_{\mathrm{FIQH}}[A,g] = 2\pi i N_{2m-1}\int_{\mathcal{B}}[\mathrm{ch}(\tilde{F})\wedge\hat{A}(\mathcal{B}) - \hat{A}(\mathcal{B})]. \tag{5.17}$$

$S'_{\mathrm{FIQH}}[A,g]$ naturally contains both a term capturing the electromagnetic response of the FIQH state and other terms that describe various different types of mixed gauge-gravitational response. The coexistence of all these terms is enforced by the properties of spin manifolds and the Atiyah-Singer index theorem, and reflects the fermionic nature of the FIQH phase. This combination also informs us that we should *not* use each of the terms to independently classify fermionic SPTs with U(1) symmetry. For bosonic systems, we can, in principle, separately study each single term in $S'_{\mathrm{FIQH}}[A,g]$ by itself, and use each of them to characterize a different class of bosonic SPTs. However, just like the quantization of the level of the U(1) CS term, we expect gauge invariance to enforce a larger quantization unit of the "level" when we isolate a single term as a bosonic response theory, as opposed to the case where that term appears in the full combination $S'_{\mathrm{FIQH}}[A,g]$ as a part of a fermionic theory. The difference in the quantization unit of the "level" between fermionic and bosonic systems will also lead to very different behaviors under dimensional reduction, the details of which will be elaborated using examples. In Sec. V B, we provide an example of the electromagnetic and gravitational response theory of FIQH states in five dimensions. In Sec. V C, we compare the example fermionic response theory with five-dimensional bosonic theories, including the BIQH state and another type of bosonic U(1) SPT state with nontrivial mixed electromagnetic and gravitational response.

B. An example of electromagnetic and gravitational response theories of FIQH states and their dimensional reduction

In this section, we restrict our discussion to the topological response theory of a five-dimensional FIQH phase, and we study its dimensional reduction to the response theory of a FIQH state in three dimensions. We start with the response theory of the FIQH phase at level $N_5 = 1$ on a five-dimensional spin manifold $\mathcal{M}^5$:

$$S_{\mathrm{FIQH}}[A,g] = 2\pi i\int_{\mathcal{B}^6}\left[\frac{1}{6}\left(\frac{\tilde{F}}{2\pi}\right)^3 - \frac{p_1}{24}\wedge\frac{\tilde{F}}{2\pi}\right], \tag{5.18}$$

where $\mathcal{B}^6$ is a six-dimensional spin manifold such that $\mathcal{M}^5 = \partial\mathcal{B}^6$. We first consider its dimensional reduction to the response theory of a FIQH state in three dimensions. In order to do so, we take the space-time manifold to be $\mathcal{M}^5 = S^2\times\mathcal{M}^3$ where $\mathcal{M}^3$ is a closed, compact three-dimensional manifold, and S^2 is a two-sphere. In this case, it is natural to consider the bounding space $\mathcal{B}^6 = S^2\times\mathcal{B}^4$ where $\mathcal{B}^4$ is a four-dimensional spin manifold such that $\mathcal{M}^3 = \partial\mathcal{B}^4$. Also, we consider the configuration with 2π flux of $\tilde{F}$ piercing the S^2 part. The response theory is then reduced to

$$S_{\mathrm{FIQH}}[A,g]|_{S^2\times\mathcal{M}^3} = 2\pi i\int_{\mathcal{B}^4}\left(\frac{\tilde{F}^2}{8\pi^2}-\frac{p_1}{24}\right) = i\int_{\mathcal{M}^3}\left[\frac{A\wedge F}{4\pi}-\frac{1}{24}\frac{1}{4\pi}\mathrm{Tr}\left(\omega\wedge d\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega\right)\right], \tag{5.19}$$

where ω is the SO(1,2) spin connection on $\mathcal{M}^3$. The first term describes the standard integer quantum Hall effect in three dimensions with unit Hall conductance. The second term, which is the gravitational Chern-Simons term, captures the gravitational response of a three-dimensional chiral state with chiral central charge $c = 1$. On the other hand, we can directly consider a five-dimensional massive Dirac fermion as a model of a five-dimensional FIQH state at level one on this background. When put on the manifold $S^2\times\mathcal{M}^3$ with 2π flux of F inside the S^2 part, the five-dimensional massive Dirac fermion effectively reduces to a three-dimensional massive Dirac fermion on $\mathcal{M}^3$ at low energies when the linear size of the S^2 part is small compared to the length scale set by the Dirac fermion mass M. The U(1) and gravitational response of the three-dimensional FIQH state is indeed given by the dimensionally reduced response theory $S_{\mathrm{FIQH}}[A,g]|_{S^2\times\mathcal{M}^3}$.

Finally, let us also remark here that the response theory (5.18) for the five-dimensional FIQH state can also be used to derive the electromagnetic and gravitational responses of a topological superconductor in four dimensions using a dimensional reduction procedure [88].

C. Comparing bosonic and fermionic systems: Quantization and dimensional reduction

As we have discussed, we can consider each term of $S_{\mathrm{FIQH}}[A,g]$ separately as a topological response theory for bosonic U(1) SPTs in five dimensions:

$$S_{\mathrm{BIQH}}[A] = 2\pi i N_5\int_{\mathcal{B}^6}\frac{1}{6}\left(\frac{\tilde{F}}{2\pi}\right)^3, \tag{5.20}$$

$$S_{\mathrm{BSPT}}[A,g] = -2\pi i N_5'\int_{\mathcal{B}^6}\frac{p_1}{24}\wedge\frac{\tilde{F}}{2\pi}. \tag{5.21}$$

$S_{\mathrm{BIQH}}[A]$ is the response theory of a five-dimensional BIQH state, and requires a quantization of level as $N_5\in 6\mathbb{Z}$ as we showed in this section and in Sec. IV. $S_{\mathrm{BSPT}}[A,g]$ characterizes an independent class of bosonic SPT states in five dimensions without a requirement of U(1) symmetry [89]. Similar to the BIQH and FIQH cases, gauge invariance requires $N_5'\int_{X^6}\frac{p_1}{24}\wedge\frac{\tilde{F}}{2\pi}\in\mathbb{Z}$ on any closed six-dimensional manifold X^6. Since p_1 and $\frac{\tilde{F}}{2\pi}$ are both cohomology classes of X^6 with integer coefficients, gauge invariance then enforces the quantization $N_5'\in 24\mathbb{Z}$. We would like to point out that previously Ref. [89] considered only closed six-dimensional manifolds that can be decomposed into products of two- and four-dimensional manifolds, and concluded that $N_5'\in 8\mathbb{Z}$. However, when we take into account more general six-dimensional manifolds, for example $C\mathbb{P}^3$, we arrive at the stronger quantization condition $N_5'\in 24\mathbb{Z}$.[2] As seen here, for both of the bosonic theories $S_{\mathrm{BIQH}}[A]$ and $S_{\mathrm{BSPT}}[A,g]$, the quantization units of their levels are larger than when these two terms appear together in the fermionic theory $S_{\mathrm{FIQH}}[A,g]$ in Eq. (5.18).

Now, let us consider a similar dimensional reduction of both $S_{\mathrm{BIQH}}[A_\mu]$ and $S_{\mathrm{BSPT}}[A_\mu,g]$ to three dimensions, as we did in the fermion case. Now, the five-dimensional spacetime manifold $\mathcal{M}^5$ is taken to be the product $S^2\times\mathcal{M}^3$ with $\mathcal{M}^3$ a three-dimensional manifold. Again, we consider the configuration with 2π flux of $\tilde{F}$ piercing the S^2 part. The dimensionally reduced theories are given by

$$S_{\mathrm{BIQH}}[A]|_{S^2\times\mathcal{M}^3} = i2\pi\frac{N_5}{2}\int_{\mathcal{M}^3}\frac{A\wedge F}{(2\pi)^2}, \tag{5.22}$$

$$S_{\mathrm{BSPT}}[A,g]|_{S^2\times\mathcal{M}^3} = -i2\pi\frac{N_5'}{24}\int_{\mathcal{M}^3}\frac{1}{4\pi}\mathrm{Tr}\left(\omega\wedge d\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega\right). \tag{5.23}$$

For the BIQH state, due to the bosonic quantization $N_5\in 6\mathbb{Z}$, we notice that the most fundamental three-dimensional BIQH state (with CS level $N_3 = 2$) cannot be realized from such a dimensional reduction from a five-dimensional BIQH state. From our analysis of the CS level of the BIQH state, it should be generally true that there are certain lower-dimensional BIQH states that cannot be realized from the dimensional reduction of higher-dimensional BIQH states. In fact, this phenomenon is not restricted to BIQH states. For the bosonic SPT states described by Eq. (5.21), due to the quantization $N_5'\in 24\mathbb{Z}$, the action $S_{\mathrm{BSPT}}[A,g]|_{S^2\times\mathcal{M}^3}$ only captures chiral bosonic states with chiral central charge $c\in 24\mathbb{Z}$. The E_8 state in (2 + 1) dimensions, which has chiral central charge $c = 8$, is absent in this dimensional reduction picture. This is in strong contrast with the fermionic theory studied in Sec. V B, in which case lower-dimensional response theories of FIQH at any level can be obtained from dimensionally reducing higher-dimensional FIQH states.

VI. ELECTROMAGNETIC RESPONSE OF BTI STATES IN ALL EVEN DIMENSIONS

In this section we construct the gauged WZ action for the boundary of BTI states in all even dimensions. Again, the action that we construct satisfies the gauging principle of Eq. (3.17). Unlike the BIQH case, however, the gauged boundary action that we find for BTI states *is* completely gauge invariant. From the form of the gauged action for the boundary of the BTI, we find that if the NLSM field on the boundary condenses in such a way that the $\mathbb{Z}_2$ symmetry of the BTI is broken, but the U(1) symmetry remains intact, then

[2] When we consider $C\mathbb{P}^3$ with the U(1) gauge field given by its fundamental line bundle, we find that $N_5'\int_{C\mathbb{P}^3}\frac{p_1}{24}\wedge\frac{\tilde{F}}{2\pi} = N_5'/6$. Combining with the result $N_5'\in 8\mathbb{Z}$ from Ref. [89], we can conclude that the gauge-invariance argument requires $N_5'\in 24\mathbb{Z}$. On the other hand, since p_1 and $\frac{\tilde{F}}{2\pi}$ are both cohomology classes with integer coefficients, any $N_5'\in 24\mathbb{Z}$ will satisfy the gauge-invariance requirement.

the boundary of the BTI can exhibit a $\mathbb{Z}_2$ symmetry-breaking quantum Hall response [recall from Sec. III that the BTI phase also has a $\mathbb{Z}_2$ symmetry such that the total symmetry group is $U(1) \rtimes \mathbb{Z}_2$].[3] We find that the boundary quantum Hall response is characterized by a CS level N_{2m-1} which is quantized in units of $\frac{m!}{2}$, i.e., the minimal boundary quantum Hall response is half that of the minimal BIQH state that can be realized intrinsically in the same space-time dimension. This boundary response implies a bulk response of the form of Eq. (2.2) with the parameter Θ_{2m} quantized as $\Theta_{2m} = 2\pi(\frac{m!}{2})$.

In Appendix A we reinterpret the gauged action constructed in this section in terms of U(1)-equivariant cohomology of the sphere S^{2m}. There we show that the problem of constructing a gauged WZ action for the boundary of the BTI phase in $2m$ dimensions is equivalent to the problem of constructing an equivariant extension of ω_{2m}, the volume form for S^{2m}, and we explicitly construct such an extension. The fact that an extension exists is mathematically equivalent to the result in this section that the gauged WZ action for the boundary of the BTI is completely gauge invariant. We also show that the forms $\Phi^{(r)}$ which appear later in this section in the counterterms of Eq. (6.25) are exactly the same forms which are needed for the construction of the equivariant extension of ω_{2m}.

We now construct the gauged WZ action for the boundary of BTI states. Recall that in the BTI case we define the integer m via $2m+1 = d+2$, so that the SPT phases we study live in $2m$ space-time dimensions and have a $(2m-1)$-dimensional boundary (the bulk space-time dimension was defined to be $d+1$). We again make use of the forms $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$, $\ell = 1,\dots,m$, defined in Eqs. (4.2). Now, however, the NLSM field has the extra component n_{2m+1}, so the relations of Eq. (4.16) are replaced with

$$\sum_{\ell=1}^{m}\left(n_{2\ell-1}^2 + n_{2\ell}^2\right) = 1 - n_{2m+1}^2, \tag{6.1a}$$

$$\sum_{\ell=1}^{m}(n_{2\ell-1}dn_{2\ell-1} + n_{2\ell}dn_{2\ell}) = -n_{2m+1}dn_{2m+1}. \tag{6.1b}$$

In this case the WZ term takes the form

$$S_{\mathrm{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m}}\int_{\mathcal{B}}\omega_{2m}, \tag{6.2}$$

where $\mathcal{B} = [0,1]\times\mathbb{R}^{d-1,1}$ is the extended boundary space-time.

[3]Our result can also be applied to systems with a symmetry of the form $U(1)\times\mathbb{Z}_2$, but only in the case that the U(1) symmetry rotates the maximal number of components of n_a as in the $U(1)\rtimes\mathbb{Z}_2$ cases considered in this paper. For example, according to Ref. [34] bosonic SPT phases in four dimensions with $U(1)\times\mathbb{Z}_2^T$ symmetry have a $(\mathbb{Z}_2)^3$ classification. However, only one of the three root phases is described by an O(5) NLSM with symmetry assignment that rotates four out of the five components of $\mathbf{n}$ [29], so this is the only case in which our technique can be applied directly. For the other cases, one must use the more general methods of Ref. [44] to gauge the U(1) symmetry.

For the BTI case it is convenient to define the forms $\Phi^{(r)}$ for $r = 0,1,\dots,m-1$ as

$$\Phi^{(r)} = \sum_{\ell_1,\dots,\ell_{m-r}=1}^{m}\mathcal{K}_{\ell_1}\wedge\cdots\wedge\mathcal{K}_{\ell_{m-r}}, \tag{6.3}$$

and in addition we define $\Phi^{(m)} = 1$, so that $\Phi^{(r)}$ is defined for all $r = 0,1,\dots,m$. Also, note that all of these forms are closed since each $\mathcal{K}_\ell$ is closed. Just as in the BIQH case, the essential ingredient in the construction of the gauged WZ action is a formula for how these forms change under a gauge transformation.

