Proximity-induced minimum radius of superconducting thin rings closed by the Josephson 0 or π junction

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Superconductivity is shown to be completely destroyed in thin mesoscopic or nanoscopic rings closed by the junction with a noticeable interfacial pair breaking and/or a Josephson coupling, if a ring's radius r is less than the minimum radius $r_{\min}$. The quantity $r_{\min}$ depends on the phase difference χ across the junction, or on the magnetic flux that controls χ in the flux-biased ring. It also depends on the Josephson and interfacial effective coupling constants, and in particular, on whether the ring is closed by the 0 or the π junction. The current-phase relation is substantially modified when the ring's radius exceeds $r_{\min}$ for some of the phase difference values, or slightly goes beyond its maximum. The modified critical temperature T_c and the temperature-dependent supercurrent near T_c are identified here as functions of the ring's radius and the magnetic flux.

I. INTRODUCTION

The superfluid flow in thin superconducting wires is known to result in pair-breaking effects, which reduce the order parameter and, in accordance with the Landau criterion, can fully destroy superconductivity at the critical value of the superfluid velocity v_s. While the Cooper pair density diminishes with increasing v_s, the supercurrent shows a nonmonotonic behavior, with its maximum value called the depairing current j_{dp} [1]. When a thin wire forms a circular loop with radius r and the absolute value of the order parameter stays spatially constant, the order parameter–superflow relation remains as it was in the straight wire. Specific features of the loop topology show up, when, for example, v_s is induced by the magnetic flux Φ penetrating the loop. The flux-induced changes of the winding number result in the oscillations of physical characteristics of the ring, in particular, of v_s and T_c, i.e., in the standard Little-Parks effect [1,2]. The effect allows a remarkably simple description within the Ginzburg-Landau (GL) approach, since the equilibrium order-parameter absolute value and v_s are spatially constant in a cylindrically symmetric thin ring.

Constant v_s is determined, along with its circulation, by a full magnetic flux through the loop: $2\pi r v_s \propto (\widetilde{\Phi} - n)$. Here $\widetilde{\Phi} = \frac{\Phi}{\Phi_0}$ is the magnetic flux in units of the superconductor flux quantum Φ_0, and n is the winding number. The critical temperature T_c, modified by the magnetic flux in the Little-Parks effect, can be found taking v_s in the last relation to be equal to the critical superfluid velocity, when the order parameter vanishes and thermodynamic potential of the ring coincides with that of the normal-metal ring. This results in the equation $r = \xi(T)|\widetilde{\Phi} - n|$, where $\xi(T)$ is the temperature-dependent coherence length of the superconducting material. Solving the equation with regard to the temperature establishes the modified T_c that depends on the magnetic flux, the winding number, and the loop radius. When the quantity $(\widetilde{\Phi} - n)$ is fixed, the superfluid velocity is inversely proportional to the radius, analogously to its dependence on the distance to the center of the Abrikosov vortex. As a result, the minimum radius $r_{\min} = \xi(T)|\widetilde{\Phi} - n|$, at which v_s takes its critical value, exists for mesoscopic and nanoscopic uninterrupted rings. Superconductivity is fully destroyed in the rings with radii $r \leqslant r_{\min}$. Since the pair breaking induced by the superflow is most

pronounced at the maximum equilibrium value $|\widetilde{\Phi} - n| = 0.5$, there is no superconductor–normal metal transition, when the magnetic flux slowly varies in the rings with $r > 0.5\xi(T)$. There are only the usual Little-Parks oscillations that occur in this case. The transition comes about under the opposite condition, i.e., in the rings with radii $r \leqslant 0.5\xi(T)$. It can be experimentally observed down to quite low temperatures, when the quantum phase transition takes place [3–5].

This paper addresses thin superconducting loops closed by the Josephson junction. It will be demonstrated theoretically that the minimum radius $r_{\min}$ of thin superconducting rings involving the junction, unlike the case of the unbroken rings, is nonzero even if v_s is much less than its critical value throughout the loop. It is the pair breaking due to the inverse proximity effect locally induced by the junction interface and by the phase difference χ across it that leads to the superconductor–normal metal phase transition in the rings of mesoscopic or nanoscopic size. The Josephson and interfacial pair breaking inevitably result in an inhomogeneous profile of the complex order parameter, which contributes considerably to the gradient term in thermodynamic potential and makes a superconducting state energetically unfavorable in the rings with $r < r_{\min}$.

The minimum radius is found to be a fraction of the temperature-dependent coherence length $\xi(T)$, up to $0.5\xi(T)$. When the ring is closed by 0 junction, the minimum radius, as a function of the phase difference, is shown to have its maxima at $\chi = (2m + 1)\pi$ (m is an integral number). For the π junction, the maxima are at $\chi = 2m\pi$. In both cases, i.e., at $\chi = m\pi$, v_s vanishes together with the supercurrent all along the ring. It is in contrast to uninterrupted rings, where the kinetic energy of the supercurrent becomes equal to the condensation energy and v_s takes its critical value at the superconductor–normal metal transition point. When r exceeds $r_{\min}$ for some of χ's values, or goes somewhat beyond its maximum, both the critical current and the current-phase relation of the junction become quite sensitive to the radius value.

The phase difference across the junction will be assumed to be controlled by the applied magnetic flux Φ_e in the flux-biased ring. The magnetic field much less than the superconductor critical fields will also be supposed. Since the inductance effects are negligibly small near the transition point $r = r_{\min}$,

where the supercurrent vanishes, the difference between the full magnetic flux Φ and the applied one Φ_e will be disregarded below. Therefore, the supercurrent hysteretic behavior due to the inductance effects [6,7] will not be considered. When the difference $r - r_{\min}$ is large or moderate, the hysteretic behavior will be shown to appear also due to the absence of the Meissner effect in thin superconducting rings. This paper mainly concerns itself with the rings of smaller sizes. Superconductivity continuously weakens in such rings and is ultimately destroyed, when the difference $r - r_{\min}$ slowly diminishes. Once the difference vanishes, the superconductor–normal metal phase transition of the second order occurs.

Within the GL theory, the coherence length $\xi(T)$ of the superconducting material is the only characteristic length of the problem in question, and the equation for the minimum radius is actually formulated for the dimensionless quantity $R = r/\xi(T)$. When the fixed value of r slightly exceeds $r_{\min}$, the quantity R can reach $R_{\min}$ as the temperature goes up making $\xi(T)$ sufficiently large. This results in the modified critical temperature T_c, which depends on r and χ, or Φ. The current-phase and current-flux relations can become quite sensitive to the temperature in the narrow vicinity of T_c, similarly to the presence of the radius dependence of the Josephson current, with r being quite close to $r_{\min}$.

The existing temperature and magnetic flux dependence of $r_{\min}$ near the transition make it possible to observe the phase transition by changing the temperature or the magnetic flux that penetrates the individual ring. Likely alternatives to make the effect discernible are the junctions with interfaces made of normal metals, magnets, or other pair-breaking materials, and/or the junctions involving unconventional superconductors. The pair breaking by the phase difference becomes noticeable for interfaces with sufficiently high transparency.

In the paper, Sec. II addresses basic equations of the GL theory that describe properties of thin superconducting rings closed by the Josephson junction. The minimum radius for such rings is obtained in Sec. III. The modified critical temperature is identified in Sec. IV. The current-phase and current-flux relations and their dependence on the ring's radius and on the temperature are found in Sec. V. Section VI concludes the paper. Appendices A and B present the analytical solutions of the equations studied.

II. BASIC EQUATIONS

Consider a superconducting circular thin ring closed by the Josephson junction. The ring's lateral dimensions are supposed to be much less than $\xi(T)$ and the magnetic penetration depth. The thickness of the junction interface is on the order of or less than the zero-temperature coherence length. Within the GL approach, the latter scale is considered to be zero. The GL free energy of the ring is represented as a sum of two terms: $\mathcal{F} = \mathcal{F}_b + \mathcal{F}_{\mathrm{int}}$. The bulk free energy per unit area of the cross section is

$$\mathcal{F}_b = \int_0^{2\pi r} ds \left[K \left| \left(\frac{d}{ds} - \frac{2\mathrm{i}e}{\hbar c} A \right) \Psi(s) \right|^2 \right.$$
$$\left. + a|\Psi(s)|^2 + \frac{b}{2}|\Psi(s)|^4 \right], \tag{1}$$

where r is the ring's radius, $s = r\varphi$ is the coordinate along the ring's circumference, and φ is the polar angle. The coefficient in front of the gradient term is here denoted as K. The vector potential is taken to be cylindrically symmetric and to have only the polar component, i.e., the s component in our notations. Such a gauge exists, for example, for the Aharonov-Bohm flux, which is delta-localized along the ring's axis, for the homogeneous magnetic field as well as for the one produced by the current in a thin circular ring.

The term $\mathcal{F}_{\mathrm{int}}$ is the interfacial free energy per unit area that can be written as

$$\mathcal{F}_{\mathrm{int}} = g_J|\Psi_{0+} - \Psi_{0-}|^2 + g(|\Psi_{0+}|^2 + |\Psi_{0-}|^2). \tag{2}$$

The junction interface is taken at $s = 0$.

Two interface invariants in (2) are determined both by the symmetry of the system and by the microscopic consideration. The latter allows one to unambiguously identify the two contributions to (2), one with the Josephson coupling of the superconducting banks, with the coupling constant g_J, the other with the interfacial pair breaking ($g > 0$) that in particular takes place in the absence of the supercurrent. In a symmetric junction $|\Psi_{0+}| = |\Psi_{0-}|$ the first term in (2) takes the form $g_J|\Psi_0|^2(1 - \cos\chi)$, which is known in describing standard symmetric tunnel junctions. In the standard case the interfacial pair breaking is negligibly small (i.e., $g = 0$), and a thin interface does not affect the superconductor at zero phase difference χ across it. Based on (1) and (2), the supercurrent through the Josephson junction can be described also beyond the tunneling approximation and taking account of the interfacial pair breaking ($g \neq 0$) [8–10]. The interfacial free energy (2) controls the corresponding anharmonic current-phase relation as well as the normalized critical current $\tilde{j}_c = j_c/j_{\mathrm{dp}}$, where $j_{\mathrm{dp}} = (8|e||a|^{3/2}K^{1/2})/(3\sqrt{3}\hbar b)$ is the depairing current deep inside the superconducting leads.

Taking the order parameter in the form $\Psi = (|a|/b)^{1/2} f(s) e^{\mathrm{i}\phi(s)}$, one can transform the GL equation for the order parameter, which follows from the bulk free energy (1), to the equation for f,

$$\frac{d^2 f}{dx^2} - \frac{i^2}{f^3} + f - f^3 = 0, \tag{3}$$

and to the current conservation condition. Here $x = s/\xi(T)$ is the dimensionless coordinate and $\xi(T) = (K/|a|)^{1/2}$. The dimensionless current density in (3) is $i = \frac{2}{3\sqrt{3}}\tilde{j} = \frac{2}{3\sqrt{3}}(j/j_{\mathrm{dp}})$.

The quantity $f(x)$ is continuous at the interface of the symmetric junction and has to satisfy the periodicity condition $f(x + 2\pi R) = f(x)$, where $R = r/\xi$ is the dimensionless radius. The boundary conditions at $x = \pm 0$ that follow from (2) and (1) can be split into the discontinuity condition for df/dx and the expression for the Josephson current via the value f_0 at the interface and the phase difference $\chi = \phi_{0-} - \phi_{0+}$:

$$\left(\frac{df}{dx} \right)_{\pm 0} = \pm g_b f_0, \quad g_b = g_\delta + 2g_\ell \sin^2 \frac{\chi}{2}, \tag{4}$$

$$i = -f^2 \left(\frac{d\phi}{dx} + \frac{2\pi\xi}{\Phi_0} A \right) = g_\ell f_0^2 \sin\chi. \tag{5}$$

Here $\Phi_0 = \frac{\pi\hbar c}{|e|}$ is the magnetic flux quantum; $g_\ell = g_J \xi(T)/K$ and $g_\delta = g\xi(T)/K$ are the dimensionless Josephson and interface effective coupling constants.

The effective coupling constants play an important role in the approach developed. Within the BCS theory, the range of variations of g_ℓ and g_δ is generally quite wide [8–10]. Thus in dirty junctions with small or moderate transparency the quantity $g_\ell > 0$ can vary from extremely small values in the tunneling limit to values that are larger than 100. The parameter g_ℓ in junctions with high transparency can be very large. While the depairing by the interface is very weak $g_\delta \ll 1$ in standard tunnel junctions with a conventional insulating barrier, a pronounced depairing $g_\delta \gg 1$ can occur in various superconductors, including s-wave superconductors, near interfaces with normal metals and/or magnets. In unconventional superconductors, significant depairing can also occur near the superconductor-insulator and superconductor-vacuum interfaces.

Once $g_b \neq 0$, the boundary conditions (4) induce an inhomogeneous equilibrium profile of the order parameter $f(x)$ in the ring, with its minimal value $f_0 < 1$ at $x = 0$. Since the supercurrent in thin wires is spatially constant, the modulus and the phase of the complex order parameter vary in space interactively, and pronounced inhomogeneities of $f(x)$ and of the gradient of the phase along the ring take place simultaneously. To find the current-phase relation based on (5), the self-consistent interface value f_0 should be determined as a function of R and χ. The corresponding solution can be obtained based on the first integral of the one-dimensional GL equation (3), analogously to other problems of this type [11].

The analytical solutions of the GL equations that describe the order-parameter profile, the magnetic flux, and the thermodynamic potential as functions of R and χ are obtained in Appendix A and used for further studies in the following sections. A periodic inhomogeneous solution for the order-parameter absolute value, with only a single minimum and a single maximum in the ring, is considered as being, as a rule, energetically the most favorable one. The minimum is induced at the junction interface $x = 0$ by the pair-breaking effects. Its derivative at $x = 0$ is nonzero and discontinuous in accordance with the boundary conditions (4). In the equilibrium, one expects the maximum to be realized at the point $x = \pi R$ diametrically opposed to $x = 0$.

The location of the minimum at the junction interface implies an interface's pair-breaking effect $g_b = g_\delta + 2g_\ell \sin^2 \frac{\chi}{2} > 0$ to take place at all phase differences, which signifies the validity of the conditions $g_\delta > 0$ and $-2g_\ell < g_\delta$, irrespective of the sign of the effective Josephson coupling constant g_ℓ. Within this framework, the solutions obtained describe at $g_\ell < 0$ the properties of thin superconducting rings closed by the π junction with a pair-breaking interface. Another possible solution, which corresponds to a proximity-enhanced superconductivity near the junction interface ($g_b < 0$), at least for some of the phase differences, will not be considered in the paper, as no evidence for thin interfaces of such type in the Josephson junctions is available at this point.

III. MINIMUM RADIUS

Superconductivity in a thin ring closed by the junction is destroyed under the condition $R < R_{\min}$. The dimensionless minimum radius $R_{\min}$ appears as a consequence of the junction's destructive effects on the adjacent superconductivity region, i.e., of the inverse proximity effects underlying the boundary conditions (4). Solutions of (3)–(5) take into account the effects.

Since the second-order phase transition takes place at $R = R_{\min}$, the quantity $f(x)$ should be very small in its vicinity. A linearization of the GL equation is known to be the simplest way for describing superconducting phenomena very close to the superconductor–normal metal transition. One could mention in this regard, for example, the problems of H_{c2} and H_{c3} [1,12,13], as well as of the proximity effects in the vicinity of the superconductor–normal metal boundaries [14,15].

However, in the issue under consideration, the spatial dependence of both the absolute value of the order parameter and the gradient of its phase play an important part. At a nonzero supercurrent density i, spatially constant in thin rings, the gauge-invariant gradient of the phase (the superfluid velocity) is proportional to $\frac{i}{f^2}$. In agreement with this relation, $v_s(x)$ is a spatially dependent quantity that does not vanish at the transition point, if the phase difference is not a multiple of π. This substantially complicates the linearization in $f(x)$ of the GL equation (3), where the supercurrent density i should be obtained via (5) that incorporates the solution f_0 taken at the interface. The quantities $v_s(x)$, i, and f_0 are, in general, nonlocal functionals of $f(x)$ that result in the complication mentioned. Because of the second term $i^2 f^{-3}$ in Eq. (3), its linearization in $f(x)$ is impossible without specifying the corresponding behavior of the supercurrent. Were $v_s(x)$ vanishingly small near the transition, the second term $i^2 f^{-3} \propto v_s^2(x) f(x)$ would be negligible in (3) as compared to the linear one. It is, however, not the case and the term, in general, cannot be disregarded.

A specific case comes about when χ is a multiple of π and the term $i^2 f^{-3}$ strictly equals zero, together with v_s and i. Quite close to the transition point one can also put $R = R_{\min}$ and neglect the cubic term in (3). The resulting solution, for $|x| \leqslant \pi R_{\min}$, is $f = f_d \cos(|x| - \pi R_{\min})$, and the boundary conditions (4) take the form $\tan(\pi R_{\min}) = g_\delta$ at $\chi = 2m\pi$ and $\tan(\pi R_{\min}) = g_\delta + 2g_\ell$ at $\chi = (2m+1)\pi$. One gets from here the quantities

$$R_{\min}(0) = \frac{1}{\pi} \arctan g_\delta, \quad R_{\min}(\pi) = \frac{1}{\pi} \arctan(g_\delta + 2g_\ell), \tag{6}$$

which satisfy the relation $R_{\min}(\pi) > R_{\min}(0)$ for zero junctions ($g_\ell > 0$), while $R_{\min}(\pi) < R_{\min}(0)$ for π junctions ($g_\ell < 0$). Under the condition $g_\delta \gg 1$ (and/or $g_\ell \gg 1$ and $\chi \approx \pi$, in case of 0 junctions) the minimum radius $r_{\min}$ approaches its upper bound $0.5\xi(T)$. Exactly the same bound to $r_{\min}$ arises in uninterrupted mesoscopic thin rings, where $r_{\min} = \xi(T)|\widetilde{\Phi} - n|$ and the maximum equilibrium value of $|\widetilde{\Phi} - n|$ is 0.5. It is also noted that the quantity $r_{\min}(0)$ is related to the minimum length $L_{\min}$ of a thin straight superconducting

wire or a thin film symmetrically sandwiched between identical pair-breaking walls: $L_{\min} = 2\pi r_{\min}(0) = 2\xi(T) \arctan g_\delta$ [16–18].