Claim. Under a gauge transformation $b_\ell \to e^{i\xi}b_\ell$ we have $\Phi^{(r)} \to \Phi^{(r)} + \delta_\xi\Phi^{(r)}$ with

$$\delta_\xi\Phi^{(r)} = -(m-r)n_{2m+1}dn_{2m+1}\wedge\Phi^{(r+1)}\wedge d\xi. \tag{6.4}$$

Proof. Using the symmetry of the summand of $\Phi^{(r)}$ under the exchange of any two of the indices $\ell_1,\dots,\ell_{m-r}$, we first find that

$$\delta_\xi\Phi^{(r)} = (m-r)\sum_{\ell_1,\dots,\ell_{m-r}=1}^{m}\left(n_{2\ell_1-1}dn_{2\ell_1-1} + n_{2\ell_1}dn_{2\ell_1}\right)\wedge d\xi\wedge\mathcal{K}_{\ell_2}\wedge\cdots\wedge\mathcal{K}_{\ell_{m-r}}. \tag{6.5}$$

Now, we can move $d\xi$ all the way to the right by commuting it past the two-forms $K_{\ell_2},\dots,K_{\ell_{m-r}}$. This gives

$$\delta_\xi\Phi^{(r)} = (m-r)\sum_{\ell_1=1}^{m}\left(n_{2\ell_1-1}dn_{2\ell_1-1} + n_{2\ell_1}dn_{2\ell_1}\right)\wedge\Phi^{(r+1)}\wedge d\xi, \tag{6.6}$$

where we used the fact that

$$\Phi^{(r+1)} = \sum_{\ell_2,\dots,\ell_{m-r}=1}^{m}\mathcal{K}_{\ell_2}\wedge\cdots\wedge\mathcal{K}_{\ell_{m-r}}. \tag{6.7}$$

Finally, we can do the sum over ℓ_1 using the second relation of Eqs. (6.1), and this gives the final formula of Eq. (6.4). ■

In terms of the form $\Phi^{(0)}$ we can write the volume form on S^{2m} as

$$\omega_{2m} = \frac{1}{(m-1)!}\left[\sum_{\ell_1,\dots,\ell_m=1}^{m}\mathcal{J}_{\ell_1}\wedge\mathcal{K}_{\ell_2}\wedge\cdots\wedge\mathcal{K}_{\ell_m}\wedge dn_{2m+1} + \frac{n_{2m+1}}{m}\Phi^{(0)}\right]. \tag{6.8}$$

The last term in this expression is just the term

$$n_{2m+1}dn_1\wedge dn_2\wedge\cdots\wedge dn_{2m-1}\wedge dn_{2m}, \tag{6.9}$$

but rewritten using the formula

$$dn_1\wedge dn_2\wedge\cdots\wedge dn_{2m-1}\wedge dn_{2m} = \frac{1}{m!}\sum_{\ell_1,\dots,\ell_m=1}^{m}\mathcal{K}_{\ell_1}\wedge\cdots\wedge\mathcal{K}_{\ell_m}. \tag{6.10}$$

We are now in a position to construct the properly gauged action step by step as in Sec. IV on the BIQH system. We demonstrate the first few steps in the construction and then write the final answer. To start we have

$$\delta_\xi \omega_{2m} = -\frac{1}{(m-1)!} dn_{2m+1} \wedge \Phi^{(1)} \wedge d\xi = -\frac{1}{(m-1)!} d(n_{2m+1}\Phi^{(1)} \wedge d\xi). \quad (6.11)$$

This is computed using Eq. (6.4) for the case $r = 0$ combined with the formula

$$\delta_\xi \left(\sum_{\ell_1,\dots,\ell_m=1}^{m} \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \cdots \wedge \mathcal{K}_{\ell_m} \wedge dn_{2m+1} \right) = -\left(1 - n_{2m+1}^2\right) dn_{2m+1} \wedge \Phi^{(1)} \wedge d\xi, \quad (6.12)$$

which is easily proven using Eqs. (4.3) and (6.1). Then, we have

$$\delta_\xi S_{\mathrm{WZ}}[\mathbf{n}] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-1)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}\Phi^{(1)} \wedge d\xi. \quad (6.13)$$

We therefore choose the first counterterm to be

$$S_{ct}^{(1)}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-1)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}\Phi^{(1)} \wedge A. \quad (6.14)$$

The total gauged WZ action is now

$$S'_{\mathrm{gauged,WZ}}[\mathbf{n},A] = S_{\mathrm{WZ}}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n},A], \quad (6.15)$$

and under a gauge transformation we find

$$\delta_\xi S'_{\mathrm{gauged,WZ}}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^2 dn_{2m+1} \wedge \Phi^{(2)} \wedge d\xi \wedge A. \quad (6.16)$$

Next, we integrate by parts using the formula

$$d\left(\frac{1}{3} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge A\right) = n_{2m+1}^2 dn_{2m+1} \wedge \Phi^{(2)} \wedge d\xi \wedge A - \frac{1}{3} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge F, \quad (6.17)$$

to find (neglecting boundary terms)

$$\delta_\xi S'_{\mathrm{gauged,WZ}}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \frac{1}{3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^3 \Phi^{(2)} \wedge d\xi \wedge F. \quad (6.18)$$

We should then take the second counterterm to be

$$S_{ct}^{(2)}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-2)!} \frac{1}{3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^3 \Phi^{(2)} \wedge A \wedge F. \quad (6.19)$$

To see the full structure of the counterterms it is necessary to go one step further. At this point, the total gauged action is

$$S''_{\mathrm{gauged,WZ}}[\mathbf{n},A] = S_{\mathrm{WZ}}[\mathbf{n}] + S_{ct}^{(1)}[\mathbf{n},A] + S_{ct}^{(2)}[\mathbf{n},A], \quad (6.20)$$

and under a gauge transformation we have

$$\delta_\xi S''_{\mathrm{gauged,WZ}}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-3)!} \frac{1}{3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^4 dn_{2m+1} \wedge \Phi^{(3)} \wedge d\xi \wedge A \wedge F. \quad (6.21)$$

We again integrate by parts and show

$$\delta_\xi S''_{\mathrm{gauged,WZ}}[\mathbf{n},A] = -\frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-3)!} \frac{1}{5\cdot 3} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^5 \Phi^{(3)} \wedge d\xi \wedge F^2. \quad (6.22)$$

Note that the denominator contains the *double factorial* $5!! = 5\cdot 3 = 5\cdot 3\cdot 1$. In general, we find that all of the counterterms contain a double factorial. Then, the third counterterm takes the form

$$S_{ct}^{(3)}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-3)!} \frac{1}{5!!} \int_{\mathbb{R}^{d-1,1}} n_{2m+1}^5 \Phi^{(3)} \wedge A \wedge F^2. \quad (6.23)$$

At this point, the pattern is clear. Continuing with this procedure we find that a total of m counterterms are needed to construct a gauged boundary action which satisfies Eq. (3.17), and the final gauged action is *completely gauge invariant*. It takes the form

$$S_{\mathrm{WZ,gauged}}[\mathbf{n},A] = S_{\mathrm{WZ}}[\mathbf{n}] + \sum_{r=1}^{m} S_{ct}^{(r)}[\mathbf{n},A], \quad (6.24)$$

where the rth counterterm is

$$S_{ct}^{(r)}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(m-r)!} \frac{1}{(2r-1)!!} \int_{\mathbb{R}^{d-1,1}} (n_{2m+1})^{2r-1} \times \Phi^{(r)} \wedge A \wedge F^{r-1}, \quad (6.25)$$

where $(2r-1)!!$ is the double factorial

$$(2r-1)!! = (2r-1)(2r-3)\dots(3)(1). \quad (6.26)$$

The final counterterm is just

$$S_{ct}^{(m)}[\mathbf{n},A] = \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(2m-1)!!} \int_{\mathbb{R}^{d-1,1}} (n_{2m+1})^{2m-1} A \wedge F^{m-1}, \quad (6.27)$$

and its change under a gauge transformation comes only from the transformation of A [the last component n_{2m+1} of the NLSM field does not transform under U(1)]. This explains why the final gauged action is completely gauge invariant: the change due to the transformation of A in the last term cancels the transformation from the previous counterterm in the action, and there are no further changes in the last term which remain to be canceled.

Now, let us show that the boundary of a BTI phase exhibits a $\mathbb{Z}_2$ symmetry-breaking response when the field n_a condenses in such a way that it preserves the U(1) symmetry, but breaks the $\mathbb{Z}_2$ symmetry. The only possible way for n_a to condense and fulfill these requirements is to have

$$n_{2m+1} = \pm 1, \tag{6.28a}$$

$$n_a = 0, \quad \forall\, a \neq 2m+1. \tag{6.28b}$$

In this case, all terms in $S_{\text{WZ,gauged}}[\mathbf{n},A]$ vanish except for the final counterterm ($r = m$), which gives the boundary electromagnetic response

$$S_{\text{eff,bdy}}[A] = \pm \frac{2\pi k}{\mathcal{A}_{2m}} \frac{1}{(2m-1)!!} \int_{\mathbb{R}^{d-1,1}} A \wedge F^{m-1}, \tag{6.29}$$

where we used $0! = 1$ and $\Phi^{(m)} = 1$. Now, we use the formulas

$$\mathcal{A}_{2m} = \frac{2\pi^m \sqrt{\pi}}{\Gamma\left(m + \frac{1}{2}\right)} \tag{6.30}$$

and

$$(2m-1)!! = \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right) \tag{6.31}$$

to find

$$S_{\text{eff,bdy}}[A] = \pm \frac{1}{2} \frac{k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-1,1}} A \wedge F^{m-1}. \tag{6.32}$$

Comparing to Eq. (2.1), we see that this is a CS response with level

$$N_{2m-1} = \pm \left(\frac{m!}{2}\right) k, \tag{6.33}$$

which is exactly *half* the response of the BIQH state which appears intrinsically in the same space-time dimension (which we calculated in Sec. IV). As we discussed in Sec. III, this boundary CS response is equivalent to a bulk electromagnetic response of the form of Eq. (2.2) with response parameter

$$\Theta_{2m} = 2\pi \left(\frac{m!}{2}\right) k. \tag{6.34}$$

However, we should recall from the discussion in Sec. III that the BTI phase with $k = 2$ is smoothly connected to the phase with $k = 0$. More generally, the BTI phase with $\theta = 2\pi k$ is smoothly connected to the phase with $\theta = 2\pi(k \pm 2)$. This means that the single nontrivial BTI phase is represented by the choice $k = 1$.

Finally, we note that the boundary of the BTI can be driven into the $\mathbb{Z}_2$ symmetry-breaking phase without explicitly breaking the $\mathbb{Z}_2$ symmetry. This can be done by adding a term $\mu\, n_{2m+1}^2$ to the Lagrangian. This term is invariant under the full $\text{U}(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI but, for $\mu > 0$ and sufficiently large, will drive the system into a phase in which the $\mathbb{Z}_2$ symmetry is spontaneously broken and $n_a = \pm\delta_{a,2m+1}$ (i.e., $n_{2m+1} = \pm 1$ and $n_a = 0$ for $a \neq 2m+1$).

VII. APPLICATIONS

In this section, we explore several applications of the results obtained so far. We start with the observation that the gauged boundary action for the BIQH state in $2m-1$ space-time dimensions can be used as building block to construct a bosonic analog of a Weyl, or chiral, semimetal in *any* even dimension. We refer to this state as a bosonic chiral semimetal (BCSM). We write an effective theory for this state in any even dimension d, compute its electromagnetic response, and compare this response with the response of an ordinary fermionic chiral semimetal. This construction represents a generalization to higher even dimensions of the work in Ref. [16] that constructed a bosonic analog of a *Dirac* semimetal in three dimensions.

As a second application, we show that the boundary theory of the BTI exhibits a bosonic analog of the parity anomaly of a single Dirac fermion in odd dimensions. As we discuss below, this is closely related to the fact (derived in Sec. VI) that the boundary theory of the BTI can exhibit a half-quantized BIQH state when the $\mathbb{Z}_2$ symmetry of the BTI is broken *spontaneously* at the boundary. This situation is clearly analogous to the time-reversal symmetry-breaking half-quantized integer quantum Hall state which appears on the surface of the familiar electron topological insulator [1]. This leads us to argue that the boundary theory for a BTI state in $2m$ dimensions cannot exist intrinsically in $2m-1$ dimensions without breaking (partially or fully) the symmetry of the BTI state.

Finally, we perform a detailed study of $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundary of BTI states. We use a dimensional reduction formula for NLSMs with WZ term, analogous to the dimensional reduction formula for theta terms that we derive in Appendix D, to show that a $\mathbb{Z}_2$ symmetry-breaking domain wall on the boundary of a BTI state in $2m$ dimensions hosts a lower-dimensional theory which is identical to the boundary theory of the BIQH state in $2m-1$ dimensions. We show that the U(1) anomaly of the theory on the domain wall is exactly canceled by an inflow of charge from the two $\mathbb{Z}_2$-breaking regions on either side of the domain wall. This calculation is an important consistency check for our results on the response of BIQH and BTI states, and also provides a clear example of the phenomenon of anomaly inflow in the context of bosonic SPT phases.

A. Bosonic analog of a Weyl semimetal in any even dimension

In this section, we describe how a bosonic analog of a Weyl semimetal can be constructed in any even space-time dimension d using two copies of an $\text{O}(d+2)$ NLSM with Wess-Zumino (WZ) term. Before discussing the bosonic analog, let us first review the basic construction of a Weyl (or more generally a chiral) semimetal of fermions in any even dimension d. Note that our construction here still assumes a pointlike structure of the Fermi surface even in higher dimensions, as opposed to the recent construction in Ref. [90] using Weyl sheets in six space-time dimensions. We consider a Dirac fermion Ψ in d dimensions. To write an action for a Dirac fermion, we need the gamma matrices γ^μ, $\mu = 0,\ldots,d-1$, which obey the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ [and we choose $\eta = \text{diag}(1,-1,\ldots,-1)$]. When d is even, we have an extra element $\overline{\gamma}$ of the Clifford algebra which anticommutes with the other gamma matrices and can be chosen to satisfy $\overline{\gamma}^\dagger = \overline{\gamma}$ and $\overline{\gamma}^2 = \mathbb{I}$ ($\overline{\gamma}$ is the higher-dimensional analog of γ^5 in $d = 4$). In the basis (known as the Weyl basis in $d = 4$) in which $\overline{\gamma}$ takes the block diagonal form

$$\overline{\gamma} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{7.1}$$

the fermion Ψ breaks up into chiral and antichiral parts as

$$\Psi = (\Psi_+,\Psi_-)^T. \tag{7.2}$$

Now a minimal, two-node chiral (or Weyl) semimetal (CSM) in d dimensions is described at low energies by chiral fermions $\Psi_\pm$ separated in momentum by $2\mathbf{B}$ and in energy by $2B_t$, where $\mathbf{B} = (B_1,\ldots,B_{d-1})$ should be thought of as a vector in a $(d-1)$-dimensional momentum space (or Brillouin zone). We assume here that the components B_μ ($\mu = 0,\ldots,d-1$, $B_0 = B_t$) are constant, although the results below are expected to hold approximately if the components B_μ are slowly varying functions of x^μ. In addition, both chiral fermions carry charge e of an external U(1) gauge field A_μ. Using the extra gamma matrix $\overline{\gamma}$, an action capturing this low-energy physics takes the form

$$S_{\mathrm{CSM}}[\Psi,A,B] = \int d^d x\, i\overline{\Psi}(\not{\partial} - ie\not{A} - i\not{B}\overline{\gamma})\Psi, \tag{7.3}$$

where $\overline{\Psi} = \Psi^\dagger\gamma^0$ and we used the Feynman slash notation $\not{\partial} = \gamma^\mu\partial_\mu$, etc. In addition, we have assumed that the separation of $\Psi_\pm$ in momentum and energy is symmetric about $B_\mu = 0$, so that $\Psi_\pm$ is located at $\pm B_\mu$ in momentum/energy space. We also note here that in this low-energy description, the chiral fermion fields $\Psi_\pm$ couple only to the linear combinations $eA_\mu \pm B_\mu$ of the vector fields A_μ and B_μ. This feature will be important later in our construction of a bosonic analog of the CSM.