Moving over to a more general description, one could proceed by taking into account the nonlinear second term, but neglecting from the very beginning the cubic term in (3). The corresponding simplified solution is expressed via the inverse trigonometric functions. Alternatively, one can find the minimum radius by introducing the corresponding simplifications in the exact solution of the GL equations (3)–(5) obtained in Appendix A. The former approach works well just at the transition point and it does not apply to the vicinity of the transition, which will be studied in Sec. V. Since the exact solution is in any case required for this paper, it is also used for the present purpose in Appendix B, where the dimensionless minimum radius, obtained as a function of the phase difference as well as of the Josephson and interface effective coupling constants, is shown to be described by the expression [19]

$$R_{\min} = \frac{1}{2\pi} \arccos \left(\frac{1 - g_\ell^2 \sin^2 \chi - g_b^2}{\sqrt{\left(1 - g_\ell^2 \sin^2 \chi - g_b^2\right)^2 + 4g_b^2}} \right). \quad (7)$$

As defined in (4), $g_b = g_\delta + 2g_\ell \sin^2 \frac{\chi}{2}$. The simple analytical expression (7), describing the minimum radius $R_{\min}$ as a function of χ, g_ℓ, and g_δ, is one of the prime results of the paper.

The continuous type of the phase transition is confirmed at sufficiently small $R - R_{\min}$ by the order-parameter behavior $f_0^2 \propto (R - R_{\min})$. The free energy linearly vanishes with $(R - R_{\min})$ near the transition point

$$\widetilde{\mathcal{F}} \approx -\frac{4\pi}{3}(R - R_{\min}). \quad (8)$$

It is the result of a strong competition between the interfacial and bulk contributions, which takes place in mesoscopic and nanoscopic rings in the presence of the inverse proximity effects.

As follows from (7) for the rings closed by the 0 junction, the phase dependence of $R_{\min}$ becomes noticeable when the interface pair breaking is not too strong $g_\delta \lesssim 1$. The larger the strength of the Josephson coupling $g_\ell > 0$, the more pronounced the modulation of $R_{\min}$ with the phase difference that takes place in a wide $(g_\ell^2 \lesssim 1)$ or narrow $(g_\ell^2 \gg 1)$ vicinity of $\chi = 2m\pi$. The difference between $R_{\min}(0)$ and $R_{\min}(\pi)$ and, therefore, the phase dependence of $R_{\min}$ as a whole become negligibly small, when $g_\delta \gg 1$ [see also (6)]. Since $R_{\min}$ increases with g_ℓ for 0 junctions, its minima stay at $\chi = 2\pi m$ and do not depend on g_ℓ. Its maxima are at $\chi = (2m + 1)\pi$ and depend on both coupling constants g_ℓ and g_δ.

As $g_\ell < 0$ in π junctions and $g_b > 0$ is assumed, a finite Josephson coupling term $2g_\ell \sin^2 \frac{\chi}{2}$ reduces the strength of the pair breaking by the junction interface characterized by the parameter g_b. Correspondingly, the quantity $R_{\min}$ decreases with increasing strength of the Josephson coupling $|g_\ell|$. Its minima occur at $\chi = (2m + 1)\pi$ and depend on both values g_ℓ and g_δ, while the maxima are at $\chi = 2m\pi$ and depend solely on g_δ. Since the supercurrent (5) and the superfluid velocity vanish at $\chi = m\pi$, the results regarding the extrema of $R_{\min}$ agree with (6) found for $\chi = 0, \pi$ within the linearized description. In
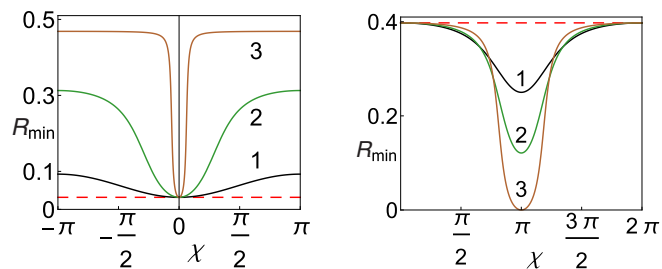

FIG. 1. $R_{\min}(\chi)$ for the rings closed by 0 junction (left panel) and π junction (right panel). Left panel: $g_\delta = 0.1$ and (1) $g_\ell = 0.1$, (2) $g_\ell = 0.7$, (3) $g_\ell = 5$, and $g_\ell = 0$ (dashed curve). Right panel: $g_\delta = 3$ and (1) $g_\ell = -1$, (2) $g_\ell = -1.3$, (3) $g_\ell = -1.5$, and $g_\ell = 0$ (dashed curve).

accordance with (8), free energy has its minimum at $\chi = 2m\pi$ for the 0 junction, and at $\chi = (2m + 1)\pi$ for the π junction.

The minimum radius $R_{\min}$ as a function of the phase difference, for the thin rings closed by 0 junctions and π junctions, is shown in the left and right panels of Fig. 1, respectively. The minimum radius has been found above by comparing the superconducting rings with various radii at fixed χ. When the Josephson coupling induces a noticeable phase dependence of $R_{\min}$, the critical value $R = R_{\min}(\chi)$ is not necessarily to be the minimum radius of the superconducting rings with $\chi' \neq \chi$.

In the flux-biased rings the minimum radius actually depends on $\widetilde{\Phi}$, rather than on χ. The relationship between the quantities χ and $\widetilde{\Phi}$ can be obtained by integrating the first expression for the current in (5) along the ring. Since $\oint_C \mathbf{A} \cdot \xi dx = \Phi$ is the full magnetic flux penetrating the ring, one gets

$$\chi + 2\pi(\widetilde{\Phi} - n) = -2i \int_0^{\pi R} \frac{dx}{f^2(x)}. \quad (9)$$

Here n is the winding number and the relation $f(x) = f(-x)$ has been used.

The integration on the right-hand side in (9), taken with the solution for the order parameter, results in (A10), which is valid at an arbitrary ring's size. The expression (A10) is substantially simplified at $R = R_{\min}(\chi)$, as shown in Appendix B [19]:

$$\widetilde{\Phi} - n = -\frac{1}{2\pi} \left(\chi + \text{sgn}(g_\ell \sin \chi) \right.$$

$$\left. \times \arccos \frac{1 + g_b^2 - g_\ell^2 \sin^2 \chi}{\sqrt{\left(1 + g_b^2 - g_\ell^2 \sin^2 \chi\right)^2 + 4g_b^2 g_\ell^2 \sin^2 \chi}} \right). \quad (10)$$

The relationships between the magnetic flux and the phase difference, obtained in the paper, can strongly deviate from the linear dependence $\chi = -2\pi(\widetilde{\Phi} - n)$ taking place in thick rings closed by the Josephson junction, in the absence of noticeable inductance effects [7]. Due to the Meissner effect, the supercurrent vanishes along with v_s in the depth of a thick ring, while the order parameter remains finite. On the contrary, in thin rings the supercurrent vanishes at $R = R_{\min}$ together with the order parameter, while v_s, in general, stays finite. As can be seen in (9), where the right-hand side represents
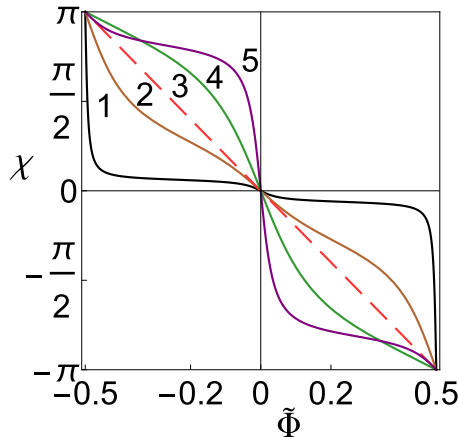
FIG. 2. $\chi(\widetilde{\Phi})$ in the rings of minimum radius. 0 junctions: $g_\delta = 0.1$ and (1) $g_\ell = 5$, (2) $g_\ell = 0.7$, (3) $g_\ell \ll 1$. π junctions: $g_\delta = 3$ and (3) $|g_\ell| \ll 1$, (4) $g_\ell = -1$, (5) $g_\ell = -1.5$.

circulation of v_s, the superfluid velocity is responsible for the nonlinear character of the relationship (10). The integral $\int dx/f^2(x)$ on the right-hand side of (9) diverges at the transition point. However, the integral multiplied by the supercurrent $i = g_\ell f_0^2 \sin \chi$ remains finite even in the limit $f(x) \to 0$, in accordance with the relation $f_0 \leqslant f(x)$. Both sides in (9) take zero values, along with v_s, when χ is a multiple of π. In this particular case the relation (10) acquires the familiar linear character and $|\widetilde{\Phi}|$ takes an integer or a half-integer number. For example, for $|\chi| = \pi$ (and $n = 0$) one gets $|\widetilde{\Phi}| = \frac{1}{2}$. Therefore, the free energy (8) has its minimum at $\widetilde{\Phi} = 0$ in the case of the 0 junction, and at $|\widetilde{\Phi}| = \frac{1}{2}$ in the case of the π junction. This agrees with the emergence of the spontaneous magnetic flux penetrating the ring closed by the π junction [20].