The quasitopological part of the electromagnetic response of the CSM follows directly from the *axial anomaly* of a Dirac fermion in d dimensions [91]. This is because this response is generated by attempting to remove the coupling to B_μ from the action via the chiral rotation

$$\Psi \to e^{i\xi\overline{\gamma}}\Psi, \tag{7.4}$$

with the parameter ξ chosen as

$$\xi = B_\mu x^\mu. \tag{7.5}$$

This transformation removes the coupling to B_μ from the action. The physical interpretation of this transformation is that it moves the two cones of the chiral semimetal to the origin of the Brillouin zone. However, the path-integral measure is not invariant under this transformation. Instead, the change in the path-integral measure generates a new term in the action of the form ("f" stands for fermionic)

$$S^{(f)}_{\mathrm{eff}}[A,B] = -\frac{2}{(\frac{d}{2})!}\left(\frac{e}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^{d-1,1}} \xi\, (F)^{\frac{d}{2}}. \tag{7.6}$$

Noting that $d\xi = B_\mu dx^\mu \equiv B$ (for constant B_μ), and integrating by parts gives the final form of the chiral semimetal response

$$S^{(f)}_{\mathrm{eff}}[A,B] = \frac{2}{(\frac{d}{2})!}\left(\frac{e}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^{d-1,1}} B\wedge A\wedge (F)^{\frac{d}{2}-1}. \tag{7.7}$$

It is also interesting to consider the form (7.6) of the semimetal response (before integrating by parts), as it has the form of the "Chern character" terms discussed earlier in the paper, but with a space-time-dependent angle $\xi = B_\mu x^\mu$ appearing in the integrand.

So, under the chiral transformation of Eq. (7.4), the CSM action of Eq. (7.3) transforms as

$$S_{\mathrm{CSM}}[\Psi,A,B] \to S_{\mathrm{CSM}}[\Psi,A,0] + S^{(f)}_{\mathrm{eff}}[A,B], \tag{7.8}$$

where we again emphasize that the term $S^{(f)}_{\mathrm{eff}}[A,B]$ was generated by the change in the path-integral measure under the chiral transformation of Eq. (7.4). Thus, we can say that the electromagnetic response of the CSM with nonzero separation vector B_μ differs from the response of a CSM with separation vector $B_\mu = 0$ (i.e., a system where the two chiral parts of the Dirac fermion sit at the same point in momentum space) by the term $S^{(f)}_{\mathrm{eff}}[A,B]$ from Eq. (7.7). For $d = 2$ and 4, the responses are

$$S^{(f)}_{\mathrm{eff}}[A,B] = \frac{e}{\pi}\int_{\mathbb{R}^{1,1}} B\wedge A \tag{7.9}$$

and

$$S^{(f)}_{\mathrm{eff}}[A,B] = \frac{e^2}{4\pi^2}\int_{\mathbb{R}^{3,1}} B\wedge A\wedge F, \tag{7.10}$$

respectively. We see that the general expression of Eq. (7.7) agrees with the known expressions in low dimensions [91,92].

Having reviewed the basic properties of fermionic chiral semimetals, we are now ready to describe our construction of a bosonic analog of a CSM (BCSM). To motivate our construction, we note that the low-energy theory of the CSM has (at least) two essential properties which we use as a guide to construct the BCSM model. The first property is that the CSM model is constructed from two building blocks, namely, the chiral fermion theories with fields $\Psi_\pm$, such that each building block *on its own* would have an anomaly in the U(1) symmetry which sends $\Psi_s \to e^{i\xi_s}\Psi_s$, $s = \pm$. The second property (already noted above) is that the two building blocks $\Psi_\pm$ couple only to the linear combinations $eA_\mu \pm B_\mu$ of vector fields. This property, combined with the axial anomaly of the Dirac fermion, is responsible for the form of the CSM response shown in Eq. (7.7). We now describe the construction of a bosonic theory with very similar properties.

Our low-energy theory for a BCSM in d dimensions (d even) consists of two copies of the O$(d+2)$ NLSM with WZ term, i.e., two copies of the boundary theory of the BIQH state in $d+1$ dimensions. To understand this system, we briefly recall a few facts from Sec. IV about the boundary theory of the BIQH state. The boundary of the BIQH state in $2m-1$ dimensions is described by an O$(2m)$ NLSM with WZ term. Here, the dimension d is related to m by $d = 2m-2$ as we are constructing a model using the boundary theory for the BIQH state. Finally, recall that under a U(1) transformation the NLSM field transforms as in Eq. (3.13) (in units where the boson charge e is set to 1). We showed that the properly gauged boundary action had an anomaly in this U(1) symmetry, with the anomaly given explicitly by Eq. (4.33).

To construct an effective theory for a bosonic semimetal in d dimensions we use two copies of the boundary theory of the BIQH state. We label the fields of the two copies by $\mathbf{n}_\pm$, or $b_{\ell,\pm}$, when written in terms of bosons, and we take the two copies to have opposite level on their WZ term, $k_\pm = \pm k$. Finally, in the effective theory we model the separation of the two copies in momentum/energy space by coupling the fields $b_{\ell,\pm}$ to the linear combinations $A_\mu \pm B_\mu$ of the external U(1) gauge field A_μ and the momentum/energy shift field B_μ. Then, our action for the BCSM theory takes the form

$$\tilde{S}_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B] = S_{\rm gauged}[\mathbf{n}_+,A+B] + S_{\rm gauged}[\mathbf{n}_-,A-B], \quad (7.11)$$

where $S_{\rm gauged}[\mathbf{n},A]$ is the properly gauged action for one O$(d+2)$ NLSM with WZ term and coupled to the external field A (as constructed in Sec. IV). We put a tilde on $\tilde{S}_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B]$ because, as we now discuss, this action has an inconsistency and must be modified.

Suppose that the vector field B_μ, which is a constant in the context of the chiral semimetal, instead had a nontrivial space-time dependence, i.e., $dB \neq 0$. In this case, the action in Eq. (7.11) is *not* invariant under the U(1) gauge transformation $b_{\ell,\pm} \to e^{i\theta} b_{\ell,\pm}$, $A \to A + d\theta$. Instead, under this transformation one can show that the change in the action of Eq. (7.11) is

$$\delta_\theta \tilde{S}_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B] = -\frac{k}{(2\pi)^{m-1}} \sum_{p=0}^{m-1} \binom{m-1}{p} [1+(-1)^{m-p}] \times \int_{\mathbb{R}^{d-1,1}} d\theta \wedge (dA)^p \wedge B \wedge (dB)^{m-2-p}, \quad (7.12)$$

where $2m-1 = d+1$. This equation requires some explanation. To compute it we used the relation (4.32) for the U(1) anomaly for each gauged WZ theory in Eq. (7.11) (but coupled to the combinations of fields $A \pm B$ instead of A alone), then expanded the powers $(dA \pm dB)^{m-1}$ using the binomial expansion, and finally performed an integration by parts to move one derivative off of B and onto θ.

So, in the presence of a space-time-dependent B_μ, our putative semimetal model is not invariant under U(1) gauge transformations. To remedy this, we modify the action by adding the counterterm

$$S_{ct}[A,B] = \frac{k}{(2\pi)^{m-1}} \sum_{p=0}^{m-1} \binom{m-1}{p} [1+(-1)^{m-p}] \times \int_{\mathbb{R}^{d-1,1}} A \wedge (dA)^p \wedge B \wedge (dB)^{m-2-p}. \quad (7.13)$$

The change in this counterterm under $A \to A + d\theta$ exactly compensates for the change in Eq. (7.11) under the U(1) gauge transformation, and so the modified BCSM action

$$S_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B] = \tilde{S}_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,B] + S_{ct}[A,B] \quad (7.14)$$

is completely gauge invariant even in the presence of a space-time dependent B_μ. The counterterm $S_{ct}[A,B]$ is the analog in our bosonic theory of the *Bardeen counterterm* that one adds to the theory of a Dirac fermion coupled to vector and axial vector gauge fields to ensure conservation of the vector current in the quantum theory [93]. Since this counterterm is absolutely necessary for the more general case of a space-time dependent B_μ, we argue that one should include it even in the simple semimetal setting in which we take B_μ to be a constant. If we now restrict to the case of a constant B_μ, then only the $p = m-2$ term in the counterterm survives, and the counterterm reduces to

$$S_{ct}[A,B] \to -\frac{2k}{(2\pi)^{m-1}}(m-1) \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2}, \quad (7.15)$$

where we used $\binom{m-1}{m-2} = m-1$.

To compute the response of the modified BCSM theory in Eq. (7.14), we attempt to remove the coupling to B from the action via the chiral transformation

$$b_{\ell,\pm} \to e^{\pm i\xi} b_{\ell,\pm}, \quad (7.16)$$

where $\xi = B_\mu x^\mu$ as in the fermionic case. Note that this transformation takes the opposite sign for the two copies of the NLSM theory: this is the analog in the bosonic theory of the chiral transformation of Eq. (7.4) that we performed in the fermionic case. Using the U(1) anomaly for the boundary theory of the BIQH state from Eq. (4.33), we find that under this transformation the original effective action for the BCSM state transforms as

$$\tilde{S}_{\rm BCSM}[\mathbf{n}_+,\ \mathbf{n}_-,A,B] \to \tilde{S}_{\rm BCSM}[\mathbf{n}_+,\mathbf{n}_-,A,0] + \tilde{S}^{(b)}_{\rm eff}[A,B], \quad (7.17)$$

where

$$\tilde{S}^{(b)}_{\rm eff}[A,B] = -\frac{2k}{(2\pi)^{m-1}} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2}. \quad (7.18)$$

However, this is not the end of the story as the full action for the BCSM state contains the counterterm $S_{ct}[A,B]$. When we combine Eq. (7.18) with the counterterm (neglecting those parts of the counterterm containing dB), then we obtain the final expression for the response of the BCSM,

$$S^{(b)}_{\rm eff}[A,B] = -2km\left(\frac{e}{2\pi}\right)^{m-1} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{m-2} \quad (7.19)$$

or, in terms of d,

$$S^{(b)}_{\rm eff}[A,B] = -2k\left(\frac{d}{2}+1\right)\left(\frac{e}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^{d-1,1}} B \wedge A \wedge (dA)^{\frac{d}{2}-1}, \quad (7.20)$$

where we have restored the charge e of the bosons. This equation is the final form of the response of our BCSM model.

If we set $k = 1$ and compare the BCSM response in Eq. (7.20) to the fermionic CSM response in Eq. (7.7), then we see that the response of the BCSM in d dimensions is larger by a factor of $(\frac{d}{2}+1)!$. To understand this number recall that our BCSM model in d dimensions is constructed from two copies of the boundary state for a BIQH state in $d+1$ dimensions. Setting $d+1 = 2m-1$, we see that $(\frac{d}{2}+1)! = m!$, so we find that the coefficients for the response of the bosonic and fermionic semimetals in d dimensions differ by exactly the same factor we found in Sec. IV for the coefficients for the response of BIQH and FIQH states in one dimension higher.

We can also see from Eq. (7.20) that *at the level of the electromagnetic response*, the BCSM theory at level k is equivalent to k copies of the BCSM theory at level 1. However, as a quantum field theory we certainly expect the theory at level k to be distinct from k copies of the theory at level 1. This can be seen very clearly in the case where $d = 2$. In this case, the BCSM model consists of two copies of an O(4) NLSM with WZ terms at levels k and $-k$, respectively. The O(4) NLSM can be reformulated in terms of a 2×2 SU(2) matrix field, and so (when the coupling constant for the NLSM takes on a particular value) the O(4) NLSM with WZ term at level k is equivalent to the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. Now, the $SU(2)_k$ theory is distinct from k copies of the $SU(2)_1$ theory (this can be seen by comparing central charges), and so we conclude that even in the simplest case of two dimensions, the BCSM model at level k is distinct (as a quantum field theory) from k copies of the BCSM model at level 1. However, it is entirely possible that k copies of the BCSM model at level 1 could flow under the renormalization group to the BCSM model at level k. In the simple $d = 2$ case discussed in this paragraph, this flow is allowed by Zamolodchikov's c theorem [94].

B. Bosonic analog of the parity anomaly on the boundary of BTI phases

In this section, we show that the half-quantized BIQH on the BTI boundary, which we derived in Sec. VI, represents a bosonic analog of the parity anomaly [45–49], which is an anomaly that is usually associated to massless Dirac fermions in odd dimensions. To start, we give a brief review of the parity anomaly in the fermionic case before explaining the bosonic analog.

To understand the parity anomaly for Dirac fermions in odd dimensions, consider a theory of a single massless Dirac fermion Ψ with U(1) symmetry in $2m-1$ dimensions. We can couple Ψ to an external electromagnetic field A and then integrate out Ψ to obtain the partition function

$$Z[A] = \int [D\Psi][D\overline{\Psi}]e^{iS[\Psi,A]}, \tag{7.21}$$

and the effective action for the external field A,

$$S_{\text{eff}}[A] = -i\ln(Z[A]). \tag{7.22}$$

The action $S[\Psi,A]$ (with Ψ a massless fermion) has an additional discrete symmetry, which is time-reversal symmetry when the space-time dimension equals 3 mod 4, or charge-conjugation (particle-hole) symmetry [95] when the space-time dimension equals 1 mod 4 (in Euclidean space-time the discrete symmetry in any dimension can be chosen to be reflection of a single coordinate). However, when one proceeds to calculate the effective action $S_{\text{eff}}[A]$, one finds that there is no choice of regularization procedure which yields an action $S_{\text{eff}}[A]$ which has this extra discrete symmetry and is also gauge invariant. In other words, one can choose to preserve either the discrete symmetry or gauge invariance, but not both. For example, when Pauli-Villars regularization is used to compute $S_{\text{eff}}[A]$, the mass terms for the regulator fermions explicitly break the discrete symmetry, and this results in the appearance of a term in $S_{\text{eff}}[A]$ which also breaks the discrete symmetry.