The magnetic flux dependence $\chi(\widetilde{\Phi})$ in the minimum radius rings is shown in Fig. 2 for $|\widetilde{\Phi}| < 0.5$, $|\chi| < \pi$. If the Josephson coupling is sufficiently weak $|g_\ell| \ll 1$, then the phase difference χ depends almost linearly on the magnetic flux $\widetilde{\Phi}$, as follows from (10) and seen in the dashed curve in Fig. 2. At the same time, curves 1 and 5 in Fig. 2, which correspond to 0 and π junctions with comparatively large Josephson coupling strengths $|g_\ell|$, demonstrate opposite signs of pronounced deviations from the linear behavior. Curves 1 and 5 show that the phase difference varies weakly in the wide region of $\widetilde{\Phi}$, while it undergoes abrupt changes in the narrow vicinities of the half-integer (integer) values of $\widetilde{\Phi}$ in 0 junctions (π junctions).

A combined consideration of (7) and (10) results in the periodic dependence of the minimum radius on the magnetic flux $R_{\min}(\widetilde{\Phi})$, which is depicted in Fig. 3. The stronger the Josephson coupling strength is, the more intense the modulation of $R_{\min}$ with the magnetic flux. In the case of the 0 junction (π junction), the maxima of $R_{\min}$ and of the pair-breaking parameter g_b occur at the magnetic flux half-integer (integer) values.

IV. MODIFIED CRITICAL TEMPERATURE

Since temperature has been incorporated in the definitions of a number of dimensionless quantities, it implicitly enters
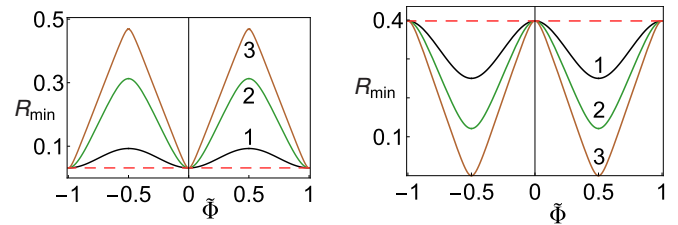

FIG. 3. $R_{\min}(\widetilde{\Phi})$ for the rings with 0 junctions (left panel) and π junctions (right panel) under the same set of parameters as in the corresponding panels in Fig. 1.

all the results obtained above. Thus the temperature dependent coherence length $\xi(T)$, as the characteristic length of the inverse proximity effects, is included in the dimensionless radius $R(T) = r/\xi(T)$. When $R(T)$ is taken at fixed r, it decreases with diminishing τ and, at least formally, vanishes at $\tau = 0$. Here $\tau = 1 - (T/T_{c0})$, and T_{c0} is an unperturbed critical temperature of the superconducting material.

On the other hand, the right-hand side of (7), i.e., $R_{\min}$, depends on τ via the temperature-dependent effective coupling constants $g_\ell = [g_J \xi(T)]/K$ and $g_\delta = [g \xi(T)]/K$. They increase with $\xi(T)$, when the temperature draws near to T_c, as a consequence of an increasing influence of the junction interface and, therefore, the boundary conditions (4), on the ring's properties as a whole. When τ goes down, the dimensionless radius decreases down to 0 at $\tau = 0$, while the quantity $R_{\min}$ increases up to 0.5, as seen in (6) and (7).

Therefore, if the ring's radius R initially exceeds $R_{\min}$ and temperature goes up, making $\xi(T)$, g_ℓ, and g_δ sufficiently large and τ small, the quantities R and $R_{\min}$ will inevitably become equal to each other. The temperature T_c, at which the equality takes place, is the modified critical temperature of the superconducting transition in the ring closed by the junction. In other words, the equality $R = R_{\min}$, with $R_{\min}$ given in (7), can also be considered as the equation for the proximity-modified critical temperature $T_c < T_{c0}$ that depends on the ring's radius and the phase difference. The critical temperature shift $\Delta T_c = T_c - T_{c0}$ is discernible within the mean-field theory, when exceeding the fluctuation region near T_{c0}. This is the case for mesoscopic or nanoscopic rings, when the ring's radius slightly differs from the minimum one. For a sufficiently large R the shift is negligibly small.

In order to reveal the temperature dependence in the equation $R = R_{\min}$, let us introduce an auxiliary "low-temperature" GL coherence length $\xi_0^{\mathrm{GL}} = \xi(T)\sqrt{\tau}$ [21] and move on to a rescaled dimensionless radius $\widetilde{R} = r/\xi_0^{\mathrm{GL}} = R/\sqrt{\tau}$. The effective "low-temperature" coupling constants of the GL theory are defined as $g_{\ell 0} = (g_J \xi_0^{\mathrm{GL}})/K$, $g_{\delta 0} = (g \xi_0^{\mathrm{GL}})/K$. With these definitions, one gets from (7) the relationship

$$\widetilde{R} = \frac{1}{2\pi\sqrt{\tau}} \arccos \frac{\tau - g_{\ell 0}^2 \sin^2 \chi - g_{b0}^2}{\sqrt{\left(\tau - g_{\ell 0}^2 \sin^2 \chi - g_{b0}^2\right)^2 + 4g_{b0}^2 \tau}}, \quad (11)$$

which can be used either as the expression for the temperature-dependent minimum radius $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, \chi)$, or as the equation for the critical temperature shift $\tau_c(\widetilde{R}, \chi) = 1 - \frac{T_c(\widetilde{R}, \chi)}{T_{c0}} = -\frac{\Delta T_c(\widetilde{R}, \chi)}{T_{c0}}$ in the ring with radius $\widetilde{R}$.
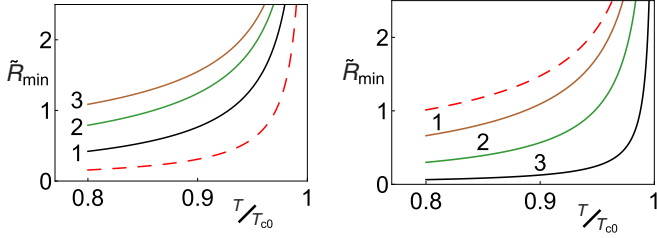
FIG. 4. $\widetilde{R}_{\min}(\frac{T}{T_{c0}})$ at $\chi = 0, \pi$ for rings with 0 junction (left panel) and π junction (right panel). Left panel: $g_{\delta 0} = 0.1$, $\chi = \pi$, and (1) $g_{\ell 0} = 0.1$, (2) $g_{\ell 0} = 0.4$, (3) $g_{\ell 0} = 5$; $\chi = 0$ (dashed curve). Right panel: $g_{\delta 0} = 3$, $\chi = \pi$, and (1) $g_{\ell 0} = -1.2$, (2) $g_{\ell 0} = -1.4$, (3) $g_{\ell 0} = -1.48$; $\chi = 0$ (dashed curve).

At a given temperature, the extrema of $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, \chi)$ take the form

$$\widetilde{R}_{\min}\left(\frac{T}{T_{c0}}, \pi\right) = \frac{1}{2\pi \sqrt{\tau}} \arccos \frac{\tau - (g_{\delta 0} + 2g_{\ell 0})^2}{\tau + (g_{\delta 0} + 2g_{\ell 0})^2}, \quad (12)$$

$$\widetilde{R}_{\min}\left(\frac{T}{T_{c0}}, 0\right) = \frac{1}{2\pi \sqrt{\tau}} \arccos \frac{\tau - g_{\delta 0}^2}{\tau + g_{\delta 0}^2}. \quad (13)$$

The temperature dependence of the extrema of $\widetilde{R}_{\min}(\chi)$ near T_c is depicted in Fig. 4 for 0 junctions (the left panel) and π junctions (the right panel). For 0 junctions, the temperature-dependent minimal value $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, 0)$ is independent of $g_{\ell 0}$, being the same for $g_{\delta 0} = 0.1$ and various values of $g_{\ell 0}$ considered in the left panel in Fig. 4 (the dashed curve). The solid curves in the Fig. 4 left panel demonstrate the temperature-dependent maximum $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, \pi)$ at various values of $g_{\ell 0}$ for $g_{\delta 0} = 0.1$. For π junctions, the temperature-dependent maximum $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, 0)$ is independent of $g_{\ell 0}$, being the same for $g_{\delta 0} = 3$ and various values of $g_{\ell 0}$ considered in the right panel of Fig. 4 (the dashed curve). The solid curves in the right panel of Fig. 4 demonstrate the temperature-dependent minimum $\widetilde{R}_{\min}(\frac{T}{T_{c0}}, \pi)$, taken at various negative values of $g_{\ell 0}$ for $g_{\delta 0} = 3$.