At this point, it helps to look at a specific example. We choose the case of a massless Dirac fermion Ψ in three space-time dimensions with U(1) symmetry and $\mathbb{Z}_2^T$ time-reversal symmetry, which was the case originally studied in Refs. [45,46]. This case also applies directly to the study of the surface of the familiar electron topological insulator in four dimensions. Because of the U(1) symmetry, Ψ can be coupled to the external field A. To discuss the transformation of Ψ under time reversal, it is convenient [8] to choose the gamma matrices in the "mostly minus" metric to be $\gamma^0 = \sigma^z$, $\gamma^1 = i\sigma^y$, and $\gamma^2 = -i\sigma^x$, where σ^a, $a = x,y,z$, are the three Pauli matrices (and recall that a single Dirac fermion in three dimensions has two components). With this choice, the time-reversal transformation of Ψ takes the form

$$\mathbb{Z}_2^T : \Psi(t,\mathbf{x}) \to i\sigma^y\Psi(-t,\mathbf{x}), \tag{7.23}$$

while the components A_μ of A transform as

$$\mathbb{Z}_2^T : A_0(t,\mathbf{x}) \to A_0(-t,\mathbf{x}), \tag{7.24}$$

$$A_i(t,\mathbf{x}) \to -A_i(-t,\mathbf{x}), \quad i = 1,2. \tag{7.25}$$

In the absence of a mass term for Ψ the action $S[\Psi,A]$ for Ψ minimally coupled to A has time-reversal symmetry in addition to the U(1) symmetry. However, when Pauli-Villars regularization is used to compute $S_{\text{eff}}[A]$, one finds that $S_{\text{eff}}[A]$ contains the time-reversal-breaking CS term for A.[4] In addition, the level of this CS term is equal to $\pm\frac{1}{2}$, which is half of the minimum Hall conductance possible for free fermions in three dimensions (i.e., the CS term with level $\pm\frac{1}{2}$ is like a half-quantized integer quantum Hall state of fermions). One can think of the parity anomaly as a quantum version of the spontaneous breaking of a discrete symmetry. Indeed, the value of the induced CS term in $S_{\text{eff}}[A]$ is determined by the sign of the mass of the Pauli-Villars regulator fermion, and this choice of sign is arbitrary in the same way that the choice of a particular point on the vacuum manifold of a "Mexican hat" potential is arbitrary.

To demonstrate that a bosonic analog of the parity anomaly occurs in the boundary of a BTI phase, we first need to discuss the symmetries of the BTI theory coupled to A. As we discussed in Sec. III, the NLSM field n_a transforms under the $\mathbb{Z}_2^T$ or $\mathbb{Z}_2^C$ symmetry of the BTI as shown in Eq. (3.14). Recall that in the case where the $\mathbb{Z}_2$ symmetry is time reversal, we also need to send $t \to -t$ in the argument of the NLSM field n_a and in the action. For a space-time of dimension d (which in our convention is the dimension of the boundary of the SPT phase), the field A transforms under time reversal and

[4] In a more precise treatment, Pauli-Villars regularization leads to an effective action which is proportional to the Atiyah-Potodi-Singer *eta invariant* of the Dirac operator [47]. In certain cases, the expression in terms of the eta invariant can then be replaced with the simpler expression in terms of a CS term with half-quantized level. However, this more precise treatment using the eta invariant still gives an effective action that breaks the time-reversal symmetry of the original Lagrangian for Ψ and A.

charge conjugation as

$$\mathbb{Z}_2^T : A_0(t,\mathbf{x}) \to A_0(-t,\mathbf{x}), \tag{7.26}$$

$$A_i(t,\mathbf{x}) \to -A_i(-t,\mathbf{x}), \quad i = 1,\ldots,d-1 \tag{7.27}$$

and

$$\mathbb{Z}_2^C : A_\mu(t,\mathbf{x}) \to -A_\mu(t,\mathbf{x}), \forall\, \mu \tag{7.28}$$

where $\mathbf{x} = (x^1,\ldots,x^{d-1})$ denotes the spatial coordinates.

The gauged WZ action of Eq. (6.24) for the boundary of the BTI phase has the $\mathbb{Z}_2^C$ (for m odd) or $\mathbb{Z}_2^T$ (for m even) symmetry of the BTI, in addition to the invariance under U(1) gauge transformations. To see this, we simply note that the counterterms from Eq. (6.25) transform under these two $\mathbb{Z}_2$ symmetries as

$$\mathbb{Z}_2^T : S_{ct}^{(r)}[\mathbf{n},A] \to (-1)^m S_{ct}^{(r)}[\mathbf{n},A] \tag{7.29}$$

and

$$\mathbb{Z}_2^C : S_{ct}^{(r)}[\mathbf{n},A] \to (-1)^{m+1} S_{ct}^{(r)}[\mathbf{n},A]. \tag{7.30}$$

So, the gauged WZ action for the BTI boundary has $\mathbb{Z}_2^T$ symmetry for m even and $\mathbb{Z}_2^C$ symmetry for m odd.

Now, although the gauged WZ action for the BTI boundary has the $\mathbb{Z}_2$ symmetry, we have seen in Sec. VI that it is possible to add the symmetry-allowed term $\mu\, n_{2m+1}^2$ to the Lagrangian and drive the boundary of the BTI into a phase in which the $\mathbb{Z}_2$ symmetry is *spontaneously* broken by the condensate $n_a = \pm\delta_{a,2m+1}$. In addition, we showed that when the field n_a condenses in this way it leads to a CS term in the effective action for A on the boundary of the BTI phase. The CS term in $2m-1$ dimensions breaks $\mathbb{Z}_2^T$ symmetry for m even, and $\mathbb{Z}_2^C$ symmetry for m odd, so the effective action for A does not have the $\mathbb{Z}_2$ symmetry of the BTI phase. We also saw that the CS level turned out to be quantized in *half-integer* multiples of $m!$.

Since the CS term in the effective action for the boundary breaks the $\mathbb{Z}_2$ symmetry of the BTI phase, and since the boundary also exhibits a "half" BIQH response, we conclude that the boundary theory of the BTI phase exhibits an anomaly in the $\mathbb{Z}_2$ symmetry which is almost an exact analog of the parity anomaly of a Dirac fermion in odd dimensions.

There is, however, one important difference between the bosonic analog of the parity anomaly discussed here and the original parity anomaly for Dirac fermions. The difference is the fact that in the bosonic case the spontaneous breaking of the $\mathbb{Z}_2$ symmetry is a classical effect, whereas in the original fermionic case the $\mathbb{Z}_2$ symmetry is broken spontaneously only at the quantum level (by the choice of the sign of the mass of the regulator fermions). One likely explanation for this difference is as follows. Since the NLSM description of the bosonic SPT phase is an effective field-theory description, i.e., it does not involve the microscopic degrees of freedom in the SPT phase, it is entirely possible that the quantum anomaly of any putative microscopic description of the SPT phase is already captured at the classical level in the effective NLSM description of the phase. This is, in fact, exactly the way in which quantum anomalies of fermionic theories are captured at the classical level in effective descriptions of those fermionic theories in terms of gauged WZ actions [39,40]. In addition, we have already seen in this paper that the pertubative U(1) anomaly on the boundary of BIQH states is completely captured at the classical level in the gauged WZ description of the BIQH boundary. For this reason, we believe that the bosonic analog of the parity anomaly discussed here is a bona fide quantum effect that occurs in the boundary theory of a BTI phase, and that this anomaly would appear as a true quantum anomaly in a more microscopic description of the boundary of the BTI. We are therefore led to argue that, due to this anomaly, the boundary theory of a $2m$-dimensional BTI phase cannot be realized intrinsically in $2m-1$ dimensions without breaking either the U(1) or the $\mathbb{Z}_2$ symmetry of the BTI phase.

Finally, let us describe one more way of understanding the bosonic analog of the parity anomaly in the specific case of the boundary theory of the four-dimensional BTI. As we know, the boundary theory is an O(5) NLSM with WZ term in three dimensions. Let us investigate what happens in this theory when we thread a 2π delta function flux of the gauge field at a point in space. This method of analysis is known to expose the parity anomaly in the theory of a single massless Dirac fermion in three dimensions (see, for example, Ref. [14]) and so it should work in this case as well. For simplicity, we consider a deformation of the O(5) theory in which we set $n_5 = 0$ [this deformation preserves the U(1) and time-reversal symmetry]. In this case, the WZ term at level k reduces to a theta term for the four-component field $(n_1,\ldots,n_4)$ with the theta angle equal to $\theta = \pi k$. In particular, for $k = 1$ (which represents the nontrivial BTI phase in four dimensions) the WZ term with level $k = 1$ reduces to a theta term with theta angle $\theta = \pi$.

Threading a 2π delta function flux at a point $\mathbf{x}_0$ in space will cause the phase of both bosons $b_1 = n_1 + i n_2$ and $b_2 = n_3 + i n_4$ to wind by 2π about $\mathbf{x}_0$, i.e., there is a vortex centered at $\mathbf{x}_0$ in the phase of both bosons. In Appendix B of Ref. [16], two of us performed a detailed study of vortex configurations of a *single* boson b_1 or b_2 in the O(4) NLSM with theta term and with $\theta = \pi$. In particular, we quantized global fluctuations of the theory over such a vortex background and showed that the ground state of these fluctuations was doubly degenerate, with the two degenerate states having charges $\pm\frac{1}{2}$. This analysis confirmed the arguments of Ref. [29] that a vortex in a single boson b_1 or b_2 should carry charge $\pm\frac{1}{2}$. As stated above, threading a 2π flux of A_μ at $\mathbf{x}_0$ induces a vortex in *both* b_1 and b_2 at that point. This composite object has an integer charge and so is naively gauge invariant, however, this composite object can actually be shown to be a fermion [16,29,96]. This fact clearly shows the anomaly in theory, as there should be no local fermionic particle with integer charge in a system made up of bosons of unit charge. The existence of a fermion with unit charge in the boundary theory of the BTI can also be inferred from the presence of a CS term at level $N_3 = 1$ in the response action for the BTI boundary using an argument from Ref. [18].

C. $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundary of BTI

We close this section by analyzing the physics of a domain wall between two opposite $\mathbb{Z}_2$ symmetry-breaking regions on the boundary of a BTI state in $2m$ dimensions. In particular, we derive the low-energy theory that exists on the domain wall, and we show that this theory has a U(1) anomaly which is exactly canceled by the contributions of the CS response actions for the $\mathbb{Z}_2$ symmetry-breaking regions on either side of the domain wall. We find that the theory which lives on the domain wall is identical to the boundary theory for the BIQH phase in $2m-1$ dimensions, and so this demonstration of anomaly cancellation for domain-wall configurations on the BTI boundary provides a nice consistency check between our gauged actions for BIQH and BTI surfaces.

To start, recall from Sec. VI that the boundary of a BTI phase in $2m$ dimensions, which is described by an $O(2m+1)$ NLSM with WZ term at level k, can exhibit a $\mathbb{Z}_2$ symmetry-breaking response of the form ($d = 2m-1$ is again the boundary dimension)

$$S_{\text{eff}}[A] = \pm\frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{d-1,1}} A\wedge F^{m-1}, \qquad (7.31)$$

when the NLSM field n_a condenses as in Eq. (6.28), i.e., $n_{2m+1} = \pm 1$ and all other components of $\mathbf{n}$ equal to zero. As we discussed earlier, this particular condensation pattern is the only one which preserves the U(1) symmetry of the BTI phase while breaking the $\mathbb{Z}_2$ symmetry.

We now consider a domain-wall configuration between opposite $\mathbb{Z}_2$-breaking regions on the boundary. Let $(x^0,\ldots,x^{d-1})$ be the boundary space-time coordinates. We study a configuration of the system in which n_{2m+1} condenses as $n_{2m+1} = 1$ in the region $x^{d-1} > 0$, and as $n_{2m+1} = -1$ in the region $x^{d-1} < 0$. Hence, the domain wall is in the x^{d-1} direction. Then, on the two sides of the domain wall, the electromagnetic response is given by

$$S_{\text{eff},+}[A] = \frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{H}_+^{d-1,1}} A\wedge F^{m-1} \qquad (7.32)$$

and

$$S_{\text{eff},-}[A] = -\frac{1}{2}\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{H}_-^{d-1,1}} A\wedge F^{m-1}, \qquad (7.33)$$

respectively, where $\mathbb{H}_+^{d-1,1}$ denotes the half-space $\{x\in\mathbb{R}^{d-1,1}|\ x^{d-1} > 0\}$, and similarly for $\mathbb{H}_-^{d-1,1}$. If we perform a gauge transformation, then the change in the total effective action is

$$\delta_\xi S_{\text{eff},+}[A] + \delta_\xi S_{\text{eff},-}[A] = \frac{k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{d-2,1}} \xi F^{m-1}, \qquad (7.34)$$

where the integration is over the domain wall which is just the space $\mathbb{R}^{d-2,1}$ sitting at $x^{d-1} = 0$. Note also that the contributions from $S_{\text{eff},\pm}[A]$ add instead of subtract due to the fact that the domain wall is on the opposite side of the two regions (the domain wall lies to the right of the region $\mathbb{H}_+^{d-1,1}$ and to the left of the region $\mathbb{H}_-^{d-1,1}$, so when we integrate the total derivative the boundary terms coming from each integral appear with opposite signs).