As seen in (11), the upper bound on the shift of the critical temperature is $\tau_c \leqslant 1/(4\widetilde{R}^2)$. While $\xi(T)$ diverges at $T = T_{c0}$, it stays finite at $T = T_c < T_{c0}$. Taking jointly the upper bound obtained and the simplest condition $\tau_c \ll 1$ for the GL theory to be applied, one gets $4\widetilde{R}^2 \gg 1$, which also agrees with the applicability domain of the GL theory.

One notes, for example, that under the particular conditions $\tau_c \ll \min(1, g_{b0}^2 + g_{\ell 0}^2 \sin^2 \chi)$, the solution of Eq. (11) takes the form

$$\tau_c(\widetilde{R}, \chi) = -\frac{\Delta T_c}{T_{c0}} \approx \frac{\pi^2}{4\left(\pi \widetilde{R} + \frac{g_{b0}(\chi)}{g_{b0}^2 + g_{\ell 0}^2 \sin^2 \chi}\right)^2}. \quad (14)$$

The critical temperature shift $\tau_c(\widetilde{R}, \chi)$ as a function of the phase difference is shown in Fig. 5 for the rings with $\widetilde{R} = 5$, closed by the 0 junction (left panel) and π junction (right panel). For the 0 junction (π junction) the shift takes its maxima at
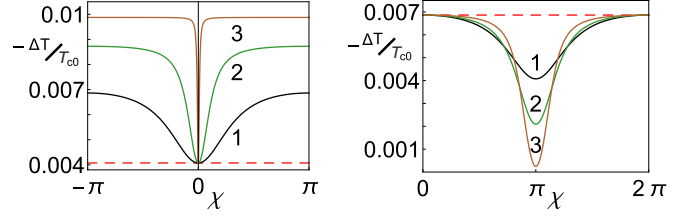

FIG. 5. $-\frac{\Delta T_c}{T_{c0}}$ as a function of χ in rings with 0 junction (left panel) and π junction (right panel); $\widetilde{R} = 5$. Left panel: $g_{\delta 0} = 0.1$ and (1) $g_{\ell 0} = 0.1$, (2) $g_{\ell 0} = 0.4$, (3) $g_{\ell 0} = 5$; $g_{\ell 0} = 0$ (dashed line). Right panel: $g_{\delta 0} = 0.3$ and (1) $g_{\ell 0} = -0.1$, (2) $g_{\ell 0} = -0.13$, (3) $g_{\ell 0} = -0.148$; $g_{\ell 0} = 0$ (dashed line).

$\chi = (2m + 1)\pi$ ($\chi = 2m\pi$) and minima at $\chi = 2m\pi$ [$\chi = (2m + 1)\pi$].

The dependence of the magnetic flux $\Phi_c(\widetilde{R}, \chi)$, taken at the modified critical temperature, can be obtained by extracting the temperature dependence in Eq. (10) and substituting for τ the solution of (11) $\tau_c(\widetilde{R}, \chi)$. This allows one to get the relative shift of the critical temperature $-\frac{\Delta T_c}{T_{c0}}$ as a periodic function of the magnetic flux $\widetilde{\Phi}$, shown in Fig. 6 for the 0 junction (the left panel) and the π junction (the right panel). As expected, the shift of T_c obtained is comparatively small but can lie beyond the fluctuation region near T_c, in a wide range of $g_{\delta 0}$ and $g_{\ell 0}$ variations.

V. RADIUS-DEPENDENT JOSEPHSON CURRENT

A noticeable dependence of the Josephson current on the ring's radius appears when r becomes close to the minimum radius. Since $r_{\min}$ depends on the phase difference, not only the critical current, but also the current-phase relation of the junction can be strongly modified, when r either exceeds the minimum radius for some of phase differences, or goes slightly over its maximum. With increasing r at $r \gtrsim \xi(T)$, the current-phase relation quickly approaches the one describing the junctions included in asymptotically large rings, or in straight long superconducting leads.

The normalized circulating supercurrent $\tilde{j} = j/j_{dp}$ is depicted as a function of the phase difference in the main panel of Fig. 7 for various ring's radii and for the 0 junction with $g_\ell = g_\delta = 0.1$. The numerical results have been obtained by carrying out the evaluation of the supercurrent (5) with the exact self-consistent formulas of Appendix A. For the set of
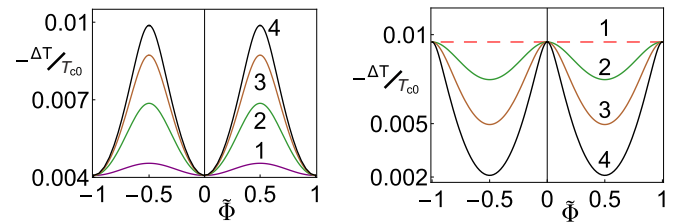

FIG. 6. $-\frac{\Delta T_c}{T_{c0}}$ as a periodic function of $\widetilde{\Phi}$ in a ring with radius $\widetilde{R} = 5$, closed by 0 junction (left panel) and π junction (right panel). Left panel: $g_{\delta 0} = 0.1$ and (1) $g_{\ell 0} = 0.01$, (2) $g_{\ell 0} = 0.1$, (3) $g_{\ell 0} = 0.4$, (4) $g_{\ell 0} = 5$. Right panel: $g_{\delta 0} = 3$ and (1) $g_{\ell 0} = 0$, (2) $g_{\ell 0} = -1.3$, (3) $g_{\ell 0} = -1.43$, (4) $g_{\ell 0} = -1.48$.
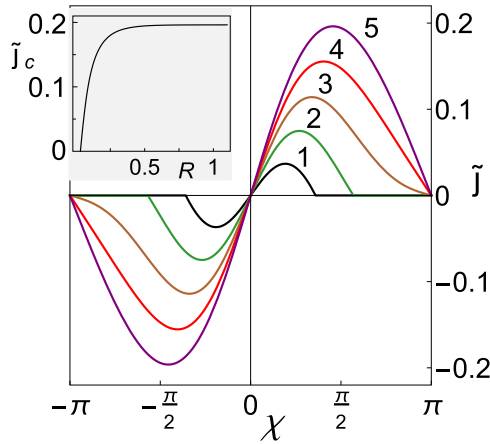
FIG. 7. $\tilde{j}(\chi)$ for 0 junction: ($g_\ell = g_\delta = 0.1$) included in the rings with various R: (1) $R = 0.05$, (2) $R = 0.07$, (3) $R = 0.1$, (4) $R = 0.16$, (5) $R \gg 1$. Inset: The critical current $\tilde{j}_c$ as a function of R.

parameters chosen, one gets from (7) $r_{\min}(0) \approx 0.0317\xi(T)$ and $r_{\min}(\pi) \approx 0.093\xi(T)$. Curves 1–3 in Fig. 7 correspond to the condition $r_{\min}(0) < r \lesssim r_{\min}(\pi)$, while curve 5 describes the current-phase relation of the same junction included in a large ring. The dependence of the critical current on the ring's radius is shown for the same set of parameters in the inset in Fig. 7. The critical current vanishes at $r = r_{\min}(0)$, while at $r \geqslant \xi(T)$ its value is quite close to the asymptotic one.

The current-phase relations of the π junction with $g_\ell = -0.1$ and $g_\delta = 0.3$, which closes the rings with the same set of radii, are shown in the main panel of Fig. 8. For curves 1–3 in Fig. 8, the π junction destroys superconductivity in the rings in a vicinity of $\chi = 2m\pi$, while the supercurrent still survives at phase differences closer to $\chi = (2m + 1)\pi$. The dependence of the critical current on the ring's radius is similar to the case of the 0 junction. For $g_\ell = -0.1$ and $g_\delta = 3$, the critical current vanishes at $r = r_{\min}(\pi) \approx 0.0317\xi(T)$, while at $r \geqslant \xi(T)$ its value is quite close to the asymptotic one.
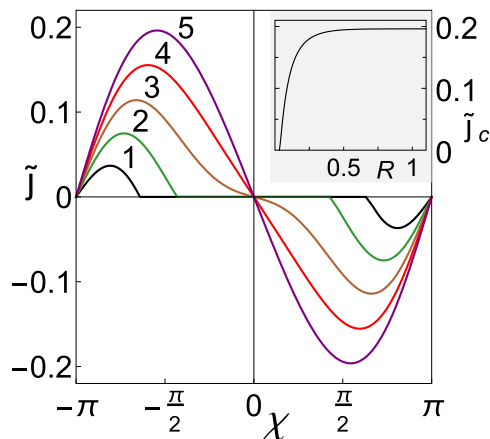

FIG. 8. $\tilde{j}(\chi)$ for π junction ($g_\ell = -0.1$ and $g_\delta = 0.3$) included in the rings with various R: (1) $R = 0.05$, (2) $R = 0.07$, (3) $R = 0.1$, (4) $R = 0.16$, (5) $R \gg 1$. Inset: The critical current $\tilde{j}_c$ as a function of R.
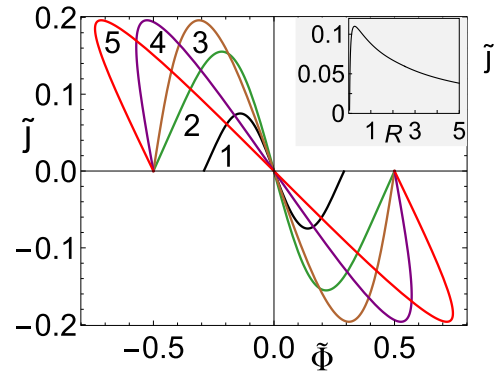

FIG. 9. $\tilde{j}(\widetilde{\Phi})$ in thin mesoscopic rings closed by 0 junction with $g_\ell = g_\delta = 0.1$, at various values of rings's radii: (1) $R = 0.07$, (2) $R = 0.16$, (3) $R = 1.03$, (4) $R = 4.00$, (5) $R = 6.50$. Inset: The supercurrent $\tilde{j}$, taken at $\widetilde{\Phi} = -0.1$, as a function of the ring's radius.