Next, we derive the theory which lives on the domain wall and show that it has a U(1) anomaly which precisely cancels the gauge transformation from Eq. (7.34). To derive this theory, we analyze the BTI surface theory in the presence of a domain wall in n_{2m+1}. Concretely, we assume that the $O(2m+1)$ NLSM field takes on the domain-wall configuration

$$\mathbf{n} = \{\sin[f(x^{d-1})]\mathbf{N}(x^0,\ldots,x^{d-2}),\cos[f(x^{d-1})]\}, \qquad (7.35)$$

where $\mathbf{N}$ is a $2m$-component unit vector which depends only on the coordinates $(x^0,\ldots,x^{d-2})$ on the domain wall, and where $f(x^{d-1})$ is a function with the limiting behavior

$$\lim_{x^{d-1}\to\infty} f(x^{d-1}) = 0, \qquad (7.36)$$

$$\lim_{x^{d-1}\to-\infty} f(x^{d-1}) = \pi. \qquad (7.37)$$

This guarantees that $\mathbf{n}$ asymptotes to a configuration with $n_{2m+1} = \pm 1$ as $x^{d-1}\to\pm\infty$. To solve for the theory which lives on the domain wall, we evaluate the $O(2m+1)$ NLSM action (with WZ term) on this configuration. Evaluating the kinetic term of the NLSM on the domain-wall configuration is simple, provided that we assume the function $f(x^{d-1})$ is sufficiently well behaved so that the integration over x^{d-1} gives a finite answer. We therefore focus our attention on the WZ term since any anomalous behavior of the domain-wall theory should come from this term. The WZ term involves an extension $\tilde{\mathbf{n}}$ of the field $\mathbf{n}$ into a fictitious extra direction with coordinate $s\in[0,1]$, and so we need to decide how to extend our domain-wall configuration into this extra direction. Here, we assume the extension takes the form

$$\tilde{\mathbf{n}} = \{\sin[f(x^{d-1})]\tilde{\mathbf{N}}(s,x^0,\ldots,x^{d-2}),\cos[f(x^{d-1})]\}, \qquad (7.38)$$

so that all of the s dependence of the extension is in the extended $2m$-component field $\tilde{\mathbf{N}}$, while the function $f(x^{d-1})$ still depends only on x^{d-1}.

We now examine how the WZ term of the $O(2m+1)$ NLSM reduces on the extended domain-wall configuration $\tilde{\mathbf{n}}$ of Eq. (7.38). The analysis is similar (but not identical) to that in Appendix D for the dimensional reduction of theta terms in NLSMs on defect configurations of the NLSM field. Recall that the WZ term takes the form

$$S_{\text{WZ}}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m}}\int_{[0,1]\times\mathbb{R}^{d-1,1}} \tilde{\mathbf{n}}^*\omega_{2m}, \qquad (7.39)$$

where ω_{2m} is the volume form for the sphere S^{2m}, and $[0,1]\times\mathbb{R}^{d-1,1}$ is the extended space-time (which we called $\mathcal{B}$ before). Using the relations

$$dn_a = \sin(f)dN_a + \cos(f)N_a df,\quad a = 1,\ldots,2m \qquad (7.40)$$

and

$$dn_{2m+1} = -\sin(f)df, \qquad (7.41)$$

one can show that on this configuration the volume form ω_{2m} for $\mathbf{n}$ reduces to

$$\omega_{2m}\to[\sin(f)]^{2m-1}df\wedge\omega_{2m-1}, \qquad (7.42)$$

where

$$\omega_{2m-1} = \sum_{a=1}^{2m}(-1)^{a-1}N_a dN_1\wedge\cdots\wedge\overline{dN_a}\wedge\cdots\wedge dN_{2m} \qquad (7.43)$$

is the volume form for N_a. Since the WZ term involves the pullback $\tilde{\mathbf{n}}^*\omega_{2m}$ of the volume form to the extended space-time, we find that the WZ term reduces as

$$S_{\mathrm{WZ}}[\mathbf{n}] \to \frac{2\pi k}{\mathcal{A}_{2m}} \int_{-\infty}^{\infty} dx^{d-1}\, f'(x^{d-1})\{\sin[f(x^{d-1})]\}^{2m-1} \times \int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^* \omega_{2m-1}$$
$$= \frac{2\pi k}{\mathcal{A}_{2m}}\left(-\frac{\sqrt{\pi}\,\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}\right)\int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^* \omega_{2m-1}$$
$$= -\frac{2\pi k}{\mathcal{A}_{2m-1}}\int_{[0,1]\times\mathbb{R}^{d-2,1}} \tilde{\mathbf{N}}^* \omega_{2m-1}. \tag{7.44}$$

We see that the theory localized on the domain wall is an O($2m$) NLSM for the field $\mathbf{N}$, with a WZ term at level $-k$. This theory also appears at the boundary theory of the BIQH phase in $2m-1$ dimensions, as discussed in Sec. IV. The extra minus sign on the level of the domain-wall theory, as compared with the level of the boundary theory of the BTI, is very important. Indeed, from our previous formula (4.32) for the U(1) anomaly of the O($2m$) NLSM with WZ term we see that, under a gauge transformation, the theory on the domain wall transforms as

$$\delta_\xi S_{\mathrm{DW}}[\mathbf{N},A] = -\frac{k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{d-2,1}} \xi F^{m-1}. \tag{7.45}$$

This exactly cancels Eq. (7.34), which was the contribution flowing into the domain wall from the $\mathbb{Z}_2$-breaking regions on either side, and so this calculation gives a nice example of anomaly inflow at the domain walls on the boundary of SPT phases. It also provides an important consistency check of the gauged WZ actions calculated in this paper for the boundaries of BIQH and BTI phases (since it relates the calculation of the BTI boundary CS response to the calculation of the anomaly of the BIQH boundary theory).

VIII. CONCLUSION

In this paper, we calculated the electromagnetic response of bosonic SPT phases with U(1) symmetry in all space-time dimensions. In particular, we focused our attention on BIQH phases in odd dimensions and BTI phases in even dimensions. To calculate the response of these phases, we used the NLSM description of bosonic SPT phases and the tool of gauged WZ actions. This enabled us to compute the coefficients N_{2m-1} and Θ_{2m} which determine, via Eqs. (2.1) and (2.2), the electromagnetic response of BIQH and BTI states in all odd and even space-time dimensions, respectively. We found that for BIQH states the coefficient N_{2m-1} was quantized in units of $m!$, and that the nontrivial BTI state in $2m$ dimensions has $\Theta_{2m} = 2\pi(\frac{m!}{2})$. This response for the BTI is equivalent to a $\mathbb{Z}_2$ symmetry-breaking quantum Hall state on the boundary of the BTI with CS level equal to $\frac{m!}{2}$, which is exactly half the response of the BIQH state which can be realized intrinsically in the same space-time dimension. In Sec. V we showed that the value of $m!$ for the CS level can be understood by studying the transformation of the CS term under large U(1) gauge transformations on general Euclidean manifolds which may or may not admit a spin structure. In that section we also applied this gauge-invariance argument to study the electromagnetic *and* gravitational responses of fermionic SPT phases with U(1) symmetry in odd space-time dimensions.

We then used our gauged WZ actions for the boundaries of the BIQH and BTI phases to further investigate the physics of BIQH and BTI states, and to construct other interesting states. In particular, we showed how two copies of the BIQH boundary theory could be used to construct an effective theory for a bosonic analog of a Weyl, or chiral, semimetal (a "bosonic chiral semimetal" or BCSM state) in any even space-time dimension. We also showed that the boundary of the BTI state exhibits a bosonic analog of the parity anomaly of a Dirac fermion in odd dimensions, and we used this fact to argue that the boundary theory of the BTI in $2m$ dimensions cannot be realized intrinsically in $2m-1$ dimensions while preserving the symmetry of the BTI state. We also explored the phenomenon of anomaly inflow at $\mathbb{Z}_2$ symmetry-breaking domain walls on the boundaries of BTI states.

From a technical point of view, one of the most interesting results of the paper is our explicit construction of gauged WZ actions for O($2m$) and O($2m+1$) NLSMs, and with the gauge group U(1). This construction allowed us to overcome the difficulties associated with calculating the electromagnetic response of bosonic SPT phases from their NLSM description. In addition, as we reviewed in Appendix A, the problem of constructing a gauged WZ action is equivalent to the mathematical problem of constructing equivariant extensions of the volume form on the target space of the NLSM. Then, from a mathematical point of view, we can say that we have succeeded in constructing an equivariant extension of the volume form ω_{2m} on S^{2m} (this is the BTI case), whereas in the case of S^{2m-1} we have constructed an extension of ω_{2m-1} which is almost, but not quite, equivariantly closed (this is the BIQH case). The fact that we could not construct an equivariant extension of ω_{2m-1} is mathematically equivalent to the statement that the boundary theory of the BIQH phase has a perturbative anomaly in the U(1) symmetry, as we expect based on physical arguments.

Our work in this paper opens up many possible directions for future investigations. In particular, it would be nice to have a physical argument along the lines of the one in Ref. [18] for why the CS level for the BIQH phase is quantized in units of $m!$ in $2m-1$ dimensions. Perhaps one can find a physical argument for this quantization by studying generalized braiding processes of extended excitations in gapped bosonic phases in higher dimensions, but we do not have any concrete suggestions as to which excitations and braiding processes might be relevant. Another possible direction would be to apply the general method of gauging WZ actions from Ref. [44] to compute the "electromagnetic" response of SPT phases with the symmetry of a non-Abelian Lie group G. From a mathematical point of view, it would also be interesting to investigate whether the theory of G-equivariant cohomology over an appropriate target manifold could be used to classify SPT phases with the symmetry of a Lie group G. Finally, it would be interesting to use the bosonic analog of the parity anomaly discussed in this paper as a guide to investigate possible dual descriptions of the boundary of BTI phases in all dimensions, analogous to the recent investigations into the dual description of the boundary of the electron topological insulator and BTI in four space-time dimensions [8–16].

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APPENDIX A: EQUIVARIANT COHOMOLOGY INTEPRETATION OF GAUGED WESS-ZUMINO ACTIONS

In this Appendix, we review the connection between the theory of gauged WZ actions and equivariant cohomology. This allows us to give a concrete mathematical interpretation of the form of the gauged WZ actions for the boundary theories of BIQH and BTI states that we derived in Secs. IV and VI of this paper. Briefly, equivariant cohomology can be thought of as a generalization of de Rham cohomology to the case where a continuous group G acts on the manifold. In the cases of interest in this paper the group G is just the group U(1) representing the charge-conservation symmetry of the SPT phases we study (i.e., the BIQH and BTI states), and this group acts on the target space of the NLSM via the rotations shown in Eq. (3.13). The connection between gauged WZ actions and equivariant cohomology has been explored in Refs. [43,97–99]. The connection was first discussed explicitly by Witten in Ref. [43] for the case of two space-time dimensions. Later, Figueroa-O'Farrill and Stanciu [98,99] considered NLSMs with a generic target space and in any space-time dimension, and they gave an explanation of the results of Ref. [44] in terms of equivariant cohomology. In addition, Wu [100] considered the equivalent mathematical problem of finding obstructions to the *equivariant extension* (to be defined below) of closed differential forms which are invariant under a group action. The result of these papers is that the problem of constructing a gauge-invariant WZ action is equivalent to the problem of constructing an equivariant extension of the volume form on the target manifold of the NLSM. We now give a brief review of equivariant cohomology and the connection to gauged WZ actions in the case where $G = \mathrm{U}(1)$, and then we apply this knowledge to give a mathematical interpretation of the counterterms of Eqs. (4.29) and (6.25) which appear in the gauged WZ actions constructed in this paper.

1. Equivariant cohomology

To introduce equivariant cohomology, we first need to recall some basic facts about calculus on manifolds. For a D-dimensional manifold $\mathcal{M}$ a vector field $\underline{V}$ in the coordinate patch with coordinates $y = (y^1, \dots, y^D)$ can be expanded as

$$\underline{V} = V^a \frac{\partial}{\partial y^a}. \tag{A1}$$

The partial derivatives $\frac{\partial}{\partial y^a}$ provide a basis for the tangent space $T_y\mathcal{M}$ of $\mathcal{M}$ at the point y, and a general vector field $\underline{V}$ is a section of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$. The differential forms dy^a provide a basis which is dual to the basis provided by $\frac{\partial}{\partial y^a}$, i.e., the dy^a form a basis for the cotangent space $T_y^*\mathcal{M}$ at the point y. A general differential p-form α is a section of the bundle whose fiber over the point y is $\bigwedge^p(T_y^*\mathcal{M})$, the pth exterior power of $T_y^*\mathcal{M}$.

Now, for any vector field $\underline{V}$ and p-form $\alpha = \frac{1}{p!}\alpha_{b_1\dots b_p} dy^{b_1} \wedge \cdots \wedge dy^{b_p}$ we can define the operator $i_{\underline{V}}$, called *interior multiplication* by $\underline{V}$, by

$$i_{\underline{V}}\alpha = \frac{1}{(p-1)!} V^a \alpha_{ab_2\dots b_p} dy^{b_2} \wedge \cdots \wedge dy^{b_p}. \tag{A2}$$

So, $i_{\underline{V}}$ takes a p-form to a $(p-1)$-form. For later use we also note that applying the interior multiplication twice gives zero, $i_{\underline{V}}^2 = 0$, and that $i_{\underline{V}} f = 0$ for any function (zero form) on $\mathcal{M}$. The Lie derivative $\mathcal{L}_{\underline{V}}$ of any differential form α along the vector field $\underline{V}$ is then given by *Cartan's formula*

$$\mathcal{L}_{\underline{V}}\alpha = d(i_{\underline{V}}\alpha) + i_{\underline{V}}(d\alpha) \tag{A3}$$

or simply

$$\mathcal{L}_{\underline{V}} = d i_{\underline{V}} + i_{\underline{V}} d \tag{A4}$$

in operator form.

We are now ready to introduce U(1)-equivariant cohomology over $\mathcal{M}$. To start, we pick some vector field $\underline{V}$ which generates a U(1) action, or circle action, on the manifold. This can be understood concretely in terms of the flow generated by $\underline{V}$ as follows. First, recall that a vector field $\underline{V}$ generates a flow on the manifold via the set of differential equations

$$\frac{dy^a(t)}{dt} = V^a(y^1, \dots, y^D), \quad a = 1, \dots, D. \tag{A5}$$

The condition that $\underline{V}$ generate a U(1) action on the manifold means that this flow carries each point on $\mathcal{M}$ along a closed path, and each point takes the same amount of "time" t to return to its initial position. Now, define the modified exterior derivative

$$\tilde{d} = d - i_{\underline{V}}. \tag{A6}$$

Note that $\tilde{d}$ takes a p-form to a linear combination of a $(p+1)$-form and a $(p-1)$-form. If we compute the square of $\tilde{d}$ then we find that

$$\tilde{d}^2 = -\mathcal{L}_{\underline{V}}, \tag{A7}$$

which means that $\tilde{d}^2 = 0$ on the subspace of forms which have a vanishing Lie derivative along $\underline{V}$. It is therefore possible to define the cohomology of the operator $\tilde{d}$ in this subspace of differential forms in the same way that one defines the ordinary de Rham cohomology of the exterior derivative d.