The magnetic flux dependence of the Josephson current can be found, in the case of the 0 junction, combining the phase dependence of the supercurrent shown in Fig. 7 with such a dependence of the magnetic flux given by (A10). Unlike the current-phase relation, the current-magnetic flux relation in thin superconducting rings substantially depends on the radius even at large R. In the absence of the Meissner screening, the circulation of v_s described on the right-hand side of (9) enters and can considerably change the relation between Φ and χ. As was shown in Sec. III, the circulation of v_s noticeably modifies the Φ-χ relation even at small r, i.e., quite close to the transition point $r = r_{\min}$, where the superfluid velocity does not vanish. At sufficiently large r, which enters the upper limit of the integration in (9), the "v_s term" in (9) increases $\propto r$ and has a profound influence on the Φ-χ relation. While at small r the dependence $\Phi(\chi)$ is a monotonic one within the period 2π, resulting in the single-valued inverse function $\chi(\Phi)$, in the rings with comparatively large radii a nonmonotonic dependence $\Phi(\chi)$ can appear and lead to a multivalued dependence of χ [and, therefore, of $j(\chi)$] on the full magnetic flux Φ.

The resulting supercurrent–magnetic flux relation for the 0 junction is shown in Fig. 9. The relation between the magnetic flux and the phase difference for curves 1 and 2 is almost linear, as could be expected from discussing it in Sec. III in the case of $g_\ell = g_\delta = 0.1 \ll 1$ and at a sufficiently small R. However, the ring's radii that correspond to curves 4 and 5 are already large enough for the v_s term to substantially shift the magnetic flux, when the supercurrent is comparatively large. When χ is a multiple of π, the effect of the v_s term vanishes together with v_s and j. This ultimately results in the multivalued supercurrent–magnetic flux relation, as seen in curves 4 and 5. Such a behavior implies also a nonmonotonic radius dependence of the supercurrent at a fixed magnetic flux, as seen in the inset in Fig. 9.

The magnetic flux dependence of the supercurrent flowing along the rings closed by the π junction is demonstrated in Fig. 10. Curve 1 represents the supercurrent–magnetic flux relation for the ring's radius satisfying the conditions $R_{\min}(0.5) < R < R_{\min}(0)$. Superconductivity is destroyed in
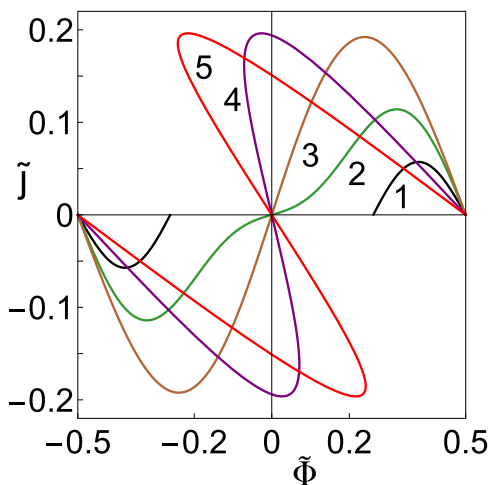
FIG. 10. $\tilde{\jmath}(\widetilde{\Phi})$ in thin mesoscopic rings closed by π junction with $g_\ell = -0.1$ and $g_\delta = 0.3$, at various values of rings's radii: (1) $R = 0.06$, (2) $R = 0.10$, (3) $R = 0.40$, (4) $R = 4.00$, (5) $R = 6.50$.

FIG. 11. $J(\widetilde{\Phi})$ at $\widetilde{R} = 5$, $g_{\ell 0} = g_{\delta 0} = 0.1$, taken at various temperatures: (1) $\tau = 0.005$, (2) $\tau = 0.006$, (3) $\tau = 0.007$, (4) $\tau = 0.008$. Inset: The multivalued current-magnetic flux relation at $g_{\ell 0} = g_{\delta 0} = 0.1$, $\widetilde{R} = 15$, and $\tau = 0.1$.

such a ring at small values of the full magnetic flux $\widetilde{\Phi}$, while it exists, and a finite supercurrent flows, in a vicinity of $\widetilde{\Phi} = 0.5$.

As known, the spontaneous supercurrent and self-magnetic-flux arise in the superconducting rings closed by the π junction [20,22–24]. The spontaneous flux is a nontrivial solution of the equation $\Phi = \frac{1}{c}LI$, where the total supercurrent is considered as a function of the magnetic flux. Since the inductance contribution is of importance for the effect, the magnetic-field term $\frac{1}{2c^2}LI^2$ should be added to the free energy of the flux-biased ring, identified in Appendix A. A nontrivial solution is, as a rule, energetically more favorable than at $\Phi = 0$. As the supercurrent vanishes at $\Phi = 0.5\Phi_0$, for sufficiently large L the solution with a comparatively small supercurrent and a flux close to half a flux quantum exists. On the contrary, there is no nontrivial solution of the equation $\Phi = \frac{1}{c}LI(\Phi)$ for sufficiently small L. When the difference $R - R_{\min}$ diminishes, the minimum inductance for the nontrivial solution to appear increases since the superconductivity region in a vicinity of half a flux quantum, as well as the supercurrent within the region, is reduced by the inverse proximity effects (see Fig. 10). Therefore, when the effects are noticeable and the temperature draws near to T_c, the spontaneous supercurrent can disappear at a sufficiently small nonzero value of $R - R_{\min}$, i.e., below T_c.

Consider now the evolution of the current–magnetic flux relation with temperature. The temperature dependence of the supercurrent $j = \tilde{\jmath} j_{\text{dp}}$ originates not only from the dimensionless radius $R = \widetilde{R}\tau^{1/2}$ and from the effective coupling constants $g_{\ell(\delta)} = g_{\ell 0(\delta 0)}/\sqrt{\tau}$, but also from the temperature dependence of the depairing current $j_{\text{dp}} \propto \tau^{3/2}$. For extracting the temperature dependence of j, it is convenient to switch over to a new dimensionless quantity $J = j/j_{\text{dp}}^{GL}(0)$, where $j_{\text{dp}}^{GL}(0)$ is the so-called zero-temperature depairing current of the GL theory $j_{\text{dp}}^{GL}(0) = \tau^{-3/2}j_{\text{dp}}$. The quantity $j_{\text{dp}}^{GL}(0)$ is known to exceed 2–3 times the real zero-temperature depairing current $j_{\text{dp}}(0)$. For example, the equality $j_{\text{dp}}(0) \approx 0.385 j_{\text{dp}}^{GL}(0)$ follows from microscopic results for the junctions, involving conventional diffusive superconductors [25].
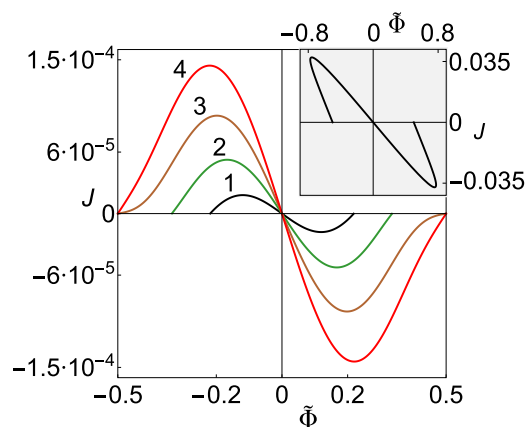
The magnetic flux dependence of J, taken near T_c at various values of τ, is shown in Fig. 11 for the ring with $\widetilde{R} = 5$ and $g_{\ell(\delta),0} = 0.1$. The supercurrent takes quite small values, as it is taken in units of $j_{\text{dp}}^{GL}(0)$ and not of j_{dp}. At substantially larger values of $\widetilde{R}$ and J, the multivalued current-magnetic flux relation can arise with varying temperature, and hence τ, as shown in the inset for the case $\tau = 0.1 \ll 1$ and $\widetilde{R} = 15$.

This paper has focused on the mean-field results within the GL theory. The current-phase relation of the junction in the superconducting ring can also be affected both by classical fluctuations of the order parameter in or close to the fluctuation region near T_c, and by quantum fluctuations, which are of importance at very low temperatures. The former case is still an open field for further study, while a number of results have already been obtained regarding the latter one [26].

VI. CONCLUSIONS

The problem of the destructive effects of the Josephson junction on superconductivity of mesoscopic or nanoscopic rings has been solved in this paper within the GL theory. The superconducting state is shown to take place when the ring's radius exceeds a minimum radius $r_{\min}$ that depends on the phase difference or the magnetic flux, as well as on the temperature and the effective Josephson and interface coupling constants. Depending on the junction transparency and/or the strength of the pair breaking by the junction interface, the minimum radius can become a noticeable fraction of the temperature-dependent coherence length, up to $0.5\xi(T)$. The superconductor–normal metal phase transition that takes place at $r = r_{\min}$ is shown to be of the second order. The magnetic flux and temperature dependence of $r_{\min}$ allow an observation of the transition under slowly varying magnetic field or temperature. The minimum radius increases when the temperature draws near to T_c, resulting in the equality $r = r_{\min}(T_c)$ at the modified critical temperature.

When the ring's radius slightly exceeds $r_{\min}$, not only the critical temperature of the superconducting state, but also the current-phase and current–magnetic flux relations can be noticeably modified by the inverse proximity effects

in the ring. The specific features of these characteristics have been determined both for 0 and π junctions, and their dependence on the ring's radius as well as on the temperature have been obtained. A substantial evolution of the magnetic flux dependence of the Josephson current with increasing radius of thin rings has been demonstrated to persist even at large ring's radii and result in a multivalued behavior, while the current-phase dependence stays almost unchanged at $r \gtrsim \xi(T)$, approaching the one that describes the junctions in asymptotically large rings. The identified multivaluedness of the supercurrent's magnetic flux dependence is related to the presence of the corresponding equilibrium and metastable states and to possible transitions between them, which result in the hysteretic behavior of the supercurrent. The hysteretic properties, however, lie outside the scope of this paper. They appear in the rings with comparatively large radii and take place irrespective of the proximity effects' strength, while the paper mainly concerns itself with the proximity-induced effects in the rings of smaller sizes.

Currents flowing in individual nanoscopic and mesoscopic superconducting rings can be experimentally determined using a number of methods. In addition to methods based on electrical transport measurement technique employing direct electrical contacts with the system [4], there also are the noninvasive ones that use micromechanical torsional magnetometers [27,28] or measure the ring's susceptibility [29,30]. The present sensitivity and accuracy of the experimental techniques as well as modern technological developments for fabrication of superconducting nanorings, nanocylinders, and nano-SQUIDs [4,31] represent the basis for possible observations of the theoretical predictions of this paper.

<div align="center">ACKNOWLEDGMENT</div>

APPENDIX A: SOLUTIONS OF THE GL EQUATIONS

The parameter $\mathcal{E}$, related to the first integral of the GL equation (3) and defined as

$$\mathcal{E} = \left(\frac{df(x)}{dx}\right)^2 + \frac{i^2}{f^2(x)} + f^2(x) - \frac{1}{2}f^4(x), \qquad \text{(A1)}$$

is spatially constant, when taken for the solutions $f(x)$.

Taking $x = 0$ in (A1) and making use of (4) and (5), one can express $\mathcal{E}$ via f_0 and the parameters of the superconducting ring closed by the junction:

$$\mathcal{E} = \left(1 + g_b^2 + g_\ell^2 \sin^2\chi\right)f_0^2 - \frac{1}{2}f_0^4. \qquad \text{(A2)}$$

Equation (A1) can be also rewritten in the form

$$\left(\frac{df}{dx}\right)^2 = \frac{1}{2f^2}(f^2 - f_+^2)(f^2 - f_d^2)(f^2 - f_-^2). \qquad \text{(A3)}$$

The quantities $t_- = f_-^2$, $t_d = f_d^2$, and $t_+ = f_+^2$ satisfy the following set of equations:

$$t_- + t_d + t_+ = 2, \qquad t_- t_d t_+ = 2i^2,$$

$$t_d t_- + t_d t_+ + t_- t_+ = 2\mathcal{E}. \qquad \text{(A4)}$$

Apart from the boundary and periodic conditions, the solution of Eq. (A3) is characterized by three formal extrema f_-, f_d, f_+ with vanishing first derivatives $\frac{df}{dx}$. When the ring closed by the junction is in the equilibrium, only one of the extrema represents an actual (maximum) value of $f(x)$ in the ring, while the other two are just auxiliary quantities. Indeed, a periodic inhomogeneous solution $t(x) = f^2(x)$ with only a single minimum and a single maximum in the ring should energetically be the most favorable one. The minimum $t_0 = f_0^2$ is induced at the junction interface $x = 0$ by the pair-breaking effects. In contrast to the extrema described by (A3) and (A4), the derivative at $x = 0$ is nonzero and discontinuous in accordance with the boundary conditions (4). In the equilibrium, one expects the maximum of $t(x)$ to coincide with one of the roots of the right-hand side of (A3) (let it be t_d), and to be realized at the point $x = \pi R$ diametrically opposed to $x = 0$. In general, either all three roots t_-, t_d, and t_+ take real values, or only one is real and the other two are the complex conjugate of each other. As the left-hand side of (A3) takes nonnegative values, one concludes that the minimum that does not actually show up in the ring (let it be t_-) has to be real in both cases. Since one of the other two extrema is the real quantity t_d, both of them also have to be real.

Let $t(x)$ have a minimum t_0 at $x = 0$ and a maximum t_d at $x = \pi R$. A nonnegative value of the left-hand side of (A3) entails the existence of one more maximum t_+. Assuming that $t_- \leqslant t_0 \leqslant t(x) \leqslant t_d \leqslant t_+$, the solution of (A3) can be represented in the region $|x| \leqslant \pi R$ as

$$|x| = \sqrt{\frac{2}{t_+ - t_-}}\left[F\left(\arcsin\sqrt{\frac{t - t_-}{t_d - t_-}}\,\middle|\,\frac{t_d - t_-}{t_+ - t_-}\right)\right.$$

$$\left. - F\left(\arcsin\sqrt{\frac{t_0 - t_-}{t_d - t_-}}\,\middle|\,\frac{t_d - t_-}{t_+ - t_-}\right)\right]. \qquad \text{(A5)}$$

Here $F(\varphi|m)$ is the elliptic integral of the first kind. The notations of arguments of elliptic integrals vary in literature and here the definitions of the Mathematica book are used [32]. Making use of the addition theorem for $F(\varphi|m)$ [33], the solution (A5) can be rewritten in the form

$$|x| = \sqrt{\frac{2}{t_+ - t_-}}\,F\left(\psi\,\middle|\,\frac{t_d - t_-}{t_+ - t_-}\right), \qquad \text{(A6)}$$

where

$$\sin\psi = \frac{\sqrt{(t - t_-)(t_d - t_0)(t_+ - t_0)} - \sqrt{(t_0 - t_-)(t_d - t)(t_+ - t)}}{(t_d - t_-)(t_+ - t_-) - (t - t_-)(t_0 - t_-)}\sqrt{t_+ - t_-}. \qquad \text{(A7)}$$

The solution (A6), (A7) contains four parameters t_-, t_d, t_+, and t_0. They are linked to each other by the system of three equations (A4), where expressions (5) and (A2) should be substituted for $\mathcal{E}$ and i. The fourth equation is obtained taking $x = \pi R$ and $t(\pi R) = t_d$ in (A6):

$$\pi R = \sqrt{\frac{2}{t_+ - t_-}} F\left(\arcsin \sqrt{\frac{(t_+ - t_-)(t_d - t_0)}{(t_d - t_-)(t_+ - t_0)}} \middle| \frac{t_d - t_-}{t_+ - t_-} \right).$$

(A8)

When the relation (A8) is taken into account, the solution (A6) and (A7) is simplified and can be reduced to the form

$$|x| = \pi R - \sqrt{\frac{2}{t_+ - t_-}} F\left(\arcsin \sqrt{\frac{(t_+ - t_-)(t_d - t)}{(t_d - t_-)(t_+ - t)}} \middle| \frac{t_d - t_-}{t_+ - t_-} \right), \quad |x| \leqslant \pi R.$$

(A9)

A joint solution of (A4) and (A8) represents t_-, t_0, t_d, and t_+, as well as the whole of the inhomogeneous profile of the order parameter (A9), as functions of the phase difference χ and of the dimensionless radius of the ring $R = r/\xi(T)$.