Given this structure, one can then try to understand what kinds of objects are closed under the action of $\tilde{d}$. From the definition of $\tilde{d}$ it is clear that a differential form of a definite degree will not, in general, be closed under the action of $\tilde{d}$. Instead, an equivariantly closed "form" α is actually a formal linear combination of differential forms of different degrees, i.e., a section of the bundle whose fiber over the point y is the exterior algebra $\bigwedge(T_y^*\mathcal{M}) = \bigoplus_{r=0}^{D} \bigwedge^r(T_y^*\mathcal{M})$. For the purposes of this paper, we are interested in the case where α is a sum of a form of degree D (the highest possible degree form on the manifold), and several other forms whose parity (even or odd) is the same as that of the form of degree D. In this case, we can expand α as

$$\alpha = \sum_{r=0}^{D'} \alpha^{(D-2r)}, \tag{A8}$$

where $\alpha^{(D-2r)}$ is a differential form of degree $D-2r$ and

$$D' = \begin{cases} \frac{D}{2} & D = \text{ even}, \\ \frac{D-1}{2} & D = \text{ odd}. \end{cases} \tag{A9}$$

The condition $\tilde{d}\alpha = 0$ then implies that the forms $\alpha^{(D-2r)}$ obey the set of equations

$$i_{\underline{V}}\alpha^{(D-2r)} = d\alpha^{(D-2r-2)}, \quad r = 0,\dots,D'-1 \tag{A10a}$$

$$i_{\underline{V}}\alpha^{(D-2D')} = 0. \tag{A10b}$$

In these equations, the second line is trivially satisfied in the case that D is even since in that case $D-2D' = 0$ and so $\alpha^{(D-2D')}$ is just a function. The relation $d\alpha^{(D)} = 0$ is also trivially satisfied since $\alpha^{(D)}$ is a highest-degree form on $\mathcal{M}$, and so we have not included it in the set of equations for the forms that make up α. The form α constructed in this way is known as an *equivariant extension* of the form $\alpha^{(D)}$. We now move on and discuss the connection between these ideas and the theory of gauged WZ actions.

2. Connection to gauged WZ actions

To understand the connection between equivariant cohomology and gauged WZ actions, consider a general NLSM with D-dimensional target space $\mathcal{M}$ (a closed, compact manifold). We denote the NLSM field by $\boldsymbol{\phi} = (\phi^1,\dots,\phi^D)$, so $\boldsymbol{\phi}$ labels a point on $\mathcal{M}$. We formulate this NLSM on a space-time manifold $\partial\mathcal{B}$ of dimension $D-1$, where $\mathcal{B}$ is an extended space-time of dimension D. So, the NLSM field $\boldsymbol{\phi}$ is a map from $\partial\mathcal{B}$ to $\mathcal{M}$. Finally, let $\alpha^{(D)}(\boldsymbol{\phi})$ be the volume form on the target space $\mathcal{M}$. Then, a WZ term for this NLSM takes the form (we absorb any constant factors needed for consistency of the WZ term into the definition of $\alpha^{(D)}$)

$$S_{\rm WZ}[\phi] = \int_{\mathcal{B}} \tilde{\boldsymbol{\phi}}^*\alpha^{(D)}, \tag{A11}$$

where $\tilde{\boldsymbol{\phi}}$ is an extension of $\boldsymbol{\phi}$ into $\mathcal{B}$ and $\tilde{\boldsymbol{\phi}}^*\alpha^{(D)}$ again denotes the pullback of $\alpha^{(D)}$ to $\mathcal{B}$ via the map $\tilde{\boldsymbol{\phi}}$. In what follows, we again omit the pullback symbols $\boldsymbol{\phi}^*$ and $\tilde{\boldsymbol{\phi}}^*$ for notational simplicity.

Now, we suppose that the NLSM has a U(1) symmetry and we attempt to probe this symmetry by coupling the system to the external field A. The transformation of the field $\boldsymbol{\phi}$ under the U(1) symmetry is generated by a vector field $\underline{V}$, i.e., under an infinitesimal gauge transformation the field $\boldsymbol{\phi}$ transforms as

$$\phi^a \to \phi^a + \xi V^a, \tag{A12}$$

where ξ is a small gauge transformation parameter. More generally, a differential p-form $\beta = \frac{1}{p!}\beta_{a_1\dots a_p} d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_p}$ on $\mathcal{M}$ transforms under a small gauge transformation as

$$\beta \to \beta + \mathcal{L}_{\xi\underline{V}}\beta, \tag{A13}$$

where $\mathcal{L}_{\xi\underline{V}}$ is the Lie derivative along the "small" vector field $\xi\underline{V}$. We can now use this more general geometric formulation to try and gauge the WZ term. We should mention that in the case of a U(1) symmetry it suffices to study the change in the action under infinitesimal gauge transformations since there is only one gauge transformation parameter ξ (as opposed to the non-Abelian case where there are several parameters ξ_J with J indexing the generators of the Lie group).

Under a small gauge transformation, the WZ term transforms as

$$\delta_\xi S_{\rm WZ}[\phi] = \int_{\mathcal{B}} \mathcal{L}_{\xi\underline{V}}\alpha^{(D)} = \int_{\partial\mathcal{B}} \xi(i_{\underline{V}}\alpha^{(D)}), \tag{A14}$$

where we used the Lie derivative formula (A13), the fact that $d\alpha^{(D)} = 0$, and the property $i_{\xi\underline{V}} = \xi i_{\underline{V}}$ of the interior multiplication. This change can be canceled by a term

$$S_{ct}^{(1)}[\phi,A] = \int_{\partial\mathcal{B}} A\wedge\alpha^{(D-2)}, \tag{A15}$$

where $\alpha^{(D-2)}$ is some $(D-2)$-form, provided that $\alpha^{(D-2)}$ satisfies the equation

$$i_{\underline{V}}\alpha^{(D)} = d\alpha^{(D-2)}. \tag{A16}$$

To see this, consider the change in $S_{ct}^{(1)}[\phi,A]$ when $A\to A + d\xi$. We find a term

$$\int_{\partial\mathcal{B}} d\xi\wedge\alpha^{(D-2)} = -\int_{\partial\mathcal{B}} \xi d\alpha^{(D-2)}, \tag{A17}$$

where we performed an integration by parts and ignored boundary terms (since $\partial\mathcal{B}$ has no boundary). At this point, the candidate gauged WZ action takes the form

$$S'_{\rm WZ,gauged}[\phi,A] = S_{\rm WZ}[\phi] + S_{ct}^{(1)}[\phi,A]. \tag{A18}$$

Now, under a small gauge transformation we find

$$\delta_\xi S'_{\rm WZ,gauged}[\phi,A] = \int_{\partial\mathcal{B}} A\wedge(\mathcal{L}_{\xi\underline{V}}\alpha^{(D-2)}), \tag{A19}$$

which can be reduced to

$$\delta_\xi S'_{\rm WZ,gauged}[\phi,A] = \int_{\partial\mathcal{B}} \xi F\wedge(i_{\underline{V}}\alpha^{(D-2)}), \tag{A20}$$

with the help of Eq. (A16), the property $i_{\underline{V}}^2 = 0$, and an integration by parts. This change can then be canceled by a term

$$S_{ct}^{(2)}[\phi,A] = \int_{\partial\mathcal{B}} A\wedge F\wedge\alpha^{(D-4)}, \tag{A21}$$

where $\alpha^{(D-4)}$ is some $(D-4)$-form that satisfies the equation

$$i_{\underline{V}}\alpha^{(D-2)} = d\alpha^{(D-4)}, \tag{A22}$$

and so on.

Proceeding in this way, we find that a gauge-invariant WZ term can be constructed if and only if there exist differential forms $\alpha^{(D-2r)}$, $r = 1,\dots,D'$, such that together with the volume form $\alpha^{(D)}$ they satisfy Eqs. (A10). Thus, we find that the problem of constructing a gauge-invariant WZ action is exactly the same as the problem of constructing an equivariant extension of the volume form $\alpha^{(D)}$ on the target space manifold $\mathcal{M}$. We now use this information to reinterpret the gauged WZ actions for the boundary theories of the BIQH and BTI phases.

3. Application to BIQH and BTI boundary theories

In the BIQH and BTI cases, the vector field $\underline{V}$ which generates the U(1) gauge transformations is

$$\underline{V} = \sum_{\ell=1}^{m}\left(-n^{2\ell}\frac{\partial}{\partial n^{2\ell-1}} + n^{2\ell-1}\frac{\partial}{\partial n^{2\ell}}\right). \tag{A23}$$

Now, the NLSMs which describe the boundary of the BIQH and BTI have target spaces S^{2m-1} and S^{2m}, respectively. We now consider the mathematical problem of constructing equivariant extensions of the volume forms ω_{2m-1} and ω_{2m} for these two manifolds. In the BTI case, we will see that an equivariant extension of ω_{2m} exists, and we will give an explicit formula for it. On the other hand, in the BIQH case we will attempt to construct an equivariant extension of ω_{2m-1}, but we will find that it is not quite closed under the action of $\tilde{d} = d - i_{\underline{V}}$. This gives a mathematical interpretation of the U(1) anomaly that we found for the boundary theory of the BIQH phase.

We start with the BTI case. Recall from our study of the boundary theory of the BTI that the construction of the gauged WZ action involved the forms $\Phi^{(r)}$ from Eq. (6.3). If we apply interior multiplication by $\underline{V}$ to these forms, we find

$$i_{\underline{V}}\Phi^{(r)} = (m-r)n_{2m+1}dn_{2m+1}\wedge\Phi^{(r+1)}, \tag{A24}$$

which bears a close resemblance to Eq. (6.4). In addition, for the volume form ω_{2m} we have

$$i_{\underline{V}}\omega_{2m} = \frac{1}{(m-1)!}d(n_{2m+1}\Phi^{(1)}). \tag{A25}$$

We now use these relations to construct an equivariant extension of ω_{2m}, i.e., a solution of Eqs. (A10) with $\alpha^{(D)} = \omega_{2m}$ (so $D = 2m$). To start, we need a form $\alpha^{(2m-2)}$ which satisfies

$$i_{\underline{V}}\omega_{2m} = d\alpha^{(2m-2)}, \tag{A26}$$

and from Eq. (A25) the answer is obviously

$$\alpha^{(2m-2)} = \frac{1}{(m-1)!}n_{2m+1}\Phi^{(1)}. \tag{A27}$$

Next, we need a form $\alpha^{(2m-4)}$ such that

$$i_{\underline{V}}\alpha^{(2m-2)} = d\alpha^{(2m-4)}, \tag{A28}$$

and Eq. (A24) tells us exactly how to find such a form. Proceeding in this way we eventually find that an equivariant extension of ω_{2m} exists and is given explicitly by

$$\tilde{\omega}_{2m} = \omega_{2m} + \sum_{r=1}^{m}\frac{1}{(m-r)!(2r-1)!!}(n_{2m+1})^{2r-1}\Phi^{(r)}. \tag{A29}$$

The terms appearing in the equivariantly closed form $\tilde{\omega}_{2m}$ are exactly the same as the terms which appear multiplying the factors $A\wedge F^{r-1}$ in the counterterms of Eq. (6.25) for the gauged action of the BTI boundary. So, our construction of a gauged WZ action for the BTI boundary is equivalent to the construction of an equivariant extension of the volume form ω_{2m} on S^{2m}.

Moving on to the BIQH phase, we recall that in the BIQH case the construction of the gauged WZ action involved the forms $\Omega^{(r)}$ defined in Eq. (4.13). Applying the interior multiplication by $\underline{V}$ to these forms gives

$$i_{\underline{V}}\Omega^{(r)} = \tfrac{1}{2}d\Omega^{(r+1)}, \tag{A30}$$

which bears a close resemblance to Eq. (4.14). We also saw that the volume form ω_{2m-1} for S^{2m-1} could be written in terms of the $\Omega^{(r)}$ as $\omega_{2m-1} = \frac{1}{(m-1)!}\Omega^{(0)}$. Using this fact, and Eq. (A30), we can then attempt to construct an equivariant extension of ω_{2m-1}, using the same procedure as in the BTI case. In this way, we find a candidate for an equivariant extension of ω_{2m-1}, which is given explicitly by

$$\tilde{\omega}_{2m-1} = \omega_{2m-1} + \frac{1}{(m-1)!}\sum_{r=1}^{m-1}\frac{1}{2^r}\Omega^{(r)}. \tag{A31}$$

However, this object is not quite closed under the action of $\tilde{d} = d - i_{\underline{V}}$, and instead we find that

$$\tilde{d}\tilde{\omega}_{2m-1} = -\frac{1}{(m-1)!}\frac{1}{2^{m-1}}. \tag{A32}$$

In fact, what has happened is that the second line of Eqs. (A10) fails to hold in this case. This failure of $\tilde{\omega}_{2m-1}$ to be equivariantly closed is the mathematical reason for why the BIQH boundary action is not gauge invariant, but instead has a pertubative anomaly in the U(1) symmetry.

It turns out that there is a simple mathematical explanation for why an equivariant extension of ω_{2m-1} does not exist in this case.[5] For the U(1) symmetry considered in this paper [see Eq. (3.13)], the action of the group U(1) on S^{2m-1} is free, i.e., only the identity element of U(1) leaves all the points in S^{2m-1} fixed. In this case, the U(1)-equivariant cohomology of S^{2m-1} is equal to the ordinary de Rham cohomology of the quotient manifold $S^{2m-1}/\mathrm{U}(1)$ (see, for example, Ref. [101]). Now, for the specific U(1) symmetry we have chosen the quotient is just $S^{2m-1}/\mathrm{U}(1) = C\mathbb{P}^{m-1}$, and we know that the cohomology ring of $C\mathbb{P}^{m-1}$ is generated by the Kähler two-form K (which we will meet in Appendix B). This means that only the even-dimensional cohomology groups of $C\mathbb{P}^{m-1}$ are nontrivial. On the other hand, the volume form of S^{2m-1} is a $(2m-1)$-form, i.e., a form of *odd* degree. Since

[5]This explanation was pointed out to us by M. Stone and we thank him for sharing it with us.

the U(1)-equivariant cohomology of S^{2m-1} is equivalent to the ordinary cohomology of $C\mathbb{P}^{m-1}$, we conclude that an equivariant extension of ω_{2m-1} does not exist for this U(1) symmetry (if such an extension did exist, then it would imply the existence of a nontrivial closed form of odd degree on $C\mathbb{P}^{m-1}$, but no such form exists).

APPENDIX B: CHERN CHARACTER ON $C\mathbb{P}^m$

In this Appendix, we compute the integral

$$\int_X \left(\frac{F}{2\pi}\right)^m \tag{B1}$$

for the specific case of $X = C\mathbb{P}^m$ (complex projective space with m complex dimensions). When the field strength F satisfies the Dirac quantization condition of Eq. (5.5) in Sec. V we find that the integral can be equal to one. This answer is already well known, but it provides a nice example of the need for the peculiar quantization of the CS level on generic manifolds, as we discussed in Sec. V.