In the case of a flux-biased ring the relationship (9) between the phase difference and the magnetic flux should be used. The integration on the right-hand side of (9) can be carried out after inserting the order parameter profile (A9). The result contains the elliptic integral of the third kind and takes the form

$$\widetilde{\Phi} - n = -\left\{ \frac{\chi}{2\pi} + i\left[\frac{1}{t_+} R + \frac{\sqrt{2}(t_+ - t_d)}{\pi t_+ t_d \sqrt{t_+ - t_-}} \Pi\left(\frac{t_+(t_d - t_-)}{t_d(t_+ - t_-)}; \arcsin \sqrt{\frac{(t_+ - t_-)(t_d - t_0)}{(t_d - t_-)(t_+ - t_0)}} \middle| \frac{t_d - t_-}{t_+ - t_-} \right) \right] \right\}.$$

(A10)

When the solutions of (A4), (A8) for t_-, t_d, t_+, and t_0, along with the second expression in (5) for the supercurrent, are inserted on the right-hand side of (A10), the latter represents the quantity $\widetilde{\Phi} - n$ as a function of χ and R. As follows from (A10), the magnetic flux equals an integral (a half-integral) number, when the phase difference is an even (odd) multiple of π, since the supercurrent vanishes at $\chi = m\pi$. Since the change of the winding number by one is, by definition, the change of the order-parameter phase $\phi(x)$ by 2π after it has gone around the loop, the changes of χ by a multiple of 2π and of the winding number by an integer are unambiguously interrelated in (A10). The 2π-periodic dependence of t_-, t_d, t_+, and t_0 on χ that follows from (A4), (A8) and (5), (A2) should be noted here.

If the critical current is small as compared to the depairing current and the ring's radius satisfies the condition $R \lesssim 1$, the quantity $\widetilde{\Phi} - n$ in (A10), as a function of χ and R, can be inversed resulting in a single-valued function $\chi(\widetilde{\Phi} - n, R)$. This allows one to obtain all the quantities as functions of $\widetilde{\Phi} - n$ and R. Similarly to the unbroken rings, the dependence of the winding number on the magnetic flux should be determined from the minimization of thermodynamic potential at a fixed $\widetilde{\Phi}$. In this case one gets the physical quantities in the equilibrium state as periodic functions of $\widetilde{\Phi}$ with unit period: $\chi(\widetilde{\Phi} + 1) = \chi(\widetilde{\Phi}) + 2\pi$.

While the superconducting state in the ring closed by the junction is inevitably inhomogeneous unless the effective coupling constants g_ℓ and g_δ vanish, the equilibrium state of the unbroken cylindrically symmetric thin ring is characterized by the spatially constant absolute value of the order parameter. However, an inhomogeneous profile of the order parameter can arise as an unstable (metastable) state of the uninterrupted rings [34–36]. Such a nonuniform solution follows from (A8)–(A10) in the limit $t_0 \to t_-$ and $\chi \to 0$ [37]. For the unbroken rings, the three extrema t_-, t_d, and t_+ can be calculated, as functions of $\widetilde{\Phi}$ and R, based on the first equation in (A4), as well as on (A8) and (A10). The inhomogeneous solution does not always exist in the unbroken axially symmetric rings, for example, at sufficiently small magnetic fluxes, since the condition $\chi \equiv 0$ makes the equation (A10) to be substantially more restrictive than in the case of rings with a junction. The junction breaks the ring's axial symmetry, which modifies the nonuniform solution and stabilizes it. Some other examples of its stabilization in rings with the broken symmetry have been discussed earlier [38–40].

The spatial integration can also be taken analytically in the expression for the bulk thermodynamic potential, with the solution (A9) of the GL equation inserted in (1). As a result, one gets the free energy in the form

$$\widetilde{\mathcal{F}} = g_b t_0 - \frac{\sqrt{2}}{3}\sqrt{(t_+ - t_0)(t_d - t_0)(t_0 - t_-)} - \frac{\pi R}{3}\left[2t_+ - \left(1 + g_b^2 + g_\ell^2 \sin^2 \chi\right)t_0 + \frac{1}{2}t_0^2 \right]$$

$$+ \frac{2\sqrt{2}}{3}\left[\sqrt{t_+ - t_-}\, E\left(\arcsin \sqrt{\frac{(t_+ - t_-)(t_d - t_0)}{(t_d - t_-)(t_+ - t_0)}} \middle| \frac{t_d - t_-}{t_+ - t_-} \right) - \sqrt{\frac{(t_d - t_0)(t_0 - t_-)}{(t_+ - t_0)}} \right],$$

(A11)

where $\widetilde{\mathcal{F}} = (b\mathcal{F})/(2K^{1/2}|a|^{3/2})$, $\mathcal{F} = \mathcal{F}_b + \mathcal{F}_{\text{int}}$, and $E(\varphi|m)$ is the elliptic integral of the second kind. Similarly to Eq. (A10) for the magnetic flux, the free energy (A11) is given as a function of χ and R, as the quantities t_-, t_d, t_+, and t_0 are the solutions of (A4) and (A8).

Thermodynamic potential (A11) takes into account both the bulk and the proximity-modified interface contributions. Near the transition point $R = R_{\text{min}}$ the two contributions strongly compete with each other and the result is described by formula (8) of the paper.

Since a joint solution of Eqs. (A4), (A8), and (A10) allows one to get the supercurrent (5) as a function of R and of the full magnetic flux Φ, the relation

$$\Phi_e = \Phi - \frac{1}{c} L I \qquad (A12)$$

gives in this case the applied magnetic flux $\Phi_e(\Phi, R)$. Here $I = \mathcal{A} j$ is the total supercurrent, $\mathcal{A}$ is the cross section's area, and L is the inductance. As discussed in Sec. I, the inductance effects can be safely ignored whenever $R - R_{\min}$ is sufficiently small. The main focus of the paper is on a relatively small $R - R_{\min}$, when the self-field effects can be mostly disregarded.

APPENDIX B: DERIVATION OF MINIMUM RADIUS

An effect of the ring's size R on the inhomogeneous solution of the GL equation is described by Eq. (A8). Were the first argument of the elliptic integral on the right-hand side of (A8) arbitrarily small at a finite $t(x)$, the equation (A8) would allow superconductivity in the rings with very small radii, on the scale of the GL theory. On account of the conditions $0 \leqslant t_- \leqslant t_0 \leqslant t(x) \leqslant t_d \leqslant t_+$, the first argument vanishes only at $t_d = t_0$, i.e., for the uniform order parameter. However, such a profile is incompatible with the proximity-induced boundary conditions (4) unless the effective coupling constants g_ℓ, g_δ vanish. Indeed, substituting the equality $t_d = t_0$, as well as (5) and (A2), in (A4), one gets the homogeneous normal-metal state, where $t_d = t_0 = t_- = 0$ and $t_+ = 2$.

In order to obtain the minimum radius $R_{\min}$, one should find not only the limits of the individual extrema t_d, t_-, and t_+ at the transition ($t_0 \to 0$), but also of the combinations of these quantities, which form the arguments of the elliptic integral in (A8). Small deviations from the individual limits have to be considered for this purpose. As seen in (A2) and (5), both $\mathcal{E}$ and

i are the small parameters, when the minimum of $t(x)$ is small, $t_0 \ll 1$. It is the case, when the radius of a superconducting ring only slightly exceeds the minimum radius. Up to the first-order terms in $\mathcal{E}$ and i, the solutions of Eq. (A4) are

$$t_{d(-)} = \frac{\mathcal{E}}{2} \pm \sqrt{\frac{\mathcal{E}^2}{4} - i^2}, \quad t_+ = 2 - \mathcal{E}. \qquad (B1)$$

It follows from (B1), as well as from the conditions $t_0 \leqslant t(x) \leqslant t_d$, that superconductivity is destroyed throughout the ring simultaneously, when the minimal value of the order parameter f_0 vanishes. The absence of an isolated phase slip center $f_0 = 0$ at $x = 0$ is a direct consequence of the boundary conditions (4).

Substituting expressions (A2) and (5) for $\mathcal{E}$ and i in (B1), one finds in the limit $t_0 \to 0$ that the second argument of the elliptic integral in (A8) vanishes while the first argument remains finite. The elliptic integral of the first kind coincides with its first argument under such conditions, so that (A8) transforms into Eq. (7) for the minimum radius.

If the phase difference χ across the junction is controlled by the magnetic flux penetrating the superconducting ring, the minimum radius actually depends on $\widetilde{\Phi}$, rather than on χ. The relation (A10) between the quantities χ and $\widetilde{\Phi}$ substantially simplifies at $R = R_{\min}(\chi)$. Taking the limit $t_0 \to 0$ in (A10), one can make the substitution $t_0 = t_- = t_d = 0$ and $t_+ = 2$ everywhere except for the extrema combinations, the calculation of which requires taking account of small deviations from the presented individual limits. On top of that, the expression (5) for the supercurrent and the relation $\Pi(n; \varphi | 0) = \frac{1}{\sqrt{1-n}} \arctan(\sqrt{1-n} \tan \varphi)$ should be used. The remaining combinations of quantities t_d, t_0, and t_- can be found in the limit $t_0 \to 0$ using (B1), (A2), and (5). As a result, one obtains Eq. (10) of the paper.

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