To compute the integral in Eq. (B1), we are going to need some background information about the complex projective space $C\mathbb{P}^m$. We choose to follow the discussion in Ref. [55]. Note that in this section we depart from previous notation and use an overline $\bar{z}$, and not a star, to denote the complex conjugate of a complex number z. For $C\mathbb{P}^m$ the second Betti number is $b_2 = \dim[H_2(X,\mathbb{R})] = 1$, meaning that $C\mathbb{P}^m$ has a single nontrivial two-cycle. This two-cycle, which we call $\mathcal{C}$, is essentially a copy of $C\mathbb{P}^1$. To understand this two-cycle, and the element of $H^2(X,\mathbb{R})$ which is dual to it, first introduce the Kähler form K on $C\mathbb{P}^m$,

$$K = \frac{i}{2} g_{ab}\, dz^a \wedge d\bar{z}^b, \tag{B2}$$

where

$$g_{ab} = \frac{1}{\mathcal{D}^2}[\mathcal{D}\delta_{ab} - \bar{z}_a z_b] \tag{B3}$$

and

$$\mathcal{D} = 1 + \sum_{c=1}^{m} z_c \bar{z}_c. \tag{B4}$$

Here, z_a, $a = 1,\dots,m$, are complex coordinates which each take values on the whole complex plane $\mathbb{C}$. The indices of z_a can be raised and lowered with δ_{ab} and δ^{ab}, and as usual there is an implied sum over any index which appears once in a lower position and once in an upper position in any expression. The quantity g_{ab} is known as the Fubini-Study metric and it satisfies $\bar{g}_{ab} = g_{ba}$. In addition, we have $dK = 0$, so K is closed. That K is closed follows immediately from the fact that it can be written as

$$K = \frac{i}{2}\partial\bar{\partial}\ln(\mathcal{D}), \tag{B5}$$

where $\partial \equiv \partial_{z^a} dz^a$, $\bar{\partial} \equiv \partial_{\bar{z}^a} d\bar{z}^a$ are the Dolbeault operators (on any Kähler manifold one has $K = \frac{i}{2}\partial\bar{\partial}\rho$, where the function ρ is called the Kähler potential). Since the exterior derivative decomposes as $d = \partial + \bar{\partial}$, and since the Dolbeault operators satisfy $\partial^2 = \bar{\partial}^2 = \{\partial,\bar{\partial}\} = 0$, we immediately see that $dK = 0$. Hence, the Kähler form is closed. However, it is not exact, and so it can be used to construct nontrivial configurations of F on $C\mathbb{P}^m$.

The Kähler form K is a representative of the nontrivial element of $H^2(X,\mathbb{R})$. In the coordinate patch that we have chosen [in which K takes the form shown in Eq. (B2)], we can take the nontrivial two-cycle $\mathcal{C}$ to be any *one* of the m complex planes whose coordinates are z_a. For example, let us take $\mathcal{C}$ to be the z_1 plane. In that plane (with all other $z_a = 0$) we have

$$K \to \left(\frac{i}{2}\right)\frac{dz^1 \wedge d\bar{z}^1}{(1 + z_1\bar{z}_1)^2}. \tag{B6}$$

If we introduce the real coordinates x_1 and x_2 by $z_1 = x_1 + ix_2$, then we have $dz^1 \wedge d\bar{z}^1 = -2i dx^1 \wedge dx^2$, and integrating K over the (x_1,x_2) plane gives

$$\int_{\mathcal{C}} K = \int \frac{dx^1 \wedge dx^2}{\left(1 + x_1^2 + x_2^2\right)^2} = \pi. \tag{B7}$$

We learn from this that a normalized form with unit flux through $\mathcal{C}$ is $\frac{K}{\pi}$, so we should set $\frac{F}{2\pi} = \frac{K}{\pi}$ or just

$$F = 2K \tag{B8}$$

in order to satisfy the Dirac quantization condition of Eq. (5.5).

Now, in order to compute the integral in Eq. (B1) we need to do the integral

$$\int_{C\mathbb{P}^m} K^m, \tag{B9}$$

so we need to compute the wedge product of K with itself m times. We have

$$K^m = \left(\frac{i}{2}\right)^m \frac{1}{\mathcal{D}^{2m}} dz^{a_1} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge dz^{a_m} \wedge d\bar{z}^{b_m} \prod_{r=1}^{m}\left[\mathcal{D}\delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r}\right]. \tag{B10}$$

To simplify this, first note that

$$dz^{a_1} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge dz^{a_m} \wedge d\bar{z}^{b_m} = \epsilon^{a_1\dots a_m}\epsilon^{b_1\dots b_m} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m. \tag{B11}$$

Now, we have to contract $\epsilon^{a_1\dots a_m}\epsilon^{b_1\dots b_m}$ with the product

$$\prod_{r=1}^{m}\left[\mathcal{D}\delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r}\right]. \tag{B12}$$

When expanded out, this product contains 2^m terms. However, most of these terms contain two or more factors of $\bar{z}_{a_r} z_{b_r}$, for example, a term might contain two of them such as $\bar{z}_{a_1} z_{b_1} \bar{z}_{a_2} z_{b_2}$. All such terms with two or more factors of $\bar{z}_{a_r} z_{b_r}$ will vanish when contracted with $\epsilon^{a_1\dots a_m}\epsilon^{b_1\dots b_m}$ because of the antisymmetry of the Levi-Civita symbol, so we only have to worry about terms with zero or one factor of $\bar{z}_{a_r} z_{b_r}$. The term with no factors of $\bar{z}_{a_r} z_{b_r}$ is

$$\mathcal{D}^m \prod_{r=1}^{m} \delta_{a_r b_r}, \tag{B13}$$

and we have

$$\epsilon^{a_1\dots a_m}\epsilon^{b_1\dots b_m}\mathcal{D}^m \prod_{r=1}^{m} \delta_{a_r b_r} = m!\mathcal{D}^m. \tag{B14}$$

Then, there are m terms which each have a single factor of $\bar{z}_{a_r} z_{b_r}$. The first such term is

$$-\bar{z}_{a_1} z_{b_1} \mathcal{D}^{m-1} \prod_{r=2}^{m} \delta_{a_r b_r}, \tag{B15}$$

and we find

$$\begin{aligned} -\epsilon^{a_1 \dots a_m} \epsilon^{b_1 \dots b_m} \bar{z}_{a_1} z_{b_1} \mathcal{D}^{m-1} \prod_{r=2}^{m} \delta_{a_r b_r} \\ = -\mathcal{D}^{m-1} (m-1)! \sum_{c=1}^{m} z_c \bar{z}_c. \end{aligned} \tag{B16}$$

So, all together we find that (recalling that there are m terms with one factor of $\bar{z}_{a_r} z_{b_r}$ and they all give an identical contribution)

$$\epsilon^{a_1 \dots a_m} \epsilon^{b_1 \dots b_m} \prod_{r=1}^{m} \left[\mathcal{D} \delta_{a_r b_r} - \bar{z}_{a_r} z_{b_r} \right] = m! \mathcal{D}^{m-1}, \tag{B17}$$

where we used $(\mathcal{D} - \sum_{c=1}^{m} z_c \bar{z}_c) = 1$. We finally obtain

$$K^m = m! \left(\frac{i}{2} \right)^m \frac{dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^m \wedge d\bar{z}^m}{\mathcal{D}^{m+1}}. \tag{B18}$$

To do the integral over $C\mathbb{P}^m$ we now introduce $2m$ real coordinates x_j, $j = 1, \dots, 2m$, defined by $z_j = x_{2j-1} + i x_{2j}$. Let $r^2 = \sum_{j=1}^{2m} x_j^2$. The integral becomes

$$\begin{aligned} \int_{C\mathbb{P}^m} K^m &= m! \int d^{2m}x \frac{1}{(1+r^2)^{m+1}} \\ &= m!\, \mathcal{A}_{2m-1} \int_0^\infty dr\, \frac{r^{2m-1}}{(1+r^2)^{m+1}} \\ &= m!\, \mathcal{A}_{2m-1} \frac{1}{2m} \\ &= \pi^m, \end{aligned} \tag{B19}$$

where we used spherical coordinates on $\mathbb{R}^{2m}$ to do the integral. Setting $F = 2K$, we then find that

$$\int_{C\mathbb{P}^m} \left(\frac{F}{2\pi} \right)^m = 1. \tag{B20}$$

APPENDIX C: FROM BIQH TO BTI STATES VIA DIMENSIONAL REDUCTION

In this Appendix, we discuss a dimensional reduction procedure which allows one to generate a BTI state in $2m-2$ dimensions from a BIQH state in $2m-1$ dimensions. The procedure is carried out at the level of the effective action $S_{\text{eff}}[A]$ and is similar, but not equivalent, to the procedure used in Ref. [60] to obtain the time-reversal invariant topological insulator in four dimensions from an integer quantum Hall state of fermions in five dimensions.

To start, we imagine separately gauging the U(1) symmetry associated with each species of "boson" b_ℓ ($\ell = 1, \dots, m$) in the NLSM description of the BIQH state in $2m-1$ dimensions. That is, we consider an O($2m$) NLSM describing a BIQH state, and we study this state with a U(1)^m symmetry which acts on the bosons as

$$b_\ell \to e^{i\xi_\ell} b_\ell, \quad \ell = 1, \dots, m \tag{C1}$$

where ξ_ℓ are a set of m-independent gauge transformation parameters. We then couple this system to m U(1) gauge fields $A_\mu^{(\ell)}$.

In this paper, we have not calculated the response of the O($2m$) NLSM when this U(1)^m subgroup is gauged. However, from our results in this paper we can make an argument for what the general form should be. The effective response action $S_{\text{eff}}[A^{(1)}, \dots, A^{(m)}]$ should have at least two properties: (i) it should reduce to a CS response with level $N_{2m-1} = m!$ for the gauge field A if we set $A^{(\ell)} = A \;\forall\; \ell$, and (ii) it should be invariant under any permutation of the labels ℓ of the different gauge fields. The second property follows from the fact that the action for the O($2m$) NLSM is invariant under any permutation of the labels ℓ of the bosons b_ℓ. This fact is not completely obvious, and so we prove it now.

The O($2m$) NLSM with theta term or WZ term is invariant under the action of the *alternating* group A_{2m} of even signature permutations of the labels $a = 1, \dots, 2m$ of the components n_a of the NLSM field $\mathbf{n}$. Now, the permutations of the labels $\ell = 1, \dots, m$ of the bosons b_ℓ consist of two transpositions in the *symmetric* group S_{2m}. This is because a permutation which swaps ℓ with ℓ' must swap $n_{2\ell-1}$ with $n_{2\ell'-1}$ and $n_{2\ell}$ with $n_{2\ell'}$. Since the signature of a permutation σ in the symmetric group is given by $\text{sgn}(\sigma) = (-1)^{N_T}$, with N_T the number of transpositions in σ, it immediately follows that the permutations of the boson labels ℓ are contained within the group A_{2m}. This proves property (ii).

Using properties (i) and (ii), we can now argue that the response action for a gauged U(1)^m symmetry must take the form

$$\begin{aligned} S_{\text{eff}}&[A^{(1)}, \dots, A^{(m)}] \\ &= \frac{k}{(2\pi)^{m-1} m!} \int_{\mathcal{M}} (A^{(1)} \wedge dA^{(2)} \wedge \dots \wedge dA^{(m)} \\ &\quad + \text{permutations}). \end{aligned} \tag{C2}$$

If $\mathcal{M}$ has no boundary, then we can integrate by parts and write this simply as

$$\begin{aligned} S_{\text{eff}}&[A^{(1)}, \dots, A^{(m)}] \\ &= \frac{k}{(2\pi)^{m-1}} \int_{\mathcal{M}} A^{(m)} \wedge dA^{(1)} \wedge \dots \wedge dA^{(m-1)}, \end{aligned} \tag{C3}$$

or we could choose some other ordering of the gauge fields $A^{(\ell)}$. We now describe the dimensional reduction procedure which allows one to derive the electromagnetic response for the BTI state in $2m-2$ dimensions from this response action for the BIQH state in $2m-1$ dimensions. For concreteness, we work on flat space-time with spatial coordinates x^j, $j = 1, \dots, 2m-2$.

To obtain the BTI state from the higher-dimensional BIQH state, we first compactify the space by wrapping the x^{2m-2} direction into a circle, which turns the space $\mathbb{R}^{2m-2}$ into the "cylinder" $\mathbb{R}^{2m-3} \times S^1$. We then thread a π flux of the gauge field $A^{(m)}$ through the hole in the cylinder, and finally shrink the circumference of the cylinder to zero. This leaves us with a response action for a phase in $2m-2$ space-time dimensions.

Mathematically, this procedure assumes the following configuration of gauge fields $A^{(\ell)}$: (i) $A^{(\ell)}$, $\ell = 1,\dots,m-1$, are independent of x^{2m-2} and have their last component $A^{(\ell)}_{\mu=2m-2}$ equal to zero, (ii) the components $A^{(m)}_\mu$, $\mu = 0,\dots,2m-3$, of the mth gauge field are equal to zero, and (iii) the last component of $A^{(m)}$ satisfies $\int dx^{2m-2}\, A^{(m)}_{2m-2} = \pi$.

Under these assumptions, the effective action for the BIQH phase with gauged $\mathrm{U}(1)^m$ symmetry reduces as

$$S_{\text{eff}}[A^{(1)},\dots,A^{(m)}] \to \frac{\pi k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{2m-3,1}} dA^{(1)}\wedge\cdots\wedge dA^{(m-1)}. \quad \text{(C4)}$$

If we now break the remaining $\mathrm{U}(1)^{m-1}$ symmetry of this phase down to U(1) by setting $A^{(\ell)} = A$ for $\ell = 1,\dots,m-1$, then we obtain (with $F = dA$)

$$S_{\text{eff}}[A^{(1)},\dots,A^{(m)}] \to \frac{\pi k}{(2\pi)^{m-1}}\int_{\mathbb{R}^{2m-3,1}} F^{m-1}, \quad \text{(C5)}$$

which is a response action of the form of Eq. (2.2) with response parameter $\Theta_{m-1} = \pi(m-1)!k$. This is exactly the response action for a BTI phase in $2m-2$ dimensions, so the dimensional reduction procedure described here does allow one to obtain the BTI response action from the BIQH response action in one higher dimension.

The main difference between the dimensional reduction procedure shown here and the procedure used in Ref. [60] is that in our procedure we only thread π flux of *one* flavor ℓ of gauge field $A^{(\ell)}$ through hole in the cylinder. On the other hand, the procedure in Ref. [60] [in which there is only a single U(1) gauge field A] is equivalent to threading π flux of *all* the gauge fields $A^{(\ell)}$. This second method *does not* give the proper quantization of the parameter Θ_{m-1} for the BTI phase in one lower dimension. The answer turns out to be too large by a factor of m. The physical reason for the different dimensional reduction procedure needed to go from BIQH to BTI states can be seen from our alternative calculation in Sec. IV of the BIQH response. There we showed that threading 2π flux of the external gauge field A generates a vortex in all m bosons b_ℓ, and so it creates m excitations. This explains why the more familiar dimensional reduction procedure of threading π flux for A gives an answer which is m times too large.

APPENDIX D: DIMENSIONAL REDUCTION FORMULAS FOR THETA TERMS IN NONLINEAR SIGMA MODELS

In this Appendix, we derive a general dimensional reduction formula for theta terms of $\mathrm{O}(D+2)$ NLSMs in $D+1$ dimensions. The formula shows how the theta term of the NLSM can reduce to a theta term for a lower-dimensional NLSM when evaluated on a "defect configuration" of the NLSM field. The formula we derive applies to any space-time dimension $D+1$, and defects of any codimension, with the simplest cases being vortices and hedgehog defects. The physical content of the dimensional reduction formula can be summarized in the following way: a topological defect of (spatial) codimension $(q+1)$ in an $\mathrm{O}(D+2)$ NLSM with theta term can trap an $\mathrm{O}(D-q+1)$ NLSM with theta term in its core. The theta angle of the lower-dimensional NLSM is related to that of the original NLSM in a simple way which we calculate below. We use a special case of this formula in the last subsection of Sec. IV to study vortices in the NLSM description of the BIQH state, but the general result presented in this appendix should be very useful for working with these models. Dimensional reduction of topological terms in NLSMs was also considered in Appendix C of Ref. [79], but to the best of our knowledge the general formula presented in this appendix has not appeared before in the literature. Finally, we also remark that the formula presented here can also be used when WZ terms are present, as the form of the WZ term is similar to the form of the theta term.

To start, recall the theta term

$$S_\theta[\mathbf{n}] = \frac{\theta}{\mathcal{A}_{D+1}}\int_{\mathbb{R}^{D,1}} \mathbf{n}^*\omega_{D+1} \quad \text{(D1)}$$

for a NLSM with field $\mathbf{n}(t,\mathbf{x})$ in $D+1$ space-time dimensions, where t represents the time, and $\mathbf{x} = (x^1,\dots,x^D)$ represents the spatial coordinates. Here, ω_{D+1} is the volume form for the sphere S^{D+1} which was introduced in Eq. (3.6). The integral is over $(D+1)$-dimensional Minkowski space-time $\mathbb{R}^{D,1}$. To describe the defect configurations considered here, we first decompose the total space-time as

$$\mathbb{R}^{D,1} = \mathbb{R}^{q+1}\times\mathbb{R}^{D-(q+1),1}, \quad \text{(D2)}$$

and we further decompose the first factor into $(q+1)$-dimensional spherical coordinates as

$$\mathbb{R}^{q+1} = [0,\infty)\times S^q. \quad \text{(D3)}$$

Here, q is a positive integer which is going to be related to the codimension of the defect in the NLSM field.

We introduce coordinates $r\in[0,\infty)$ and $\mathbf{s} = (s^1,\dots,s^q)$ to parametrize $\mathbb{R}^{q+1} = [0,\infty)\times S^q$. The precise nature of the coordinates $\mathbf{s}$ for S^q will not be important to us here. We also use t and $\mathbf{y} = (y^1,\dots,y^{D-(q+1)})$ to denote the remaining coordinates on $\mathbb{R}^{D-(q+1),1}$. The defect configuration we consider takes the form

$$\mathbf{n}(t,\mathbf{x}) = \{\sin[f(r)]\mathbf{N}(t,\mathbf{y}), \cos[f(r)]\mathbf{m}(\mathbf{s})\}, \quad \text{(D4)}$$

where $\mathbf{N}$ is a $(D-q+1)$-component unit vector which depends only on the coordinates $(t,\mathbf{y})$ for $\mathbb{R}^{D-(q+1),1}$, $\mathbf{m}$ is a $(q+1)$-component unit vector which depends only on the coordinates $\mathbf{s}$ for S^q, and where $f(r)$ is a function obeying the boundary conditions

$$f(0) = \frac{\pi}{2}, \quad \text{(D5)}$$

$$\lim_{r\to\infty} f(r) = 0. \quad \text{(D6)}$$

Physically, this form of $\mathbf{n}$ describes a defect of spatial codimension $q+1$ in which the field $\mathbf{m}$ takes on a nontrivial configuration on the sphere S^q. The field $\mathbf{N}$ then describes a lower-dimensional NLSM which lives in the core of this defect, and the core size is controlled by the profile of the function $f(r)$. The nontriviality of the configuration of $\mathbf{m}$ is captured by the winding number n_q of $\mathbf{m}$ on S^q:

$$n_q = \frac{1}{\mathcal{A}_q}\int_{S^q} \mathbf{m}^*\omega_q. \quad \text{(D7)}$$

After some algebra, one can show that the pullback $\mathbf{n}^*\omega_{D+1}$ of the volume form for the original NLSM field $\mathbf{n}$ will reduce on this configuration as

$$\mathbf{n}^*\omega_{D+1} \to (-1)^{(D-q)(q+1)+1}\{\sin[f(r)]\}^{D-q}\{\cos[f(r)]\}^q \times f'(r)dr \wedge \mathbf{m}^*\omega_q \wedge \mathbf{N}^*\omega_{D-q}. \quad \text{(D8)}$$

This formula can be derived from the formula for $\mathbf{n}^*\omega_{D+1}$ by using the fact that wedge products of the differential of any coordinate with itself will vanish. This fact strongly constrains the terms which survive in $\mathbf{n}^*\omega_{D+1}$ once one assumes that $\mathbf{n}$ is in the defect configuration of Eq. (D4). Now, we just need to do the integrals over the radial direction (parametrized by r) and the sphere S^q to find the reduced theta term for $\mathbf{N}$. For the radial integral we have

$$\begin{aligned} I_r &\equiv -\int_0^\infty dr\ \{\sin[f(r)]\}^{D-q}\{\cos[f(r)]\}^q f'(r) \\ &= \int_0^{\frac{\pi}{2}} df\ [\sin(f)]^{D-q}[\cos(f)]^q \\ &= \frac{\Gamma\left(\frac{D-q+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{D}{2}+1\right)}. \end{aligned} \quad \text{(D9)}$$

Combining this with Eq. (D7) for the winding of the defect in the $\mathbf{m}$ field, we find that the theta term of Eq. (D1) for $\mathbf{n}$ reduces as

$$\begin{aligned} S_\theta[\mathbf{n}] &\to (-1)^{(D-q)(q+1)}\frac{\theta I_r}{\mathcal{A}_{D+1}}\int_{S^q}\mathbf{m}^*\omega_q\int_{\mathbb{R}^{D-(q+1),1}}\mathbf{N}^*\omega_{D-q} \\ &= \frac{\theta_{\text{eff}}}{\mathcal{A}_{D-q}}\int_{\mathbb{R}^{D-(q+1),1}}\mathbf{N}^*\omega_{D-q}, \end{aligned} \quad \text{(D10)}$$

where the effective theta angle for the lower-dimensional NLSM is

$$\theta_{\text{eff}} = (-1)^{(D-q)(q+1)}n_q\ \theta, \quad \text{(D11)}$$

and where we used the formula

$$\frac{\Gamma\left(\frac{D-q+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{D}{2}+1\right)}\frac{\mathcal{A}_q}{\mathcal{A}_{D+1}} = \frac{1}{\mathcal{A}_{D-q}}. \quad \text{(D12)}$$

So, we see that on this defect configuration the original theta term for $\mathbf{n}$ has reduced to a theta term for the field $\mathbf{N}$ which lives in the core of the defect. In addition, from Eq. (D11) we see that the theta angle θ_{eff} for this lower-dimensional NLSM is simply related to the original theta angle by a sign factor $(-1)^{(D-q)(q+1)}$ and by multiplication by the winding number n_q of the defect in $\mathbf{m}$.

APPENDIX E: ELECTROMAGNETIC RESPONSE OF O(2) NLSM IN ONE SPACE-TIME DIMENSION

In this Appendix, we derive the electromagnetic response of the O(2) NLSM with theta term, which represents an analog of the BIQH state in 1 space-time dimension. In the last subsection of Sec. IV we presented an alternative derivation of the electromagnetic response of the O($2m$) NLSM with theta term at $\theta = 2\pi k$ in $2m-1$ dimensions, in which we were able to relate the level N_{2m-1} of the CS term in the response for the O($2m$) NLSM to the level N_1 for the response of the O(2) NLSM at $\theta = 2\pi k$. Specifically, we found that the two levels were related as

$$N_{2m-1} = (m!)N_1. \quad \text{(E1)}$$

We now derive the formula

$$N_1 = -k \quad \text{(E2)}$$

for the O(2) NLSM with $\theta = 2\pi k$, which we then use to complete our alternative derivation at the end of Sec. IV of the formula $N_{2m-1} = -(m!)k$ for the CS response of the O($2m$) NLSM with $\theta = 2\pi k$.

We begin the derivation by parametrizing the O(2) field as $\mathbf{n} = \{\cos(\varphi),\sin(\varphi)\}$ or as $b_1 = e^{i\varphi}$ in terms of the boson $b_1 = n_1 + in_2$. In terms of the angular variable φ the action for the O(2) NLSM with theta term takes the form

$$S[\mathbf{n}] = \int_0^T dt\ \left\{\frac{1}{2g}(\partial_t\varphi)^2 + \frac{\theta}{2\pi}\partial_t\varphi\right\}. \quad \text{(E3)}$$

Here, we have made the calculation as concrete as possible by considering a finite-time interval $[0,T)$, and we assume periodic boundary conditions for the boson b_1 in the time direction. This leaves open the possibility that φ can wind around the time direction, i.e., we can have configurations in which $\varphi(t+T) = \varphi(t) + 2\pi n$ for an integer n. As in the higher-dimensional cases, we will be interested in the limit $g \to \infty$. In this one-dimensional case, this limit just projects onto the ground state (or states) of this quantum mechanical system (in higher dimensions $g \to \infty$ corresponds to the disordered phase of the model).

The U(1) symmetry which acts on b_1 as $b_1 \to e^{i\xi}b_1$ then acts on φ as

$$\text{U}(1):\ \varphi \to \varphi + \xi. \quad \text{(E4)}$$

We would now like to couple φ to a U(1) gauge field $A = A_t dt$. For the boundary conditions we are considering we can write A_t in the general form

$$A_t = \overline{A}_t + \delta A_t, \quad \text{(E5)}$$

where $\overline{A}_t = \frac{1}{T}\int_0^T dt\ A_t$ and $\int_0^T dt\ \delta A_t = 0$. This is equivalent to the statement that the closed form A can be written as an exact part plus a piece which has a nonvanishing integral around the nontrivial one-cycle in the time direction since we assumed periodic boundary conditions in time. We can always remove the exact part δA_t from A_t by a small U(1) gauge transformation $\varphi \to \varphi + \xi$, $A \to A + d\xi$ with $\int d\xi = 0$. Therefore, we will just work with the constant part $\overline{A}_t$ in what follows.

The gauged O(2) NLSM action is obtained by the standard minimal coupling procedure

$$\begin{aligned} S_{\text{gauged}}[\mathbf{n},A] &= \int_0^T dt\ \left\{\frac{1}{2g}(\partial_t\varphi - \overline{A}_t)^2 + \frac{\theta}{2\pi}(\partial_t\varphi - \overline{A}_t)\right\}, \end{aligned} \quad \text{(E6)}$$

however, there is one subtle point here. This action is invariant under small and large U(1) gauge transformations, where by a large U(1) gauge transformation we mean a transformation in which $\int d\xi \neq 0$. Now, if we only cared about invariance under small U(1) gauge transformations, we could just as well have used the action

$$S'_{\text{gauged}}[\mathbf{n},A] = \int_0^T dt\,\left\{\frac{1}{2g}(\partial_t\varphi - \overline{A}_t)^2 + \frac{\theta}{2\pi}\partial_t\varphi\right\}, \quad \text{(E7)}$$

which *does not* involve minimal coupling inside the theta term. This form is more relevant in cases in which one is interested in enforcing certain discrete symmetries *at the expense* of large U(1) gauge invariance, as could be the case in the investigation of global anomalies in discrete symmetries of this theory at $\theta = \pi$ (compare with the discussion for fermionic systems in one dimension in Ref. [95]). This could be relevant for studies of the boundary states of SPT phases in two space-time dimensions. In our case, however, we are interested in the O($2m$) NLSM in $2m-1$ dimensions as a low-energy description of a bosonic lattice model which can be coupled to a *compact* U(1) gauge field, and so we gauge the theory in such a way as to preserve this large U(1) gauge invariance. With these remarks in mind, we now proceed with the computation.

From Eq. (E6) the momentum conjugate to φ is $p = \frac{1}{g}(\partial_t\varphi - \overline{A}_t) + \frac{\theta}{2\pi}$, and the Hamiltonian is

$$H = \frac{g}{2}\left(p - \frac{\theta}{2\pi}\right) + p\overline{A}_t. \quad \text{(E8)}$$

To quantize, we impose the commutation relations $[\varphi,p] = i$ (we set $\hbar = 1$ here), and we use the Schrödinger representation $p = -i\partial_\varphi$. The eigenfunctions of p and H are then the Fourier modes $\psi_n(\varphi) = \frac{1}{\sqrt{2\pi}}e^{in\varphi}$, $n \in \mathbb{Z}$. We now restrict ourselves to the case of $\theta = 2\pi k$ and $g \to \infty$, which is the case for which we are trying to calculate the electromagnetic response. Then, the ground state is $\psi_k(\varphi) = \frac{1}{\sqrt{2\pi}}e^{ik\varphi} \equiv \langle\varphi|\,\text{G.S.}\rangle$, and the partition function (vacuum-to-vacuum transition function) in this case is

$$Z[A] = \langle \text{G.S.}|e^{-iHT}|\text{G.S.}\rangle = e^{-ikT\overline{A}_t} \quad \text{(E9)}$$

or in terms of the original field $A = A_t dt$,

$$Z[A] = e^{-ik\int_0^T dt\;A_t} = e^{-ik\int A}. \quad \text{(E10)}$$

The effective action is then

$$S_{\text{eff}}[A] = -i\ln(Z[A]) = -k\int A, \quad \text{(E11)}$$

from which Eq. (E2) immediately follows.

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